5.5 Déjà vu: two applications revisited

We return to the applications from the theory of branching processes and from collective risk theory discussed in Sections 1.4 and 1.5, respectively. As it turns out, we are now able to provide asymptotic estimates of the relevant quantities under more general model assumptions.

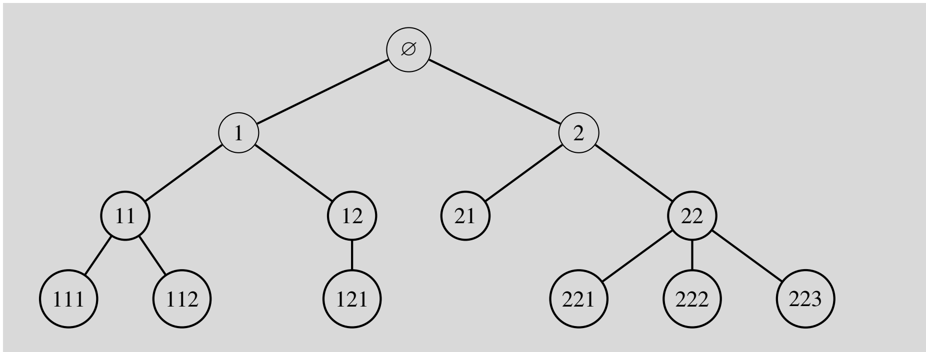

Fig. 5.1 A random tree with Ulam-Harris labeling.

5.5.1 Age-dependent branching processes

Consider a population of individuals originating from one ancestor born at time 0. The individuals are labeled by elements of $\mathbb{V} := \{\varnothing\} \cup \bigcup_{n\ge 1} \mathbb{N}^n$, called *Ulam-Harris tree*, in the following way so as to give full information about the genealogy. Use $\mathsf{v}_1...\mathsf{v}_n$ as shorthand notation for $(\mathsf{v}_1,...,\mathsf{v}_n) \in \mathbb{N}^n$. The ancestor gets label $\varnothing$ (the root) and any member of the n^{th} generation $(n \ge 1)$ is labeled by an element of $\mathbb{N}^n$ subject to the following constraints: If label $\mathsf{v}_1...\mathsf{v}_n$ has been assigned to a population member, then the same holds true for $\mathsf{v}_1...\mathsf{v}_{n-1}$ (the mother) as well as any $\mathsf{v}_1...\mathsf{v}_{n-1}k$ for $1 \le k < \mathsf{v}_n$ (siblings) [☞ Fig. 5.1]. Let $\{(T_\mathsf{v}, N_\mathsf{v}) : \mathsf{v} \in \mathbb{V}\}$ be a family of iid random vectors taking values in $\mathbb{R}_> \times \mathbb{N}_0$ and interpret T_v as the random lifetime of the (potential) individual v, while N_v denotes its number of children. Both variables are assumed to have finite mean. We further assume, although this could be relaxed, that all children are born at the end of the mother's lifetime.

As in Section 1.4, we are interested in the population size process $(Z(t))_{t\ge 0}$, called *age-dependent branching process* which satisfies [compare (1.16)]

$$Z(t) \;:=\; \mathbf{1}_{\{T_\varnothing > t\}} \;+\; \mathbf{1}_{\{T_\varnothing \le t\}} \sum_{k=1}^{N_\varnothing} Z_k(t - T_\varnothing), \quad t \ge 0, \tag{5.21}$$

where the $(Z_k(t))_{t\ge 0}$, $k \ge 1$, are iid copies of $(Z(t))_{t\ge 0}$ and independent of $(T,N) := (T_\varnothing, N_\varnothing)$. They represent the subpopulation size processes pertaining to the (potential) children of $\varnothing$. As a consequence of (5.21), the mean function $M(t) := \mathbb{E}Z(t)$ satisfies the standard renewal equation

$$\begin{aligned} M(t) \;&=\; \mathbb{P}(T > t) + \int_{[0,t]} \sum_{n\ge 0} n M(t-s)\, \mathbb{P}(N = n | T = s)\, \mathbb{P}(T \in ds) \\ &=\; \mathbb{P}(T > t) + \int_{[0,t]} M(t-s)\, h(s)\, \mathbb{P}(T \in ds) \\ &=\; \mathbb{P}(T > t) + \int_{[0,t]} M(t-s)\, F(ds), \quad t \ge 0 \end{aligned}$$

with $h(s) := \mathbb{E}(N|T = s)$ for $s \ge 0$ and $F(ds) := h(s)\mathbb{P}(T \in ds)$, the latter being clearly admissible and having total mass $\|F\| = \mathbb{E}N$. The asymptotic behavior of $M(t)$ as $t \to \infty$ is now easily derived with the help of Theorem 5.7 and depends on whether $\mathbb{E}N < 1$, $= 1$, or > 1. Owing to a quite different behavior, the age-dependent branching process $(Z(t))_{t\ge 0}$ is called *subcritical* in the first case, *critical* in the second, and *supercritical* in the last one. Suppose that F has characteristic exponent ϑ, called *Malthusian parameter* in the present context and given as the unique positive number such that

$$\|F_\vartheta\| \;=\; \int e^{\vartheta s}\, F(ds) \;=\; \mathbb{E}Ne^{\vartheta T} \;=\; 1.$$

Note that this assumption can only fail in the subcritical case and is trivially true with $\vartheta = 0$ if $\|F\| = 1$. In the supercritical case, ϑ always exists and is negative by Rem. 5.8. Here is the result, where we restrict ourselves to the case when F is nonarithmetic.

Proposition 5.18. *Let $(Z(t))_{t\ge 0}$ be an age-dependent branching process as described before such that $\mathbb{E}T < \infty$, $\mathbb{E}N < \infty$ and $F(ds) = \mathbb{E}(N|T = s)\mathbb{P}(T \in ds)$ is nonarithmetic with characteristic exponent ϑ. Then its mean function $M(t)$ satisfies*

$$\lim_{t\to\infty} e^{\vartheta t} M(t) \;=\; \begin{cases} \dfrac{\mathbb{E}e^{\vartheta T} - 1}{\vartheta\,\mathbb{E}NTe^{\vartheta T}} & \text{in the subcritical case } (\vartheta > 0),\\[2mm] \dfrac{\mathbb{E}T}{\mathbb{E}NT} & \text{in the critical case } (\vartheta = 0),\\[2mm] \dfrac{1 - \mathbb{E}e^{\vartheta T}}{|\vartheta|\,\mathbb{E}NTe^{\vartheta T}} & \text{in the subcritical case } (\vartheta < 0).\end{cases}$$

with the usual convention that the right-hand side equals 0 *if* $\mathbb{E}NTe^{\vartheta T} = \infty$.

Proof. The result is a direct consequence of Thm. 5.7(b), the standard formula

$$\int_0^\infty e^{\vartheta x}\,\mathbb{P}(T > x)\,dx \;=\; \begin{cases} \vartheta^{-1}\mathbb{E}(e^{\vartheta T} - 1), & \text{if } \vartheta > 0,\\ \mathbb{E}T, & \text{if } \vartheta = 0,\\ |\vartheta|^{-1}\mathbb{E}(1 - e^{-\vartheta T}), & \text{if } \vartheta < 0,\end{cases}$$

that follows upon integration by parts [☞ A.1], and Lemma 4.3 which ensures the direct Riemann integrability of $x \mapsto e^{\vartheta x}\,\mathbb{P}(T > x)$ on $\mathbb{R}_\ge$ for $\vartheta \ne 0$. □

Remark 5.19. The limits in the above proposition simplify a little if N and T are further assumed to be independent, in which case $(Z(t))_{t\ge 0}$ is also called *Bellman-Harris process*. Namely, we then have $\mathbb{E}NTe^{\vartheta T} = \mathbb{E}N\,\mathbb{E}Te^{\vartheta T}$, giving limit 1 in the critical case. The cell-division process discussed in Section 1.4 is a special example for this situation that even led us to explicit results for $M(t)$ for standard exponentially distributed T [☞ (1.19)].

Another quantity of interest for the given model is the expectation $M^a(t)$, say, of the number $Z^a(t)$ of individuals alive at time t and not older than a for arbitrary $a > 0$. Since

$$Z^a(t) \;=\; \mathbf{1}_{\{t<T\le a\}}\mathbf{1}_{[0,a)}(t) + \sum_{k=1}^{N} Z_k^a(t-T)\mathbf{1}_{\{T\le t\}}, \quad t \ge 0, \tag{5.22}$$

as one can readily see (with $Z_k^a(t)$ having the obvious meaning), the mean function $M^a(t) = \mathbb{E}Z^a(t)$ satisfies the renewal equation

$$M^a(t) \;=\; \mathbb{P}(t < T \le a)\,\mathbf{1}_{[0,a)}(t) + \int_0^t M^a(t-s)\,F(ds)$$

of similar type as $M(t)$. The following result is therefore obtained in the same manner as the previous proposition and stated without proof.

Proposition 5.20. *Given the assumptions of Prop. 5.18, the function $M^a(t)$ satisfies for each $a > 0$*

$$\lim_{t\to\infty} e^{\vartheta t} M^a(t) \;=\; \begin{cases} \dfrac{\mathbb{E}(e^{\vartheta T}-1)\mathbf{1}_{\{T\le a\}}}{\vartheta\,\mathbb{E}NTe^{\vartheta T}} & \text{in the subcritical case } (\vartheta > 0), \\[2ex] \dfrac{\mathbb{E}T\mathbf{1}_{\{T\le a\}}}{\mathbb{E}NT} & \text{in the critical case } (\vartheta = 0), \\[2ex] \dfrac{\mathbb{E}(1-e^{\vartheta T})\mathbf{1}_{\{T\le a\}}}{|\vartheta|\,\mathbb{E}NTe^{\vartheta T}} & \text{in the subcritical case } (\vartheta < 0). \end{cases}$$