Gerold Alsmeyer

Random Recursive Equations and Their Distributional Fixed Points

June 17, 2011

Springer

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cdf	(cumulative) distribution function
chf	characteristic function
CLT	central limit theorem
dRi	directly Riemann integrable
FT	Fourier transform
GWP	Galton-Watson process
gf	generating function
iff	if, and only if
iid	independent and identically distributed
i.o.	infinitely often
LT	Laplace transform
MC	Markov chain
mgf	moment generating function
RDE	random difference equation
RP	renewal process
RTP	recursive tree process
RW	random walk
SFPE	stochastic fixed point equation
SLLN	strong law of large numbers
SRP	standard renewal process = zero-delayed renewal process
SRW	standard random walk = zero-delayed random walk
ui	uniformly integrable
WLLN	weak law of large numbers

Symbols

$Bern(\theta)$	Bernoulli distribution with parameter $\theta \in (0,1)$
$\beta(a,b)$	Beta distribution with parameters $a,b \in \mathbb{R}_>$
$\beta^*(a,b)$	Beta distribution of the second kind with parameters $a,b \in \mathbb{R}_>$
$Bin(n,\theta)$	Binomial distribution with parameters $n \in \mathbb{N}$ and $\theta \in (0,1)$
δ_a	Dirac distribution in a
$Exp(\theta)$	Exponential distribution with parameter $\theta \in \mathbb{R}_>$
$\Gamma(\alpha,\beta)$	Gamma distribution with parameters $\alpha,\beta \in \mathbb{R}_>$
$Geom(\theta)$	Geometric distribution with parameter $\theta \in (0,1)$
$NBin(n,\theta)$	Negative binomial distribution with parameters $n \in \mathbb{N}$ and $\theta \in \mathbb{R}_>$
$Normal(\mu,\sigma^2)$	Normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_>$
$Poisson(\theta)$	Poisson distribution with parameter $\theta \in \mathbb{R}_>$
$\mathscr{S}(\alpha,b)$	Symmetric stable law with index $\alpha \in (0,2]$ and scaling parameter $b \in \mathbb{R}_>$
$\mathscr{S}_+(\alpha,b)$	One-sided stable law with index $\alpha \in (0,1]$ and scaling parameter $b \in \mathbb{R}_>$
$Unif(a,b)$	Uniform distribution on $[a,b]$, $a < b$

Chapter 1
Introduction

Probabilists are often facing the task to determine the asymptotic behavior of a given stochastic sequence $(X_n)_{n\geq 0}$, more precisely, to prove its convergence (in a suitable sense) to a limiting variable X_∞, as $n\to\infty$, and to find or at least provide information about the distribution (law) of X_∞, denoted as $\mathscr{L}(X_\infty)$. Of course, there is no universal approach to accomplish this task, but in situations where $(X_n)_{n\geq 0}$ exhibits some kind of recursive structure, expressed in form of a so-called *random recursive equation*, one is naturally prompted to take advantage of this fact in one way or another. Often, one is led to a *distributional equation* for the limit variable X_∞ of the form

$$\mathscr{L}(X_\infty) \;=\; \mathscr{L}\left(\Psi(X_\infty(1),X_\infty(2),...)\right) \tag{1.1}$$

where $X_\infty(1),X_\infty(2),...$ are independent copies of X_∞ and Ψ denotes a random function independent of these variables. (1.1) constitutes the general form of a so-called *stochastic fixed-point equation (SFPE)*, also called *recursive distributional equation* by ALDOUS & BANDYOPADHYAY in [1]. The distribution of X_∞ is then called a *solution to the SFPE* (1.1).

To provide an introduction of a collection of interesting equations of this kind, some of them related to very classical problems in probability theory, and of the methods needed for their analysis is the main goal of this course. The present chapter is devoted to an informal discussion of a selection of examples that will help the reader to gain a first impression of what is lying ahead.

1.1 A true classic: the central limit problem

Every student with some basic knowledge in theoretical probability knows that, given a sequence of iid real-valued random variables $X,X_1,X_2,...$ with mean 0 and variance 1, the associated sequence of standardized partial sums

$$S_n^* \;:=\; \frac{X_1+...+X_n}{n^{1/2}},\quad n\geq 1$$

converges in distribution to a standard normal random variable Z as $n \to \infty$. This is the classic version of the *central limit theorem (CLT)* and most efficiently proved by making use of characteristic functions (Fourier transforms). Namely, let ϕ denote the chf of X and note that a second order Taylor expansion of ϕ at 0 gives

$$\phi(t) \; = \; 1 - \frac{t^2}{2} + o(t^2) \quad \text{as } t \to 0.$$

Since S_n^* has chf $\psi_n(t) = \phi(n^{-1/2}t)^n$, we now infer

$$\lim_{n\to\infty} \psi_n(t) \; = \; \lim_{n\to\infty} \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{2n}\right)\right)^n \; = \; e^{-t^2/2}$$

for all $t \in \mathbb{R}$ and thus the asserted convergence by Lévy's continuity theorem combined with the fact that $e^{-t^2/2}$ is the chf of the standard normal distribution.

Having solved the central limit problem for good in the classical setup of iid random variables the reader may wonder so far about its connection with random recursive equations. Let us therefore narrow our perspective by assuming that the weak convergence of $\mathscr{L}(S_n^*)$ to a limit law Q with mean 0 and unit variance has already been settled. Then the problem reduces to giving an argument that shows that Q must be the standard normal distribution. To this end, we make the observation that

$$S_{2n}^* \; = \; \frac{S_n^* + S_{n,n}^*}{2^{1/2}}, \tag{1.2}$$

where $S_{n,n}^* := n^{-1/2}(X_{n+1} + \ldots + X_{2n})$ for $n \geq 1$. Since $S_{n,n}^*$ is an independent copy of S_n^*, it follows that

$$\left(S_n^*, S_{n,n}^*\right) \; \stackrel{d}{\to} \; (Z, Z') \quad \text{as } n \to \infty$$

for two independent random variables Z, Z' with common distribution Q and then from (1.2), by the continuous mapping theorem, that

$$Z \; \stackrel{d}{=} \; \frac{Z + Z'}{2^{1/2}}, \tag{1.3}$$

where $\stackrel{d}{=}$ means equality in distribution. In terms of the chf φ, say, of Z, this equation becomes

$$\varphi(t) \; = \; \varphi\left(\frac{t}{2^{1/2}}\right)^2, \quad t \in \mathbb{R}, \tag{1.4}$$

which via iteration leads to

$$\varphi(t) \; = \; \lim_{n\to\infty} \varphi\left(\frac{t}{2^{n/2}}\right)^{2^n} \; = \; \lim_{n\to\infty} \left(1 - \frac{t^2}{2^{n+1}} + o\left(\frac{t^2}{2^{n+1}}\right)\right)^{2^n} \; = \; e^{-t^2/2},$$

for all $t \in \mathbb{R}$, when noting that φ satisfies the same Taylor expansion as ϕ given above. Hence we have proved that Q is the standard normal law.

The random recursive equation (1.2) that has worked for us here may also be written as

$$S_{2n}^* = \Psi(S_n^*, S_{n,n}^*) \quad \text{with} \quad \Psi(x,y) := \frac{x+y}{2^{1/2}}.$$

Although stated in terms of random variables, it should be noticed that its recursive property is rather in terms of distributions: The distribution of S_{2n}^* is expressed as a functional of the distribution of S_n^* (recalling that $S_{n,n}^*$ is an independent copy of this variable). Then, by taking the limit $n \to \infty$ and using the continuity of Ψ, the limiting distribution has been identified as a solution to the SFPE (1.3), viz.

$$Z \stackrel{d}{=} \Psi(Z, Z') = \frac{Z+Z'}{2^{1/2}}.$$

Under the proviso that Z (or Q) has mean 0 and unit variance, we have shown that the standard normal distribution forms the unique solution to (1.3). We note in passing that, by a simple scaling argument, $Normal(0, \sigma^2)$, the normal distribution with mean 0 and variance $\sigma^2 > 0$, is found to be the unique solution to the very same SFPE within the class of distributions with mean 0 and variance σ^2.

Generalizing in another direction, fix any $N \geq 2$ and positive integers $k_1, ..., k_N$ satisfying $k_1 + ... + k_N = N$. Define $s_1 := k_1, s_2 := k_1 + k_2, ..., s_N := k_1 + ... + k_N$ and then

$$S_n^*(j) := \frac{1}{(k_j n)^{1/2}} \sum_{m=s_{j-1}n+1}^{s_j n} X_m$$

for $j = 1, ..., N$. The latter random variables are clearly independent with $\mathscr{L}(S_n^*(j)) = \mathscr{L}(S_{k_j n}^*)$ for each $j = 1, ..., N$. Moreover, the random recursive equation

$$S_{Nn}^* = \sum_{j=1}^N T_j S_n^*(j) \tag{1.5}$$

with $T_j := (k_j/N)^{1/2}$ for $j = 1, ..., N$ holds true. Hence, the distribution of S_{Nn}^* is a functional of the distributions of $S_{k_1 n}^*, ..., S_{k_N N}^*$. By another appeal to the continuous mapping theorem, we obtain upon passing to the limit $n \to \infty$ that (under the same proviso as before)

$$Z \stackrel{d}{=} \sum_{j=1}^N T_j Z_j \tag{1.6}$$

where $Z_1, ..., Z_N$ are independent copies of Z. Equivalently,

$$\varphi(t) = \prod_{j=1}^N \varphi(T_j t), \quad t \in \mathbb{R}$$

holds for the chf φ of Z, and a similar argument as before may be employed to conclude that the standard normal law forms the unique solution to (1.6) within the class of distributions with mean 0 and unit variance. We close this section with the following natural question:

> *Under which conditions on* $(N,T_1,\ldots,T_N)$*, the parameters of the SFPE* (1.6)*, does the previous uniqueness statement remain valid?*

The restriction imposed by our construction is that N is finite and that $T_1^2,\ldots,T_N^2$ are positive rationals summing to unity. The last property is clearly necessary, for (1.6) in combination with $\mathbb{V}\mathrm{ar}\,Z=1$ entails

$$1\ =\ \mathbb{V}\mathrm{ar}\,Z\ =\ \sum_{j=1}^N \mathbb{V}\mathrm{ar}\,(T_jZ_j)\ =\ \sum_{j=1}^N T_j^2.$$

That no further restriction on $(N,T_1,\ldots,T_N)$ is needed will be shown in ??????? in a more general framework. This means that N may even be infinite and $T_1,T_2,\ldots,T_N$ any real numbers such that $\sum_{j=1}^N T_j^2=1$.

Problems

Problem 1.1. For any $\alpha\in(0,2]$ and $b>0$, the function $\phi(t)=\exp(-b|t|^\alpha)$ is the chf of a (symmetric) distribution $\mathscr{S}(\alpha,b)$ on $\mathbb{R}$, called *symmetric stable law with index* α *and scaling parameter* b. Note that $\mathscr{S}(2,b)=\mathit{Normal}(0,2b)$ and $\mathscr{S}(1,b)=\mathit{Cauchy}(b)$, the symmetric Cauchy distribution with $\lambda\!\!\lambda$-density $\frac{1}{\pi}\frac{b}{b^2+x^2}$. Prove that $\mathscr{S}(\alpha,b)$ forms a solution to the SFPE

$$X\ \stackrel{d}{=}\ \frac{X_1+\ldots+X_n}{n^{1/\alpha}} \tag{1.7}$$

for any $n\ge 2$, where $X_1,\ldots,X_n$ are independent copies of X.

Problem 1.2. Prove the following assertions for any $b>0$:

(a) The function $\mathbb{R}_\ge\ni t\mapsto\varphi(t)=\exp(-bt^\alpha)$ is the LT of a distribution $\mathscr{S}_+(\alpha,b)$ on $\mathbb{R}_\ge$, called *one-sided stable law with index* α *and scaling parameter* b, iff $\alpha\in(0,1]$.
(b) $\mathscr{S}_+(\alpha,b)$ forms a nonnegative solution to the SFPE (1.7).

Problem 1.3. Let $N\in\mathbb{N}\cup\{\infty\}$ and $T_1,\ldots,T_N\ge 0$. Find conditions on $N,T_1,\ldots,T_N$ such that $\mathscr{S}(\alpha,b)$ and $\mathscr{S}_+(\alpha,b)$ are solutions to the SFPE (1.6).

1.2 A prominent queuing example: the Lindley equation

In a single-server queuing system, the *Lindley equation* for the waiting time of a customer before receiving service provides another well-known example of a random recursive equation. To set up the model, suppose that an initially idle server is facing (beginning at time 0) arrivals of customers at random epochs $0 \le T_0 < T_1 < T_2 < \ldots$ with service requests of (temporal) size $B_0, B_1, \ldots$ Customers who find the server busy join a queue and are served in the order they have arrived (first in, first out). Typical performance measures are quantities like workload, queue length or waiting times of customers in the system. They may be studied over time (transient analysis) or in the long run (steady state analysis). Here we will focus on the time a customer spends in the queue (if there is one) before receiving service and will do so for the so-called *G/G/1-queue* specified by the following assumptions [☞ also [5]]:

(G/G/1-1) The sequence of arrival epochs $(T_n)_{n\ge 0}$ has iid positive increments $A_1, A_2, \ldots$ with finite mean λ and thus forms a renewal process with finite drift.
(G/G/1-2) The service times $B_0, B_1, \ldots$ are iid with finite positive mean μ.
(G/G/1-3) The sequences $(T_n)_{n\ge 0}$ and $(B_n)_{n\ge 0}$ are independent.
(G/G/1-4) There is one server and a waiting room of infinite capacity.
(G/G/1-5) The queue discipline is FIFO ("first in, first out").

The *Kendall notation* "G/G/1", which may be expanded by further symbols when referring to more complex systems, has the following meaning: The first letter refers to the arrival pattern, the second one to the service pattern, and the number in the third position to the number of servers (or counters). The letter "G" stands for "general" and is sometimes replaced with "GI" for "general independent". It means that both, interarrival times and service times are each iid with a general distribution.

Let W_n denote the quantity in question, that is, the waiting time of the n^{th} arriving customer before receiving service and notice that $W_0 = 0$, for the server is supposed to be idle before T_0. In order to derive Lindley's equation for W_n $(n \ge 1)$, we point out the following: Either $W_n = 0$, which happens if the n^{th} customer arrives after his predecessor has already left the system, or W_n equals the time spent in the system by the predecessor, i.e. $W_{n-1} + B_{n-1}$, minus the time A_n that elapses between the arrival of that customer and his own arrival. The first case occurs if $T_n > T_{n-1} + W_{n-1} + B_{n-1}$ or, equivalently, $W_{n-1} + B_{n-1} - A_n < 0$, while the second one occurs if $W_{n-1} + B_{n-1} - A_n \ge 0$. Consequently, the Lindley equation [47] takes the form

$$W_n \;=\; (W_{n-1} + X_n)^+ \tag{1.8}$$

for each $n \ge 1$, where $X_n := B_{n-1} - A_n$. Put also $S_0 := 0$ and $S_n = X_1 + \ldots + X_n$ for $n \ge 1$. Then $(S_n)_{n\ge 0}$ forms an ordinary zero-delayed random walk with drift $\mu - \lambda$. It is now an easy exercise [☞ Problem 1.5] to prove via iteration that

$$W_n \;=\; \max_{0\le k\le n} (S_n - S_k) \;\stackrel{d}{=}\; \max_{0\le k\le n} S_k \tag{1.9}$$

for each $n \ge 0$ and then to deduce the following result about the asymptotic behavior of W_n.

Theorem 1.4. *Under the stated assumptions, the waiting time W_n converges in distribution to $W_\infty := \max_{k\geq 0} S_k$ iff $\mu < \lambda$. In this case*

$$W_\infty \stackrel{d}{=} (W_\infty + X)^+, \qquad (1.10)$$

where X denotes a generic copy of $X_1, X_2, \ldots$ independent of W_∞. Furthermore,

$\lim_{n\to\infty} W_n = \infty$ *a.s. if* $\mu > \lambda$,

and

$0 = \liminf_{n\to\infty} W_n < \limsup_{n\to\infty} W_n = \infty$ *a.s. if* $\mu = \lambda$.

Proof. Problem 1.5 □

It is intuitively obvious (and indeed true) that asymptotic stability of waiting times, i.e. distributional convergence of the W_n, is equivalent to the asymptotic stability of the whole system in the sense that other relevant functionals like queue length or workload approach a distributional limit as well. Adopting a naive standpoint by simply ignoring random fluctuations of system behavior, we expect this to be true iff the mean service time is smaller than the mean time between two arriving customers, for then the server works faster on average than the input rate. The previous result tells us that naive thinking does indeed lead to the correct answer.

Further dwelling on the stable situation, thus assuming $\mu < \lambda$, it is natural to strive for further information on the distribution of W_∞, which in general cannot be determined explicitly [☞ Problem 1.7 for an exception]. For this purpose, the queuing background no longer matters so that we may just assume to be given a general nonnegative sequence $(W_n)_{n\geq 0}$, called *Lindley process*, of the recursive form (1.8) with iid random variables $X_1, X_2, \ldots$ with negative mean. The reader is asked in Problem 1.6 to show that then W_n always converges in distribution to $W_\infty = \max_{k\geq 0} S_k$, regardless of the distribution of W_0. This implies that the SFPE (1.10) determines the distribution G, say, of W_∞ uniquely. *Implicit renewal theory*, to be developed in Chapter 4, will enable us to determine the asymptotic behavior of the tail probabilities $\mathbb{P}(W > t)$ as $t \to \infty$ with the help of (1.10). At this point we finally note that the latter may be stated in terms of $G(t) = \mathbb{P}(W \leq t)$ as

$$G(t) \;=\; \int_{(-\infty,t]} G(t-x)\, \mathbb{P}(X \in dx), \quad t \geq 0, \qquad (1.11)$$

called *Lindley's integral equation*.

Problems

Problem 1.5. Given a G/G/1-queue as described above, prove that W_n satisfies (1.9) and then Theorem 1.4.

Problem 1.6. Given a sequence of iid real-valued random variables $X, X_1, X_2, \ldots$ with associated SRW $(S_n)_{n\ge 0}$, consider the Lindley process

$$W_n \;=\; (W_{n-1}+X_n)^+, \quad n \ge 1$$

with arbitrary initial value $W_0 \ge 0$ independent of $X_1, X_2, \ldots$ Prove the following assertions:

(a) For each $n \ge 1$, $W_n \stackrel{d}{=} M_{n-1} \vee (W_0 + S_n)$, where $M_n = \max_{0\le k\le n} S_k$ for $n \ge 0$.

(b) If $\mathbb{E}X < 0$, then $W_n \stackrel{d}{\to} W_\infty = \max_{k\ge 0} S_k$.

(c) If $\mathbb{E}X < 0$, then $\mathscr{L}(W_\infty)$ forms the unique solution to the SFPE (1.10) in the class of distributions on $\mathbb{R}_\ge$.

Problem 1.7. In the previous problem, suppose that X is integer-valued with negative mean and $\mathscr{L}(X^+) = Bern(p)$ for some $p > 0$. Prove that W_∞ has a geometric distribution. [Hint: Consider the *strictly descending ladder epochs* $(\sigma_n^<)_{n\ge 0}$, recursively defined by $\sigma_0^< := 0$ and

$$\sigma_n^< \;:=\; \inf\left\{k > \sigma_{n-1}^< : S_k < S_{\sigma_{n-1}^<}\right\}$$

for $n \ge 1$, where $\inf\emptyset := \infty$. Then write W_∞ in terms of the associated *ladder heights* $S_{\sigma_n^<}\mathbf{1}_{\{\sigma_n^<<\infty\}}$ and use that, given $\sigma_n^< < \infty$, the random vectors

$$\left(\sigma_k^< - \sigma_{k-1}^<, S_{\sigma_k^<} - S_{\sigma_{k-1}^<}\right), \quad k = 1, \ldots, n,$$

are conditionally iid [☞ Subsec. 2.2.1 in [2] for further information].]

Problem 1.8. Here is a version of the *continuous mapping theorem* that will frequently be used hereafter:

Let $\theta_1, \theta_2, \ldots$ be iid $\mathbb{R}^d$-valued $(d \ge 1)$ random variables with generic copy θ and independent of X_0. Suppose further that $X_n = \psi(X_{n-1}, \theta_n)$ for all $n \ge 1$ and a continuous function $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$. Prove that, if X_n converges in distribution to X_∞, then

$$\psi(X_{n-1}, \theta_n) \stackrel{d}{\to} \psi(X_\infty, \theta) \quad \text{and} \quad X_\infty \stackrel{d}{=} \psi(X_\infty, \theta),$$

where X_∞ and θ are independent.

1.3 A rich pool of examples: branching processes

Consider a population starting from one ancestor (generation 0) in which individuals of the same generation produce offspring independently and also independent of the current generation size. The offspring distribution, denoted as $(p_n)_{n\ge 0}$, is supposed to be the same for all population members and to have finite mean m. Under these assumptions, the generation size process $(Z_n)_{n\ge 0}$, thus $Z_0 = 0$, forms a so-called *(simple) Galton-Watson (branching) process (GWP)* and satisfies the random recursive equation

$$Z_n \;=\; \sum_{k=1}^{Z_{n-1}} X_{n,k}, \quad n \ge 1, \tag{1.12}$$

where $\{X_{n,k} : n,k \ge 1\}$ forms a family of iid integer-valued random variables with common distribution $(p_n)_{n\ge 0}$. Here Z_n denotes the size of the n^{th} generation and $X_{n,k}$ the number off children of the k^{th} member of this generation (under an arbitrary labeling of these members). To exclude the trivial case $Z_0 = Z_1 = ... = 1$, we make the standing assumption $p_1 < 1$.

A classical result, known as the *extinction-explosion principle*, states that the population either dies out ($Z_n = 0$ eventually) or explodes ($Z_n \to \infty$), i.e.

$$\mathbb{P}(\{Z_n = 0 \text{ eventually}\} \cup \{Z_n \to \infty\}) = 1.$$

Moreover, extinction occurs almost surely if $m < 1$ (subcritical case) or $m = 1$ (critical case), while $q := \mathbb{P}(Z_n = 0 \text{ eventually}) < 1$ if $m > 1$ (supercritical case). Defining the offspring gf $f(s) := \sum_{n\ge 0} p_n s^n$ for $s \in [0,1]$, q equals the minimal fixed point of f in $[0,1]$.

It is easily verified that the normalized sequence $W_n = m^{-n} Z_n$, $n \ge 0$, constitutes a nonnegative mean one martingale which therefore converges to a nonnegative limit W with $\mathbb{E}W \le 1$ by the martingale convergence theorem [☞ Problem 1.9]. If $m \le 1$, then clearly $W = 0$ a.s. holds true, but if $m > 1$ we may hope for $W > 0$ a.s. on the survival event $\{Z_n \to \infty\}$ giving that Z_n grows like a random constant times m^n on that event as $n \to \infty$. A famous result by Kesten & Stigum [43] states that this holds true iff

$$\mathbb{E}Z_1 \log Z_1 \;=\; \sum_{n\ge 1} p_n\, n \log n \;<\; \infty \tag{ZlogZ}$$

which we will assume hereafter. Then $(W_n)_{n\ge 0}$ is ui and thus $\mathbb{E}|W_n - W| \to 0$, in particular $\mathbb{E}W = \mathbb{E}W_0 = 1$.

What can be said about the distribution of W? The following argument shows that once again its distribution satisfies a SFPE. First notice that, besides (1.12), we further have

$$Z_n \;=\; \sum_{j=1}^{Z_1} Z_{n-1}(j), \quad n \ge 1, \tag{1.13}$$

where $(Z_n(j))_{n\ge 0}$ denotes the generation size process of the subpopulation stemming from the j^{th} individual in the first generation of the whole population. In fact, we can define $(Z_n(j))_{n\ge 0}$ for any $j \ge 1$ in such a way that these processes are independent copies of $(Z_n)_{n\ge 0}$ and also independent of Z_1. Then, defining $W_n(j)$ in an obvious manner, we infer

$$W_n \;=\; \frac{1}{m} \sum_{j=1}^{Z_1} W_{n-1}(j), \quad n \ge 1$$

and then, by letting $n \to \infty$, that

$$W \;=\; \frac{1}{m}\sum_{j=1}^{Z_1} W(j) \quad \text{a.s.,} \tag{1.14}$$

where $W(j)$ denotes the almost sure limit of the martingale $(W_n(j))_{n\geq 0}$. By what has been pointed out before, the $W(j)$ are independent copies of W and independent of Z_1 so that (1.14) does indeed constitute an SFPE for $\mathscr{L}(W)$. In terms of the LT $\varphi(t) := \mathbb{E}e^{-tW}$ of W, it may be restated as

$$\varphi(t) \;=\; f\circ\varphi\left(\frac{t}{m}\right), \quad t\geq 0, \tag{1.15}$$

as one can readily verify. With the help of this equation, one can further show (under (ZlogZ)) that φ is the unique solution with right derivative $\varphi'(0+) = -\mathbb{E}W = -1$ at 0 [☞ Problem 1.10]. Since distributions are determined by their LT's we hence conclude that $\mathscr{L}(W)$ is the unique solution to (1.14).

There are many other functionals in connection with GWP's that can be described by a random recursive equation. Here we confine ourselves to two further examples in the case when $m\leq 1$ in which almost certain extinction occurs. First, consider the *total population size process*

$$Y_n \;:=\; \sum_{k=0}^{n} Z_k, \quad n\geq 0$$

which satisfies

$$Y_n \;=\; 1+\sum_{j=1}^{Z_1} Y_{n-1}(j), \quad n\geq 1, \tag{1.16}$$

where $(Y_n(j))_{n\geq 0}$ denotes the total population size process associated with the GWP $(Z_n(j))_{n\geq 0}$ defined above. Plainly, Y_n increases to an a.s. finite limit Y_∞ which, by (1.16), satisfies the SFPE

$$Y_\infty \;=\; 1+\sum_{j=1}^{Z_1} Y_\infty(j). \tag{1.17}$$

Problem 1.11 shows that this equation characterizes the distribution of Y_∞ uniquely. It is not obvious at all but has been shown by DWASS [24] that $\mathscr{L}(Y_\infty)$ can be obtained explicitly, namely

$$\mathbb{P}(Y_\infty = j) \;=\; \frac{1}{j}\,p_{j,j-1}, \quad j\geq 1,$$

where $p_{ij} := \mathbb{P}(Z_1 = j|Z_0 = i)$ for $i,j\geq 0$. The proof is based on a clever analysis of the random recursive equation (1.16) in terms of gf's.

As a second example, still assuming $m\leq 1$, we mention the *extinction time* of $(Z_n)_{n\geq 0}$, viz.

$$T \;:=\; \inf\{n\geq 1 : Z_n = 0\}.$$

If $T(j)$ denotes the corresponding random variable for the GWP $(Z_n(j))_{n\geq 0}$ for each $j \geq 1$, then the following SFPE follows immediately:

$$T \;=\; 1 + \bigvee_{j=1}^{Z_1} T(j) \tag{1.18}$$

with the convention that $\bigvee_{j=1}^{0} x_j := 0$.

Problems

Problem 1.9. Given a GWP $(Z_n)_{n\geq 0}$ with one ancestor and finite offspring mean m, prove that $W_n = m^{-n} Z_n$ for $n \geq 0$ forms a nonnegative martingale.

Problem 1.10. Prove (1.15) and then, assuming (ZlogZ), that φ is the unique solution with right derivative at 0 satisfying $|\varphi'(0+)| = 1$.

Problem 1.11. Suppose $m \leq 1$ and let φ denote the LT of the final total population size Y_∞. Prove that φ satisfies the functional equation $\varphi(t) = e^{-t} f \circ \varphi(t)$ equivalent to (1.17) and that it forms the unique solution in the class of LT's of distributions. [Hint: Use the convexity of f.]

1.4 The sorting algorithm `Quicksort`

`Quicksort`, first introduced by Hoare [39, 40], is probably the nowadays most commonly used, so called *divide-and-conquer algorithm* to sort a list of n real numbers and serves as the standard sorting algorithm in UNIX-systems. Based on the general idea to successively divide a given task into subtasks of the same kind but smaller dimension, it forms a random recursive algorithm that may be briefly described as follows: Given n distinct reals $a_1,...,a_n$, which are to be sorted in increasing order, the first step is to create two sublists by first choosing an element x from the list, called *pivot*, and then to put all a_k smaller than x in the first sublist and all a_k bigger than x in the second sublist. The same procedure is then applied to the two sublists and all further on created ones as long as these contain at least two elements. Hence, the algorithm terminates when all sublists consist of only one element which are then merged to yield $a_1,...,a_n$ in increasing order. The way the pivots are chosen throughout the performance of the algorithm may be deterministic or at random, e.g. by picking any element of a given sublist with equal probability. Notice that the particular values of $a_1,...,a_n$ do not matter for the algorithm so that we may assume w.l.o.g. that $(a_1,...a_n)$ is a permutation of the numbers $1,...n$. When picking such a permutation at random, the number of key comparisons needed by `Quicksort` to output the ordered sample becomes a random variable X_n, and our goal hereafter is to study the distribution of X_n. But before proceeding we give an example first.

Example 1.12. In order to illustrate how `Quicksort` may perform on a given sample, we have depicted a permutation of the numbers $1,2,\ldots,12$. The table below shows that the algorithm needs four rounds to output the ordered sample. Each round consists of the further subdivisions of the currently given sublists with more than one element with respect to previously chosen pivots (shown in boldface). The final column of the table displays how many key comparisons are needed to complete the round.

List to be sorted	6 3 9 2 5 12 8 1 10 4 11 7	# key comparisons
Round 1	3 2 5 1 4 **6** 9 12 8 10 11 7	11
Round 2	2 1 **3** 5 4 • 8 7 **9** 12 10 11	9
Round 3	1 **2** • 4 **5** • 7 **8** • 10 11 **12**	5
Round 4	1 • • 4 • • 7 • • **10** 11 •	1

The reader may wonder about the necessity of Round 4 because the sample appears to be in correct order already after Round 3. The simple explanation is that after Round 3 we still have one sublist of length ≥ 2, namely $(10,11)$ which in the final round is assessed to be in correct order by choosing 10 as the pivot and making the one necessary comparison with the other element 11 [☞ also Figure 1.1 below].

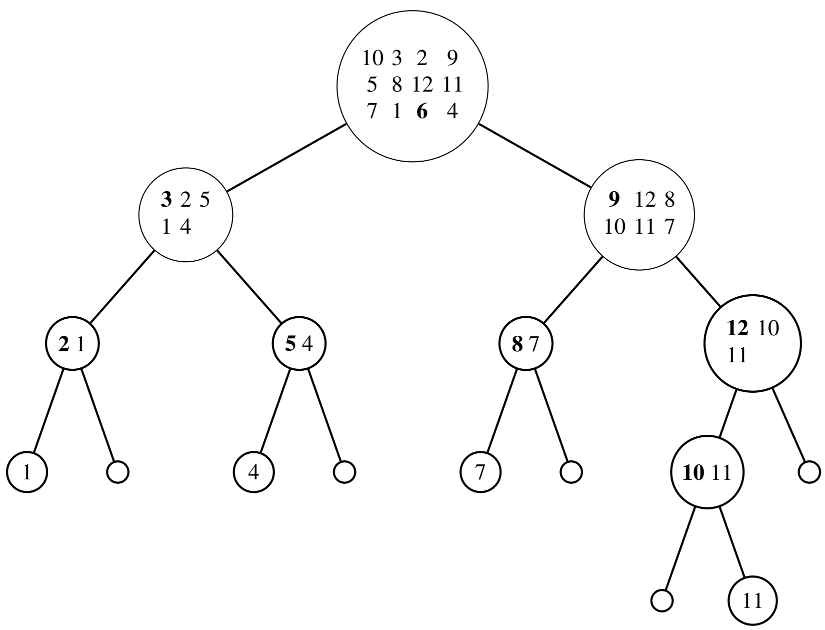

Fig. 1.1 Example 1.12: Left and right nodes of the tree are representing the respective sublists as created in the successive rounds by comparison with the pivot (shown in boldface) in the previous node.

As already announced, our performance analysis of `Quicksort` will be based on the number of key comparisons X_n needed to sort a random permutation of length

n. It seems plausible that this number is essentially proportional to the performance time and therefore the appropriate quantity to analyze.

To provide a rigorous model for X_n, let

$$\Omega_n \ := \ \{\pi \in \{1,...,n\}^n : \pi_i \neq \pi_j \text{ for } i \neq j\}$$

be the permutation group of $1,...,n$, here the set of possible inputs, and $\mathbb{P}_n$ the (discrete) uniform distribution on Ω_n. The discrete random variable $X_n : \Omega_n \to \mathbb{N}_0$ then maps any π on the number of key comparisons needed by `Quicksort` to sort π in increasing order where, for simplicity, we assume that pivots are always chosen as first elements in the appearing sublists[1]. Consequently, $Z_n(\pi) := \pi_1$ denotes the pivot in the input list and has a uniform distribution on $\{1,...,n\}$. It also gives the rank of this element in the list. The derivation of results about the distribution of X_n will be heavily based on the recursive structure of `Quicksort` which we are now going to make formally more explicit. Denote by L_n, R_n the rank tuples of the left and right sublist, respectively, created in the first round via key comparison with Z_n. Observe that these lists have lengths $Z_n - 1$ and $n - Z_n$, respectively, so that $L_n(\pi) \in \Omega_{Z_n(\pi)-1}$ and $R_n(\pi) \in \Omega_{n-Z_n(\pi)}$ for any $\pi \in \Omega_n$. After these settings the crucial random recursive equation for X_n takes the form

$$X_n \ = \ X_{Z_n-1} \circ L_n \ + \ X_{n-Z_n} \circ R_n \ + \ n-1 \tag{1.19}$$

for any $n \geq 1$, where $X_0(\emptyset) := 0$. It follows by a combinatorial argument that, given $Z_n = i$, L_n and R_n are conditionally independent and uniformly distributed on Ω_{i-1} and Ω_{n-i}, respectively [☞ Problem 1.14]. Setting $\mathbb{P}_0(X_0 \in \cdot) := \delta_0$, it hence follows that

$$\begin{aligned}
\mathbb{P}_n(X_n \in \cdot) \ &= \ \sum_{i=1}^{n} \mathbb{P}_n(Z_n = i)\, \mathbb{P}_n(X_n \in \cdot | Z_n = i) \\
&= \ \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}_n(X_{Z_n-1} \circ L_n + X_{n-Z_n} \circ R_n + n - 1 \in \cdot | Z_n = i) \\
&= \ \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}_n(X_{Z_n-1} \circ L_n \in \cdot | Z_n = i) * \mathbb{P}_n(X_{n-Z_n} \circ R_n \in \cdot | Z_n = i) * \delta_{n-1} \\
&= \ \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}_{i-1}(X_{i-1} \in \cdot) * \mathbb{P}_{n-i}(X_{n-i} \in \cdot) * \delta_{n-1}
\end{aligned}$$

for each $n \geq 1$. From now on, we assume that *all* X_n, Z_n, $n \geq 1$, are defined on just one sufficiently large probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ which further carries independent random variables $X_0', X_0'', X_1', X_1'', ...$, which are also independent of $(X_n, Z_n)_{n \geq 1}$, such that

$$X_0' = X_0'' := 0 \quad \text{and} \quad X_n \stackrel{d}{=} X_n' \stackrel{d}{=} X_n'' \quad \text{for } n \geq 1.$$

Then equation (1.19) provides us with the distributional relation

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[1] this version is sometimes referred to as *vanilla* `Quicksort`

$$X_n \stackrel{d}{=} X'_{Z_n-1} + X''_{n-Z_n} + n-1 \tag{1.20}$$

for all $n \geq 1$.

Our ultimate goal to be accomplished later [☞ ??????] is to show that a suitable normalization of X_n converges in distribution to some X_∞ and to characterize $\mathscr{L}(X_\infty)$ as the solution to a certain SFPE. At this point, we must contend ourselves with some preliminary considerations towards this result due to RÖSLER [53] including a heuristic derivation of the SFPE.

In order to gain an idea about a suitable normalization of X_n, we first take a look at its expectation. Let $H_n := \sum_{k=1}^n \frac{1}{k}$ be the n^{th} harmonic sum and

$$\gamma := \lim_{n\to\infty} (H_n - \log n) = 0.5772...$$

denote Euler's constant.

Lemma 1.13. *For each $n \geq 1$,*

$$\mathbb{E}X_n = 2(n+1)H_n - 4n \tag{1.21}$$

holds true and, furthermore,

$$\mathbb{E}X_n = 2(n+1)\log n + (2\gamma-4)n + 2\gamma + 1 + O\left(\frac{1}{n}\right) \tag{1.22}$$

as $n \to \infty$.

Proof. Taking expectations in (1.20), we obtain

$$\begin{aligned}
\mathbb{E}X_n &= n-1+\sum_{j=1}^n \mathbb{P}(Z_n=j)\left(\mathbb{E}X_{j-1}+\mathbb{E}X_{n-j}\right)\\
&= n-1+\frac{1}{n}\sum_{j=1}^n \left(\mathbb{E}X_{j-1}+\mathbb{E}X_{n-j}\right)\\
&= n-1+\frac{2}{n}\sum_{j=1}^{n-1} \mathbb{E}X_j
\end{aligned}$$

and then upon division by $n+1$ and a straightforward calculation that

$$\frac{\mathbb{E}X_n}{n+1} = \frac{\mathbb{E}X_{n-1}}{n} + \frac{2(n-1)}{n(n+1)} \tag{1.23}$$

for all $n \geq 1$. We leave it to the reader [☞ Problem 1.15] to verify this recursion and to derive (1.21) from it. The asymptotic expansion (1.22) then follows directly when using that $H_n = \log n + \gamma + (2n)^{-1} + O(n^{-2})$ as $n \to \infty$. □

The reader is asked to show in Problem 1.16 that $\mathbb{V}\mathrm{ar}\, X_n \simeq \sigma^2 n^2$ as $n \to \infty$, where $\sigma^2 := 7 - \frac{2}{3}\pi^2$. In view of this fact it is now reasonable to study the asymptotic behavior of the normalization

$$\widehat{X}_n := \frac{X_n - \mathbb{E}X_n}{n}.$$

The contraction argument due to RÖSLER [53] that proves convergence in distribution of $\widehat{X}_n$ to a limit $\widehat{X}_\infty$ with mean 0 and variance σ^2 will be postponed to ??????. Here we outline the argument that shows that $\mathscr{L}(X_\infty)$, called `Quicksort`-*distribution*, may again be characterized by an SFPE.

The argument embarks on the distributional equation (1.20), which after normalization becomes

$$\widehat{X}_n \stackrel{d}{=} \frac{Z_n - 1}{n} \widehat{X}'_{Z_n - 1} + \frac{n - Z_n}{n} \widehat{X}''_{n - Z_n} + g_n(Z_n) \tag{1.24}$$

for $n \geq 2$, where $\widehat{X}_0 = \widehat{X}_1 := 0$ and $g_n : \{1, ..., n\} \to \mathbb{R}$ is defined by

$$g_n(k) := \frac{n-1}{n} + \frac{1}{n} \left(\mathbb{E}X_{k-1} + \mathbb{E}X_{n-k} - \mathbb{E}X_n \right). \tag{1.25}$$

Notice that the $\widehat{X}'_n, \widehat{X}''_n$, $n \geq 0$, continue to be independent of $(X_n, Z_n)_{n \geq 1}$. The reader can easily verify [☞ Problem 1.17] that $Z_n/n \stackrel{d}{\to} \mathit{Unif}(0,1)$, and we will prove in ??????? that $0 \leq \sup_{n \geq 1,\, 1 \leq k \leq n} g_n(k) < \infty$ as well as

$$\lim_{n \to \infty} g_n(\lceil nt \rceil) = g(t) := 1 + 2t \log t + 2(1-t) \log(1-t)$$

for all $t \in (0,1)$ uniformly on compact subsets, where $\lceil x \rceil := \inf\{n \in \mathbb{Z} : x \leq n\}$. By combining these facts and $\widehat{X}_n \stackrel{d}{\to} \widehat{X}_\infty$, it can be deduced from (1.24) that $\mathscr{L}(\widehat{X}_\infty)$ solves the SFPE

$$\widehat{X}_\infty \stackrel{d}{=} U\widehat{X}'_\infty + (1-U)\widehat{X}''_\infty + g(U) \tag{1.26}$$

where $\widehat{X}'_\infty, \widehat{X}''_\infty$ and U are independent with $\widehat{X}'_\infty \stackrel{d}{=} \widehat{X}''_\infty \stackrel{d}{=} \widehat{X}_\infty$ and $U \stackrel{d}{=} \mathit{Unif}(0,1)$. This is the `Quicksort`-equation, and we will also show in ??????? that $\mathscr{L}(\widehat{X}_\infty)$ forms its unique solution within the class of zero mean distributions with finite variance.

Binary search trees. A binary search tree (BST) of size n is a labeled binary tree with n internal nodes generated from a permutation $\pi = (\pi_1, ..., \pi_n) \in \Omega_n$. One way to construct it is as follows: Start with π_1, store it in the root of the tree and attach to it two empty nodes, called external. Then π_2 is compared with π_1 and becomes the left descendant if $\pi_2 < \pi_1$, and the right descendant otherwise. Attach two empty nodes to the now internal node occupied by π_2. Proceed with any π_k in the same manner by moving it along internal nodes until an external one is reached where it is stored. At each internal node with value x, say, where $x \in \{\pi_1, ..., \pi_{k-1}\}$, move left if $\pi_k < x$ and right otherwise. Finish step k by attaching two external nodes to the node now occupied by π_k. After n steps all keys have been stored, giving a binary tree with n internal and $n+1$ external nodes. This is exemplified in Fig. 1.2 with the permutation from Example 1.12. As one can see, the same tree as in Fig. 1.1 is obtained when ignoring external nodes. In fact, an application of `Quicksort` *always* leads to the same result as the procedure just described when only storing the leading element of each sublist (the pivot) in the nodes.

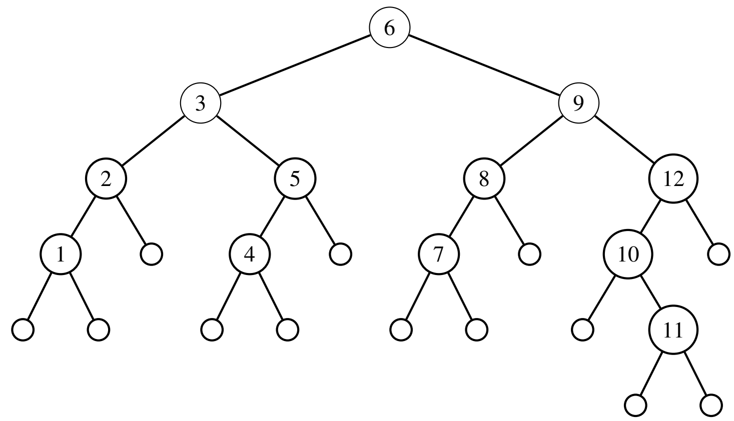

Fig. 1.2 The permutation $(6,3,9,2,5,12,8,1,10,4,11,7)$ from Example 1.12 stored in a binary search tree. External nodes are shown as empty circles.

Problems

Problem 1.14. Prove that, given $Z_n = i$, the rank tuples L_n and R_n are conditionally independent with a discrete uniform distribution on Ω_{i-1} and Ω_{n-i}, respectively.

Problem 1.15. Complete the proof of Lemma 1.13 by verifying (1.23) and then deriving (1.21) from it.

Problem 1.16. Prove that $\sigma_n^2 := \mathbb{V}\mathrm{ar} X_n$ satisfies

$$\sigma^2 \;:=\; \lim_{n\to\infty} \frac{\sigma_n^2}{n^2} \;=\; 7 - \frac{2}{3}\pi^2 \;=\; 0.4203... \tag{1.27}$$

by doing the following parts:

(a) Use (1.20) to show that

$$\sigma_n^2 \;=\; c_n - (n-1)^2 + \frac{2}{n}\sum_{k=1}^{n-1} \sigma_k^2 \tag{1.28}$$

for all $n \geq 1$, where $\mu_n := \mathbb{E}X_n$ and $c_n := \frac{1}{n}\sum_{k=1}^{n}(\mu_{k-1} + \mu_{n-k} - \mu_n)^2$.

(b) Use (1.28) to derive the recursion

$$\frac{d_n}{n+1} \;=\; \frac{d_{n-1}}{n} \;+\; \frac{2\left(c_{n-1}-(n-2)^2\right)}{n(n+1)} \tag{1.29}$$

for any $n \geq 2$, where $d_n := \sigma_n^2 + (n-1)^2 - c_n$ for $n \geq 1$. Note that $\sigma_1^2 = c_1 = d_1 = 0$.

(c) Use Lemma 1.13 to show that

$$\frac{c_n}{n^2} \;=\; \frac{4}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\log\left(\frac{k}{n}\right) + \left(1-\frac{k}{n}\right)\log\left(1-\frac{k}{n}\right)\right)^2 \;+\; o(1)$$

as $n \to \infty$ and thereby

$$\lim_{n\to\infty}\frac{c_n}{n^2} \;=\; c \;:=\; \frac{10}{3} - \frac{2}{9}\pi^2. \tag{1.30}$$

(d) Finally, combine the previous parts to infer

$$\lim_{n\to\infty}\frac{\sigma_n^2}{n^2} \;=\; 3(c-1)$$

which is easily seen to be the same as (1.27).

(e) Those readers who want to work harder should prove the stronger assertion [stated by FILL & JANSON in [31]]

$$\frac{\sigma_n^2}{n^2} \;=\; 3(c-1) - \frac{2\log n}{n} + O\left(\frac{1}{n}\right)$$

as $n \to \infty$.

Problem 1.17. Prove that $Z_n/n \stackrel{d}{\to} \mathit{Unif}(0,1)$ if Z_n has a discrete uniform distribution on $\{1,\ldots,n\}$ for each $n \geq 1$.

Problem 1.18. Let D_n denote the *depth* or *height* of a random BST with n internal nodes, thus $D_0 = D_1 = 0$. Prove that

$$D_n \;\stackrel{d}{=}\; 1 + D''_{Z_n-1} \vee D''_{n-Z_n}$$

for each $n \geq 1$, where D'_k, D''_k, $k \geq 0$, and Z_n are independent random variables such that $\mathscr{L}(D_k) = \mathscr{L}(D'_k) = \mathscr{L}(D''_k)$ for each k and $\mathscr{L}(Z_n) = \mathit{Unif}(\{1,\ldots,n\})$.

1.5 Random difference equations and perpetuities

A *random difference equation (RDE)* is probably the simplest nontrivial example of a random recursive equation, the recursion being defined by a random affine linear function $\Psi(x) = Mx + Q$ for a pair (M,Q) of real-valued random variables. More precisely, let $(M_n,Q_n)_{n\geq 1}$ be a sequence of independent copies of (M,Q), X_0 a further random variable independent of this sequence, and define the sequence $(X_n)_{n\geq 0}$ recursively by

$$X_n \;:=\; M_n X_{n-1} + Q_n, \quad n \geq 1. \tag{1.31}$$

This is the general form of a (one-dimensional) RDE and has been used in many applications to model a quantity that is subject to an intrinsic random increase or decay, given by M_n for the time interval $(n-1,n]$, and an external random in- or output of size Q_n right before time n for any $n \geq 1$. Here are some special cases:

- The choice $M \equiv 1$ leads to the classical random walk (RW)

$$X_n \;=\; X_0 + \sum_{k=1}^{n} Q_k, \quad n \geq 0$$

  with initial value (delay) X_0, which constitutes one of the most fundamental type of a random sequence.
- If $Q \equiv 0$, then we obtain its multiplicative counterpart

$$X_n \;=\; X_0 \prod_{k=1}^{n} M_k, \quad n \geq 0,$$

  called *multiplicative RW*.
- If $M \equiv \alpha$ for some constant $\alpha \neq 0$, then

$$X_n \;=\; \alpha X_{n-1} + Q_n \;=\; \ldots \;=\; \alpha^n X_0 + \sum_{k=1}^{n} \alpha^{n-k} Q_k, \quad n \geq 1$$

  is a so-called *autoregressive process of order 1*, usually abbreviated as *AR(1)*, and one of the simplest examples of a *linear times series*.
- As a particular case of an AR(1)-process consider the situation where $\alpha \in (0,1)$, $X_0 = 0$ and $Q_n = \alpha\xi_n$ with $\mathscr{L}(\xi_n) = Bern(p)$ for some $p \in (0,1)$. Then we have

$$X_n \;=\; \sum_{k=1}^{n} \alpha^k \xi_{n+1-k} \;\stackrel{d}{=}\; \sum_{k=1}^{n} \alpha^k \xi_k \;=:\; \widehat{X}_n$$

  for each $n \geq 0$, and since $\widehat{X}_n$ increases a.s. to the limit

$$\widehat{X}_\infty \;:=\; \sum_{n\geq 1} \alpha^n \xi_n,$$

  we infer that $X_n \stackrel{d}{\to} \widehat{X}_\infty$. The law of $\widehat{X}_\infty$ is called a *Bernoulli convolution* and has received considerable interest with regard to the question when it is nonsingular with respect to Lebesgue measure. The interested reader may consult the survey by PERES, SCHLAG & SOLOMYAK [52] and the references given there.

Returning to the general situation, we first note that an iteration of (1.31) leads to

$$\begin{aligned} X_n &= M_n X_{n-1} + Q_n \\ &= M_n M_{n-1} X_{n-2} + M_n Q_{n-1} + Q_n \\ &= M_n M_{n-1} M_{n-2} X_{n-3} + M_n M_{n-1} Q_{n-2} + M_n Q_{n-1} + Q_n \\ &\vdots \\ &= M_n M_{n-1} \cdot \ldots \cdot M_1 X_0 + \sum_{k=1}^{n} M_n \cdot \ldots \cdot M_{k+1} Q_k \end{aligned}$$

for each $n \geq 1$. Now use the independence assumptions and replace $(M_k, Q_k)_{1 \leq k \leq n}$ with the copy $(M_{n+1-k}, Q_{n+1-k})_{1 \leq k \leq n}$ to see that

$$X_n \stackrel{d}{=} \Pi_n X_0 + \sum_{k=1}^{n} \Pi_{k-1} Q_k \tag{1.32}$$

for any $n \geq 1$, where $(\Pi_n)_{n \geq 0}$ is the multiplicative RW associated with $(M_n)_{n \geq 1}$ and starting at $\Pi_0 = 1$.

We are interested in finding conditions that ensure the convergence in distribution of X_n, but confine ourselves at this point to some basic observations. By an appeal to the continuous mapping theorem [as stated in Problem 1.8], we infer from (1.31) that $X_n \xrightarrow{d} X_\infty$ implies the SFPE

$$X_\infty \stackrel{d}{=} M X_\infty + Q, \tag{1.33}$$

naturally the independence of (M, Q) and X_∞. Furthermore, by (1.32), it entails that $X_\infty \stackrel{d}{=} \widehat{X}_\infty$, where

$$\widehat{X}_\infty := \lim_{n \to \infty} \left(\Pi_n X_0 + \sum_{k=1}^{n} \Pi_{k-1} Q_k \right)$$

exists in the sense of distributional convergence.

It is natural to ask whether $\mathscr{L}(X_\infty)$ depends on the initial value X_0. Consider the bivariate RDE

$$(X_n, X_n') = (M_n X_{n-1} + Q_n, M_n X_{n-1}' + Q_n), \quad n \geq 1$$

with two distinct initial values X_0 and X_0'. Then

$$X_n - X_n' = M_n (X_{n-1} - X_{n-1}') = \ldots = \Pi_n (X_0 - X_0')$$

for each $n \geq 1$. Consequently, sufficient for $\mathscr{L}(X_\infty)$ to be independent of X_0 is that

$$\lim_{n \to \infty} \Pi_n = 0 \quad \text{a.s.} \tag{1.34}$$

$$\text{and} \quad \widehat{X}_\infty = \sum_{k\ge 1} \Pi_{k-1} Q_k \quad \text{exists a.s. in } \mathbb{R}. \tag{1.35}$$

The infinite series $\sum_{k\ge 1} \Pi_{k-1} Q_k$ is called *perpetuity* which is an actuarial notion for the present value of a infinite payment stream, here $Q_1, Q_2, ...$, at times 1,2,... discounted by the random products $\Pi_1, \Pi_2, ...$ The reader is asked in Problem 1.20 to show that (1.34) and (1.35) are valid if $\mathbb{E} \log |M| < 0$, $\mathbb{E}|M|^\theta < \infty$ and $\mathbb{E}|Q|^\theta < \infty$ for some $\theta > 0$. On the other hand, these conditions are far from being necessary. RDE's and perpetuities will be further discussed in Subsections 3.1.4 and 4.4.1.

Problems

Problem 1.19. Suppose that $M \ge 0$ a.s. and that $\mathbb{E} \log M$ exists, i.e. $\mathbb{E} \log^+ M < \infty$ or $\mathbb{E} \log^- M < \infty$. Prove that exactly one of the following cases occurs for the multiplicative RW $(\Pi_n)_{n\ge 0}$ and characterize them in terms of M.

$$\Pi_n = 1 \quad \text{a.s. for all } n \ge 0$$
$$\lim_{n\to\infty} \Pi_n = \infty \quad \text{a.s.}$$
$$\lim_{n\to\infty} \Pi_n = 0 \quad \text{a.s.}$$
$$0 = \liminf_{n\to\infty} \Pi_n < \limsup_{n\to\infty} \Pi_n = \infty \quad \text{a.s.}$$

Problem 1.20. Assuming $\mathbb{E} \log |M| < 0$, $\mathbb{E}|M|^\theta < \infty$ and $\mathbb{E}|Q|^\theta < \infty$ for some $\theta > 0$, prove the following assertions:

(a) There exists $\kappa \in (0, \theta]$ such that $\mathbb{E}|M|^\kappa < 1$. [Hint: Consider the function $s \mapsto \mathbb{E}|M|^s$ for $s \in [0, \theta]$.]

(b) $|\Pi_n| \to 0$ a.s. and

$$\left| \sum_{k\ge 1} \Pi_{k-1} Q_k \right| \le \sum_{k\ge 1} |\Pi_{k-1} Q_k| < \infty \quad \text{a.s.}$$

(c) The last assertion remains valid if $\mathbb{E} \log^+ |Q| < \infty$ [use a Borel-Cantelli argument].

Problem 1.21. Given an RDE $X_n = M_n X_{n-1} + Q_n$ for $n \ge 1$, prove that, if X_n converges in distribution and $\mathbb{P}(Q = 0) < 1$, then $\mathbb{P}(M = 0) = 0$.

Problem 1.22. Assuming M and Q to be constants, find all solutions to the SFPE (1.33), i.e. $X \stackrel{d}{=} MX + Q$.

1.6 A nonlinear time series model

Motivated by its relevance for the modeling of financial data, BORKOVEC & KLÜPPELBERG [16] studied the limit distribution of the following nonlinear time series model, designed to allow conditional variances to depend on past information (*conditional heteroscedasticity*) and reflecting the observations of early empirical work by MANDELBROT [51] and FAMA [28] which had shown that "that large changes in equity returns and exchange rates, with high sampling frequency, tend to be followed by large changes settling down after some time to a more normal behavior" [☞ [16, p. 1220]]. This leads to models of the form

$$X_n \;=\; \sigma_n \varepsilon_n, \quad n \geq 1, \tag{1.36}$$

where the ε_n, called *innovations*, are iid symmetric random variables and the σ_n, called *volatilities*, describe the change of the (conditional) variance. If σ_n^2 is a linear function of the p prior squared observations, viz.

$$\sigma_n^2 \;=\; \beta + \sum_{k=1}^{p} \lambda_k X_{n-k}^2, \quad n \geq 1, \tag{1.37}$$

where $\beta, \lambda_p > 0$ and $\lambda_1, ..., \lambda_{p-1} \geq 0$, we are given an *autoregressive conditionally heteroscedastic (ARCH) model of order p* as introduced by ENGLE [26]. Here we focus on the simplest case $p = 1$ and note that a combination of (1.36) and (1.37) then leads to the random recursive equation

$$X_n \;=\; \left(\beta + \lambda X_{n-1}^2\right)^{1/2} \varepsilon_n, \quad n \geq 1, \tag{1.38}$$

for some $\beta, \lambda > 0$, naturally assuming that X_0 and $(\varepsilon_n)_{n\geq 1}$ are independent. It may further be extended by adding an autoregressive term, viz.

$$X_n \;=\; \alpha X_{n-1} + \left(\beta + \lambda X_{n-1}^2\right)^{1/2} \varepsilon_n, \quad n \geq 1, \tag{1.39}$$

with $\alpha \in \mathbb{R}$, to give an *AR(1)-model with ARCH(1) errors*. This is the model actually studied in [16] and belongs to a larger class of autoregressive models with ARCH errors introduced by WEISS [61].

If X_n converges in distribution to a random variable X_∞, the latter may obviously again be described by an SFPE, namely

$$X_\infty \stackrel{d}{=} \alpha X_\infty + \left(\beta + \lambda X_\infty^2\right)^{1/2} \varepsilon \tag{1.40}$$

where ε is a copy of the ε_n and independent of X_∞. The intersting questions are, for which parameter triples (α, β, λ) convergence in distribution actually occurs, whether in this case the SFPE characterizes $\mathscr{L}(X_\infty)$, and what information the SFPE provides about the tail behavior of $\mathscr{L}(X_\infty)$.

We close this section with some observations of a more general kind, exemplified by the present model. Writing (1.39) in the form

$$X_n = \phi(X_{n-1}, \varepsilon_n), \quad n \geq 1,$$

where $\phi(x,y) = \phi_{\alpha,\beta,\lambda}(x,y) = \alpha x + (\beta + \lambda x^2)^{1/2} y$, we immediately infer, by using the independence of X_{n-1}, ε_n and the identical distribution of the innovations, that $(X_n)_{n\geq 0}$ forms a temporally homogeneous Markov chain (MC) with state space $\mathbb{R}$ and transition kernel

$$P(x,A) = \mathbb{P}((\alpha x + (\beta + \lambda x^2)^{1/2} \varepsilon \in A), \quad A \in \mathscr{B}(\mathbb{R}).$$

The continuity of $\phi(\cdot, y)$ for any $y \in \mathbb{R}$ further shows that $(X_n)_{n\geq 0}$ forms a *Feller chain*, defined by the property that

$$x \mapsto Pf(x) := \int f(y)\, P(x,dy) = \mathbb{E} f(\alpha x + (\beta + \lambda x^2)^{1/2} \varepsilon) \in \mathscr{C}_b(\mathbb{R})$$

whenever $f \in \mathscr{C}_b(\mathbb{R})$. In other words, a *Feller kernel* P maps bounded continuous functions to bounded continuous functions. Next we point out that π forms a solution to the SFPE (1.40), i.e. $X \stackrel{d}{=} \phi(X, \varepsilon)$, iff it is a a stationary distribution for $(X_n)_{n\geq 0}$. The latter means that

$$\pi P := \int P(x, \cdot)\, \pi(dx) = \pi$$

and therefore that $\mathscr{L}(X_{n-1}) = \pi$ implies $\mathscr{L}(X_n) = \pi$. Thus, to determine all solutions to the SFPE means to find all stationary distributions of the MC $(X_n)_{n\geq 0}$. Here is a lemma that sometimes provides a simple tool to check the existence of a stationary distribution for a Feller chain on $\mathbb{R}$.

Lemma 1.23. *Let $(X_n)_{n\geq 0}$ be a Feller chain on $\mathbb{R}$.*

(a) If $X_n \stackrel{d}{\to} X_\infty$, then $\mathscr{L}(X_\infty)$ is a stationary distribution.
(b) If $(X_n)_{n\geq 0}$ is tight, then there exists a stationary distribution.

Proof. Problem 1.24 □

Problems

Problem 1.24. Prove Lemma 1.23. [Hint for part (b): Show that tightness implies that $(n^{-1} \sum_{k=1}^{n} \mu P^k)_{n\geq 1}$, contains a weakly convergent subsequence, where P^k denotes the k-step transition kernel of the chain and $\mu := \mathbb{P}(X_0 \in \cdot)$. Then verify that the weak limit is necessarily a stationary distribution.]

Problem 1.25. Given the random recursive equation (1.38) with $\lambda \in (0,1)$, $\mathbb{E}\varepsilon^2 = 1$ and $\mathbb{E}X_0^2 < \infty$, prove the following assertions:

(a) $(X_n)_{n\ge 0}$ is L^2-bounded, that is $\sup_{n\ge 0}\mathbb{E}X_n^2 < \infty$.
(b) $(X_n)_{n\ge 0}$ possesses a stationary distribution which is nondegenerate.

Problem 1.26. Consider the random recursive equation (1.39) with $\alpha \neq 0$, $\mathbb{E}|\varepsilon| < \infty$ (thus $\mathbb{E}\varepsilon = 0$) and $\mathbb{E}|X_0| < \infty$.

(a) Prove that $(\alpha^{-n}X_n)_{n\ge 0}$ is a martingale.
(b) Assuming $\mathbb{E}\varepsilon^2 < \infty$ and $\mathbb{E}X_0^2 < \infty$, find a necessary and sufficient condition on $(\alpha,\beta,\lambda) \in (\mathbb{R}\backslash\{0\}) \times \mathbb{R}_> \times \mathbb{R}_>$ for $(X_n)_{n\ge 0}$ to be L^2-bounded.

1.7 A noisy voter model on a directed tree

Let $\mathbb{T}_3(n) = \bigcup_{k=0}^n \{1,2,3\}^k$ be the rooted homogenous tree of order 3 and height n in *Ulam-Harris labeling* and $\mathbb{T}_3 = \mathbb{T}_3(\infty)$ its infinite height counterpart. This means that $\{1,2,3\}^0$ consists of the root $\varnothing$ and that each vertex $\mathsf{v} = (\mathsf{v}_1,...,\mathsf{v}_k) \in \{1,2,3\}^k$ at level k ($< n$ for $\mathbb{T}_3(n)$) has exactly 3 children, labeled $(\mathsf{v}_1,...,\mathsf{v}_k,i)$ for $i = 1,2,3$ [☞ Fig. 1.3 below]. Let us write $\mathsf{v}_1...\mathsf{v}_k$ as shorthand for $(\mathsf{v}_1,...,\mathsf{v}_k)$, $|\mathsf{v}|$ for the length of v, and uv as the concatenation of u and v. Note that u is the parent node of $\mathsf{u1},\mathsf{u2},\mathsf{u3}$.

For any fixed $n \ge 1$, let $\{X_n(\mathsf{v}) : \mathsf{v} \in \{1,2,3\}^n\}$ be a family of iid *Bern*(p)-variables ($p > 0$) and $\{\xi(\mathsf{v}) : \mathsf{v} \in \mathbb{T}_3(n-1)\}$ a second family of iid *Bern*(ε)-variables ($\varepsilon > 0$ small) independent of the former one. As in ALDOUS & BANDYOPADHYAY [1, Example 13], we now define recursively

$$X_n(\mathsf{u}) \ := \ \xi(\mathsf{u}) + \mathbf{1}_{\{X_n(\mathsf{u1})+X_n(\mathsf{u2})+X_n(\mathsf{u3})\ge 2\}} \bmod 2 \tag{1.41}$$

and $X_n := X_n(\varnothing)$. A possible interpretation, reflecting the title of this subsection, is the following: Each parent node adopts the majority opinion, which can be 0 or 1, of its children, except with a small probability ε adopting the opposite opinion.

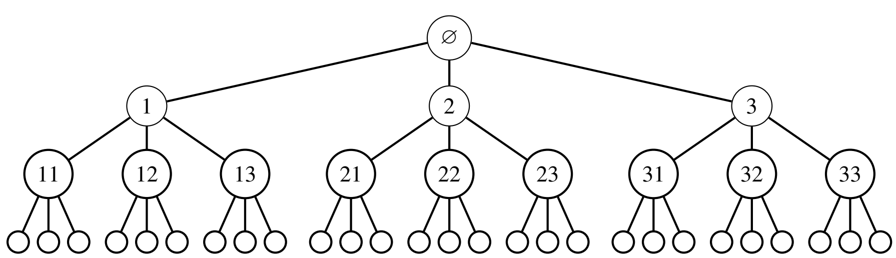

Fig. 1.3 The rooted homogenous tree $\mathbb{T}_3(3)$ with Ulam-Harris labeling

Following [1], we call the process $(X_n(\mathsf{v}))_{\mathsf{v}\in\mathbb{T}_3(n)}$ a *recursive tree process (RTP) of depth n* and note that the recursion is bottom-up because the value of $X_n(\mathsf{v})$ is defined via the values of the corresponding variables of the children $\mathsf{v}i$ for $i = 1,2,3$. Hence, the terminal or output value is $X_n(\varnothing)$.

The reader is asked in Problem 1.27 to verify the basic recursive relation

$$X_n \stackrel{d}{=} \xi + \mathbf{1}_{\{X_{n-1}+X'_{n-1}+X''_{n-1}\geq 2\}} \bmod 2 \tag{1.42}$$

for $n \geq 1$, where $X_{n-1}, X'_{n-1}, X''_{n-1}$ are iid and independent of $\xi \stackrel{d}{=} Bern(\varepsilon)$, and $\mathscr{L}(X_0(\varnothing)) = Bern(p)$. This constitutes again a random recursive equation for the X_n, but only in terms of their distributions. In other words, we are given here a mapping that maps the distribution of X_{n-1} to the distribution of X_n. Now it is readily seen that, if $\mathscr{L}(X_{n-1}) = Bern(q)$, then $\mathscr{L}(X_n) = Bern(g(q))$, where

$$g(s) := (1-\varepsilon)(s^3 + 3s^2(1-s)) + \varepsilon(1 - s^3 - 3s^2(1-s)) \tag{1.43}$$

for $s \in [0,1]$. As Fig. 1.4 shows, the function g has three fixed points $p(\varepsilon), \frac{1}{2}, 1-p(\varepsilon)$ with $p(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, if $\mathscr{L}(X_0) = Bern(q)$ with q being one of these fixed points, then $\mathscr{L}(X_n) = Bern(q)$ for all $n \geq 1$. The asymptotic behavior of X_n when $\mathscr{L}(X_0) = Bern(q)$ for $q \notin \{p(\varepsilon), \frac{1}{2}, 1-p(\varepsilon)\}$ is discussed in Problem 1.28.

Returning to the RTP $(X_n(\mathsf{v}))_{\mathsf{v}\in\mathbb{T}_3(n)}$ defined above, it follows that the marginal distributions of all $X_n(\mathsf{v})$ are the same whenever $Bern(q)$ for $q \in \{p(\varepsilon), \frac{1}{2}, 1-p(\varepsilon)\}$ is chosen as the distribution of the variables at the bottom of the tree (level n). In this case the RTP is called *invariant*, and it may be extended to an invariant RTP $(X_n(\mathsf{v}))_{\mathsf{v}\in\mathbb{T}_3}$ on the infinite tree $\mathbb{T}_3$ with the help of Kolmogorov's consistency theorem.

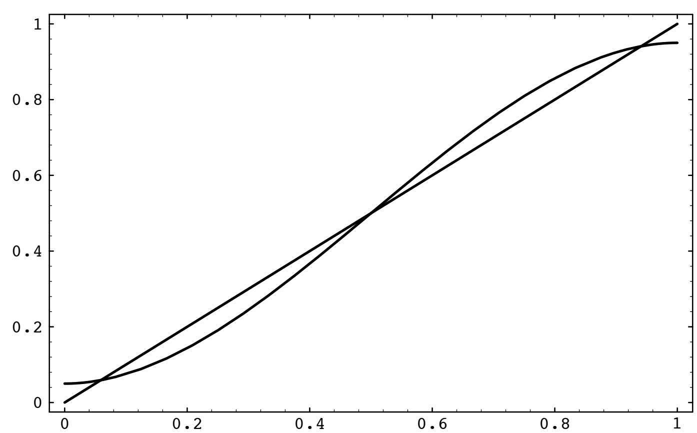

Fig. 1.4 The function $g(s) = (1-\varepsilon)(s^3 + 3s^2(1-s)) + \varepsilon(1 - s^3 - 3s^2(1-s))$ with $\varepsilon = 0.05$.

Problems

Problem 1.27. Prove (1.42) under the stated assumptions.

Problem 1.28. Let $(X_n)_{n\ge 0}$ be a sequence of Bernoulli variables satisfying (1.42) with $\mathscr{L}(\xi) = Bern(\varepsilon)$ for any small ε and $\mathscr{L}(X_0) = Bern(q)$ for any $q \in [0,1]$. Let g be defined by (1.43) with fixed points $p(\varepsilon), \frac{1}{2}, 1-p(\varepsilon)$. Prove that

$$X_n \xrightarrow{d} X_\infty, \quad \text{where } \mathscr{L}(X_\infty) \;=\; \begin{cases} Bern(p(\varepsilon)), & \text{if } q < \frac{1}{2}, \\ Bern(1/2), & \text{if } q = \frac{1}{2}, \\ Bern(1-p(\varepsilon)), & \text{if } q > \frac{1}{2}. \end{cases}$$

Problem 1.29. Here is a simpler variation of the noisy voter model on the binary trees $\mathbb{T}_2(n) = \bigcup_{k=0}^n \{1,2\}^n$, $n \ge 1$: Consider an RTP $(X_n(\mathsf{v}))_{\mathsf{v}\in\mathbb{T}_2(n)}$ of depth n with a family $\{X_n(\mathsf{v}) : \mathsf{v} \in \{1,2\}\}^n$ of iid $Bern(p)$-variables $(0 \le p \le 1)$. For any parental vertex $\mathsf{u} \in \mathbb{T}_2(n-1)$, define

$$X_n(\mathsf{u}) \;:=\; \xi(\mathsf{u}) + X_n(\mathsf{u}\zeta(\mathsf{u})) \bmod 2,$$

where $\{(\xi(\mathsf{u}), \zeta(\mathsf{u})) : \mathsf{u} \in \mathbb{T}_2(n-1)\}$ is independent of $\{X_n(\mathsf{v}) : \mathsf{v} \in \{1,2\}\}^n$ and consisting of iid random vectors with common distribution $Bern(\varepsilon) \otimes \mathrm{Unif}(\{1,2\})$ for some $\varepsilon \in (0,1)$. This means that u adopts the opinion of the randomly chosen child $\mathsf{u}\zeta(\mathsf{u})$, except with probability ε adopting the opposite opinion. Put $X_n := X_n(\varnothing)$ for $n \ge 0$, where $\mathscr{L}(X_0(\varnothing)) = Bern(p)$, and prove:

(a) For all $n \ge 1$, $X_n \stackrel{d}{=} \xi + X_{n-1} \bmod 2$.
(b) For any $p \in [0,1]$, X_n converges in distribution to $Bern(1/2)$.
(c) The RTP's defined above are invariant iff $p = 1/2$.

1.8 An excursion to hydrology: the Horton-Strahler number

The *Strahler number*[2] or *Horton-Strahler number* was first developed by two Americans, the ecologist and soil scientist HORTON [41] and the geoscientist STRAHLER [56, 57], as a measure in hydrology for stream size based on a hierarchy of tributaries[3] In this context, it is also referred to as the *Strahler stream order*. It further arises in the analysis of hierarchical biological structures (like biological trees) and of social networks. BENDER in his introductory book [11] on mathematical modeling has a nicely written section on stream networks which provides a little more background information.

[2] in German called *Fluss- oder Gewässerordnungszahl nach Strahler*

[3] defined as a river which flows into a parent river or lake instead of directly flowing into a sea or ocean.

In mathematics, the Strahler number is simply a numerical measure of the branching complexity of a finite (mathematical) tree and defined as follows (when using Ulam-Harris labeling as in the previous section): Starting at the bottom, all leaves (the sources in the hydrological context) get Strahler number 1. For any other vertex v, suppose it has children $\mathsf{v1},...,\mathsf{vk}$ with respective Strahler numbers $S(\mathsf{v1}),...,S(\mathsf{vk})$ having maximal value s, say. Then the Strahler number $S(\mathsf{v})$ at v is recursively defined as

- s if this value is attained uniquely among the $S(\mathsf{vi})$, $\mathsf{i}=1...,k$.
- $s+1$ if $S(\mathsf{vi})=S(\mathsf{vj})$ for at least two distinct $\mathsf{i},\mathsf{j}\in\{1,...,k\}$.

Finally, the Strahler number or index of the tree is defined as $S(\varnothing)$.

In the river network context, the trees are typically binary and the numbers are assigned to the edges leaving a node upwards rather than the node itself [☞ Fig. 1.5]. Of course, the nodes represent the points where two streams come together. When two streams of the same order k meet, they form a stream of order $k+1$, whereas if one of them has a lower order it is viewed as subordinate to the higher stream, the order of which thus remains unchanged. The index of a stream or river may range from 1 (a stream with no tributaries) to 12 (the most powerful river, the Amazon, at its mouth). The Ohio River is of order eight and the Mississippi River is of order 10. 80% of the streams and rivers on the planet are first or second order [☞ http://en.wikipedia.org/wiki/Strahler_number].

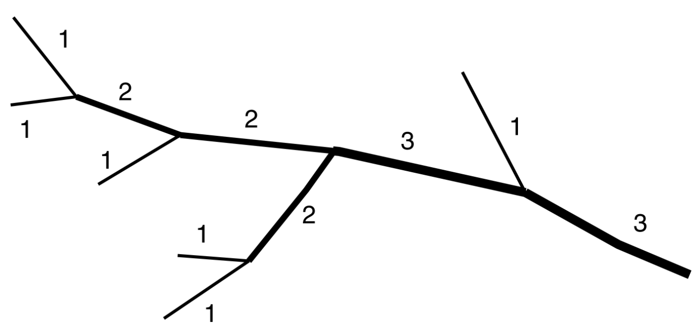

Fig. 1.5 U.S. Corps of Engineer diagram showing the Strahler stream order.
[license by http://creativecommons.org/licenses/by-sa/3.0/deed.en]

Now let us return to the mathematical framework. Given a finite tree τ, the above definition of the $(S(\mathsf{v}))_{\mathsf{v}\in\tau}$ provides us with another example of a RTP of finite depth which becomes stochastic as soon as τ is chosen by some random mechanism. For instance τ may be the realization of a Galton-Watson tree up to some finite generation. In this case, with $(Z_n)_{n\ge0}$ denoting the associated GWP, one can easily derive the following random recursive equation for the Strahler index S_n of the Galton-Watson tree up to generation n:

$$S_n \;=\; \mathbf{1}_{\{Z_1=0\}}+\mathbf{1}_{\{Z_1\ge1\}}\Big(\max_{1\le k\le Z_1} S_{n-1}(k)+\mathbf{1}_{\{N_n>1\}}\Big), \tag{1.44}$$

where $S_{n-1}(k)$ denotes the Strahler index of the subtree rooted at the k^{th} member of the first generation and $N_n := |\{1 \le k \le Z_1 : S_{n-1}(k) = \max_{1\le i\le Z_1} S_{n-1}(i)\}|$. In this formulation, only $S_{n-1}(1),...,S_{n-1}(Z_1)$ are specified and, given Z_1, conditionally i.i.d. with the same distribution as S_{n-1}. However, we can also define an infinite sequence $(S_{n-1}(k))_{k\ge1}$ of independent copies of S_{n-1} which are unconditionally independent of Z_1. This does not affect the validity of (1.44). Since N_n is then obviously a measurable function of $Z_1, S_{n-1}(1), S_{n-1}(2),...$, we see that (1.44) fits into the general form

$$S_n \ = \ \Psi(Z_1, S_{n-1}(1), S_{n-1}(2), ...)$$

for some measurable function Ψ (not depending on n).

As another example, one may consider $(S(v))_{v\in\tau}$ when τ is drawn at random from the set of binary trees with n nodes. This was done by DEVROYE & KRUSCZEWSKI [21] who showed that, if $S_n := S(\varnothing)$, then

$$\begin{aligned} \mathbb{E}S_n \ &= \ \log_4 n + O(1) \quad \text{as } n \to \infty \\ \text{and} \quad \mathbb{P}(|S_n - \log_4 n| \ge x) \ &\le \ c4^{-x} \end{aligned}$$

for all $x > 0$, $n \ge 1$ and some $c > 0$. Therefore, the distribution of S_n exhibits

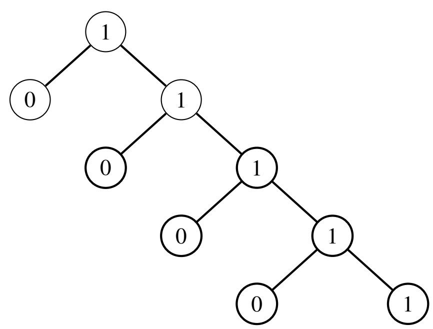

Fig. 1.6 The binary tree with 5 internal nodes having minimal Strahler number 1. Due to its shape when including external nodes (those with numbers 0) it is sometimes called *"gourmand de la vigne"*.

very sharp concentration about its mean which is approximately equal to $\log_4 n$. In connection with this result it is worthwhile to point out that the binary trees with extremal Strahler numbers are

- the single-stranded tree with n nodes and Strahler number 1 [☞ Fig. 1.6],
- the complete tree with k levels, $n = 2^k - 1$ nodes and Strahler number $S_n = k = \log_2(n+1)$ [☞ Fig. 1.7].

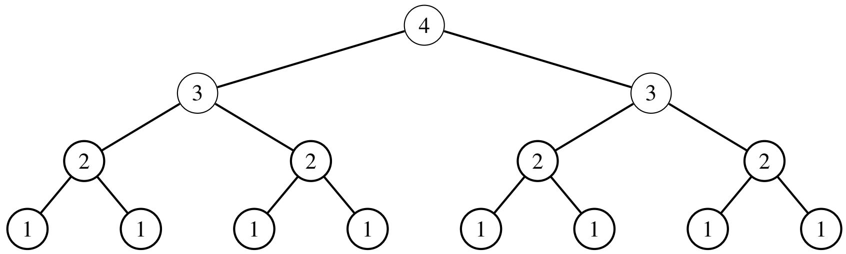

Fig. 1.7 The binary tree with $2^4 - 1 = 15$ internal nodes and maximal Strahler number 4.

Chapter 2
Renewal theory

Terminology. An additive sequence $(S_n)_{n\ge 0}$ of real-valued random variables with increments $X_1, X_2, \ldots$ is called

random walk (RW)	if $X_1, X_2, \ldots$ are iid and independent of S_0;
renewal process (RP)	if it is a RW such that $S_0, X_1, X_2, \ldots$ are nonnegative and $\mathbb{P}(X_1 > 0) > 0$;
standard random walk (SRW)	if it is a RW with $S_0 = 0$. It is also called *zero-delayed RW*;
standard renewal process (SRP)	if it is a RP with $S_0 = 0$. It is also called *zero-delayed RP*.

Given a RW $(S_n)_{n\ge 0}$, we use X for a generic copy of its increments. The initial variable S_0 is also called *delay*, the mean of X, if it exists, the *drift* of $(S_n)_{n\ge 0}$. Finally, we are given a *standard model* $(\Omega, \mathfrak{A}, (S_n)_{n\ge 0}, (\mathbb{P}_\lambda)_{\lambda\in\mathscr{P}(\mathbb{R})})$ if $(S_n)_{n\ge 0}$, defined on $(\Omega, \mathfrak{A})$, constitutes a RW under each $\mathbb{P}_\lambda$ with the same increment distribution F, say, and $\mathbb{P}_\lambda(S_0 \in \cdot) = \lambda$, hence $\mathbb{P}_\lambda(S_n \in \cdot) = \lambda * F^{*n}$ for each $n \in \mathbb{N}$, where F^{*n} denotes n-fold convolution of F.

2.1 An introduction and first results

Let us begin with a short description of the classical renewal problem: Suppose we are given an infinite supply of light bulbs which are used one at a time until they fail. Their lifetimes are denoted as $X_1, X_2, \ldots$ and assumed to be iid random variables with positive mean μ. If the first light bulb is installed at time $S_0 := 0$, then

$$S_n := \sum_{k=1}^{n} X_k \quad \text{for } n \ge 1$$

denotes the time at which the n^{th} bulb fails and is replaced with a new one. In other words, each S_n marks a renewal epoch. Due to this interpretation, a sequence

$(S_n)_{n\ge 0}$ with iid nonnegative increments having positive mean is called *renewal process (RP)*. Let $N(t)$ denote the number of renewals up to time t, that is

$$N(t) \;:=\; \sup\{n\ge 0: S_n\le t\}\quad \text{for } t\ge 0. \tag{2.1}$$

An equivalent definition is

$$N(t) \;:=\; \sum_{n\ge 1}\mathbf{1}_{[0,t]}(S_n)$$

and has the advantage that it immediately extends to general measurable subsets A of $\mathbb{R}_\ge$ by putting

$$N(A) \;:=\; \sum_{n\ge 1}\mathbf{1}_A(S_n) \;=\; \sum_{n\ge 1}\delta_{S_n}(A). \tag{2.2}$$

We see that N is in fact a *random counting measure*, also called *point process*, on $(\mathbb{R}_\ge,\mathscr{B}(\mathbb{R}_\ge))$. By further defining its *intensity measure*

$$\mathbb{U}(A) \;:=\; \mathbb{E}N(A) \;=\; \sum_{n\ge 1}\mathbb{P}(S_n\in A),\quad A\in\mathscr{B}(\mathbb{R}_\ge), \tag{2.3}$$

we arrive at the so-called *renewal measure* of $(S_n)_{n\ge 1}$ which measures the expected number of renewals in a set and is one of the central objects in renewal theory. Its “distribution function”

$$[0,\infty)\;\ni\; t\;\mapsto\;\mathbb{U}(t)\;:=\;\mathbb{U}([0,t])\;=\;\sum_{n\ge 1}\mathbb{P}(S_n\le t) \tag{2.4}$$

is called *renewal function* of $(S_n)_{n\ge 1}$ and naturally of particular interest.

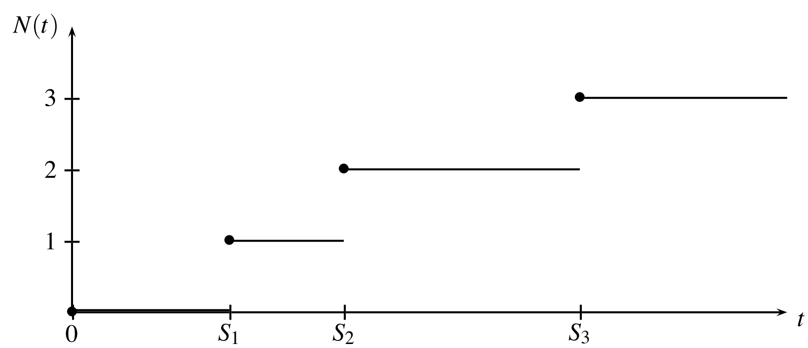

Fig. 2.1 The renewal counting process $(N(t))_{t\ge 0}$ with renewal epochs $S_1,S_2,\ldots$

Natural questions to be asked now are ...

(Q1) Is the number of renewals up to time t, denoted as $N(t)$, almost surely finite for all $t>0$? And what about its expectation $\mathbb{E}N(t)$?

(Q2) What is the asymptotic behavior of $t^{-1}N(t)$ and its expectation as $t\to\infty$, that is the long run average (expected) number of renewals per unit of time?
(Q3) What can be said about the long run behavior of $\mathbb{E}(N(t+h)-N(t))$ for any fixed $h>0$?

... with partial answers provided by the following lemma. Given a distribution F, let F^{*n} denote its n-fold convolution for $n\in\mathbb{N}$ and $F^{*0}:=\delta_0$.

Lemma 2.1. *Let $(S_n)_{n\ge 0}$ be a RP with $S_0=0$, increment distribution F, drift $\mu=\mathbb{E}S_1\in(0,\infty]$ and renewal measure $\mathbb{U}=\sum_{n\ge 1}\mathbb{P}(S_n\in\cdot)$. Then the following assertions hold true:*

(a) $N(t)<\infty$ *a.s. for all* $t\ge 0$.
(b) $\mathbb{P}(N(t)=n)=F^{*n}(t)-F^{*(n+1)}(t)$ *for all* $n\in\mathbb{N}_0$ *and* $t\ge 0$.
(c) $\mathbb{U}=\sum_{n\ge 1}F^{*n}$, *in particular* $\mathbb{U}(t)=\sum_{n\ge 1}F^{*n}(t)$ *for any* $t\ge 0$.
(d) $\mathbb{E}e^{aN(t)}<\infty$ *for all* $t\ge 0$ *and some* $a>0$.
(e) $t^{-1}N(t)\to\mu^{-1}$ *a.s. with the usual convention that* $\infty^{-1}:=0$.
(f) **[Elementary Renewal Theorem]** $\lim_{t\to\infty}t^{-1}\mathbb{U}(t)=\mu^{-1}$.

Proof. (a) follows immediately from $S_n\to\infty$ a.s.

(b) follows when noting that

$$\{N(t)=n\}=\{S_n\le t<S_{n+1}\}=\{S_n\le t\}\backslash\{S_{n+1}\le t\}$$

for all $n\in\mathbb{N}_0$ and $t\ge 0$.

(c) Here it suffices to note that $\mathscr{L}(S_n)=F^{*n}$ for all $n\in\mathbb{N}_0$.

(d) Since $\mu=\mathbb{E}S_1>0$, there exists $b>0$ such that $F(b)<1$. Consider the RP $(S_n')_{n\ge 0}$ with increments given by $X_n':=b\mathbf{1}_{\{X_n>b\}}$ for $n\in\mathbb{N}$ and renewal counting process $(N'(t))_{t\ge 0}$. Then $S_n'\le S_n$ for all $n\in\mathbb{N}_0$ implies $N(t)\le N'(t)$ for all $t\ge 0$. Now observe that, for $n\in\mathbb{N}$ and $0<t<b$,

$$\mathbb{P}(N'(t)>n)\ =\ \mathbb{P}(X_1'=...=X_n'=0)\ =\ F(b)^n$$

implying $\mathbb{E}e^{aN(t)}\le\mathbb{E}e^{aN'(t)}<\infty$ for any $a<-\log F(b)$ as one easily see. We leave it as an exercise [☞ Problem 2.3] to extend the last assertion to all $t\ge b$.

(e) Since $N(t)\to\infty$ a.s., the SLLN implies $N(t)^{-1}S_{N(t)}\to\mu$ a.s. By combining this with the obvious inequality $S_{N(t)}\le t<S_{N(t)+1}$ [use (2.1)] we find

$$\frac{S_{N(t)}}{N(t)}\ \le\ \frac{t}{N(t)}\ \le\ \frac{N(t)+1}{N(t)}\cdot\frac{S_{N(t)+1}}{N(t)+1}$$

and then obtain $t^{-1}N(t)\to\mu$ a.s. by letting t tend to ∞ in this inequality.

(f) Use $\mathbb{U}(t)=\mathbb{E}N(t)$, (e) and Fatou's lemma to infer

$$\liminf_{t\to\infty}\frac{\mathbb{U}(t)}{t}\ \ge\ \mathbb{E}\left(\liminf_{t\to\infty}\frac{N(t)}{t}\right)\ \ge\ \frac{1}{\mu}.$$

Towards a reverse estimate, notice that

$$N(t)+1=\tau(t):=\inf\{n\ge 0:S_n>t\}$$

and thus $\mathbb{U}(t)+1=\mathbb{E}\tau(t)$ for all $t\ge 0$. If $X_1,X_2,...$ are bounded by some $c>0$, in particular giving $\mu<\infty$, then we obtain with the help of Wald's identity [☞ Prop. 2.53]

$$\mathbb{E}\tau(t) \;=\; \frac{\mathbb{E}S_{\tau(t)}}{\mu} \;=\; \frac{t+\mathbb{E}(S_{\tau(t)}-t)}{\mu} \;\le\; \frac{t+c}{\mu}$$

and thereby

$$\limsup_{t\to\infty}\frac{\mathbb{U}(t)}{t} \;\le\; \limsup_{t\to\infty}\frac{\mathbb{E}\tau(t)}{t} \;\le\; \frac{1}{\mu}$$

as required. Otherwise, consider the RP $(S_{c,n})_{n\ge 0}$ with generic increment $X\wedge c$, drift $\mu_c:=\mathbb{E}(X\wedge c)>0$ and renewal measure $\mathbb{U}_c$. Plainly, $\mathbb{U}(t)\le\mathbb{U}_c(t)$ for all $t\ge 0$, whence

$$\limsup_{t\to\infty}\frac{\mathbb{U}(t)}{t} \;\le\; \limsup_{t\to\infty}\frac{\mathbb{U}_c(t)}{t} \;\le\; \frac{1}{\mu_c}$$

for any $c>0$. Finally, use $\lim_{c\to\infty}\mu_c=\mu$ to arrive at

$$\limsup_{t\to\infty}\frac{\mathbb{U}(t)}{t} \;\le\; \frac{1}{\mu}$$

which completes the proof of the lemma. □

Remark 2.2. The reader is asked in Problem 2.4 below to verify that all assertions of the previous lemma except for (b) remain valid if $(S_n)_{n\ge 0}$ has arbitrary delay distribution $F_0:=\mathbb{P}(S_0\in\cdot)$. As for part (b), it must be modified as

$$\mathbb{P}(N(t)=n) \;=\; F_0*F^{*(n-1)}(t)-F_0*F^{*n}(t)$$

for $n\in\mathbb{N}$ and $t\ge 0$, and $\mathbb{P}(N(t)=0)=1-F_0(t)$.

Problems

Problem 2.3. Let $(S_n)_{n\ge 0}$ be a RP with $S_0=0$, increment distribution $Bern(p)$ for some $p\in(0,1)$ and renewal counting process $(N(t))_{t\ge 0}$. Prove the following assertions:

(a) $\mathscr{L}(N(n))=NBin(n+1,p)$ for each $n\in\mathbb{N}_0$.
(b) For any $t\ge 0$, $\mathbb{E}e^{aN(t)}<\infty$ iff $a<-\log(1-p)$.

Problem 2.4. Prove Lemma 2.1, with part (b) modified in the form stated in Rem. 2.2, for a general delayed RP with delay distribution F_0.

2.2 An important special case: exponential lifetimes

A case of particular interest occurs when the RP $(S_n)_{n\ge 0}$ has exponential increments, i.e. $F = Exp(1/\mu)$ for some $\mu > 0$. Then S_n has a Gamma distribution with parameters n and $1/\mu$, i.e. $F^{*n} = \Gamma(n, 1/\mu)$, the $\lambda\!\!\!\lambda$-density of which equals

$$f_n(x) \;=\; \frac{x^{n-1}}{\mu^n(n-1)!}\, e^{-x/\mu}\, \mathbf{1}_{(0,\infty)}(x)$$

for each $n \in \mathbb{N}$. Since $\mathbb{U} = \sum_{n\ge 1} F^{*n}$, we find that its λ-density u, called *renewal density*, equals

$$u(x) \;=\; \sum_{n\ge 1} f_n(x) \;=\; e^{-x/\mu} \sum_{n\ge 1} \frac{x^{n-1}}{\mu^n(n-1)!} \;=\; \frac{1}{\mu}$$

for all $x > 0$, hence $\mathbb{U} = \mu^{-1}\lambda\!\!\!\lambda^+$, where $\lambda\!\!\!\lambda^+ := \lambda\!\!\!\lambda(\cdot \cap \mathbb{R}_>)$. Equivalently, the expected number of renewals in an interval $[t, t+h] \subset \mathbb{R}_\ge$ of length $h > 0$ *always* equals $\mu^{-1}h$. The reason lurking behind this phenomenon is of course the lack of memory property of the exponential distribution. Here is a heuristic argument: Suppose we start observing the RP at a time $t > 0$ and reset our clock to 0. Then renewals occur at $S_{\tau(t)} - t, S_{\tau(t)+1} - t, \ldots$ with interrenewal times $X_{\tau(t)+1}, X_{\tau(t)+2}, \ldots$ after the *delay* $R(t) := S_{\tau(t)} - t$. Proposition **??** will show that $R(t)$ and $X_{\tau(t)+1}, X_{\tau(t)+2}, \ldots$ are independent and the latter sequence further iid with $\mathscr{L}(X_{\tau(t)+1}) = Exp(1/\mu)$. Consequently, we will see the same arrival pattern as someone who starts observing the system at time 0 if $\mathscr{L}(R(t)) = Exp(1/\mu)$ as well. But this is indeed intuitively clear by the lack of memory property and may also formally be proved fairly easily [☞ Problem 2.6].

Turning to the associated renewal counting process $(N(t))_{t\ge 0}$, the previous considerations entail that $\mathscr{L}(N(t+h) - N(t)) = \mathscr{L}(N(h))$ for any $t \ge 0$ and $h > 0$ which means that $(N(t))_{t\ge 0}$ *has stationary increments*. They further provide some evidence (though not a proof) that the numbers of renewals in $[0,t]$ and $[t, t+h]$ are independent. In fact, one can more generally show that, for any choice $0 = t_0 < t_1 < \ldots < t_n < \infty$ and $n \in \mathbb{N}$, the random variables $N(t_k) - N(t_{k-1})$, $k = 1, \ldots, n$, are independent which means that $(N(t))_{t\ge 0}$ *has independent increments*. It remains to find the distribution of $N(t)$ for any $t > 0$. By Lemma 2.1(b), it follows that $p_n(t) := \mathbb{P}(N(t) = n)$ satisfies

$$p_n(t) \;=\; F^{*n}(t) - F^{*(n+1)}(t)$$

for all $t > 0$ and $n \in \mathbb{N}_0$. If $n = 0$, this yields

$$p_0(t) \;=\; 1 - F(t) \;=\; e^{-t/\mu}, \quad t > 0.$$

For $n \ge 1$, $p_n(\cdot)$ is differentiable with

$$p_n'(t) \;=\; f_n(t) - f_{n+1}(t) \;=\; e^{-t/\mu}\left(\frac{t^{n-1}}{\mu^n(n-1)!} - \frac{t^n}{\mu^{n+1}n!}\right), \quad t > 0,$$

and $p_n(0) = 0$. Consequently,

$$p_n(t) \;=\; e^{-t/\mu}\,\frac{(t/\mu)^n}{n!}, \quad t > 0,$$

and we have arrived at the following result.

Theorem 2.5. *If $(S_n)_{n\ge 0}$ is a SRP having exponential increments with parameter $1/\mu$, then the associated renewal counting process $(N(t))_{t\ge 0}$ forms a* ***homogeneous Poisson process with intensity (rate)*** *$1/\mu$, that is:*

(PP1) $N(0) = 0$.

(PP2) $(N(t))_{t\ge 0}$ *has independent increments, i.e.,*

$$N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

are independent random variables for each choice of $n \in \mathbb{N}$ and $0 < t_1 < t_2 < \ldots < t_n < \infty$.

(PP3) $(N(t))_{t\ge 0}$ *has stationary increments, i.e., $N(s+t) - N(s) \stackrel{d}{=} N(t)$ for all $s,t \ge 0$.*

(PP4) $N(t) \stackrel{d}{=} Poisson(t/\mu)$ *for each $t \ge 0$.*

If $\mu = 1$, then $(N(t))_{t\ge 0}$ is also called ***standard Poisson process****.*

Poisson processes have many nice properties some of which are stated in the Problem section below.

Problems

Problem 2.6. Let $(S_n)_{n\ge 0}$ be a RP with $S_0 = 0$, increment distribution $Exp(1/\mu)$ for some $\mu > 0$ and renewal measure $\mathbb{U} = \sum_{n\ge 1} \mathbb{P}(S_n \in \cdot)$. Let also $R(t) = S_{\tau(t)} - t$ for $t \ge 0$. Prove the following assertions:

(a) $\mathbb{P}(R(t) > s) = e^{-(s+t)/\mu} + \int_{[0,t]} e^{-(t+s-x)/\mu}\,\mathbb{U}(dx)$ for all $s > 0$.

(b) $\mathscr{L}(R(t)) = Exp(1/\mu)$ for all $t \ge 0$. [Use (a).]

Problem 2.7. Let $(N(t))_{t\ge 0}$ be a homogeneous Poisson process with intensity θ. Find the conditional distribution of $N(s)$ given $N(t) = n$ for any $0 < s < t$ and $n \in \mathbb{N}_0$.

Problem 2.8 (Superposition of Poisson processes). Given two independent homogeneous Poisson processes $(N_1(t))_{t\ge 0}$ and $(N_2(t))_{t\ge 0}$ with intensities θ_1 and θ_2, respectively, prove that the *superposition* $N(t) := N_1(t) + N_2(t)$ for $t \geq 0$ forms a homogeneous Poisson process with intensity $\theta_1 + \theta_2$.

Problem 2.9 (Thinning of Poisson processes). Given a homogeneous Poisson process $(N(t))_{t\geq 0}$ with associated SRP $(S_n)_{n\geq 0}$ of arrival epochs, let $(\xi_n)_{n\geq 1}$ be an independent sequence of iid *Bern(p)* variables for some $p \in (0,1)$. Let $(N_1(t))_{t\geq 0}$ be the *thinning* or *p-thinning* of $(N(t))_{t\geq 0}$, defined by

$$N_1(t) \ := \ \sum_{n\geq 1} \xi_n \delta_{S_n}([0,t]), \quad t \geq 0,$$

and put $N_2(t) = N(t) - N_1(t)$ for $t \geq 0$.

Problem 2.10 (Conditional equidistribution of points). Let $(N(t))_{t\geq 0}$ be a homogeneous Poisson process with intensity θ and associated SRP $(S_n)_{n\geq 0}$. Let further $(U_n)_{n\geq 1}$ be a sequence of iid *Unif*$(0,1)$ variables. Prove that

$$\mathscr{L}((S_1,...,S_n)|N(t) = n) \ = \ \mathscr{L}((tU_{(1)},...,tU_{(n)}))$$

for all $t > 0$ and $n \in \mathbb{N}$, where $(U_{(1)},...,U_{(n)})$ denotes the increasing order statistic of the random vector $(U_1,...,U_n)$. This means that, given $N(t) = n$, a sample of $S_1,...,S_n$ may be generated by throwing n points uniformly at random into the interval $[0,t]$.

2.3 Lattice-type

A more profound analysis of the renewal measure $\mathbb{U}$ of a SRP $(S_n)_{n\geq 0}$ must take into account the fact that, if X takes values only in a closed discrete subgroup of $\mathbb{R}$, thus in $\mathbb{G}_d := d\mathbb{Z}$ for some $d > 0$, then $\mathbb{U}$ puts only mass on this subgroup as well and consequently looks very different from Lebesgue measure encountered in the previous section. The subsequent definitions provide the appropriate specifications of the *lattice-type* of a distribution F on $\mathbb{R}$ and of a RW $(S_n)_{n\geq 0}$.

Definition 2.11. For a distribution F on $\mathbb{R}$, its *lattice-span* $d(F)$ is defined as

$$d(F) \ := \ \sup\{d \in [0,\infty] : F(\mathbb{G}_d) = 1\}.$$

Let $\{F_x : x \in \mathbb{R}\}$ denote the translation family associated with F, i.e., $F_x(B) := F(x+B)$ for all Borel subsets B of $\mathbb{R}$. Then F is called

– *nonarithmetic*, if $d(F) = 0$ and thus $F(\mathbb{G}_d) < 1$ for all $d > 0$.
– *completely nonarithmetic*, if $d(F_x) = 0$ for all $x \in \mathbb{R}$.
– *d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F) = d$.
– *completely d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F_x) = d$ for all $x \in \mathbb{G}_d$.

If X denotes any random variable with distribution F, thus $\mathscr{L}(X-x)=F_x$ for each $x\in\mathbb{R}$, then the previous attributes are also used for X, and we also write $d(X)$ instead of $d(F)$ and call it the lattice-span of X.

For our convenience, a nonarithmetic distribution is sometimes referred to as *0-arithmetic* hereafter, for example in the lemma below. A random variable X is nonarithmetic iff it is not a.s. taking values only in a lattice $\mathbb{G}_d$, and it is completely nonarithmetic if this is not either the case for any shifted lattice $x+\mathbb{G}_d$, i.e. any affine closed subgroup of $\mathbb{R}$. As an example of a nonarithmetic, but not completely nonarithmetic random variable we mention $X=\pi+Y$ with a standard Poisson variable Y. Then $d(X-\pi)=d(Y)=1$. If $X=\frac{1}{2}+Y$, then $d(X)=\frac{1}{2}$ and $d(X-\frac{1}{2})=1$. In this case, X is $\frac{1}{2}$-arithmetic, but not completely $\frac{1}{2}$-arithmetic. The following simple lemma provides the essential property of a completely d-arithmetic random variable $(d\ge 0)$.

Lemma 2.12. *Let X,Y be two iid random variables with lattice-span $d\ge 0$. Then $d\le d(X-Y)$ with equality holding iff X is completely d-arithmetic.*

Proof. Let F denote the distribution of X,Y. The inequality $d\le d(X-Y)$ is trivial, and since $(X+z)-(Y+z)=X-Y$, we also have $d(X+z)\le d(X-Y)$ for all $z\in\mathbb{R}$. Suppose X is *not* completely d-arithmetic. Then $d(X+z)>d$ for some $z\in\mathbb{G}_d$ and hence also $c:=d(X-Y)>d$. Conversely, if the last inequality holds true, then

$$1 \;=\; \mathbb{P}(X-Y\in\mathbb{G}_c) \;=\; \int_{\mathbb{G}_d}\mathbb{P}(X-y\in\mathbb{G}_c)\,F(dy)$$

implies

$$\mathbb{P}(X-y\in\mathbb{G}_c) \;=\; 1 \quad \text{for all } F\text{-almost all } y\in\mathbb{G}_d$$

and thus $d(X-y)\ge c>d$ for F-almost all $y\in\mathbb{G}_d$. Therefore, X cannot be completely d arithmetic. □

Definition 2.13. A RW $(S_n)_{n\ge 0}$ with increments $X_1,X_2,\ldots$ is called

– *(completely) nonarithmetic*	if X_1 is (completely) nonarithmetic.
– *(completely) d-arithmetic*	if $d>0$, $\mathbb{P}(S_0\in\mathbb{G}_d)=1$, and X_1 is (completely) d-arithmetic.

Furthermore, the lattice-span of X_1 is also called the lattice-span of $(S_n)_{n\ge 0}$ in any of these cases.

The additional condition on the delay in the d-arithmetic case, which may be restated as $d(S_0)=kd$ for some $k\in\mathbb{N}\cup\{\infty\}$, is needed to ensure that $(S_n)_{n\ge 0}$ is really concentrated on the lattice $\mathbb{G}_d$. The unconsidered case where $(S_n)_{n\geq 0}$ has d-arithmetic increments but non- or c-arithmetic delay for some $c \notin \mathbb{G}_d \cup \{\infty\}$ will not play any role in our subsequent analysis.

2.4 Uniform local boundedness and stationary delay distribution

Given a RP $(S_n)_{n\geq 0}$ with renewal measure $\mathbb{U} = \sum_{n\geq 0} \mathbb{P}(S_n \in \cdot)$ and renewal counting measure $N = \sum_{n\geq 0} \delta_{S_n}$, we now turn to question (Q3) about the asymptotic behavior of $\mathbb{U}((t,t+h]) = \mathbb{E}(N(t+h) - N(t))$ for any fixed $h > 0$. Notice that, unlike in the previous sections, summation in the definitions of $\mathbb{U}$ and N now ranges over $n \geq 0$. Denoting by λ and F the distribution of S_0 and X, we thus have

$$\mathbb{U} \;=\; \sum_{n\geq 0} \lambda * F^{*n} \;=\; \lambda * \sum_{n\geq 0} F^{*n} \;=\; \lambda * \mathbb{U}_0, \tag{2.5}$$

where $\mathbb{U}_0$ is the renewal measure of the SRP $(S_n - S_0)_{n\geq 0}$. Assuming a standard model, (2.5) may in fact also be stated as $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ for any $\lambda \in \mathscr{P}(\mathbb{R}_\geq)$ if $\mathbb{U}_\lambda$ denotes the renewal measure under $\mathbb{P}_\lambda$ and $\mathbb{U}_x$ is used for $\mathbb{U}_{\delta_x}$.

2.4.1 Uniform local boundedness

The first step towards our main results in the next sections is the following lemma which particularly shows *uniform local boundedness* of $\mathbb{U}_\lambda$, defined by

$$\sup_{t\in\mathbb{R}} \mathbb{U}([t,t+h]) \;<\; \infty \quad \text{for all } h > 0.$$

Lemma 2.14. *Let $(S_n)_{n\geq 0}$ be a RP in a standard model. Then*

$$\sup_{t\in\mathbb{R}} \mathbb{P}_\lambda(N([t,t+h]) \geq n) \;\leq\; \mathbb{P}_0(N(h) \geq n) \tag{2.6}$$

for all $h > 0$, $n \in \mathbb{N}_0$ and $\lambda \in \mathscr{P}(\mathbb{R}_\geq)$. In particular,

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda([t,t+h]) \;\leq\; \mathbb{U}_0(h) \tag{2.7}$$

and $\{N([t,t+h]) : t \in \mathbb{R}\}$ is uniformly integrable under each $\mathbb{P}_\lambda$ for all $h > 0$.

Proof. If (2.6) holds true, then the uniform integrability of $\{N([t,t+h]) : t \in \mathbb{R}\}$ is a direct consequence, while (2.7) follows by summation over n. So (2.6) is the only assertion to be proved. Fix $t \in \mathbb{R}$, $h > 0$, and define $\tau := \inf\{n \geq 0 : S_n \in [t,t+h]\}$.

Then

$$N([t,t+h]) \;=\; \begin{cases} \sum_{k\ge 0}\mathbf{1}_{[t,t+h]}(S_{\tau+k}), & \text{if } \tau<\infty,\\ 0, & \text{otherwise.}\end{cases}$$

The desired estimate now follows from

$$\begin{aligned}\mathbb{P}_\lambda(N([t,t+h])\ge n) \;&=\; \mathbb{P}_\lambda\left(\tau<\infty,\sum_{k\ge 0}\mathbf{1}_{[t,t+h]}(S_{\tau+k})\ge n\right)\\ &=\; \sum_{j\ge 0}\mathbb{P}_\lambda\left(\tau=j,\sum_{k\ge 0}\mathbf{1}_{[t,t+h]}(S_{j+k})\ge n\right)\\ &\le\; \sum_{j\ge 0}\mathbb{P}_\lambda\left(\tau=j,\sum_{k\ge 0}\mathbf{1}_{[0,h]}(S_{j+k}-S_j)\ge n\right)\\ &=\; \sum_{j\ge 0}\mathbb{P}_\lambda(\tau=j)\,\mathbb{P}_0(N(h)\ge n)\\ &=\; \mathbb{P}_\lambda(\tau<\infty)\,\mathbb{P}_0(N(h)\ge n)\end{aligned}$$

for all $n\in\mathbb{N}$ and $\lambda\in\mathscr{P}(\mathbb{R}_\ge)$. □

2.4.2 *Finite mean case: the stationary delay distribution*

As already explained in the previous section, the behavior of $\mathbb{U}((t,t+h])$ is expected to be different depending on whether the underlying RP $(S_n)_{n\ge 0}$ is arithmetic or not. We make the standing assumption hereafter that $(S_n)_{n\ge 0}$ has either lattice-span $d=0$ or $d=1$. The latter is no restriction in the arithmetic case because one may otherwise switch to the RP $(d^{-1}S_n)_{n\ge 0}$. Recall that $\mathbb{G}_d=d\mathbb{Z}$ for $d>0$ and put also $\mathbb{G}_0=\mathbb{R}$ as well as $\mathbb{G}_{d,\alpha}:=\mathbb{G}_d\cap\mathbb{R}_\alpha$ for $\alpha\in\{\ge,>\}$. Let $\lambda\!\!\lambda_0$ denote Lebesgue measure, thus $\lambda\!\!\lambda_0=\lambda\!\!\lambda$, and $\lambda\!\!\lambda_1$ counting measure on $\mathbb{Z}$. Since $\mathbb{U}$ is concentrated on $\mathbb{Z}$ in the 1-arithmetic case, it is clear that convergence of $\mathbb{U}((t,t+h])$ in this case can generally take place only as $t\to\infty$ through $\mathbb{Z}$. This should be kept in mind for the following discussion.

Intuitively, the asymptotic behavior of $\mathbb{U}((t,t+h])$ should not depend on where the RP started, that is, on the delay distribution. In a standard model, this means that the limit of $\mathbb{U}_\lambda((t,t+h])$, if it exists, should be independent of $\lambda\in\mathscr{P}(\mathbb{G}_d)$. If we can find a delay distribution ν such that $\mathbb{U}_\nu$ may be computed explicitly, in particular $\mathbb{U}_\nu((t,t+h])$ for any $h>0$ and $t\to\infty$ through $\mathbb{G}_d$, then we may hope for being able to provide a coupling argument that shows $|\mathbb{U}_\lambda((t,t+h])-\mathbb{U}_\nu((t,t+h])|\to 0$ for any $\lambda\in\mathscr{P}(\mathbb{G}_{d,\ge})$ and thus confirm the afore-mentioned intuition. For a quick assessment of what the limit of $g(h)=\lim_{\mathbb{G}_d\ni t\to\infty}\mathbb{U}_\lambda((t,t+h])$ for any $h>0$ looks like, observe that

$$g(h_1+h_2) \;=\; \lim_{\mathbb{G}_d\ni t\to\infty}\mathbb{U}_\lambda((t,t+h_1]) \;+\; \lim_{\mathbb{G}_d\ni t\to\infty}\mathbb{U}_\lambda((t+h_1,t+h_1+h_2])$$

$$= \; g(h_1) + g(h_2) \quad \text{for all positive } h_1, h_2 \in \mathbb{G}_d$$

which shows that g must be linear on $\mathbb{G}_{d,\geq}$. In combination with the elementary renewal theorem, this entails that $g(h) = h/\mu$ for all $h \in \mathbb{G}_{d,>}$, thus $g \equiv 0$ if $\mu = \infty$.

Suppose now we are given a RP $(S_n)_{n\geq 0}$ in a standard model with *finite* drift μ and increment distribution F. The first thing to note is that $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ satisfies the convolution equation

$$\mathbb{U}_\lambda \; = \; \lambda + F * \mathbb{U}_\lambda \quad \text{for any } \lambda \in \mathscr{P}(\mathbb{R}_\geq)$$

which in terms of the renewal function becomes a so-called *renewal equation* to be studied in more detail in Section 2.7, namely

$$\mathbb{U}_\lambda(t) \; = \; \lambda(t) \; + \; \int_{[0,t]} F(t-x) \; \mathbb{U}_\lambda(dx) \quad \text{for any } \lambda \in \mathscr{P}(\mathbb{R}_\geq) \tag{2.8}$$

The goal is to find a λ such that $\mathbb{U}_\lambda(t) = \mu^{-1} t$ for all $t \in \mathbb{R}_\geq$ (thus $\mathbb{U}_\lambda = \mu^{-1}\lambda\!\!\lambda_0^+$) and we will now do so by simply plugging the result into (2.8) and solving for $\lambda(t)$. Then, with $\overline{F} := 1 - F$,

$$\begin{aligned} \lambda(t) \; &= \; \frac{t}{\mu} \; - \; \frac{1}{\mu} \int_0^t F(t-x) \; dx \\ &= \; \frac{1}{\mu} \int_0^t \overline{F}(t-x) \; dx \; = \; \frac{1}{\mu} \int_0^t \overline{F}(x) \; dx \quad \text{for all } t \geq 0. \end{aligned}$$

We thus see that there is only one λ, now called F^s, that gives the desired property of $\mathbb{U}_\lambda$, viz.

$$F^s(t) \; := \; \frac{1}{\mu} \int_0^t \overline{F}(x) \; dx \; = \; \frac{1}{\mu} \int_0^t \mathbb{P}(X > x) \; dx \quad \text{for all } t \geq 0,$$

which is continuous and requires that μ is finite. To all those who prematurely lean back now let is be said that this is not yet the end of the story because there are questions still open, viz. "Is this really the answer we have been looking for if the RP is arithmetic?" and "What about the infinite mean case?"

If $(S_n)_{n\geq 0}$ is 1-arithmetic a continuous delay distribution appears to be inappropriate because it gives a continuous renewal measure. In fact, the stationary delay distribution F^s must now rather be concentrated on $\mathbb{N}$, but only give $\mathbb{U}_{F^s}(t) = \mu^{-1} t$ for $t \in \mathbb{N}_0$. By pursuing the same argument as above, but for $t \in \mathbb{N}_0$ only, one finds [☞ Problem 2.18] that F^s must satisfy

$$F^s(n) \; = \; \frac{1}{\mu} \sum_{k=0}^{n-1} \overline{F}(k) \; = \; \frac{1}{\mu} \sum_{k=1}^{n} \mathbb{P}(X \geq k) \quad \text{for all } n \in \mathbb{N}$$

as the unique solution among all distributions concentrated on $\mathbb{N}$. We summarize our findings as follows.

Proposition 2.15. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with finite drift μ and lattice-span $d\in\{0,1\}$. Define its* ***stationary delay distribution*** *F^s on $\mathbb{R}_>$ by*

$$F^s(t) := \begin{cases} \frac{1}{\mu}\int_0^t \mathbb{P}(X>x)\,dx, & \text{if } d=0,\\ \frac{1}{\mu}\sum_{k=1}^{n(t)}\mathbb{P}(X\ge k), & \text{if } d=1 \end{cases} \tag{2.9}$$

for $t\in\mathbb{R}_>$, where $n(t):=\lfloor t\rfloor=\sup\{n\in\mathbb{Z}: n\le t\}$. Then $\mathbb{U}_{F^s}=\mu^{-1}\lambda\!\!\lambda_d^+$.

Now observe that the integral equation (2.8) remains valid if λ is any locally finite measure on $\mathbb{R}_\ge$ and $\mathbb{U}_\lambda$ is still defined as $\lambda*\mathbb{U}_0$. This follows because (2.8) is linear in λ. Hence, if we drop the normalization μ^{-1} in the definition of F^s, we obtain without further ado the following extension of the previous proposition.

Corollary 2.16. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with lattice-span $d\in\{0,1\}$. Define the locally finite measure ξ on $\mathbb{R}_>$ by*

$$\xi(t) := \begin{cases} \int_0^t \mathbb{P}(X>x)\,dx, & \text{if } d=0,\\ \sum_{k=1}^{n(t)}\mathbb{P}(X\ge k), & \text{if } d=1 \end{cases} \tag{2.10}$$

for $t\in\mathbb{R}_>$ and $n(t)$ as in Prop. 2.15. Then $\mathbb{U}_\xi=\lambda\!\!\lambda_d^+$.

2.4.3 Infinite mean case: restricting to finite horizons

There is no stationary delay distribution if $(S_n)_{n\ge 0}$ has infinite mean μ, but Cor. 2.16 helps us to provide a family of delay distributions for which stationarity still yields when restricting to finite horizons, that is to time sets $[0,a]$ for $a\in\mathbb{R}_>$. As a further ingredient we need the observation that the renewal epochs in $[0,a]$ of $(S_n)_{n\ge 0}$ and $(S_{a,n})_{n\ge 0}$, where $S_{a,n}:=S_0+\sum_{k=1}^n (X_k\wedge a)$, are the same. As a trivial consequence they also have the same renewal measure on $[0,a]$, whatever the delay distribution is. But by choosing the latter appropriately, we also have a domination result on (a,∞) as the next result shows.

Proposition 2.17. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with drift $\mu=\infty$ and lattice-span $d\in\{0,1\}$. With ξ given by* (2.10) *and for $a>0$, define distributions F_a^s on $\mathbb{R}_>$ by*

$$F_a^s(t) := \frac{\xi(t\wedge a)}{\xi(a)} \quad \text{for } t\in\mathbb{R}_>. \tag{2.11}$$

Then, for all $a \in \mathbb{R}_>$, $\mathbb{U}_{F_a^s} \le \xi(a)^{-1} \lambda\!\!\lambda_d^+$ *with equality holding on* $[0,a]$.

Proof. Noting that F_a^s can be written as $F_a^s = \xi(a)^{-1}\xi - \lambda_a$, where $\lambda_a \in \mathscr{P}(\mathbb{R}_>)$ is given by

$$\lambda_a(t) \;:=\; \frac{\xi(t) - \xi(a \wedge t)}{\xi(a)} \;=\; \mathbf{1}_{(a,\infty)}(t)\frac{\xi(t) - \xi(a)}{\xi(a)} \quad \text{for all } t \in \mathbb{R}_>,$$

we infer with the help of Cor. 2.16 that

$$\mathbb{U}_{F_a^s} \;=\; \xi(a)^{-1}\mathbb{U}_\xi - \lambda_a * \mathbb{U}_0 \;\le\; \xi(a)^{-1}\mathbb{U}_\xi \;=\; \xi(a)^{-1}\lambda\!\!\lambda_d \quad \text{on } \mathbb{R}_>$$

as claimed. □

Problems

Problem 2.18. Given a 1-arithmetic RP $(S_n)_{n\ge 0}$ in a standard model with finite drift μ and increment distribution F, prove that F^s as defined in (2.9) for $d = 1$ is the unique distribution on $\mathbb{N}$ such that $\mathbb{U}_{F^s} = \mu^{-1}\lambda\!\!\lambda_1^+$.

Problem 2.19. Under the assumptions of Prop. 2.15, let μ_p and μ_p^s for $p > 0$ denote the p^{th} moment of F and F^s, respectively. Prove that

$$\mu_p^s \;:=\; \int t^p\, F^s(dt) \;=\; \begin{cases} \dfrac{\mu_{p+1}}{(p+1)\mu}, & \text{if } d = 0,\\[2ex] \dfrac{1}{\mu}\,\mathbb{E}\left(\displaystyle\sum_{n=1}^{X} n^p\right), & \text{if } d = 1. \end{cases} \tag{2.12}$$

and in the 1-arithmetic case furthermore

$$\frac{\mu_{p+1}}{(p+1)\mu} \;\le\; \frac{1}{\mu}\,\mathbb{E}\left(\sum_{n=1}^{X} n^p\right) \;\le\; \frac{\mu_{p+1}}{(p+1)\mu} + \frac{\mathbb{E}(X+1)^p}{\mu}. \tag{2.13}$$

Hence, $\mu_p^s < \infty$ iff $\mu_{p+1} < \infty$. Note also that $\mu^s = \mu_1^s$ satisfies

$$\mu^s \;=\; \frac{\mathbb{E}X(X+d)}{2\mu} \;=\; \frac{\mu_2}{2\mu} + \frac{d}{2} \;=\; \frac{\sigma^2 + \mu^2}{2\mu} + \frac{d}{2}, \tag{2.14}$$

where $\sigma^2 := \mathbb{V}\mathrm{ar}X$.

2.5 Blackwell's renewal theorem

Blackwell's renewal theorem first obtained by ERDÖS, FELLER & POLLARD [27] for arithmetic RP's and by BLACKWELL [15] for nonarithmetic ones, may be rightfully called the mother of all deeper results in renewal theory. Not only it provides an answer to question (Q3) stated in the first section of this chapter on the expected number of renewals in a finite remote interval, but is also the simpler, yet equivalent version of the *key renewal theorem* discussed in the next section that allows us to determine the asymptotic behavior of many interesting quantities in applied stochastic models.

The following notation is introduced so as to facilitate a unified formulation of subsequent results for the arithmetic and the nonarithmetic case. For $d \in \{0,1\}$, define

$$d\text{-}\lim_{t\to\infty} f(t) \;:=\; \begin{cases} \lim_{t\to\infty} f(t), & \text{if } d=0,\\ \lim_{n\to\infty} f(n), & \text{if } d=1.\end{cases}$$

Recall that $\lambda\!\!\lambda_0$ denotes Lebesgue measure on $\mathbb{G}_0 = \mathbb{R}$, while $\lambda\!\!\lambda_1$ is counting measure on $\mathbb{G}_1 = \mathbb{Z}$.

Theorem 2.20. [Blackwell's renewal theorem] *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with lattice-span $d \in \{0,1\}$ and positive drift μ. Then*

$$d\text{-}\lim_{t\to\infty} \mathbb{U}_\lambda([t,t+h]) \;=\; \mu^{-1}\lambda\!\!\lambda_d([0,h]) \tag{2.15}$$

for all $h \ge 0$ and $\lambda \in \mathscr{P}(\mathbb{G}_{d,\ge})$, where $\mu^{-1} := 0$ if $\mu = \infty$.

The result, which actually extends to RW's with positive drift as will be seen later, has been proved by many authors and using various methods. The interested reader is referred to the monography [2, Ch. 3] for a detailed historical account. Here we will employ a coupling argument which to some extent forms a blend of the proofs given by LINDVALL [48], ATHREYA, MCDONALD & NEY [6], THORISSON [58] and finally by LINDVALL & ROGERS [49], all based on coupling as well. The proof is split into several steps given in separate subsections.

2.5.1 *First step of the proof : shaking off technicalities*

1st reduction: $S_0 = 0$.
It is no loss of generality to prove (2.15) for zero-delayed RP's only. Indeed, if S_0 has distribution $\lambda \in \mathscr{P}(\mathbb{G}_{d,\ge})$, then

$$\mathbb{U}_\lambda([t,t+h]) \;=\; \int \mathbb{U}_0([t-x,t-x+h])\,\lambda(dx)$$

together with $\sup_{t\in\mathbb{R}}\mathbb{U}_0([t,t+h])\le\mathbb{U}_0([-h,h])<\infty$ [☞ Lemma 2.14] implies by an appeal to the dominated convergence theorem that (2.15) is valid for $\mathbb{U}_\lambda$ if so for $\mathbb{U}_0$.

2nd reduction: $(S_n)_{n\ge 0}$ is completely d-arithmetic ($d\in\{0,1\}$).
The second reduction that will be useful hereafter is to assume that the increment distribution is completely d-arithmetic so that, by Lemma 2.12, its symmetrization has the the same lattice-span.

Lemma 2.21. *Let $(S_n)_{n\ge 0}$ be a SRP with lattice-span $d\in\{0,1\}$ and renewal measure $\mathbb{U}$. Let $(\rho_n)_{n\ge 0}$ a SRP independent of $(S_n)_{n\ge 0}$ and with geometric increments, viz. $\mathbb{P}(\rho_1=n)=(1-\theta)^{n-1}\theta$ for some $\theta\in(0,1)$ and $n\in\mathbb{N}$. Then $(S_{\rho_n})_{n\ge 0}$ is a completely d-arithmetic SRP with renewal measure $\mathbb{U}^{(\rho)}$ satisfying $\mathbb{U}^{(\rho)}=(1-\theta)\delta_0+\theta\,\mathbb{U}$.*

Proof. First of all, let $(I_n)_{n\ge 1}$ be a sequence of iid Bernoulli variables with parameter θ independent of $(S_n)_{n\ge 0}$. Each I_n may be interpreted as the outcome of a coin tossing performed at time n. Let $(J_n)_{n\ge 0}$ be the SRP associated with $(I_n)_{n\ge 1}$ and let $(\rho_n)_{n\ge 0}$ be the sequence of copy sums associated with $\rho=\rho_1:=\inf\{n\ge 1:I_n=1\}$. Then $(\rho_n)_{n\ge 0}$ satisfies the assumptions of the lemma, and one can easily verify [☞ Problem 2.24] that $(S_{\rho_n})_{n\ge 0}$ forms a SRP. Next observe that, for each $A\in\mathscr{B}(\mathbb{R})$,

$$\mathbb{U}^{(\rho)}(A)-\delta_0(A)\;=\;\mathbb{E}\left(\sum_{n\ge 1}I_n\mathbf{1}_A(S_n)\right)\;=\;\mathbb{E}I_1\left(\mathbb{U}(A)-\delta_0(A)\right)$$

which proves the relation between $\mathbb{U}^{(\rho)}$ and $\mathbb{U}$, for $\mathbb{E}I_1=\theta$.

It remains to show that $(S_{\rho_n})_{n\ge 0}$ is completely d-arithmetic. Let $(S'_n,\rho'_n)_{n\ge 0}$ be an independent copy of $(S_n,\rho_n)_{n\ge 0}$ and put $\rho':=\rho'_1$. By Lemma 2.12, it suffices to show that the symmetrization $S_\rho-S'_{\rho'}$ is d-arithmetic. Since $c:=d(S_\rho-S'_{\rho'})\ge d$, we must only consider the case $c>0$ and then show that $\mathbb{P}(X_1\in c\mathbb{Z})=1$. But

$$1\;=\;\mathbb{P}(S_\rho-S'_{\rho'}\in c\mathbb{Z})\;=\;\sum_{m,n\ge 1}\theta^2(1-\theta)^{m+n-2}\,\mathbb{P}(S_m-S'_n\in c\mathbb{Z})$$

clearly implies $\mathbb{P}(S_m-S'_n\in c\mathbb{Z})=1$ for all $m,n\in\mathbb{N}$. Hence

$$0<\mathbb{P}(S_1-S'_1=0)=\mathbb{P}(S_2-S'_1\in c\mathbb{Z},S_1-S'_1=0)=\mathbb{P}(X_1\in c\mathbb{Z})\,\mathbb{P}(S_1-S'_1=0)$$

giving $\mathbb{P}(X_1\in c\mathbb{Z})=1$ as asserted. $\square$

2.5.2 Setting up the stage: the coupling model

Based on the previous considerations, we now assume that $(S_n)_{n\ge 0}$ is a zero-delayed completely d-arithmetic RP with drift μ. As usual, the increment distribution is denoted by F and a generic copy of the increments by X. The starting point of the coupling construction is to consider this sequence together with a second one $(S_n')_{n\ge 0}$ such that the following conditions are satisfied:

(C1) $(S_n,S_n')_{n\ge 0}$ is a bivariate RP with iid increments (X_n,X_n'), $n\ge 1$.

(C2) $(S_n'-S_0')_{n\ge 0}\stackrel{d}{=}(S_n)_{n\ge 0}$ and thus $X'\stackrel{d}{=}X$.

(C3) $S_0'\stackrel{d}{=}F^s$ if $\mu<\infty$, and $S_0'\stackrel{d}{=}F_a^s$ for some $a>0$ if $\mu=\infty$.

Here F^s and F_a^s denote the stationary delay distribution and its truncated variant defined in (2.9) and (2.11), respectively. By the results in Section 2.5, the renewal measure $\mathbb{U}'$ of $(S_n')_{n\ge 0}$ satisfies $\mathbb{U}'([t,t+h])=\mu^{-1}\lambda\!\!\lambda_d([0,h])$ for all $t,h\in\mathbb{R}_>$ if $\mu<\infty$, and $\mathbb{U}'([t,t+h])\le\xi(a)^{-1}\lambda\!\!\lambda_d([0,h])$ for all $t,h\in\mathbb{R}_>$ if $\mu=\infty$ where $\xi(a)$ tends to ∞ as $a\to\infty$. Hence $\mathbb{U}'$ satisfies (2.15) in the finite mean case and does so approximately for sufficiently large a if $\mu=\infty$. The idea is now to construct a third RP $(S_n'')_{n\ge 0}$ from the given two which is a copy of $(S_n')_{n\ge 0}$ and such that S_n'' is equal or at least almost equal to S_n for all $n\ge T$, T an a.s. finite stopping time for $(S_n,S_n')_{n\ge 0}$, called *coupling time*. This entails that the *coupling process* $(S_n'')_{n\ge 0}$ has renewal measure $\mathbb{U}'$ while simultaneously being close to $\mathbb{U}$ on remote intervals because with high probability such intervals contain only renewal epochs S_n'' for $n\ge T$.

Having outlined the path towards the asserted result we must now complete the specification of the above bivariate model so as to facilitate a successful coupling. But the only unspecified component of the model is the joint distribution of (X,X') for which the following two alternatives will be considered:

(C4a) X and X' are independent or, equivalently, $(S_n)_{n\ge 0}$ and $(S_n')_{n\ge 0}$ are independent.

(C4b) $X'=Y\,\mathbf{1}_{[0,b]}(|X-Y|)+X\,\mathbf{1}_{(b,\infty)}(|X-Y|)$, where Y is an independent copy of X an b is chosen so large that $G_b:=\mathbb{P}(X-Y\in\cdot\,||X-Y|\le b)$ is d-arithmetic (and thus nontrivial).

The existence of b with $d(G_b)=d$ follows from the fact that $G:=\mathbb{P}(X-Y\in\cdot)$ is d-arithmetic together with $G_b\stackrel{w}{\to}G$.

Condition (C4a) is clearly simpler than (C4b) and will serve our needs in the finite mean case in which the symmetrization X_1-X_1' is integrable with mean zero and also d-arithmetic. Hence we infer from Thm. 2.22 below that $(S_n-S_n')_{n\ge 0}$ is (topologically) recurrent on $\mathbb{G}_d$.

On the other hand, if $\mu=\infty$, the difference of two independent X,X' fails to be integrable, while under (C4b) we have $X-X'=(X-Y)\mathbf{1}_{[-b,b]}(X-Y)$ which is again symmetric with mean zero and d-arithmetic by choice of b. Once again we hence infer the recurrence of the symmetric RW $(S_n-S_n')_{n\ge 0}$ on $\mathbb{G}_d$.

We close this subsection with the recurrence theorem for centered RW's needed here to guarantee successful coupling. The proof is omitted because it cannot be given shortly and is of no importance for our purposes. It may be found e.g. in [2, Ch. 2].

Theorem 2.22. *Any SRW $(S_n)_{n\ge 0}$ with lattice-span $d \in \{0,1\}$ and drift zero is (topologically) recurrent on $\mathbb{G}_d$, that is*

$$\mathbb{P}(|S_n - x| < \varepsilon \text{ infinitely often}) = 1$$

for any $x \in \mathbb{G}_d$ and $\varepsilon > 0$.

2.5.3 Getting to the point: the coupling process

In the following suppose that (C1–3) and (C4a) are valid if $\mu < \infty$, while (C4a) is replaced with (C4b) if $\mu = \infty$. Fix any $\varepsilon > 0$ if F is nonarithmetic, while $\varepsilon = 0$ if F has lattice-span $d > 0$. Since $(S_n - S_n')_{n\ge 0}$ is recurrent on $\mathbb{G}_d$ (recall that the delay distribution of S_0' is also concentrated on $\mathbb{G}_d$) we infer the a.s. finiteness of the *ε-coupling time*

$$T \;:=\; \inf\{n \ge 0 : |S_n - S_n'| \le \varepsilon\}$$

and define the *coupling process* $(S_n'')_{n\ge 0}$ by

$$S_n'' \;:=\; \begin{cases} S_n', & \text{if } n \le T, \\ S_n - (S_T - S_T'), & \text{if } n \ge T \end{cases} \quad \text{for } n \in \mathbb{N}_0, \tag{2.16}$$

which may also be stated as

$$S_n'' \;:=\; \begin{cases} S_n', & \text{if } n \le T, \\ S_T' + \sum_{k=T+1}^{n} X_k, & \text{if } n > T \end{cases} \quad \text{for } n \in \mathbb{N}_0. \tag{2.17}$$

The subsequent lemma accounts for the intrinsic properties of this construction.

Lemma 2.23. *Under the stated assumptions, the following assertions hold true for the coupling process $(S_n'')_{n\ge 0}$:*

(a) $(S_n'')_{n\ge 0} \stackrel{d}{=} (S_n')_{n\ge 0}$.
(b) $|S_n'' - S_n| \le \varepsilon$ *for all* $n \ge T$.

Proof. We only need to show (a) because (b) is obvious from the definition of the coupling process and the coupling time. Since T is a stopping time for the bivariate

RP $(S_n,S'_n)_{n\ge 0}$, Problem 2.25 shows that $X_{T+1},X_{T+2},...$ are iid with the same distribution as X and further independent of $T,(S_n,S'_n)_{0\le n\le T}$. But this easily seen to imply assertion (a), namely

$$\begin{aligned}
&\mathbb{P}(S''_0\in B_0,X''_j\in B_j \text{ for } 1\le j\le n)\\
&= \sum_{k=0}^{n}\mathbb{P}(T=k,S'_0\in B_0,X'_j\in B_j \text{ for } 1\le j\le k)\,\mathbb{P}(X_j\in B_j \text{ for } k<j\le n)\\
&\quad +\mathbb{P}(T>n,S'_0\in B_0,X'_j\in B_j \text{ for } 1\le j\le n)\\
&= \sum_{k=0}^{n}\mathbb{P}(T=k,S'_0\in B_0,X'_j\in B_j \text{ for } 1\le j\le k)\,\mathbb{P}(X'_j\in B_j \text{ for } k<j\le n)\\
&\quad +\mathbb{P}(T>n,S'_0\in B_0,X'_j\in B_j \text{ for } 1\le j\le n)\\
&= \mathbb{P}(S'_0\in B_0,X'_j\in B_j \text{ for } 1\le j\le n)
\end{aligned}$$

for all $n\in\mathbb{N}$ and $B_1,...,B_n\in\mathscr{B}(\mathbb{R}_\ge)$. □

Before moving on to the finishing argument, let us note that a coupling with a.s. finite 0-coupling time is called *exact coupling*, while we refer to an *ε-coupling* otherwise.

2.5.4 The final touch

As usual, let $N(I)$ denote the number of renewals S_n in I, and let $N''(I)$ be the corresponding variable for the coupling process $(S''_n)_{n\ge 0}$. Define further $N_k(I) := \sum_{j=0}^{k}\mathbf{1}_I(S_n)$ and $N''_k(I)$ in a similar manner. Fix any $h>0$, $\varepsilon\in(0,h/2)$, and put $I:=[0,h]$, $I_\varepsilon:=[\varepsilon,h-\varepsilon]$, and $I^\varepsilon:=[-\varepsilon,h+\varepsilon]$. The following proof of (2.15) focusses on the slightly more difficult nonarithmetic case, i.e. $d=0$ hereafter. We first treat the case $\mu<\infty$.

A. The finite mean case. By Lemma 2.23(a), $(S''_n)_{n\ge 0}$ has renewal measure $\mathbb{U}'$ which in turn equals $\mu^{-1}\lambda\!\!\lambda_0^+$ by our model assumption (C3). It follows from the coupling construction that

$$\{S''_n\in t+I_\varepsilon\}\ \subset\ \{S_n\in t+I\}\ \subset\ \{S''_n\in t+I^\varepsilon\}$$

for all $t\in\mathbb{R}_\ge$ and $n\ge T$. Consequently,

$$N''(t+I_\varepsilon)-N_T(t+I)\ \le\ N(t+I)\ \le\ N''(t+I^\varepsilon)+N_T(t+I)$$

and therefore, by taking expectations,

$$\mathbb{U}'(t+I_\varepsilon)-\mathbb{E}N_T(t+I)\ \le\ \mathbb{U}(t+I)\ \le\ \mathbb{U}'(t+I^\varepsilon)+\mathbb{E}N_T(t+I) \tag{2.18}$$

for all $t \in \mathbb{R}_{\ge}$. But $\mathbb{U}'(t+I_\varepsilon) = \mu^{-1}(h-2\varepsilon)$ and $\mathbb{U}'(t+I^\varepsilon) = \mu^{-1}(h+2\varepsilon)$ for all $t > \varepsilon$. Moreover, the uniform integrability of $\{N(t+I) : t \in \mathbb{R}\}$ [☞ Lemma 2.14] in combination with $N_T(t+I) \le N(t+I)$ and $\lim_{t\to\infty} N_T(t+I) = 0$ a.s. entails

$$\lim_{t\to\infty} \mathbb{E}N_T(t+I) = 0.$$

Therefore, upon letting t tend to infinity in (2.18), we finally arrive at

$$\frac{h-2\varepsilon}{\mu} \;\le\; \liminf_{t\to\infty} \mathbb{U}(t+I) \;\le\; \limsup_{t\to\infty} \mathbb{U}(t+I) \;\le\; \frac{h+2\varepsilon}{\mu}.$$

As ε can be made arbitrarily small, we have proved (2.15).

B. The infinite mean case. Here we have $\mathbb{U}' \le \xi(a)^{-1}\lambda_0^+$ where a may be chosen so large that $\xi(a)^{-1} \le \varepsilon$. Since validity of (2.18) remains unaffected by the drift assumption, we infer by just using the upper bound

$$\limsup_{t\to\infty} \mathbb{U}(t+I) \;\le\; \xi(a)^{-1}(h+2\varepsilon) \;\le\; \varepsilon(h+2\varepsilon)$$

and thus again the assertion, for ε can be made arbitrarily small. This completes our coupling proof of Blackwell's theorem. □

Problems

Problem 2.24. Let $(S_n)_{n\ge 0}$ and $(\rho_n)_{n\ge 0}$ be two independent SRP's such that $\rho_1, \rho_2, \ldots$ take values in $\mathbb{N}_0$.

(a) Prove that $(S_{\rho_n})_{n\ge 0}$ forms a SRP as well.
(b) Find the distribution of S_{ρ_1} if $(S_n)_{n\ge 0}$ has exponential increments and $\mathscr{L}(\rho_1) = Geom(\theta)$ for some $\theta \in (0,1)$.

Problem 2.25. Let $(S_n)_{n\ge 0}$ be a RW adapted to a filtration $(\mathscr{F}_n)_{n\ge 0}$ such that $\mathscr{F}_n$ is independent of $(X_k)_{k>n}$ for each $n \in \mathbb{N}_0$. Let T be an a.s. finite stopping time with respect to $(\mathscr{F}_n)_{n\ge 0}$.

(a) Prove that X_{T+1} is independent of $\mathscr{F}_T$ and $X_{T+1} \stackrel{d}{=} X_1$.
(b) Use (a) and an induction to infer that $(X_{T+n})_{n\ge 1}$ is a sequence of iid random variables independent of $\mathscr{F}_T$.

2.6 The key renewal theorem

Given a RP $(S_n)_{n\ge 0}$ in a standard model with drift μ and lattice-span d, the simple observation

$$\mathbb{U}_\lambda([t-h,t]) \;=\; \int \mathbf{1}_{[0,h]}(t-x)\,\mathbb{U}_\lambda(dx) \;=\; \mathbf{1}_{[0,h]} * \mathbb{U}_\lambda(t)$$

for all $t \in \mathbb{R}$, $h \in \mathbb{R}_>$ and $\lambda \in \mathscr{P}(\mathbb{R}_\geq)$ shows that the nontrivial part of Blackwell's renewal theorem may also be stated as

$$d\text{-}\lim_{t\to\infty} \mathbf{1}_{[0,h]} * \mathbb{U}_\lambda(t) \;=\; \frac{1}{\mu}\int \mathbf{1}_{[0,h]}\,d\lambda\!\!\lambda_d \tag{2.19}$$

for all $h \in \mathbb{R}_>$ and $\lambda \in \mathscr{P}(\mathbb{G}_{d,\geq})$, in other words, as a limiting result for convolutions of indicators of compact intervals with the renewal measure. This raises the question, supported further by numerous applications [☞ e.g. [2, Ch. 1]], to which class $\mathscr{R}$ of functions $g : \mathbb{R} \to \mathbb{R}$ an extension of (2.19) in the sense that

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}_\lambda(t) \;=\; \frac{1}{\mu}\int g\,d\lambda\!\!\lambda_d \quad \text{for all } g \in \mathscr{R} \tag{2.20}$$

is possible. Obviously, all finite linear combinations of indicators of compact intervals are elements of $\mathscr{R}$. By taking monotone limits of such step functions, one can further easily verify that $\mathscr{R}$ contains any g that vanishes outside a compact interval I and is Riemann integrable on I. On the other hand, in view of applications a restriction to functions with compact support appears to be undesirable and calls for appropriate conditions on g that are not too difficult to check in concrete examples. In the nonarithmetic case one would naturally hope for $\lambda\!\!\lambda_0$-integrability as being a sufficient condition, but unfortunately this is not generally true. The next subsection specifies the notion of *direct Riemann integrability*, first introduced and thus named by Feller [30], and provides also a discussion of necessary and sufficient conditions for this property to hold. Assertion (2.20) for functions g of this kind, called *key renewal theorem*, is proved in Subsection 2.6.2.

2.6.1 Direct Riemann integrability

Definition 2.26. Let g be a real-valued function on $\mathbb{R}$ and define, for $\delta > 0$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
I_{n,\delta} &:= (\delta n, \delta(n+1)],\\
m_{n,\delta} &:= \inf\{g(x) : x \in I_{n,\delta}\}, \quad M_{n,\delta} := \sup\{g(x) : x \in I_{n,\delta}\}\\
\underline{\sigma}(\delta) &:= \delta \sum_{n\in\mathbb{Z}} m_{n,\delta} \quad \text{and} \quad \overline{\sigma}(\delta) := \delta \sum_{n\in\mathbb{Z}} M_{n,\delta}.
\end{aligned}$$

The function g is called *directly Riemann integrable (dRi)* if $\underline{\sigma}(\delta)$ and $\overline{\sigma}(\delta)$ are both absolutely convergent for all $\delta > 0$ and

$$\lim_{\delta\to 0}(\overline{\sigma}(\delta)-\underline{\sigma}(\delta)) \;=\; 0.$$

The definition reduces to ordinary Riemann integrability if the domain of g is only a compact interval instead of the whole line. In the case where $\int_{-\infty}^{\infty} g(x)\,dx$ may be defined as the limit of such ordinary Riemann integrals $\int_{-a}^{b} g(x)\,dx$ with a,b tending to infinity, the function g is called *improperly Riemann integrable*. An approximation of g by upper and lower step functions having integrals converging to a common value is then still only taken over compact intervals which are made bigger and bigger. However, in the above definition such an approximation is required to be possible *directly* over the whole line and therefore of a more restrictive type than improper Riemann integrability.

The following lemma, partly taken from [5, Prop. V.4.1], collects a whole bunch of necessary and sufficient criteria for direct Riemann integrability.

Proposition 2.27. *Let g be an arbitrary real-valued function on $\mathbb{R}$. Then the following two conditions are necessary for direct Riemann integrability:*

(dRi-1) *g is bounded and $\lambda\!\!\lambda_0$-a.e. continuous.*
(dRi-2) *g is $\lambda\!\!\lambda_d$-integrable for all $d \ge 0$.*

Conversely, any of the following conditions is sufficient for g to be dRi:

(dRi-3) *For some $\delta > 0$, $\underline{\sigma}(\delta)$ and $\overline{\sigma}(\delta)$ are absolutely convergent, and g satisfies (dRi-1).*
(dRi-4) *g has compact support and satisfies (dRi-1).*
(dRi-5) *g satisfies (dRi-1) and $f \le g \le h$ for dRi functions f,h.*
(dRi-6) *g vanishes on $\mathbb{R}_<$, is nonincreasing on $\mathbb{R}_\ge$ and $\lambda\!\!\lambda_0$-integrable.*
(dRi-7) *$g = g_1 - g_2$ for nondecreasing functions g_1, g_2 and $f \le g \le h$ for dRi functions f,h.*
(dRi-8) *g^+ and g^- are dRi.*

Proof. (a) Suppose that g is dRi. Then the absolute convergence of $\underline{\sigma}(1)$ and $\overline{\sigma}(1)$ ensures that g is bounded, for

$$\sup_{x\in\mathbb{R}}|g(x)| \;\le\; \sup_{n\in\mathbb{Z}}(|m_n^1|+|M_n^1|) \;<\; \infty.$$

That g must also be $\lambda\!\!\lambda_0$-a.e. continuous is a standard fact from Lebesgue integration theory but may also be quickly assessed as follows: If g fails to have this property then, with $g_*(x) := \liminf_{y\to x} g(y)$ and $g^*(x) := \limsup_{y\to x} g(y)$, we have

$$\alpha \;:=\; \lambda\!\!\lambda_0(\{g^* \ge g_* + \varepsilon\}) > 0 \quad \text{for some } \varepsilon > 0.$$

As $m_{n,\delta} \le g_*(x) \le g^*(x) \le M_{n,\delta}$ for all $x \in (n\delta,(n+1)\delta)$, $n \in \mathbb{Z}$ and $\delta > 0$, it follows that

$$\overline{\sigma}(\delta) - \underline{\sigma}(\delta) \ \ge\ \int (g^*(x) - g_*(x))\, \lambda\!\!\lambda_0(dx) \ \ge\ \varepsilon\alpha \quad \text{for all } \delta > 0$$

which contradicts direct Riemann integrability. We have thus proved necessity of (dRi-1).

As for (dRi-2), it suffices to note that, with

$$\underline{\phi}(\delta) \ :=\ \delta \sum_{n\in\mathbb{Z}} |m_{n,\delta}| \quad \text{and} \quad \delta\overline{\phi}(\delta) \ :=\ \sum_{n\in\mathbb{Z}} |M_{n,\delta}|,$$

we have $\int |g(x)|\, \lambda\!\!\lambda_0(dx) \le \underline{\phi}(1) + \overline{\phi}(1)$ and $\int |g(x)|\, \lambda\!\!\lambda_d(dx) \le \underline{\phi}(d) + \overline{\phi}(d)$ for each $d > 0$.

(b) Turning to the sufficient criteria, put

$$g_\delta \ :=\ \sum_{n\in\mathbb{Z}} m_{n,\delta} \mathbf{1}_{I_{n,\delta}} \quad \text{and} \quad g^\delta \ :=\ \sum_{n\in\mathbb{Z}} M_{n,\delta} \mathbf{1}_{I_{n,\delta}} \quad \text{for } \delta > 0. \tag{2.21}$$

If (dRi-3) holds true, then $g_\delta \uparrow g$ and $g^\delta \downarrow g$ $\lambda\!\!\lambda_0$-a.e. as $\delta \downarrow 0$ by the $\lambda\!\!\lambda_0$-a.e. continuity of g. Hence the monotone convergence theorem implies (using $-\infty < \underline{\sigma}(\delta) \le \overline{\sigma}(\delta) < \infty$)

$$\underline{\sigma}(\delta) \ =\ \int g_\delta \, d\lambda\!\!\lambda_0 \ \uparrow\ \int g \, d\lambda\!\!\lambda_0 \quad \text{and} \quad \overline{\sigma}(\delta) \ =\ \int g^\delta \, d\lambda\!\!\lambda_0 \ \uparrow\ \int g \, d\lambda\!\!\lambda_0$$

proving that g is dRi.

Since each of (dRi-4) and (dRi-5) implies (dRi-3), there is nothing to prove under these conditions.

Assuming (dRi-6), the monotonicity of g on $\mathbb{R}_\ge$ gives

$$M_{n,\delta} = g(n\delta+) \quad \text{and} \quad m_{n,\delta} = g((n+1)\delta) \ge M_{n,\delta} \quad \text{for all } n \in \mathbb{N}_0,\ \delta > 0.$$

Consequently,

$$\begin{aligned} 0 \ \le\ \underline{\sigma}(\delta) \ &\le\ \int_0^\infty g(x)\, dx \ \le\ \overline{\sigma}(\delta) \\ &\le\ \delta g(0) + \underline{\sigma}(\delta) \ \le\ \int_0^\infty g(x)\, dx \ +\ \delta g(0) \ <\ \infty \end{aligned}$$

and therefore $\overline{\sigma}(\delta) - \underline{\sigma}(\delta) \le \delta g(0) \to 0$ as $\delta \to 0$.

Assuming (dRi-7) the monotonicity of g_1 and g_2 ensures that g has at most countably many discontinuities and is thus $\lambda\!\!\lambda_0$-a.e. continuous. g is also bounded because $f \le g \le h$ for dRi function f,h. Hence (dRi-5) holds true.

Finally assuming (dRi-8), note first that g^-, g^+ both satisfy (dRi-1) because this is true for g. Moreover,

$$0 \le g^{\pm} \le (g^{\delta})^{+} + (g_{\delta})^{-} \le \sum_{n\in\mathbb{Z}} \big(|M_{n,\delta}| + |m_{n,\delta}|\big)\, \mathbf{1}_{I_{n,\delta}} \quad \text{for all } \delta > 0$$

whence g^{-}, g^{+} both satisfy (dRi-5). □

For later purposes, we give one further criterion for direct Riemann integrability, but leave the simple proof to the reader [☞ Problem 2.35 and also Problem 4.8 for an extension].

Lemma 2.28. *Let g be a function on $\mathbb{R}$ that vanishes on $\mathbb{R}_{<}$ and is nondecreasing on $\mathbb{R}_{\ge}$. Then $g_\theta(x) := e^{\theta x} g(x)$ is dRi for any $\theta \in \mathbb{R}$ such that g_θ is $\lambda\!\!\lambda_0$-integrable.*

Given a measurable function $f : \mathbb{R} \to \mathbb{R}$ and a standard exponential random variable X, define the *exponential smoothing of f* by

$$\overline{f}(t) := \int_{(-\infty,t]} e^{-(t-x)} f(x)\, \lambda\!\!\lambda_0(dx) = \mathbb{E} f(t-X), \quad t \in \mathbb{R}, \tag{2.22}$$

whenever this function is well-defined, which is obviously the case if $f \in L^1$. It then also has the same integral because

$$\int \overline{f}\, d\lambda\!\!\lambda_0 = \mathbb{E}\left(\int f(t-X)\, d\lambda\!\!\lambda_0\right) = \int f\, d\lambda\!\!\lambda_0. \tag{2.23}$$

In Chapter 4, we will use the fact that *exponential smoothing* of a L^1-function always provides us with a dRi function. This is stated as Lemma 2.30 after the following auxiliary result.

Lemma 2.29. *Suppose that $f \in L^1$ satisfies $f \ge 0$ and $f(t+\varepsilon) \ge r(\varepsilon) f(t)$ for all $t \in \mathbb{R}$, $\varepsilon > 0$ and a function $r : \mathbb{R}_{>} \to \mathbb{R}_{\ge}$ satisfying $\lim_{\varepsilon \downarrow 0} r(\varepsilon) = 1$. Then f is dRi.*

Proof. W.l.o.g. let r be nonincreasing. Then

$$r(\delta) f(n\delta) \le r(x-n\delta) f(n\delta) \le f(x) \le \frac{f((n+1)\delta)}{r((n+1)\delta - x)} \le \frac{f((n+1)\delta)}{r(\delta)}$$

for all $n \in \mathbb{Z}$, $\delta > 0$ and $x \in I_{n,\delta}$ implies

$$r(\delta) f(n\delta) \le m_{n,\delta} \le \frac{1}{\delta} \int_{I_{n,\delta}} f\, d\lambda\!\!\lambda_0 \le M_{n,\delta} \le r(\delta)^{-1} f((n+1)\delta)$$

and therefore

$$\delta r(\delta)\sum_{n\in\mathbb{Z}} f(n\delta) \;\le\; \underline{\sigma}(\delta) \;\le\; \int f\,d\lambda\!\!\lambda_0 \;\le\; \overline{\sigma}(\delta) \;\le\; \frac{\delta}{r(\delta)}\sum_{n\in\mathbb{Z}} f(n\delta)$$

for any $\delta > 0$. Hence, $\delta\sum_{n\in\mathbb{Z}} f(n\delta)$ stays bounded as $\delta \downarrow 0$ and so

$$\overline{\sigma}(\delta) - \underline{\sigma}(\delta) \;\le\; \left(\frac{1}{r(\delta)} - r(\delta)\right)\delta\sum_{n\in\mathbb{Z}} f(n\delta) \;\xrightarrow{\delta\downarrow 0}\; 0$$

as required. □

Lemma 2.30. *For each $f \in L^1$, its exponential smoothing $\overline{f}$ is dRi.*

Proof. By considering f^+ and f^-, we may assume w.l.o.g. that $f \ge 0$. Then

$$\begin{aligned}\overline{f}(t+\varepsilon) \;&=\; e^{-\varepsilon}\int_{(-\infty,t+\varepsilon]} e^{-(t-x)} f(x)\;\lambda\!\!\lambda_0(dx)\\ &\ge\; e^{-\varepsilon}\int_{(-\infty,t]} e^{-(t-x)} f(x)\;\lambda\!\!\lambda_0(dx) \;=\; e^{-\varepsilon}\overline{f}(t)\end{aligned}$$

for all $t \in \mathbb{R}$ and $\varepsilon > 0$, whence we may invoke the previous lemma to infer that f is dRi. □

2.6.2 The key renewal theorem: statement and proof

We are now ready to formulate and prove the announced extension of Blackwell's renewal theorem. In allusion to its eminent importance in applications Smith [54] called it *key renewal theorem*. The proof presented here is essentially due to FELLER [30].

Theorem 2.31. [Key renewal theorem] *Let $(S_n)_{n\ge 0}$ be a RP with drift μ, lattice-span $d \in \{0,1\}$ and renewal measure $\mathbb{U}$. Then*

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}(t) \;=\; \frac{1}{\mu}\int_{\mathbb{R}_\ge} g\; d\lambda\!\!\lambda_d \tag{2.24}$$

for every dRi function $g : \mathbb{R} \to \mathbb{R}$ vanishing on the negative halfline.

Listing non- and d-arithmetic case separately, (2.24) takes the form

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}(t) \;=\; \frac{1}{\mu}\int_0^\infty g(x)\;dx \tag{2.25}$$

if $d=0$ where the right-hand integral is meant as an improper Riemann integral. In the case $d=1$, we have accordingly

$$\lim_{n\to\infty} g*\mathbb{U}(n) \;=\; \frac{d}{\mu}\sum_{n\ge 0} g(n) \tag{2.26}$$

and, furthermore, for any $a\in\mathbb{R}$,

$$d\text{-}\lim_{t\to\infty} g*\mathbb{U}(nd+a) \;=\; \frac{1}{\mu}\sum_{n\ge 0} g(n+a), \tag{2.27}$$

because $g(\cdot+a)$ is clearly dRi as well.

Proof. We restrict ourselves to the more difficult nonarithmetic case. Given a dRi function g vanishing on $\mathbb{R}_<$, let g_δ, g^δ be as in (2.21) for $\delta>0$. Plainly, these functions vanish on $\mathbb{R}_<$ as well, so that we have

$$g_\delta\le g\le g^\delta,\quad \underline{\sigma}(\delta)=\int_0^\infty g_\delta(x)\,dx \quad\text{and}\quad \overline{\sigma}(\delta)=\int_0^\infty g^\delta(x)\,dx.$$

Fix any $\delta\in(0,1)$ and $m\in\mathbb{N}$ large enough such that $\sum_{n>m}|M_{n,\delta}|<\delta$. Then, using inequality (2.7), we infer

$$g^\delta*\mathbb{U}(t) \;=\; \sum_{n\ge 0} M_{n,\delta}\mathbb{U}(t-I_{n,\delta}) \;\le\; \sum_{n=0}^{m} M_{n,\delta}\,\mathbb{U}(t-n\delta-I_{0,\delta}) \;+\; \delta\,\mathbb{U}(1)$$

and therefore with Blackwell's theorem

$$\begin{aligned}\limsup_{t\to\infty} g^\delta*\mathbb{U}(t) \;&\le\; \sum_{n=0}^{m} M_{n,\delta}\lim_{t\to\infty}\mathbb{U}(t-n\delta-I_{0,\delta}) \;+\; \delta\,\mathbb{U}(1)\\ &=\; \frac{\delta}{\mu}\sum_{n=0}^{m} M_{n,\delta} \;+\; \delta\,\mathbb{U}(1)\\ &\le\; \frac{1}{\mu}\int_0^\infty g^\delta(x)\,dx \;+\; \frac{\delta^2}{\mu} \;+\; \delta\,\mathbb{U}(1)\\ &=\; \frac{1}{\mu}\overline{\sigma}(\delta) \;+\; \frac{\delta^2}{\mu} \;+\; \delta\,\mathbb{U}(1).\end{aligned} \tag{2.28}$$

Consequently, as $g*\mathbb{U}\le g^\delta*\mathbb{U}$ for all $\delta>0$,

$$\limsup_{t\to\infty} g*\mathbb{U}(t) \;\le\; \lim_{\delta\downarrow 0}\limsup_{t\to\infty} g^\delta*\mathbb{U}(t) \;\le\; \frac{1}{\mu}\int_0^\infty g(x)\,dx.$$

Replace g with $-g$ in the above estimation to obtain

$$\liminf_{t\to\infty} g*\mathbb{U}(t) \;\ge\; \frac{1}{\mu}\int_0^\infty g(x)\,dx.$$

This completes the proof of (2.24). □

Remark 2.32. In the 1-arithmetic and thus discrete case, the convolution $g*\mathbb{U}$ may actually be considered as a function on the discrete group $\mathbb{Z}$ and thus requires g to be considered on this set only which reduces it to a sequence $(g_n)_{n\in\mathbb{Z}}$. Doing so merely absolute summability, i.e. $\sum_{n\in\mathbb{Z}}|g_n|<\infty$, is needed instead of direct Riemann integrability. With this observation the result reduces to a straightforward consequence of Blackwell's renewal theorem and explains that much less attention has been paid to it in the literature.

Remark 2.33. The following counterexample shows that in the nonarithmetic case $\lambda\!\!\!\lambda_0$-integrability of g does not suffice to ensure (2.24). Consider a distribution F on $\mathbb{R}_\geq$ with positive mean $\mu=\int x\,F(dx)$ and renewal measure $\mathbb{U}=\sum_{n\geq 0}F^{*n}$. The function $g:=\sum_{n\geq 1}n^{1/2}\mathbf{1}_{[n.n+n^{-2})}$ is obviously $\lambda\!\!\!\lambda_0$-integrable, but

$$g*\mathbb{U}(n)\ =\ \sum_{k\geq 0}g*F^{*k}(n)\ \geq\ g*F^{*0}(n)\ =\ g(n)\ =\ n^{1/2}$$

diverges to ∞ as $n\to\infty$. Here the atom at 0, which any renewal measure of a SRP possesses, already suffices to demonstrate that $g(x)$ must not have unbounded oscillations as $x\to\infty$. But there are also examples of renewal measures with no atom at 0 (thus pertaining to a delayed RP) such that the key renewal theorem fails to hold for $\lambda\!\!\!\lambda_0$-integrable g. FELLER [30, p. 368] provides an example of a $\lambda\!\!\!\lambda_0$-continuous distribution F with finite positive mean such that $\mathbb{U}=\sum_{n\geq 1}F^{*n}$ satisfies $\limsup_{t\to\infty}g*\mathbb{U}(t)=\infty$ for some $\lambda\!\!\!\lambda_0$-integrable g.

Example 2.34. **[Forward and backward recurrence times]** Let $(S_n)_{n\geq 0}$ be a SRP with increment distribution F, lattice-span $d\in\{0,1\}$ and finite drift μ. For $t\geq 0$, let $\tau(t):=\inf\{n\geq 0: S_n>t\}$ denote the *first passage time* beyond level t and consider the first renewal epoch after t and the last renewal epoch before t, more precisely

$$R(t):=S_{\tau(t)}-t\quad\text{and}\quad\widehat{R}(t):=t-S_{\tau(t)-1}$$

called *forward* and *backward recurrence time*, respectively. Other names for $R(t)$, depending on the context in which it is discussed, are *overshoot, excess (over the boundary)* or *residual waiting time*. Other names for $\widehat{R}(t)$ are *age* and *spent waiting time*. We are interested in the asymptotic behavior of $R(t)$ and $\widehat{R}(t)$. It follows by a standard renewal argument that

$$\begin{aligned}\mathbb{P}(R(t)>r)\ &=\ \int_{[0,t]}\mathbb{P}(X>t+r-x)\ \mathbb{U}(dx)\\ \text{and}\quad\mathbb{P}(\widehat{R}(t)>r)\ &=\ \int_{[0,t]}\mathbb{P}(X>t-x)\,\mathbf{1}_{(r,\infty)}(t-x)\ \mathbb{U}(dx)\end{aligned}$$

for all $r,t\geq 0$. To both right-hand expressions the key renewal theorem applies and yields that

$$R(t)\stackrel{d}{\to}R(\infty)\quad\text{and}\quad\widehat{R}(t)+d\stackrel{d}{\to}R(\infty)\tag{2.29}$$

as $t \to \infty$ (through $\mathbb{Z}$ if $d=1$), where $\mathscr{L}(R(\infty)) = F^s$. Details are left as an exercise to the reader [☞ Problem 2.36].

Problems

Problem 2.35. Prove Lemma 2.28.

Problem 2.36. Under the assumptions of Example 2.34, prove (2.29) by filling in the details of the argument outlined there. Then proceed in a similar manner to find the asymptotic joint distribution of $(R(t), \widehat{R}(t))$ and of $X_{\tau(t)} = R(t) + \widehat{R}(t)$ as $t \to \infty$ (through $\mathbb{Z}$ if $d=1$). Do the results persist if the distribution of S_0 is arbitrarily chosen from $\mathscr{P}(\mathbb{G}_{d,\geq})$?

Problem 2.37. Still in the situation of Example 2.34, suppose that $F = Exp(\theta)$ for some $\theta > 0$. Compute the asymptotic joint distribution of $(R(t), \widehat{R}(t))$ and of $X_{\tau(t)} = R(t) + \widehat{R}(t)$ in this case.

2.7 The renewal equation

Almost every renewal quantity may be described as the solution to a convolution equation of the general form

$$\Psi \;=\; \psi \;+\; \Psi * Q, \tag{2.30}$$

where Q is a given locally finite measure and ψ a given locally bounded function on $\mathbb{R}_{\geq}$ (standard case) or $\mathbb{R}$ (general case). For reasons that will become apparent soon it is called *renewal equation*. If $\psi = 0$, then (2.30) is also a well-known object in harmonic analysis where its solutions are called *Q-harmonic functions*. It has been studied in the more general framework of Radon measures on separable locally compact Abelian groups by CHOQUET & DENY [19] and is therefore also known as the *Choquet-Deny equation*. Here we will focus on the standard case where functions and measures vanish on the negative halfline. Eq. (2.30) then takes the form

$$\Psi(x) \;=\; \psi(x) \;+\; \int_{[0,x]} \Psi(x-y)\, Q(dy), \quad x \in \mathbb{R}_{\geq}, \tag{2.31}$$

and is called *standard renewal equation* because it is the one encountered in most applications. Regarding the total mass of Q, a renewal equation is called *defective* if $\|Q\| < 1$, *proper* if $\|Q\| = 1$, and *excessive* if $\|Q\| > 1$.

2.7.1 Getting started

Some further notation is needed hereafter and therefore introduced first. Recall that Q is assumed to be locally finite, thus $Q(t)=Q([0,t])<\infty$ for all $t\in\mathbb{R}_\geq$. We denote its mean value by $\mu(Q)$ and its mgf by ϕ_Q, that is

$$\mu(Q) \;:=\; \int_{\mathbb{R}_\geq} x\,Q(dx)$$

and

$$\phi_Q(\theta) \;:=\; \int_{\mathbb{R}_\geq} e^{\theta x}\,Q(dx).$$

The latter function is nondecreasing and convex on its natural domain

$$\mathbb{D}_Q \;:=\; \{\theta\in\mathbb{R}:\phi_Q(\theta)<\infty\}$$

for which one of the four alternatives

$$\mathbb{D}_Q \;=\; \emptyset,\ (-\infty,\theta^*),\ (-\infty,\theta^*],\ \text{or } \mathbb{R}$$

with $\theta^*\in\mathbb{R}$ must hold. If $\mathbb{D}_Q$ has interior points, then ϕ_Q is infinitely often differentiable on $\operatorname{int}(\mathbb{D}_Q)$ with n^{th} derivative given by

$$\phi_Q^{(n)}(\theta) \;=\; \int_{\mathbb{R}_\geq} x^n e^{\theta x}\,Q(dx) \quad \text{for all } n\in\mathbb{N}.$$

In the following we will focus on measures Q on $\mathbb{R}_\geq$, called *admissible*, for which $\mu(Q)>0$, $Q(0)<1$ and $\mathbb{D}_Q\neq\emptyset$ holds true. Note that the last condition is particularly satisfied if $\|Q\|<\infty$ or, more generally, Q is uniformly locally bounded, i.e.

$$\sup_{t\geq 0} Q([t,t+1]) \;<\; \infty.$$

Moreover, ϕ_Q is increasing and strictly convex for such Q. Hence, there exists at most one value $\vartheta\in\mathbb{D}_Q$ such that $\phi_Q(\vartheta)=1$. It is called the *characteristic exponent of Q* hereafter.

Let $\mathbb{U}:=\sum_{n\geq 0}Q^{*n}$ with $Q^{*0}:=\delta_0$ be the renewal measure of Q. Put further

$$Q_\theta(dx) \;:=\; e^{\theta x}\,Q(dx)$$

again a locally finite measure for any $\theta\in\mathbb{R}$, and let $\mathbb{U}_\theta$ be its renewal measure.[1] Then

$$\mathbb{U}_\theta(dx) \;=\; \sum_{n\geq 0} Q_\theta^{*n}(dx) \;=\; \sum_{n\geq 0} e^{\theta x}\,Q^{*n}(dx) \;=\; e^{\theta x}\,\mathbb{U}(dx). \tag{2.32}$$

[1] The reader will notice here a notational conflict because in all previous and almost all subsequent sections $\mathbb{U}_\theta=\delta_\theta*\mathbb{U}$. On the other hand, whenever the current definition is meant, this will be clearly pointed out and should therefore not lead to any confusion.

Moreover, $\phi_{Q_\theta} = \phi_Q(\cdot+\theta)$ and $\phi_{\mathbb{U}_\theta} = \phi_{\mathbb{U}}(\cdot+\theta)$.

Lemma 2.38. *Given an admissible measure Q on $\mathbb{R}_\geq$, the following assertions hold true for any $\theta \in \mathbb{R}$:*

(a) Q_θ^{*n} *is admissible for all $n \in \mathbb{N}$.*
(b) $\mathbb{U}_\theta$ *is locally finite, that is $\mathbb{U}_\theta(t) < \infty$ for all $t \in \mathbb{R}_\geq$.*
(c) $\lim_{n\to\infty} Q^{*n}(t) = 0$ *for all $t \in \mathbb{R}_\geq$.*

Proof. Assertion (a) is trivial when noting that $Q_\theta^{*n}(0) = Q^{*n}(0) = Q(0)^n$ for all $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. As for (b), it clearly suffices to show that $\mathbb{U}_\theta$ is locally finite for *some* $\theta \in \mathbb{R}$. To this end note that $\mathbb{D}_Q \neq \emptyset$ implies $\phi_Q(t) \to 0$ as $t \to -\infty$ and thus the existence of $\theta \in \mathbb{R}$ such that $\|Q_\theta\| = \phi_Q(\theta) < 1$. Hence $\mathbb{U}_\theta$ is the renewal measure of the defective probability measure Q_θ and thus finite, for

$$\|\mathbb{U}_\theta\| \;=\; \sum_{n\geq 0} \|Q_\theta^{*n}\| \;=\; \sum_{n\geq 0} \|Q_\theta\|^n \;=\; \frac{1}{1-\phi_Q(\theta)} \;<\; \infty.$$

Finally, the local finiteness of $\mathbb{U} = \mathbb{U}_0$ gives $\mathbb{U}(t) = \sum_{n\geq 0} Q^{*n}(t) < \infty$ for all $t \in \mathbb{R}_\geq$ from which (c) directly follows. □

2.7.2 Existence and uniqueness of a locally bounded solution

We are now ready to prove the fundamental theorem about existence and uniqueness of solutions in the standard case (2.31) under the assumption that the measure Q is regular and the function ψ is *locally bounded* on $\mathbb{R}_\geq$, i.e.

$$\sup_{x\in[0,t]} |\psi(x)| \;<\; \infty \quad \text{for all } t \in \mathbb{R}_\geq.$$

Before stating the result let us note that n-fold iteration of equation (2.31) leads to

$$\Psi(x) \;=\; \sum_{k=0}^{n} \psi * Q^{*k}(x) \;+\; \Psi * Q^{*(n+1)}(x)$$

which in view of part (c) of the previous lemma suggests that $\Psi = \psi * \mathbb{U}$ forms the unique solution of (2.31).

Theorem 2.39. *Let Q be an admissible measure on $\mathbb{R}_\geq$ and $\psi : \mathbb{R}_\geq \to \mathbb{R}$ a locally bounded function. Then there exists a unique locally bounded solution Ψ of the renewal equation* (2.31)*, viz.*

$$\Psi(x) \;=\; \psi * \mathbb{U}(x) \;=\; \int_{[0,x]} \psi(x-y)\,\mathbb{U}(dy), \quad x \in \mathbb{R}_{\geq}$$

where $\mathbb{U}$ *denotes the renewal measure of* Q*. Moreover,* Ψ *is nondecreasing if the same holds true for* ψ*.*

Proof. Since $\mathbb{U}$ is locally finite, the local boundedness of ψ entails the same for the function $\psi * \mathbb{U}$, and the latter function satisfies (2.31) as

$$\psi * \mathbb{U} \;=\; \psi * \delta_0 \;+\; \left(\sum_{n\geq 1} \psi * Q^{*(n-1)}\right) * Q \;=\; \psi \;+\; (\psi * \mathbb{U}) * Q.$$

Moreover, $\psi * \mathbb{U}$ is nondecreasing if ψ has this property.

Turning to uniqueness, suppose we have two locally bounded solutions Ψ_1, Ψ_2 of (2.31). Then its difference Δ, say, satisfies the very same equation with $\psi \equiv 0$, that is $\Delta = \Delta * Q$. By iteration,

$$\Delta \;=\; \Delta * Q^{*n} \quad \text{for all } n \in \mathbb{N}.$$

Since Δ is locally bounded, it follows upon setting $\|\Delta\|_{x,\infty} := \sup_{y\in[0,x]} |\Delta(x)|$ and an appeal to Lemma 2.38(c) that

$$|\Delta(x)| \;=\; \lim_{n\to\infty} |\Delta * Q^{*n}(x)| \;\leq\; \|\Delta\|_{x,\infty} \lim_{n\to\infty} Q^{*n}(x) \;=\; 0 \quad \text{for all } x \in \mathbb{R}_{\geq}$$

which proves $\Psi_1 = \Psi_2$. □

The following version of the Choquet-Deny lemma is a direct consequence of the previous result.

Corollary 2.40. *If* Q *is an admissible measure on* $\mathbb{R}_{\geq}$*, then* $\Psi \equiv 0$ *is the only locally bounded solution to the Choquet-Deny equation* $\Psi = \Psi * Q$*.*

2.7.3 Asymptotics

Continuing with a study of the asymptotic behavior of solutions $\psi * \mathbb{U}$ a distinction of the cases $\|Q\| < 1$, $\mathbb{Q}\| = 1$, and $\|Q\| > 1$ is required. Put $I_d := \{0\}$ if $d = 0$, and $I_d := [0, d)$ if $d > 0$.

We begin with the defective case when $\phi_Q(0) = \|Q\| < 1$ and thus $\mathbb{U}$ is finite with total mass $\|U\| = (1 - \phi_Q(0))^{-1}$.

Theorem 2.41. *Given a defective renewal equation of the form* (2.31) *with locally bounded* ψ *such that* $\psi(\infty) := \lim_{x\to\infty}\psi(x) \in [-\infty,\infty]$ *exists, the same holds true for* $\Psi = \psi * \mathbb{U}$*, namely*

$$\Psi(\infty) \;=\; \frac{\psi(\infty)}{1-\phi_Q(0)}.$$

Proof. If $\psi(\infty) = \infty$, then the local boundedness of ψ implies $\inf_{x\ge 0}\psi(x) > -\infty$. Consequently, by an appeal to Fatou's lemma,

$$\begin{aligned}\liminf_{x\to\infty}\Psi(x) \;&=\; \liminf_{x\to\infty}\int_{[0,x]}\psi(x-y)\;\mathbb{U}(dy)\\ &\ge\; \int_{\mathbb{R}_\ge}\liminf_{x\to\infty}\mathbf{1}_{[0,x]}(y)\,\psi(x-y)\;\mathbb{U}(dy)\\ &=\; \psi(\infty)\,\|U\| \;=\; \infty.\end{aligned}$$

A similar argument shows $\limsup_{x\to\infty}\Psi(x) = -\infty$ if $\psi(\infty) = -\infty$. But if $\psi(\infty)$ is finite then ψ is necessarily bounded and we obtain by the dominated convergence theorem that

$$\lim_{x\to\infty}\Psi(x) \;=\; \int_{\mathbb{R}_\ge}\lim_{x\to\infty}\mathbf{1}_{[0,x]}(y)\,\psi(x-y)\;\mathbb{U}(dy) \;=\; \psi(\infty)\,\|\mathbb{U}\| \;=\; \frac{\psi(\infty)}{1-\phi_Q(0)}$$

as claimed. □

Turning to the case where $Q \ne \delta_0$ is a probability distribution on $\mathbb{R}_\ge$ (proper case) a statement about the asymptotic behavior of solutions $\psi * \mathbb{U}$ can be directly deduced with the help of the key renewal theorem 2.31.

Theorem 2.42. *Given a proper renewal equation of the form* (2.31) *with dRi function* ψ*, it follows for all* $a \in I_d$ *that*

$$d\text{-}\lim_{x\to\infty}\Psi(x+a) \;=\; \frac{1}{\mu(Q)}\int_{\mathbb{R}_\ge}\psi(x+a)\;\lambda\!\!\lambda_d(dx), \tag{2.33}$$

where d *denotes the lattice-span of* Q*.*

Our further investigations will rely on the subsequent lemma which shows that a renewal equation preserves its structure under the exponential transform $Q(dx) \mapsto Q_\theta(dx) = e^{\theta x}Q(dx)$ for any $\theta \in \mathbb{R}$. Plainly, Q_θ is a probability measure iff θ equals the characteristic exponent of Q. Given a function ψ on $\mathbb{R}_\ge$, put

$$\psi_\theta(x) \;:=\; e^{\theta x}\psi(x), \quad x \in \mathbb{R}_\ge$$

for any $\theta \in \mathbb{R}$.

Lemma 2.43. *Let Q be an admissible measure on $\mathbb{R}_\geq$, $\psi : \mathbb{R}_\geq \to \mathbb{R}$ a locally bounded function and Ψ any solution to the pertinent renewal equation* (2.31). *Then, for any $\theta \in \mathbb{R}$, Ψ_θ forms a solution to* (2.31) *for the pair (ψ_θ, Q_θ), i.e.*

$$\Psi_\theta \ = \ \psi_\theta + \Psi_\theta * Q_\theta. \tag{2.34}$$

Moreover, if $\Psi = \psi * \mathbb{U}$, then $\Psi_\theta = \psi_\theta * \mathbb{U}_{Q_\theta}$ is the unique locally bounded solution to (2.34).

Proof. For the first assertion, it suffices to note that $\Psi = \psi + \Psi * Q$ obviously implies (2.34), for

$$e^{\theta x}\Psi(x) \ = \ e^{\theta x}\psi(x) \ + \ \int_{[0,x]} e^{\theta(x-y)}\,\Psi(x-y)\,e^{\theta y}\,Q(dy)$$

for all $x \in \mathbb{R}_\geq$. Since Q_θ is admissible for any $\theta \in \mathbb{R}$, the second assertion follows by Thm. 2.39. □

With the help of this lemma we are now able to derive the following general result on the asymptotic behavior of $\psi * \mathbb{U}$ for a standard renewal equation of the form (2.31). It covers the excessive as well as the defective case.

Theorem 2.44. *Given a renewal equation of the form* (2.31) *with admissible Q with lattice-span d and locally bounded function ψ, the following assertions hold true for its unique locally bounded solution $\Psi = \psi * \mathbb{U}$:*

(a) *If $\theta \in \mathbb{R}$ is such that $\|Q_\theta\| < 1$ and $\psi_\theta(\infty)$ exists, then*

$$\lim_{x\to\infty} e^{\theta x}\,\Psi(x) \ = \ \frac{\psi_\theta(\infty)}{1-\phi_Q(\theta)} \tag{2.35}$$

(b) *If Q possesses a characteristic exponent ϑ, then*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\,\Psi(x+a) \ = \ \frac{1}{\mu(Q_\vartheta)} \int_{\mathbb{R}_\geq} e^{\vartheta x}\,\psi(x+a)\,\lambda\!\!\lambda_d(dx) \tag{2.36}$$

for all $a \in I_d$ if ψ_ϑ is dRi.

Proof. All assertions are direct consequences of the previous results. □

Remark 2.45. If $1 < \|Q\| < \infty$ in the previous theorem, then $\mathbb{D}_Q \supset (-\infty, 0]$ and the continuity ϕ_Q together with $\lim_{\theta\to-\infty}\phi_Q(\theta) = 0$ *always* ensures the existence of

$\vartheta < 0$ with $\phi_Q(\vartheta) = \|Q_\vartheta\| = 1$ by the intermediate value theorem. On the other hand, if Q is an infinite admissible measure, then it is possible that $\phi_Q(\theta) < 1$ *for all* $\theta \in \mathbb{D}_Q$.

There is yet another situation uncovered so far where further information on the asymptotic behavior of $\psi * \mathbb{U}$ may be obtained. Suppose that, for some $\theta \in \mathbb{R}$, $\psi_\theta(\infty)$ exists but is nonzero and that Q_θ is defective. Then Thm. 2.41 provides us with

$$\Psi_\theta(\infty) \;=\; \lim_{x\to\infty} e^{\theta x}\Psi(x) \;=\; \frac{\psi_\theta(\infty)}{1-\phi_Q(\theta)} \;\neq\; 0$$

which in turn raises the question whether the rate of convergence of $\Psi_\theta(x)$ to $\Psi_\theta(\infty)$ may be studied by finding a renewal equation satisfied by the difference $\Psi_\theta^0 := \Psi_\theta(\infty) - \Psi_\theta$. An answer is given by the next theorem for which $\theta = 0$ is assumed without loss of generality. For $d \in \mathbb{R}_\geq$ and $\theta \in \mathbb{R}$, let us define

$$e(d,\theta) \;:=\; \begin{cases} \theta, & \text{if } d = 0, \\ (e^{\theta d}-1)/d, & \text{if } d > 0. \end{cases} \tag{2.37}$$

which is a continuous function on $\mathbb{R}_\geq \times \mathbb{R}$.

Theorem 2.46. *Given a defective renewal equation of the form* (2.31) *with locally bounded* ψ *such that* $\psi(\infty) \neq 0$*, it follows that* $\Psi^0 := \Psi(\infty) - \Psi$ *forms the unique locally bounded solution to the renewal equation* $\Psi^0 = \widehat{\psi} + \Psi^0 * Q$ *with*

$$\widehat{\psi}(x) \;:=\; \psi^0(x) + \psi(\infty)\,\frac{Q((x,\infty))}{1-\phi_Q(0)}, \quad x \in \mathbb{R}.$$

Furthermore, if Q *has characteristic exponent* ϑ *(necessarily positive) and lattice-span* d*, then*

$$d\text{-}\lim_{x\to\infty} \Psi_\vartheta^0(x+a) \;=\; \frac{e^{\vartheta a}}{\mu(Q_\vartheta)}\left(\frac{\psi(\infty)}{e(d,\vartheta)} + \int_{\mathbb{R}_\geq} e^{\vartheta y}\,\psi^0(y+a)\;\lambda\!\!\lambda_d(dy)\right) \tag{2.38}$$

for any $a \in I_d$ *provided that* $\widehat{\psi}_\vartheta$ *is dRi.*

Proof. A combination of

$$\Psi(\infty) - \int_{[0,x]} \Psi(\infty)\,Q(dx) \;=\; \Psi(\infty)\,Q((x,\infty)) \;=\; \frac{\psi(\infty)\,Q((x,\infty))}{1-\phi_Q(0)} \;=\; \widehat{\psi}(x) - \psi^0(x)$$

and $\Psi = \psi + \Psi * Q$ shows the asserted renewal equation for Ψ^0. By the previous results, we then infer under the stated conditions on $\widehat{\psi}$ and Q that

$$d\text{-}\lim_{x\to\infty} \Psi_\vartheta^0(x+a) \;=\; \frac{1}{\mu(Q_\vartheta)} \int_{\mathbb{R}_\geq} \widehat{\psi}_\vartheta(y+a)\;\lambda\!\!\lambda_d(dy) \quad \text{for any } a \in [0,d).$$

Hence it remains to verify that the right-hand side equals the right-hand side of (2.38).

Let us first consider the case $d=0$: Using $\phi_Q(\vartheta)=1$, we find that

$$\begin{aligned}\int_{\mathbb{R}_\geq} e^{\vartheta y}(\widehat{\psi}(y)-\psi^0(y))\,\lambda\!\!\lambda_0(dy) &= \frac{\psi(\infty)}{1-\phi_Q(0)}\int_{\mathbb{R}_\geq} e^{\vartheta y}Q((y,\infty))\,\lambda\!\!\lambda_0(dy)\\ &= \frac{\psi(\infty)}{\vartheta(1-\phi_Q(0))}\int_{\mathbb{R}_\geq}(e^{\vartheta y}-1)\,Q(dy) = \frac{\psi(\infty)}{\vartheta}\end{aligned}$$

which is the desired result.

If $d>0$ and $a\in[0,d)$, use $Q((y+a,\infty))=Q((y,\infty))$ for any $y\in d\mathbb{Z}$ to see that

$$\begin{aligned}\int_{d\mathbb{N}_0} e^{\vartheta y}(\widehat{\psi}(y+a)-\psi^0(y+a))\,\lambda\!\!\lambda_d(dy) &= \frac{\psi(\infty)}{1-\phi_Q(0)}\int_{d\mathbb{N}_0} e^{\vartheta y}Q((y,\infty))\,\lambda\!\!\lambda_d(dy)\\ &= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{n\geq 0}\sum_{k>n} e^{\vartheta nd}Q(\{kd\})\\ &= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{k\geq 1}Q(\{kd\})\sum_{n=0}^{k-1}e^{\vartheta nd}\\ &= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{k\geq 1}\frac{e^{\vartheta kd}-1}{e^{\vartheta d}-1}\,Q(\{kd\})\\ &= \frac{\psi(\infty)}{(1-\phi_Q(0))e(d,\vartheta)}\sum_{k\geq 0}(e^{\vartheta kd}-1)\,Q(\{kd\}) = \frac{\psi(\infty)}{e(d,\vartheta)}.\end{aligned}$$

The proof is herewith complete. □

It is worthwile to give the following corollary that provides information on the behavior of the renewal function $\mathbb{U}(t)$ pertaining to an admissible measure Q that possesses a characteristic exponent $\vartheta\neq 0$. The proper renewal case $\vartheta=0$ will be considered more carefully later in the upcoming section.

Corollary 2.47. *Let Q be an admissible measure on $\mathbb{R}_\geq$ with lattice-span d and characteristic exponent ϑ. Then its renewal function $\mathbb{U}(x)$ satisfies*

(a) *in the defective case ($\vartheta>0$):*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\left(\frac{1}{1-\phi_Q(0)}-\mathbb{U}(x)\right) = d\text{-}\lim_{x\to\infty}\mathbb{U}((x,\infty)) = \frac{1}{\mu(Q_\vartheta)e(d,\vartheta)}.$$

(b) *in the excessive case ($\vartheta<0$):*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\,\mathbb{U}(x) = \frac{1}{\mu(Q_\vartheta)|e(d,\vartheta)|}$$

Proof. Since $\mathbb{U}(x) = I(x) + \mathbb{U} * Q(x)$ for $x \in \mathbb{R}_\geq$ with $I := \mathbf{1}_{[0,\infty)}$, we infer from Thm. 2.46 that in the defective case $\mathbb{U}^0(x) = \|U\| - \mathbb{U}((x,\infty))$ satisfies the renewal equation

$$\mathbb{U}^0(x) \;=\; \widehat{I}(x) + \mathbb{U}^0 * Q(x) \quad \text{with} \quad \widehat{I}(x) \;:=\; \|\mathbb{U}\|\, Q((x,\infty)).$$

The function $\widehat{I}_\vartheta$ is dRi by Lemma 2.28 because $\widehat{I}$ is nondecreasing on $\mathbb{R}_\geq$ and $\int_0^\infty \vartheta e^{\vartheta y} Q((y,\infty))\,dy = \phi_Q(\vartheta) - \phi_Q(0) < \infty$. Hence we obtain the asserted result by an appeal to (2.38) of Thm. 2.46.

In the excessive case, $\vartheta < 0$ implies that $I_\vartheta(x) = e^{\vartheta x}\mathbf{1}_{[0,\infty)}(x)$ is dRi so that, by (2.36) of Thm. 2.44(b),

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\,\mathbb{U}(x) \;=\; \frac{1}{\mu(Q_\vartheta)} \int_{\mathbb{R}_\geq} e^{\vartheta x}\, \lambda\!\!\lambda_d(dx) \;=\; \frac{1}{\mu(Q_\vartheta)|e(d,\vartheta)|}$$

as claimed. □

2.8 Renewal function and first passage times

Let $(S_n)_{n\geq 0}$ be a RP in a standard model with increment distribution F, lattice-span $d \in \{0,1\}$, *finite* drift μ and renewal measure $\mathbb{U}_\lambda$ under $\mathbb{P}_\lambda$. Recall that $\tau(t) = \inf\{n \geq 0 : S_n > t\}$ denotes the associated first passage time beyond level t for $t \geq 0$. The rather crude asymptotic $t^{-1}\mathbb{U}_0(t) \to \mu^{-1}$, known as the elementary renewal theorem [☞ Lemma 2.1(f)], in combination with $d\text{-}\lim_{t\to\infty}(\mathbb{U}_0(t+h) - \mathbb{U}_0(t)) = \mu^{-1}\lambda\!\!\lambda_d((0,h])$ for any $h > 0$ from Blackwell's theorem 2.20 provides some evidence for the assertion that

$$\mathbb{U}_0(t) \;=\; \frac{t}{\mu} + \Delta + o(1) \quad \text{as } t \to \infty \text{ through } \mathbb{G}_d,$$

where Δ denotes a suitable constant depending on F. This will now in fact be derived via another standard renewal equation and requires the assumption that F has finite variance $\sigma^2 = \mathbb{E}(X-\mu)^2$. The result will also lead to an asymptotic expansion of $\mathbb{E}_0\tau(t)$ up to vanishing terms as $t \to \infty$ because of the simple but important relationship between renewal function and mean first passage times.

Lemma 2.48. *Given a RP $(S_n)_{n\geq 0}$ in a standard model with associated first passage times $\tau(t)$, the identity*

$$\mathbb{U}_\lambda(t) \;=\; \mathbb{E}_\lambda \tau(t) \tag{2.39}$$

holds true for any $t \geq 0$ and $\lambda \in \mathscr{P}(\mathbb{R}_\geq)$.

Proof. It suffices to note that

$$\mathbb{E}_\lambda \tau(t) \;=\; \sum_{n\ge 0} \mathbb{P}_\lambda(\tau(t)>n) \;=\; \sum_{n\ge 0} \mathbb{P}_\lambda(S_n \le t) \;=\; \mathbb{U}_\lambda(t)$$

for any $t \ge 0$ and $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$. □

Observe that, by an appeal to Wald's identity [☞ Prop. 2.53],

$$\mathbb{U}_0(t) \;=\; \frac{1}{\mu}\,\mathbb{E}_0 S_{\tau(t)} \;=\; \frac{t}{\mu} + \frac{1}{\mu}\,\mathbb{E}(S_{\tau(t)} - t) \;\ge\; \frac{t}{\mu}$$

for all $t \ge 0$. Therefore, the function $\Psi(t) := \mathbb{U}_0(t) - \mathbb{U}_{F^s}(t)$ is nonnegative, locally bounded, vanishes on $\mathbb{R}_<$ and equals $U_0(t) - \mu^{-1}t$ for $t \in \mathbb{G}_{d,\ge}$. By (2.8), it satisfies the renewal equation

$$\Psi(t) \;=\; \psi(t) \;+\; \int_{[0,t]} \Psi(t-x)\,F(dx), \quad t \ge 0,$$

where

$$\psi(t) \;:=\; \delta_0(t) - F^s(t) \;=\; \overline{F}^s(t) \quad t \ge 0,$$

is clearly locally bounded. Consequently, Ψ forms the unique locally bounded solution to the renewal equation by Thm. 2.39 and must hence equal $\psi * \mathbb{U}_0$. The function ψ is dRi by (dRi-6) of Prop. 2.27, for it is nonincreasing on $\mathbb{R}_\ge$ and satisfies

$$\int_0^\infty \overline{F}^s(t)\,dt \;=\; \mu^s \;=\; \frac{\sigma^2+\mu^2}{2\mu} + \frac{d}{2} \;<\; \infty \tag{2.40}$$

by (2.14). The following result, giving the announced second order approximation of the renewal function, is now easily inferred with the help of Thm. 2.42.

Theorem 2.49. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with lattice-span $d \in \{0,1\}$, drift μ and finite increment variance σ^2. Then*

$$d\text{-}\lim_{t\to\infty} \left(\mathbb{U}_\lambda(t) - \frac{t}{\mu}\right) \;=\; \frac{d}{2\mu} + \frac{\mu^2+\sigma^2}{2\mu^2} - \frac{\mu_0}{\mu} \tag{2.41}$$

for any $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$ having finite mean μ_0.

Proof. The result follows from the previous considerations if $\lambda = \delta_0$ and for general λ with finite mean μ_0 upon using $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$. Details are left to the reader [☞ Problem 2.51]. □

Regarding the forward recurrence time $R(t) = S_{\tau(t)} - t$, let us finally point out that a combination of the previous result with (2.39), (2.40) and Wald's identity [☞ Prop. 2.53] implies

$$d\text{-}\lim_{t\to\infty} \mathbb{E}_0 R(t) \;=\; d\text{-}\lim_{t\to\infty} \Big(\mu\,\mathbb{E}_0\tau(t) - t\Big) \;=\; d\text{-}\lim_{t\to\infty} \Big(\mu\,\mathbb{U}_0(t) - t\Big) \;=\; \mu^s.$$

But we further know from (2.29) that $R(t)$ converges in distribution to F^s, whence the following result is immediate.

Corollary 2.50. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with lattice-span $d \in \{0,1\}$, drift μ and finite increment variance σ^2. Then the family of forward recurrence times $\{R(t) : t \ge 0\}$ is ui.*

Problems

Problem 2.51. Prove Thm. 2.49.

2.9 An intermezzo: random walks, stopping times and ladder variables

Before giving a brief account of the most important extensions of previous results to random walks on the line with positive drift, we collect some important facts about random walks and stopping times including the crucial concept of *ladder variables*. We skip some of the proofs and refer instead to [2].

In the following, let $(S_n)_{n\ge 0}$ be a RW in a standard model with increments $X_1, X_2, \ldots$ and increment distribution F. For convenience, it may take values in any $\mathbb{R}^d$, $d \ge 1$. We will use $\mathbb{P}$ for probabilities that do not depend on the distribution of S_0. Let further $(\mathscr{F}_n)_{n\ge 0}$ be a filtration such that

(F1) $(S_n)_{n\ge 0}$ is adapted to $(\mathscr{F}_n)_{n\ge 0}$, i.e., $\sigma(S_0, \ldots, S_n) \subset \mathscr{F}_n$ for all $n \in \mathbb{N}_0$.
(F2) $\mathscr{F}_n$ is independent of $(X_{n+k})_{k\ge 1}$ for each $n \in \mathbb{N}_0$.

Let also $\mathscr{F}_\infty$ be the smallest σ-field containing all $\mathscr{F}_n$. Condition (F2) ensures that $(S_n)_{n\ge 0}$ is a temporally homogeneous Markov chain with respect to $(\mathscr{F}_n)_{n\ge 0}$, viz.

$$\mathbb{P}(S_{n+1} \in B | \mathscr{F}_n) \;=\; \mathbb{P}(S_{n+1} \in B | S_n) \;=\; F(B - S_n) \quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n \in \mathbb{N}_0$, $\lambda \in \mathscr{P}(\mathbb{R}^d)$ and $B \in \mathscr{B}(\mathbb{R}^d)$. A more general, but in fact equivalent statement is that

$$\mathbb{P}((S_{n+k})_{k\ge 0} \in C | \mathscr{F}_n) \;=\; \mathbb{P}((S_{n+k})_{k\ge 0} \in C | S_n) \;=\; \mathbf{P}(S_n, C) \quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n \in \mathbb{N}_0$, $\lambda \in \mathscr{P}(\mathbb{R})$ and $C \in \mathscr{B}(\mathbb{R}^d)^{\mathbb{N}_0}$, where

$$\mathbf{P}(x, C) \;:=\; \mathbb{P}_x((S_k)_{k\ge 0} \in C) \;=\; \mathbb{P}_0((S_k)_{k\ge 0} \in C - x) \quad \text{for } x \in \mathbb{R}^d.$$

Let us recall that, if τ is any stopping time with respect to $(\mathscr{F}_n)_{n\ge 0}$, also called $(\mathscr{F}_n)$-*time* hereafter, then

$$\mathscr{F}_\tau \;=\; \{A\in\mathscr{F}_\infty : A\cap\{\tau\le n\}\in\mathscr{F}_n \text{ for all } n\in\mathbb{N}_0\},$$

and the random vector $(\tau,S_0,\ldots,S_\tau)\mathbf{1}_{\{\tau<\infty\}}$ is $\mathscr{F}_\tau$-measurable. The following basic result combines the strong Markov property and temporal homogeneity of $(S_n)_{n\ge 0}$ as a Markov chain with its additional *spatial homogeneity* owing to its iid increments.

Proposition 2.52. *Under the stated assumptions, let τ be a $(\mathscr{F}_n)$-time. Then, for all $\lambda\in\mathscr{P}(\mathbb{R}^d)$, the following equalities hold $\mathbb{P}_\lambda$-a.s. on $\{\tau<\infty\}$:*

$$\mathbb{P}((S_{\tau+n}-S_\tau)_{n\ge 0}\in\cdot|\mathscr{F}_\tau) = \mathbb{P}((S_n-S_0)_{n\ge 0}\in\cdot) = \mathbb{P}_0((S_n)_{n\ge 0}\in\cdot). \quad (2.42)$$

$$\mathbb{P}((X_{\tau+n})_{n\ge 1}\in\cdot|\mathscr{F}_\tau) = \mathbb{P}((X_n)_{n\ge 1}\in\cdot). \quad (2.43)$$

If $\mathbb{P}_\lambda(\tau<\infty)=1$, then furthermore (under $\mathbb{P}_\lambda$)

(a) $(S_{\tau+n}-S_\tau)_{n\ge 0}$ *and* $\mathscr{F}_\tau$ *are independent.*

(b) $(S_{\tau+n}-S_\tau)_{n\ge 0}\stackrel{d}{=}(S_n-S_0)_{n\ge 0}$.

(c) $X_{\tau+1},X_{\tau+2},\ldots$ *are iid with the same distribution as* X_1.

Proof. It suffices to prove (2.43) for which we pick any $k\in\mathbb{N}_0$, $n\in\mathbb{N}$, $B_1,\ldots,B_n\in\mathscr{B}(\mathbb{R}^d)$ and $A\in\mathscr{F}_\tau$. Using $A\cap\{\tau=k\}\in\mathscr{F}_k$ and (F2), it follows for each $\lambda\in\mathscr{P}(\mathbb{R}^d)$ that

$$\begin{aligned}&\mathbb{P}_\lambda(A\cap\{\tau=k,X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n\})\\ &\quad=\;\mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n)\\ &\quad=\;\mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_1\in B_1,\ldots,X_n\in B_n),\end{aligned}$$

and this yields the desired conclusion. □

We continue with a statement of two very useful identities originally due to A. Wald [60] for the first and second moment of stopped sums S_τ for finite mean stopping times τ, known as *Wald's equations* or *Wald's identities*. The first of these has already been used before.

Proposition 2.53. [Wald's equations] *Let $(S_n)_{n\ge 0}$ be a SRW adapted to a filtration $(\mathscr{F}_n)_{n\ge 0}$ satisfying (F1) and (F2). Let further τ be an a.s. finite $(\mathscr{F}_n)$-time and suppose that $\mu:=\mathbb{E}X_1$ exists. Then*

$$\mathbb{E}S_\tau \;=\; \mu\,\mathbb{E}\tau \qquad \textbf{(Wald's equation)}$$

provided that either the X_n are a.s. nonnegative, or τ has finite mean. If the X_n have also finite variance σ^2, then furthermore

$$\mathbb{E}(S_\tau - \mu\tau)^2 \;=\; \sigma^2\,\mathbb{E}\tau \qquad\qquad \textbf{(Wald's 2nd equation)}$$

for any $(\mathscr{F}_n)$-time τ with finite mean.

Proof. ☞ [2, Prop. 2.11 and 2.12]. □

Assuming $S_0 = 0$ hereafter, let us now turn to the concept of formally copying a stopping time τ for $(S_n)_{n\ge1}$. The latter means that there exist $B_n \in \mathscr{B}(\mathbb{R}^{nd})$ for $n \ge 1$ such that

$$\tau \;=\; \inf\{n \ge 1 : (S_1,...,S_n) \in B_n\}, \tag{2.44}$$

where as usual $\inf\emptyset := \infty$. With the help of the B_n we can copy this stopping rule to the *post-τ process* $(S_{\tau+n} - S_\tau)_{n\ge1}$ if $\tau < \infty$. For this purpose put $S_{n,k} := S_{n+k} - S_n$,

$$\begin{aligned} \mathbf{S}_{n,k} &:= (S_{n+1} - S_n,...,S_{n+k} - S_n) \;=\; (S_{n,1},...,S_{n,k}) \quad \text{and} \\ \mathbf{X}_{n,k} &:= (X_{n+1},...,X_{n+k}) \end{aligned}$$

for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Definition 2.54. Let τ be a stopping time for $(S_n)_{n\ge1}$ as in (2.44). Then the sequences $(\tau_n)_{n\ge1}$ and $(\sigma_n)_{n\ge0}$, defined by $\sigma_0 := 0$ and

$$\tau_n \;:=\; \begin{cases} \inf\{k \ge 1 : \mathbf{S}_{\sigma_{n-1},k} \in B_k\}, & \text{if } \sigma_{n-1} < \infty \\ \infty, & \text{if } \sigma_{n-1} = \infty \end{cases} \quad \text{and} \quad \sigma_n \;:=\; \sum_{k=1}^{n} \tau_k$$

for $n \ge 1$ (thus $\tau_1 = \tau$) are called the *sequence of formal copies of* τ and its associated *sequence of copy sums*, respectively.

The following proposition summarizes the most important properties of the τ_n, σ_n and $S_{\sigma_n}\mathbf{1}_{\{\sigma_n<\infty\}}$.

Proposition 2.55. *Given the previous notation, put further $\beta := \mathbb{P}(\tau < \infty)$ and $\mathbf{Z}_n := (\tau_n, \mathbf{X}_{\sigma_{n-1},\tau_n})$ for $n \in \mathbb{N}$. Then the following assertions hold true:*

(a) $\sigma_0, \sigma_1, ...$ *are stopping times for* $(S_n)_{n\ge0}$.
(b) τ_n *is a stopping time with respect to* $(\mathscr{F}_{\sigma_{n-1}+k})_{k\ge0}$ *and* $\mathscr{F}_{\sigma_{n-1}}$*-measurable for each* $n \in \mathbb{N}$.
(c) $\mathbb{P}(\tau_n \in \cdot|\mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\tau < \infty)$ *a.s. on* $\{\sigma_{n-1} < \infty\}$ *for each* $n \in \mathbb{N}$.
(d) $\mathbb{P}(\tau_n < \infty) = \mathbb{P}(\sigma_n < \infty) = \beta^n$ *for all* $n \in \mathbb{N}$.

(e) $\mathbb{P}(\mathbf{Z}_n \in \cdot, \tau_n < \infty | \mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\mathbf{Z}_1 \in \cdot, \tau_1 < \infty)$ *a.s. on* $\{\sigma_{n-1} < \infty\}$ *for all* $n \in \mathbb{N}$.

(f) *Given* $\sigma_n < \infty$*, the random vectors* $\mathbf{Z}_1, ..., \mathbf{Z}_n$ *are conditionally iid with the same distribution as* $\mathbf{Z}_1$ *conditioned upon* $\tau_1 < \infty$.

(g) *If* $G := \mathbb{P}((\tau, S_\tau) \in \cdot | \tau < \infty)$*, then* $\mathbb{P}((\sigma_n, S_{\sigma_n}) \in \cdot | \sigma_n < \infty) = G^{*n}$ *a.s. for all* $n \in \mathbb{N}$.

In the case where τ *is a.s. finite* ($\beta = 1$)*, this implies further:*

(h) $\mathbf{Z}_n$ *and* $\mathscr{F}_{\sigma_{n-1}}$ *are independent for each* $n \in \mathbb{N}$.

(i) $\mathbf{Z}_1, \mathbf{Z}_2, ...$ *are iid.*

(j) $(\sigma_n, S_{\sigma_n})_{n \geq 0}$ *forms a SRW taking values in* $\mathbb{N}_0 \times \mathbb{R}^d$.

Proof. The simple proof of (a) and (b) is left to the reader. Assertion (c) and (e) follow from (2.42) when observing that, on $\{\sigma_{n-1} < \infty\}$,

$$\tau_n = \sum_{k \geq 0} \mathbf{1}_{\{\tau_n > k\}} = \sum_{k \geq 0} \prod_{j=1}^{k} \mathbf{1}_{B_j^c}(\mathbf{S}_{\sigma_{n-1},j}) \quad \text{and} \quad \mathbf{Z}_n \mathbf{1}_{\{\tau_n < \infty\}}$$

are measurable functions of $(S_{\sigma_{n-1},k})_{k \geq 0}$. Since $\mathbb{P}(\tau_n < \infty) = \mathbb{P}(\tau_1 < \infty, ..., \tau_n < \infty)$, we infer (d) by an induction over n and use of (c). Another induction in combination with (d) gives assertion (f) once we have proved that

$$\begin{aligned} &\mathbb{P}((\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \mathbf{Z}_{n+1} \in B, \sigma_{n+1} < \infty) \\ &\qquad = \mathbb{P}((\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \sigma_n < \infty) \mathbb{P}(\mathbf{Z}_1 \in B, \tau < \infty) \end{aligned}$$

for all $n \in \mathbb{N}$ and A_n, B from the σ-fields obviously to be chosen here. But with the help of (e), this is inferred as follows:

$$\begin{aligned} &\mathbb{P}((\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \mathbf{Z}_{n+1} \in B, \sigma_{n+1} < \infty) \\ &\quad = \mathbb{P}((\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \mathbf{Z}_{n+1} \in B, \sigma_n < \infty, \tau_{n+1} < \infty) \\ &\quad = \int_{\{(\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \sigma_n < \infty\}} \mathbb{P}(\mathbf{Z}_{n+1} \in B, \tau_{n+1} < \infty | \mathscr{F}_{\sigma_n})\, d\mathbb{P} \\ &\quad = \mathbb{P}((\mathbf{Z}_1, ..., \mathbf{Z}_n) \in A_n, \sigma_n < \infty) \mathbb{P}(\mathbf{Z}_1 \in B, \tau < \infty). \end{aligned}$$

Assertion (g) is a direct consequence of (f), and the remaining assertion (h),(i) and (j) in the case $\beta = 1$ are just the specializations of (e),(f) and (g) to this case. □

The most prominent sequences of copy sums in the theory of RW's are obtained by looking at the record epochs and record values of a RW $(S_n)_{n \geq 0}$, or its reflection $(-S_n)_{n \geq 0}$. They also provide a key tool for the extension of renewal theory to random walks with positive drift

Definition 2.56. Given a SRW $(S_n)_{n\ge0}$, the stopping times

$$\sigma^> := \inf\{n\ge1 : S_n>0\},\quad \sigma^\ge := \inf\{n\ge1 : S_n\ge0\},$$
$$\sigma^< := \inf\{n\ge1 : S_n<0\},\quad \sigma^\le := \inf\{n\ge1 : S_n\le0\},$$

are called *first strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epoch*, respectively, and

$$S_1^> := S_{\sigma^>}\mathbf{1}_{\{\sigma^><\infty\}},\quad S_1^\ge := S_{\sigma^\ge}\mathbf{1}_{\{\sigma^\ge<\infty\}},$$
$$S_1^< := S_{\sigma^<}\mathbf{1}_{\{\sigma^<<\infty\}},\quad S_1^\le := S_{\sigma^\le}\mathbf{1}_{\{\sigma^\le<\infty\}}$$

their respective *ladder heights*. The associated sequences of copy sums $(\sigma_n^>)_{n\ge0}$, $(\sigma_n^\ge)_{n\ge0}$, $(\sigma_n^<)_{n\ge0}$ and $(\sigma_n^\le)_{n\ge0}$ are called *sequences of strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epochs*, respectively, and

$$S_n^> := S_{\sigma_n^>}\mathbf{1}_{\{\sigma_n^><\infty\}},\ n\ge0,\quad S_n^\ge := S_{\sigma_n^\ge}\mathbf{1}_{\{\sigma_n^\ge<\infty\}},\ n\ge0,$$
$$S_n^< := S_{\sigma_n^<}\mathbf{1}_{\{\sigma_n^<<\infty\}},\ n\ge0,\quad S_n^\le := S_{\sigma_n^\le}\mathbf{1}_{\{\sigma_n^\le<\infty\}},\ n\ge0$$

the respective *sequences of ladder heights*.

Plainly, if $(S_n)_{n\ge0}$ has nonnegative (positive) increments, then $\sigma_n^\ge = n$ ($\sigma_n^> = n$) for all $n\in\mathbb{N}$. Moreover, $\sigma_n^\ge = \sigma_n^>$ and $\sigma_n^\le = \sigma_n^<$ a.s. for all $n\in\mathbb{N}$ in the case where the increment distribution is continuous, for then $\mathbb{P}(S_m=S_n)=0$ for all $m,n\in\mathbb{N}$.

The following proposition provides some basic information on the ladder variables and is a consequence of the SLLN and Prop. 2.55.

Proposition 2.57. *Let $(S_n)_{n\ge0}$ be a nontrivial SRW. Then the following assertions are equivalent:*

(a) $\left(\sigma_n^\alpha, S_{\sigma_n^\alpha}\right)_{n\ge0}$ *is a SRW taking values in $\mathbb{N}_0\times\mathbb{R}$ for any $\alpha\in\{>,\ge\}$ (resp. $\{<,\le\}$).*

(b) $\sigma^\alpha<\infty$ *a.s. for $\alpha\in\{>,\ge\}$ (resp. $\{<,\le\}$).*

(c) $\limsup_{n\to\infty} S_n=\infty$ *a.s. (resp. $\liminf_{n\to\infty} S_n=-\infty$ a.s.)*

Proof. It clearly suffices to prove equivalence of the assertions outside parentheses. The implications “(a)⇒(b)” and “(c)⇒(b)” are trivial, while “(b)⇒(a)” follows from Prop. 2.55(j). This leaves us with a proof of “(a),(b)⇒(c)”. But $\mathbb{E}S^>>0$ in combination with the SLLN applied to $(S_n^>)_{n\ge0}$ implies

$$\limsup_{n\to\infty} S_n \ \ge\ \lim_{n\to\infty} S_n^> = \infty \quad \text{a.s.}$$

and thus the assertion. □

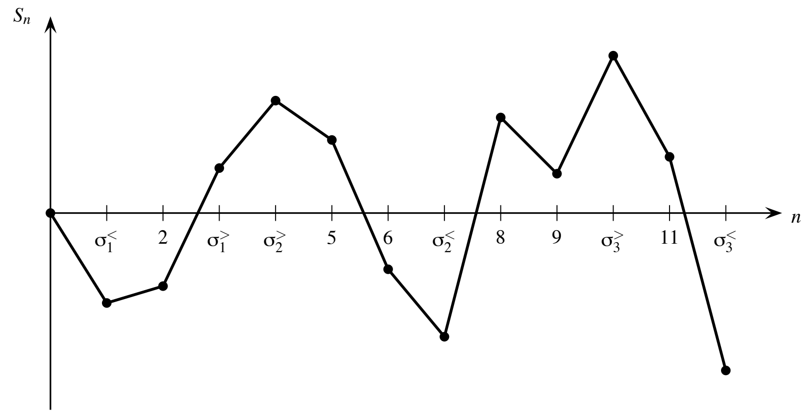

Fig. 2.2 Path of a RW with strictly ascending ladder epochs $\sigma_1^> = 3$, $\sigma_2^> = 4$ and $\sigma_3^> = 10$, and strictly descending ladder epochs $\sigma_1^< = 1$, $\sigma_2^< = 5$ and $\sigma_3^< = 12$.

If $\mathbb{E}X_1 > 0$ (resp. < 0) we thus have that $\sigma^>, \sigma^\geq$ (resp. $\sigma^<, \sigma^\leq$) are a.s. finite whence the associated sequences of ladder epochs and ladder heights each constitute nondecreasing (resp. nonincreasing) zero-delayed RW's. Much deeper information, however, is provided by Prop. 2.59 below which discloses a quite unexpected duality between ascending and descending ladder epochs that will enable us to derive a further classification of RW's as to their asymptotic behavior including the *Chung-Fuchs theorem* on the asymptotic behavior of a RW with drift zero. We pause for the following lemma about the lattice-type of a ladder height.

Lemma 2.58. *Let $(S_n)_{n\geq 0}$ be a nontrivial SRW and σ an a.s. finite first ladder epoch. Then $d(X_1) = d(S_\sigma)$.*

Proof. ☞ [2, Lemma 2.33] or Problem 2.62. □

Proposition 2.59. *Given a SRW $(S_n)_{n\geq 0}$ with first ladder epochs $\sigma^\geq$, $\sigma^>$, $\sigma^\leq$, $\sigma^<$, the following assertions hold true:*

$$\mathbb{E}\sigma^\geq = \frac{1}{\mathbb{P}(\sigma^< = \infty)} \quad \textit{and} \quad \mathbb{E}\sigma^> = \frac{1}{\mathbb{P}(\sigma^\leq = \infty)}, \tag{2.45}$$

$$\mathbb{P}(\sigma^\leq = \infty) = (1-\kappa)\mathbb{P}(\sigma^< = \infty), \tag{2.46}$$

where

$$\kappa := \sum_{n\geq 1}\mathbb{P}(S_1>0,\ldots,S_{n-1}>0,S_n=0) = \sum_{n\geq 1}\mathbb{P}(\sigma^{\leq}=n,S_1^{\leq}=0).$$

Proof. ☞ [2, Prop. 2.15]. □

We close this section with the announced classification of RW's that provides us with a good understanding of their long-run behavior.

Theorem 2.60. *Let $(S_n)_{n\geq 0}$ be a nontrivial SRW. Then exactly one of the following three cases holds true:*

(i) $\sigma^{\leq},\sigma^{<}$ *are both defective and* $\mathbb{E}\sigma^{\geq},\mathbb{E}\sigma^{>}$ *are both finite.*
(ii) $\sigma^{\geq},\sigma^{>}$ *are both defective and* $E\sigma^{\leq},E\sigma^{<}$ *are both finite.*
(iii) $\sigma^{\geq},\sigma^{>},\sigma^{\leq},\sigma^{<}$ *are all a.s. finite with infinite expectation.*

In terms of the asymptotic behavior of S_n as $n\to\infty$, these three alternatives are characterized as follows:

(i) $\lim_{n\to\infty}S_n=\infty$ *a.s.*
(ii) $\lim_{n\to\infty}S_n=-\infty$ *a.s.*
(iii) $\liminf_{n\to\infty}S_n=-\infty$ *and* $\limsup_{n\to\infty}S_n=\infty$ *a.s.*

Finally, if $\mu := EX_1$ exists, thus $EX^+<\infty$ or $EX^-<\infty$, then (i), (ii), and (iii) are equivalent to $\mu>0$, $\mu<0$, and $\mu=0$, respectively.

The last stated fact that alternative (iii) occurs for any SRW with drift $\mu=0$ is usually referred to as the ***Chung-Fuchs theorem***.

Proof. Notice first that $\mathbb{P}(X_1=0)<1$ is equivalent to $\kappa<1$, whence (2.46) ensures that $\sigma^{>},\sigma^{\geq}$ as well as $\sigma^{<},\sigma^{\leq}$ are always defective simultaneously in which case the respective dual ladder epochs have finite expectation by (2.45). Hence, if neither (a) nor (b) holds true, the only remaining alternative is that all four ladder epochs are a.s. finite with infinite expectation. By combining the three alternatives for the ladder epochs just proved with Prop. 2.57, the respective characterizations of the behavior of S_n for $n\to\infty$ are immediate.

Suppose now that $\mu=\mathbb{E}X_1$ exists. In view of Prop. 2.57 it then only remains to verify that (iii) holds true in the case $\mu=0$. But any of the alternatives (i) or (ii) would lead to the existence of a ladder epoch σ such that $\mathbb{E}\sigma<\infty$ and S_σ is a.s. positive or negative. On the other hand, $\mathbb{E}S_\sigma=\mu\,\mathbb{E}\sigma=0$ would follow by an appeal to Wald's identity which is impossible. Hence $\mu=0$ entails (iii). □

The following definition gives names to the three above alternatives (i), (ii) and (iii) that may occur for a RW $(S_n)_{n\geq 0}$.

Definition 2.61. A RW $(S_n)_{n\ge 0}$ is called

– *positive divergent* if $\lim_{n\to\infty} S_n = \infty$ a.s.
– *negative divergent* if $\lim_{n\to\infty} S_n = -\infty$ a.s.
– *oscillating* if $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$ a.s.

Problems

Problem 2.62. Prove Lemma 2.58.

2.10 Two-sided renewal theory: a short path to extensions

Now we are ready to extend some of the previously stated renewal theorems to RW's with positive drift. The basic idea is as simple as effective and based upon the use of the embedded RP of strictly ascending ladder heights.

For most of the following derivations it suffices to consider the zero-delayed case when $S_0 = 0$, for the result in the general case then usually follows by a straightforward argument. So let $(S_n)_{n\ge 0}$ be a SRW with increment distribution F, positive drift μ and embedded SRP $(S_n^>)_{n\ge 0}$ of strictly ascending ladder heights the drift of which we denote by $\mu^> = \mathbb{E}S_1^>$. Since $\mathbb{E}\sigma^>$ is finite, Wald's equation implies

$$\mu^> \;=\; \mathbb{E}S_1^> \;=\; \mathbb{E}S_{\sigma^>} \;=\; \mu\,\mathbb{E}\sigma^>$$

even if $\mu = \infty$. As before, let $\mathbb{U} = \sum_{n\ge 0} F^{*n} = \sum_{n\ge 0} \mathbb{P}(S_n \in \cdot)$ be the renewal measure of $(S_n)_{n\ge 0}$ so that $\mathbb{U}(A)$ gives the expected number of visits of the RW to $A \in \mathscr{B}(\mathbb{R})$. We remark that it is not clear at this point whether $\mathbb{U}(A)$ is always finite for any bounded B as in the renewal case. The renewal measure of $(S_n^>)_{n\ge 0}$ is denoted $\mathbb{U}^>$.

2.10.1 The key tool: cyclic decomposition

Let σ be an a.s. finite stopping time for $(S_n)_{n\ge 1}$ with associated sequence $(\sigma_n)_{n\ge 0}$ of copy sums. Denote by $\mathbb{U}^{(\sigma)}$ the renewal measure of the RW $(S_{\sigma_n})_{n\ge 0}$ and define the *pre-σ occupation measure of* $(S_n)_{n\ge 0}$ by

$$\mathbb{V}^{(\sigma)}(A) \;:=\; \mathbb{E}\left(\sum_{n=0}^{\sigma-1} \mathbf{1}_A(S_n)\right) \quad \text{for } A \in \mathscr{B}(\mathbb{R}), \tag{2.47}$$

which has total mass $\|V^{(\sigma)}\| = \mathbb{E}\sigma$ and is hence finite if σ has finite mean. The next lemma shows that $\mathbb{U}$ and $\mathbb{U}^{(\sigma)}, \mathbb{V}^{(\sigma)}$ are related in a nice way and holds true even without any assumption on the drift of $(S_n)_{n\geq 0}$.

Lemma 2.63. [Cyclic decomposition formula] *Under the stated assumptions,*

$$\mathbb{U} \;=\; \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}$$

for any a.s. finite stopping time σ for $(S_n)_{n\geq 1}$.

Proof. Using cyclic decomposition with the help of the σ_n, we obtain

$$\begin{aligned}
\mathbb{U}(A) \;&=\; \mathbb{E}\left(\sum_{k\geq 0}\mathbf{1}_A(S_k)\right) \;=\; \sum_{n\geq 0}\mathbb{E}\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1}\mathbf{1}_A(S_k)\right)\\
&=\; \sum_{n\geq 0}\int_{\mathbb{R}} E\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1}\mathbf{1}_{A-x}(S_k - S_{\sigma_n})\middle| S_{\sigma_n} = x\right)\,\mathbb{P}(S_{\sigma_n}\in dx)\\
&=\; \sum_{n\geq 0}\int_{\mathbb{R}}\mathbb{V}^{(\sigma)}(A-x)\,\mathbb{P}(S_{\sigma_n}\in dx)\\
&=\; \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}(A)\quad \text{for all } A\in\mathscr{B}(\mathbb{R}),
\end{aligned}$$

where (2.42) of Prop. 2.52 has been utilized in the penultimate line. □

Specializing to $\sigma = \sigma^{>}$ and writing $\mathbb{V}^{>}$ for $\mathbb{V}^{(\sigma^{>})}$, the cyclic decomposition formula takes the form

$$\mathbb{U} \;=\; \mathbb{V}^{>} * \mathbb{U}^{>}. \tag{2.48}$$

We thus have a convolution formula for the renewal measure $\mathbb{U}$ that involves a finite measure concentrated on $\mathbb{R}_{\leq}$, viz. $\mathbb{V}^{>}$, and the renewal measure of a SRP, namely $\mathbb{U}^{>}$, for which the asymptotic behavior has been found in the previous sections. Various results for RP's including Blackwell's theorem and the key renewal theorem will now easily be extended to RW's with positive drift with help of this formula.

If $(S_n)_{n\geq 0}$, given in a standard model, has arbitrary initial distribution λ, then Lemma 2.63 in combination with $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ immediately implies

$$\mathbb{U}_\lambda \;=\; \lambda * \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)} \;=\; \mathbb{V}^{(\sigma)} * \mathbb{U}_\lambda^{(\sigma)} \tag{2.49}$$

where $\mathbb{V}^{(\sigma)}, \mathbb{U}^{(\sigma)}$ are defined as before under $\mathbb{P}_0$.

Returning to the zero-delayed situation, let us further note that a simple computation shows that

$$\mathbb{V}^{(\sigma)} \;=\; \sum_{n\geq 0}\mathbb{P}(\sigma > n, S_n\in\cdot) \tag{2.50}$$

[☞ Problem 2.69] and that, for any real- or complex-valued function f

$$\int f\, d\mathbb{V}^{(\sigma)} \;=\; \sum_{n\ge 0}\int_{\{\sigma>n\}} f(S_n)\, d\mathbb{P} \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma-1} f(S_n)\right) \tag{2.51}$$

whenever one of the three expressions exist.

2.10.2 Uniform local boundedness and stationary delay distribution

The following lemma showing uniform local boundedness of the renewal measure for any random walk with positive drift is the partial extension of Lemma 2.14. A full extension by extending the argument given there is stated as Problem 2.71.

Lemma 2.64. *Let $(S_n)_{n\ge 0}$ be a RW with positive drift in a standard model. Then $\mathbb{U}_\lambda$ is uniformly locally bounded for each $\lambda \in \mathscr{P}(\mathbb{R})$, in fact*

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda([t,t+h]) \;\le\; \mathbb{E}\sigma^> \,\mathbb{U}_0^>(h) \tag{2.52}$$

for all $h>0$.

Proof. For any $\lambda \in \mathscr{P}(\mathbb{R})$ and $h>0$, the cyclic decomposition formula (2.49) with $\sigma = \sigma^>$ in combination with $\|\mathbb{V}^>\| = \mathbb{E}\sigma^>$ and

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda^>([t,t+h]) \;\le\; \mathbb{U}_0^>(h)$$

by Lemma 2.14 yields

$$\begin{aligned}\mathbb{U}_\lambda([t,t+h]) \;&=\; \mathbb{V}^> * \mathbb{U}_\lambda^>([t,t+h])\\ &=\; \int \mathbb{U}_\lambda^>([t-x,t-x+h])\, \mathbb{V}^>(dx)\\ &\le\; \mathbb{E}\sigma^> \,\mathbb{U}_0^>(h)\end{aligned}$$

as claimed. □

Cyclic decomposition also allows us to generalize the results from Subections 2.4.2 and 2.4.3 about the stationary delay distribution. This is accomplished by considering F^s, F_a^s and ξ as defined there, but for the associated ladder height RP $(S_n^>)_{n\ge 0}$. Hence we put

$$\xi(t) \;:=\; \begin{cases}\int_0^t \mathbb{P}(S_1^> > x)\, dx, & \text{if } d=0,\\ \sum_{k=1}^{n(t)} \mathbb{P}(S_1^> \ge k), & \text{if } d=1\end{cases} \tag{2.53}$$

for $t \in \mathbb{R}_{\geq}$ (with $n(t)$ as in Prop. 2.15) and then again F_a^s by (2.11) for $a \in \mathbb{R}_>$. If $S_1^>$ has finite mean $\mu^>$ and hence ξ is finite, then let F^s be its normalization, i.e. $F^s = (\mu^>)^{-1}\xi$. Recall from Lemma 2.58 that $S_1^>$ and X_1 are of the same lattice-type.

Theorem 2.65. *Let $(S_n)_{n\geq 0}$ be a RW in a standard model with positive drift μ and lattice-span $d \in \{0,1\}$. Then the following assertions hold with ξ, F_a^s and F^s as defined in* (2.53) *and thereafter.*

(a) $\mathbb{U}_\xi^+ = \mathbb{E}\sigma^> \lambda_d^+$.
(b) $\mathbb{U}_{F_a^s}^+ \leq \xi(a)^{-1}\mathbb{E}\sigma^> \lambda_d^+$ *for all* $a \in \mathbb{R}_>$.
(c) *If μ is finite, then* $\mathbb{U}_{F^s}^+ = \mu^{-1}\lambda_d^+$.

Proof. First note that $\mu > 0$ implies $\mathbb{E}\sigma^> < \infty$ [☞ Thm. 2.60] and $\mu^> = \mathbb{E}S_1^> = \mu\,\mathbb{E}\sigma^>$ by Wald's identity. By (2.49), $\mathbb{U}_\lambda = \mathbb{V}^> * \mathbb{U}_\lambda^>$ for any distribution λ and this obviously extends to arbitrary locally finite measures λ. Therefore,

$$\begin{aligned}\mathbb{U}_\xi(A) &= \mathbb{V}^> * \mathbb{U}_\xi^>(A) = \int_{\mathbb{R}_\leq} \mathbb{U}_\xi^>(A-x)\,\mathbb{V}^>(dx)\\ &= \int_{\mathbb{R}_\leq} \lambda_d^+(A-x)\,\mathbb{V}^>(dx) = \mathbb{V}^>(\mathbb{R}_\leq)\,\lambda_d^+(A)\\ &= \mathbb{E}\sigma^>\,\lambda_d^+(A) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}_\geq)\end{aligned}$$

where Cor. 2.16, the translation invariance of λ_d and $A - x \subset \mathbb{R}_\geq$ for all $x \in \mathbb{R}_\leq$ have been utilized. Hence assertion (a) is proved. As (b) and (c) are shown in a similar manner, we omit supplying the details again and only note for (c) that, if $\mu < \infty$, $\mathbb{U}_{F^s}^+ = (\mu^>)^{-1}\,\mathbb{E}\sigma^>\,\lambda_d^+$ really equals $\mu^{-1}\lambda_d^+$ because $\mu^> = \mu\,\mathbb{E}\sigma^>$ as mentioned above. □

2.10.3 Extensions of Blackwell's and the key renewal theorem

Extensions of the two main renewal theorems to RW's with positive drift are now obtained in a straightforward manner by combination of these results for the ladder height RP with cyclic decomposition.

Theorem 2.66. [Blackwell's renewal theorem] *Let $(S_n)_{n\geq 0}$ be a RW in a standard model with lattice-span $d \geq 0$ and positive drift μ. Then*

$$d\text{-}\lim_{t\to\infty} \mathbb{U}_\lambda([t,t+h]) = \mu^{-1}\lambda_d([0,h]) \quad \textit{and} \tag{2.54}$$

$$\lim_{t\to-\infty} \mathbb{U}_\lambda([t,t+h]) = 0 \tag{2.55}$$

for all $h > 0$ and $\lambda \in \mathscr{P}(\mathbb{G}_d)$, where $\mu^{-1} := 0$ if $\mu = \infty$.

Proof. By another use of cyclic decomposition in combination with Lemma 2.64, Blackwell's theorem for $\mathbb{U}_\lambda^>$ and of $\mu\,\|\mathbb{V}^>\| = \mu\,\mathbb{E}\sigma^> = \mu^>$ if $\mu < \infty$, it follows with the help of the dominated convergence theorem that

$$\begin{aligned} d\text{-}\lim_{t\to\infty} \mathbb{U}_\lambda([t,t+h]) \;&=\; \int d\text{-}\lim_{t\to\infty} \mathbb{U}_\lambda^>([t-x,t-x+h])\,\mathbb{V}^>(dx) \\ &=\; \frac{\|\mathbb{V}^>\|\,\lambda\!\!\lambda_d([0,h])}{\mu^>} \;=\; \frac{\lambda\!\!\lambda_d([0,h])}{\mu} \end{aligned}$$

for any $h > 0$, i.e. (2.54). But (2.55) follows analogously, for $\mathbb{U}^>$ vanishes on the negative halfline giving $\lim_{t\to-\infty} \mathbb{U}_\lambda^>([t,t+h]) = 0$. □

Theorem 2.67. [Key renewal theorem] *Let $(S_n)_{n\ge 0}$ be a RW with positive drift μ, lattice-span $d \in \{0,1\}$ and renewal measure $\mathbb{U}$. Then*

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}(t) \;=\; \frac{1}{\mu}\int g\, d\lambda\!\!\lambda_d \quad \text{and} \tag{2.56}$$

$$\lim_{t\to-\infty} g * \mathbb{U}(t) \;=\; 0 \tag{2.57}$$

for every dRi function $g : \mathbb{R} \to \mathbb{R}$.

Proof. Given a dRi function $g : \mathbb{R} \to \mathbb{R}$, we leave it to the reader [☞ Problem 2.72] to verify that Thm. 2.31 still applies and yields

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}^>(t) \;=\; \frac{1}{\mu^>}\int_{-\infty}^{\infty} g(x)\, \lambda\!\!\lambda_d(dx)$$

as well as

$$\lim_{t\to-\infty} g * \mathbb{U}^>(t) \;=\; 0.$$

In particular, $g * \mathbb{U}^>$ is a bounded function. Then use cyclic decomposition and the dominated convergence theorem to infer that

$$g * \mathbb{U}(t) \;=\; \int g * \mathbb{U}^>(t-x)\, \mathbb{V}^>(dx)$$

has the asserted limits as $t \to \infty$ (through $\mathbb{G}_d$) and $t \to -\infty$. □

We finish with a brief look at the renewal function $\mathbb{U}(t) = \mathbb{U}((-\infty,t])$ for a SRW $(S_n)_{n\ge 0}$ with positive drift μ. We know that $(\mu^>)^{-1}t \le \mathbb{U}^>(t) \le (\mu^>)^{-1}t + C$ for all $t \ge 0$ and a suitable constant $C \in \mathbb{R}_>$, whence cyclic decomposition provides us with the estimate

$$\mathbb{U}(t) \;=\; \int_{\mathbb{R}_\leq} \mathbb{U}^>(t-x)\,\mathbb{V}^>(dx) \begin{cases} \geq\; \mu^{-1}t \;+\; \int_{\mathbb{R}_\leq} |x|\,\mathbb{V}^>(dx) \\ \leq\; \mu^{-1}t \;+\; \int_{\mathbb{R}_\leq} |x|\,\mathbb{V}^>(dx) \;+\; C\,\mathbb{E}\sigma^> \end{cases}$$

for all $t \geq 0$. As a consequence, $\mathbb{U}(t) < \infty$ for some/all $t \in \mathbb{R}$ holds iff (use (2.51))

$$\int_{\mathbb{R}_\leq} |x|\,\mathbb{V}^>(dx) \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma^>-1} |S_n|\right) \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma^>-1} S_n^-\right) \;<\; \infty.$$

A nontrivial result not derived here states [☞ [2, Cor. 6.25]] that this is equivalent to the moment condition

$$\mu_2^- := \mathbb{E}(X^-)^2 < \infty.$$

2.10.4 *An application: Tail behavior of* $\sup_{n\geq 0} S_n$ *in the negative drift case*

In Applied Probability, the supremum of a SRW with negative drift (or, equivalently, the minimum of a SRW with positive drift) pops up in various contexts like the ruin problem in insurance mathematics or the asymptotic analysis of queuing models. An instance already encountered is Lindley's equation

$$W \stackrel{d}{=} (W+X)^+$$

for a random variable X with negative mean and independent of W. As explained in Section 1.2, the law of W equals the equilibrium distribution of a customer's waiting time in a G/G/1-queue before proceeding to the server if $X = B - A$, the difference of a generic service time B and a generic interarrival time A. If $(S_n)_{n\geq 0}$ denotes a SRW with increments $X_1, X_2, \ldots$ which are copies of X, then

$$W \stackrel{d}{=} \sup_{n\geq 0} S_n$$

as stated in Theorem 1.4 [☞ also Problem 1.6(b)]. The following classical result, which may already be found in FELLER's textbook [30, Ch. XII, (5.13)], provides the exact first-order asymptotics for $\mathbb{P}(W > t)$ as $t \to \infty$ under an exponential moment condition.

Theorem 2.68. *Let* $(S_n)_{n\geq 0}$ *be a SRW with negative drift* μ*, lattice-span* $d \in \{0,1\}$*,* $\mathbb{E}e^{\vartheta S_1} = 1$ *and* $\mu_\vartheta := \mathbb{E}e^{\vartheta S_1^>} S_1^> \mathbf{1}_{\{\sigma^> < \infty\}} < \infty$ *for some* $\vartheta > 0$*. Then*

$$d\text{-}\lim_{t\to\infty} e^{\vartheta t}\, \mathbb{P}\left(\sup_{n\geq 0} S_n > t\right) \;=\; \frac{\mathbb{P}(\sigma^> = \infty)}{e(d,\vartheta)\mu_\vartheta} \;\in\; \mathbb{R}_> \qquad (2.58)$$

with $e(d,\theta)$ *as defined in* (2.37). *If* $\mu_\vartheta = \infty$, *the result remains true when interpreting the right-hand side of* (2.58) *as* 0.

For an alternative approach to this result via implicit renewal theory, we refer to Subsection 4.4.2. Let us further point out that the increments of $(S_n)_{n\ge 0}$ may take values in $\mathbb{R}\cup\{-\infty\}$ as one can readily see from the following proof. In this case, $e^{-\infty} := 0$ as usual.

Proof. By Prop. 2.59, $\mathbb{P}(\sigma^> = \infty) = (\mathbb{E}\sigma^{\le})^{-1} > 0$, for $\mu < \infty$ implies $\mathbb{E}\sigma^{\le} < \infty$. Further, $\mathbb{P}(\sigma^> = \infty) < 1$ and $\mu_\vartheta > 0$, for $\mathbb{E}e^{\vartheta S_1} = 1$ ensures $\mathbb{P}(S_1 > 0) > 0$. Consequently, $Q_> := \mathbb{P}(S_1^> \in \cdot, \sigma^> < \infty)$ is nonzero and defective, i.e. $0 < \|Q_>\| < 1$, and the associated renewal measure

$$\mathbb{U}^> \;=\; \sum_{n\ge 0} \mathbb{P}(S_n^> \in \cdot, \sigma_n^> < \infty) \;=\; \sum_{n\ge 0} Q_>^{*n}$$

[use Prop. 2.55(g) for the second equality] a finite measure.

Since $\mathbb{E}e^{\vartheta S_1} = 1$, the sequence $(e^{\vartheta S_n})_{n\ge 0}$ constitutes a nonnegative martingale with mean one. Let $(\mathscr{F}_n)_{n\ge 0}$ denote its natural filtration and $\mathscr{F}_\infty := \sigma(S_n : n \ge 0)$. Define a new probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathscr{F}_\infty)$ by

$$\widehat{\mathbb{P}}(A) := \mathbb{E}e^{\vartheta S_n}\mathbf{1}_A \quad \text{for } A \in \mathscr{F}_n \text{ and } n \ge 0.$$

As one easily see, $X_1, X_2, \dots$ are still iid under $\widehat{\mathbb{P}}$ with the same lattice-span d, common distribution $\widehat{Q}(B) := \mathbb{E}e^{\vartheta S_1}\mathbf{1}_B(S_1)$ for $B \in \mathscr{B}(\mathbb{R})$, and mean $\widehat{\mu} := \mathbb{E}e^{\vartheta S_1}S_1$. Equivalently, $(S_n)_{n\ge 0}$ is still a SRW with drift $\widehat{\mu}$. The latter is positive because $\phi(\theta) := \mathbb{E}e^{\theta S_1}$ is a convex function on $[0,\vartheta]$ with $\phi(0) = \phi(\vartheta) = 1$ and negative (right) derivative μ at 0. We further infer that

$$1 \;=\; \widehat{\mathbb{P}}(\sigma^> < \infty) \;=\; \sum_{n\ge 1} \widehat{\mathbb{P}}(\sigma^> = n) \;=\; \sum_{n\ge 1} \mathbb{E}e^{\vartheta S_n}\mathbf{1}_{\{\sigma^> = n\}} \;=\; \mathbb{E}e^{\vartheta S_1^>}.$$

Now observe that, with $g(x) := \mathbf{1}_{(-\infty,0)}(x)$,

$$\begin{aligned}\mathbb{P}\left(\sup_{n\ge 0} S_n > t\right) \;&=\; \sum_{n\ge 0} \mathbb{P}(S_n^> > t,\, \sigma_n^> < \infty,\, \sigma_{n+1}^> - \sigma_n^> = \infty)\\ &=\; \mathbb{P}(\sigma^> = \infty)\,\mathbb{U}^>((t,\infty))\\ &=\; \mathbb{P}(\sigma^> = \infty)\, g * \mathbb{U}^>(t)\end{aligned}$$

implying

$$e^{\vartheta t}\,\mathbb{P}\left(\sup_{n\ge 0} S_n > t\right) \;=\; \mathbb{P}(\sigma^> = \infty)\, g_\vartheta * \mathbb{U}_\vartheta^>(t),$$

where, as in Section 2.7, $g_\vartheta(x) = e^{\vartheta x} g(x)$ and $\mathbb{U}^>_\vartheta(dx) = e^{\vartheta x}\mathbb{U}^>(dx)$, the latter being the renewal measure of $(S^>_n)_{n\ge 0}$ under $\widehat{\mathbb{P}}$. Since g_ϑ is easily seen to be dRi, the assertion finally follows by an appeal to the key renewal theorem 2.67. □

Problems

Problem 2.69. Prove (2.50).

Problem 2.70. Prove that the cyclic decomposition formula remains true for any a.s. finite σ that is independent of $(S_n)_{n\ge 0}$ and called *randomized stopping time* for this RW. [Hint: Consider a SRP $(\sigma_n)_{n\ge 0}$ independent of $(S_n)_{n\ge 0}$ with $\mathscr{L}(\sigma_1) = \mathscr{L}(\sigma)$.]

Problem 2.71. Let $(S_n)_{n\ge 0}$ be a RW with positive drift in a standard model. As in the renewal case, put $N(A) := \sum_{n\ge 0} \mathbf{1}_A(S_n)$ for $A \in \mathscr{B}(\mathbb{R})$. Then

$$\sup_{t\in\mathbb{R}} \mathbb{P}_\lambda(N([t,t+h]) \ge n) \;\le\; \mathbb{P}_0(N([-h,h]) \ge n) \tag{2.59}$$

for all $h > 0$, $n \in \mathbb{N}_0$ and $\lambda \in \mathscr{P}(\mathbb{R})$. In particular,

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda([t,t+h]) \;\le\; \mathbb{U}_0([-h,h]) \tag{2.60}$$

and $\{N([t,t+h]) : t \in \mathbb{R}\}$ is uniformly integrable under each $\mathbb{P}_\lambda$ for all $h > 0$. [Hint: Generalize the proof of Lemma 2.14.]

Problem 2.72. Prove that the (one-sided) key renewal theorem 2.31 remains valid if g is dRi, but not necessarily vanishing on the negative halfline, and that $g * \mathbb{U}(t) \to 0$ as $t \to -\infty$ holds true in this case as well.

Chapter 3
Iterated random functions

This chapter is devoted to a rather short introduction of the general theory of iterations of iid random Lipschitz functions, also called *iterated function systems*. They may be viewed as a particular class of Markov chains on a topological state space for which stability results are usually deduced via appropriate contraction conditions on the occurring class of random functions.

3.1 The model, definitions, some basic observations and examples

In order to provide an appropriate framework for the subsequent considerations, we begin with a formal definition of the special class of Markov chains to be studied here. The relevance in connection with random recursive equations, the actual topic of this course, becomes immediately apparent by the way these chains are defined in (3.1) below.

Although all examples encountered so far have been Markov chains on $\mathbb{R}$ or $\mathbb{R}^m$, we have chosen to take a more general approach here by allowing the state space to be any complete separable metric space $(\mathbb{X},d)$ endowed with the Borel σ-field $\mathscr{B}(\mathbb{X})$. The reader will hopefully acknowledge that this appears to be quite natural and does not make our life more complicated. Nevertheless it may be useful to point out the following facts:

Convergence in distribution for random elements $X,X_1,X_2,...$ in $(\mathbb{X},\mathscr{B}(\mathbb{X}))$ is still defined in the usual manner, i.e. $X_n \stackrel{d}{\to} X$ if

$$\lim_{n\to\infty} \mathbb{E}f(X_n) \;=\; \mathbb{E}f(X) \quad \text{for all } f \in \mathscr{C}_b(\mathbb{X}),$$

where $\mathscr{C}_b(\mathbb{X})$ denotes the space of bounded continuous functions $f:\mathbb{X}\to\mathbb{R}$. Uniqueness of the limit distribution is guaranteed by the fact that this space is *measure-determining*, i.e., two bounded measures λ_1,λ_2 on $(\mathbb{X},\mathscr{B}(\mathbb{X}))$ are equal whenever

$$\int_{\mathbb{X}} f(x)\,\lambda_1(dx) \;=\; \int_{\mathbb{X}} f(x)\,\lambda_2(dx) \quad \text{for all } f \in \mathscr{C}_b(\mathbb{X}).$$

Finally, the Portmanteau theorem remains valid as well. For further information on convergence of probability measures on metric spaces we refer to the classic monograph by BILLINGSLEY [12].

3.1.1 *Definition of an iterated function system and its canonical model*

The formal definition of an iterated function system is first, followed by the discussion of some measurability aspects and the specification of a canonical model.

Definition 3.1. Let $(\mathbb{X},d)$ be a complete separable metric space with Borel-σ-field $\mathfrak{B}(\mathbb{X})$. A temporally homogeneous Markov chain $(X_n)_{n\geq 0}$ with state space $\mathbb{X}$ is called *iterated function system (IFS) of iid Lipschitz maps* if it satisfies a recursion of the form

$$X_n = \Psi(\theta_n, X_{n-1}) \tag{3.1}$$

for $n \geq 1$, where

(IFS-1) $X_0, \theta_1, \theta_2, \ldots$ are independent random elements on a common probability space $(\Omega, \mathfrak{A}, \mathbb{P})$;

(IFS-2) $\theta_1, \theta_2, \ldots$ are identically distributed with common distribution Λ and taking values in a measurable space $(\Theta, \mathscr{A})$;

(IFS-3) $\Psi : (\Theta \times \mathbb{X}, \mathscr{A} \otimes \mathscr{B}(\mathbb{X})) \to (\mathbb{X}, \mathscr{B}(\mathbb{X}))$ is jointly measurable and Lipschitz continuous in the second argument, that is

$$d(\Psi(\theta,x),\Psi(\theta,y)) \leq C_\theta\, d(x,y)$$

for all $x,y \in \mathbb{X}$, $\theta \in \Theta$ and a suitable $C_\theta \in \mathbb{R}_{\geq}$.

A natural way to generate an IFS is to first pick an iid sequence $\Psi_1, \Psi_2, \ldots$ of random elements from the space $\mathscr{C}_{Lip}(\mathbb{X})$ of Lipschitz self-maps on $\mathbb{X}$ and to then produce a Markov chain $(X_n)_{n\geq 0}$ by picking an initial value X_0 and defining

$$X_n \;=\; \Psi_n \circ \ldots \circ \Psi_1(X_0) \tag{3.2}$$

for each $n \geq 1$. In the context of the above definition, $\Psi_n = \Psi(\theta_n, \cdot)$, but it becomes a measurable object only if we endow $\mathscr{C}_{Lip}(\mathbb{X})$ with a suitable σ-field. Therefore we continue with a short description of what could be called the canonical model of an IFS which particularly meets the last requirement.

Let $\mathbb{X}_0 := \{x_1, x_2, ...\}$ be a countable dense subset of $\mathbb{X}$ and $\mathscr{L}(\mathbb{X}_0, \mathbb{X})$ the "sequence" space of all mappings from $\mathbb{X}_0$ to $\mathbb{X}$. The latter clearly forms a complete separable metric space, for instance, when choosing

$$\rho(\psi_1, \psi_2) = \sum_{n\geq 1} \frac{1}{2^n} \frac{d(\psi_1(x_n), \psi_2(x_n))}{1 + d(\psi_1(x_n), \psi_2(x_n))}$$

for $\psi_1, \psi_2 \in \mathscr{L}(\mathbb{X}_0, \mathbb{X})$ as a metric. We endow $\mathscr{L}(\mathbb{X}_0, \mathbb{X})$ with the product σ-field $\mathscr{B}(\mathbb{X})^{\mathbb{X}_0}$ generated by the product topology. Finally, we define the *Lipschitz constant of* ψ as

$$L(\psi) \;:=\; \sup_{x,y\in\mathbb{X}, x\neq y} \frac{d(\psi(x), \psi(y))}{d(x,y)} \tag{3.3}$$

with the convention $L(\psi) := 0$ if ψ is constant. The following lemma is taken from [22].

Lemma 3.2. *Given the previous notation, the following assertions hold true:*

(a) $\mathscr{C}_{Lip}(\mathbb{X})$ *is a Borel subset of* $\mathscr{L}(\mathbb{X}_0, \mathbb{X})$.
(b) *The mapping* $\psi \mapsto L(\psi)$ *is a Borel function on* $\mathscr{C}_{Lip}(\mathbb{X})$.
(c) *The mapping* $(\psi, x) \mapsto \psi(x)$ *is a Borel function on* $\mathscr{C}_{Lip}(\mathbb{X}) \times \mathbb{X}$.

Proof. The map $L_0 : \mathscr{L}(\mathbb{X}_0, \mathbb{X}) \to [0, \infty]$, defined by

$$L_0(\psi) \;:=\; \sup_{x,y\in\mathbb{X}_0, x\neq y} \frac{d(\psi(x), \psi(y))}{d(x,y)},$$

is clearly a Borel function, for it is the supremum of countably many continuous functions, namely

$$\mathscr{L}(\mathbb{X}_0, \mathbb{X}) \;\ni\; \psi \;\mapsto\; \frac{d(\psi(x), \psi(y))}{d(x,y)}$$

for $(x,y) \in \mathbb{X}_0^2$, $x \neq y$. Now observe that, if $L_0(\psi) < \infty$, then ψ has a unique extension to a Lipschitz function on $\mathbb{X}$ with $L(\psi) = L_0(\psi)$ because $\mathbb{X}_0$ is dense in $\mathbb{X}$. Conversely, the restriction of any Lipschitz continuous ψ to $\mathbb{X}_0$ satisfies $L_0(\psi) = L(\psi)$ whence we conclude

$$\mathscr{C}_{Lip}(\mathbb{X}) \;=\; \{\psi : L_0(\psi) < \infty\} \;\in\; \mathscr{B}(\mathbb{X})^{\mathbb{X}_0}$$

(by unique embedding) as well as the measurability of $\psi \mapsto L(\psi)$.

In oder to prove (c), let $x_1, x_2, ...$ be an enumeration of the elements of $\mathbb{X}_0$ and $B_\varepsilon(x)$ the open ε-ball with center x. For $n, k \in \mathbb{N}$, define

$$A_{n,1} := B_{1/n}(x_1) \quad \text{and} \quad A_{n,k} := B_{1/n}(x_k) \cap \bigcup_{j=1}^{k-1} B_{1/n}(x_j)^c \text{ for } k \geq 2.$$

Then each $(A_{n,k})_{k\geq 1}$ forms a measurable partition of $\mathbb{X}$. For any $\psi : \mathbb{X} \to \mathbb{X}$, put

$$\psi_n(x) \;:=\; \sum_{k\geq 1} \psi(x_k)\,\mathbf{1}_{A_{n,k}}(x) \quad \text{for } n \geq 1.$$

Then the mapping $(\psi,x) \mapsto \psi_n(x)$ is measurable from $\mathscr{L}(\mathbb{X}_0,\mathbb{X}) \times \mathbb{X}$ to $\mathbb{X}$ and its retraction to $\mathscr{C}_{Lip}(\mathbb{X}) \times \mathbb{X}$ converges pointwise to the evaluation map $(\psi,x) \mapsto \psi(x)$ which gives the desired result. □

In view of the previous lemma we can choose the following *canonical model* for an IFS of iid Lipschitz maps: Let $\mathscr{A}$ be the Borel σ-field on $\Theta := \mathscr{C}_{Lip}(\mathbb{X})$, more precisely $\mathscr{A} := \mathscr{B}(\mathbb{X})^{\mathbb{X}_0} \cap \mathscr{C}_{Lip}(\mathbb{X})$, and let $(X_0,\theta) = (X_0,\theta_1,\theta_2,...)$ be the identity map on the product space $(\mathbb{X} \times \mathscr{C}_{Lip}(\mathbb{X})^{\mathbb{N}}, \mathscr{B}(\mathbb{X}) \otimes \mathscr{A}^{\mathbb{N}})$, so that θ_n denotes the n^{th} projection for each n, taking values in $(\Theta,\mathscr{A})$. If we choose an infinite product distribution $\Lambda^{\mathbb{N}}$ on $(\mathscr{C}_{Lip}(\mathbb{X})^{\mathbb{N}}, \mathscr{A}^{\mathbb{N}})$ and $\mathbb{P}_x := \delta_x \otimes \Lambda^{\mathbb{N}}$ on $(\Omega,\mathfrak{A}) := (\mathbb{X} \times \mathscr{C}_{Lip}(\mathbb{X})^{\mathbb{N}}, \mathscr{B}(\mathbb{X}) \otimes \mathscr{A}^{\mathbb{N}})$, these projections are iid with common distribution F and independent of X_0 under any $\mathbb{P}_\lambda := \int \mathbb{P}_x\,\lambda(dx)$, $\lambda \in \mathscr{P}(\mathbb{X})$. Finally, define $\Psi : (\mathscr{C}_{Lip}(\mathbb{X}) \times \mathbb{X}, \mathscr{A} \otimes \mathscr{B}(\mathbb{X}))$ by $\Psi(\theta,x) := \theta(x)$ and $X_n := \Psi(\theta_n,X_0)$. Then

$$(\Omega,\mathfrak{A},(X_n)_{n\geq 0},(\mathbb{P}_\lambda)_{\lambda\in\mathscr{P}(\mathbb{X})}) \tag{3.4}$$

provides a canonical model for the IFS $(X_n)_{n\geq 0}$ of iid Lipschitz maps in which

$$\Psi_n := \Psi(\theta_n,\cdot), \quad n \geq 0$$

is a sequence of iid random elements in $\mathscr{C}_{Lip}(\mathbb{X})$ independent of X_0 under each $\mathbb{P}_\lambda$. As a Markov chain, $(X_n)_{n\geq 0}$ has one-step transition kernel

$$P(x,B) \;=\; \mathbb{P}_x(\Psi(\theta_1,X_0) \in B) \;=\; \Lambda(\Psi(\cdot,x) \in B), \quad B \in \mathscr{B}(\mathbb{X}), \tag{3.5}$$

which is easily seen to be Fellerian [☞ Problem 3.12]. In the following, we will always assume a standard model of the afore-mentioned type be given and write Ψ_n for $\Psi(\theta_n,\cdot)$, thus

$$X_n \;=\; \Psi_n(X_{n-1}) \;=\; \Psi_n \circ ...\Psi_1(X_0)$$

as already stated in (3.2). We further put $L_n := L(\Psi_n)$ for $n \geq 1$, by Lemma 3.2(b) a random variable taking values in $\mathbb{R}_{\geq}$, and note that $L_1, L_2, ...$ are iid under each $\mathbb{P}_\lambda$ with a distribution independent of λ. Therefore we use $\mathbb{P}$ for probabilities not depending on the initial distribution of the Markov chain.

3.1.2 Lipschitz constants, contraction properties and the top Liapunov exponent

In view of the fact that $(\mathscr{C}_{Lip}(\mathbb{X}),\circ)$ forms a multiplicative semigroup and thus $\Psi_{n:k} := \Psi_n \circ \ldots \circ \Psi_k \in \mathscr{C}_{Lip}(\mathbb{X})$ for any $n \ge 1$ and $1 \le k \le n$, it is natural to ask about how the Lipschitz constant $L(\Psi_{n:1})$ of $\Psi_{n:1}$ relates to those of its factors $\Psi_1,\ldots,\Psi_n$. The following simple lemma is basic for our analysis.

Lemma 3.3. *For any* $\psi_1,\psi_2 \in \mathscr{C}_{Lip}(\mathbb{X})$,

$$L(\psi_1 \circ \psi_2) \le L(\psi_1)\cdot L(\psi_2).$$

Proof. Problem 3.13. □

As an immediate consequence of this lemma, we infer that

$$L(\Psi_{n:1}) \ \le\ L(\Psi_{n:k+1})L(\Psi_{k:1}) \quad \text{for any } 1 \le k < n \tag{3.6}$$

$$\text{and} \quad L(\Psi_{n:1}) \ \le\ \prod_{k=1}^{n} L_k \quad \text{for any } n \ge 1. \tag{3.7}$$

An important consequence of (3.6) is that it entails the existence of the so-called *(top) Liapunov exponent* with the help of Kingman's subadditive ergodic theorem, the latter being stated without proof in an Appendix to this chapter. The following result is due to FURSTENBERG & KESTEN [32] for linear maps and to ELTON [25] for Lipschitz maps.

Theorem 3.4. [Furstenberg-Kesten, Elton] *Let* $(X_n)_{n\ge 0}$ *be an IFS of iid Lipschitz maps with Lipschitz constants* $L_1,L_2,\ldots$ *satisfying* $\mathbb{E}\log^+ L_1 < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{n}\log L(\Psi_{n:1}) \ =\ \inf_{n\ge 1} \frac{\mathbb{E}\log L(\Psi_{n:1})}{n} \ =:\ \ell \quad \text{a.s.}$$

where $\ell \in \mathbb{R}\cup\{-\infty\}$ *is called* ***(top) Liapunov exponent*** *of* $(X_n)_{n\ge 0}$. *If* ℓ *is finite, then the convergence holds in* L^1 *as well.*

Proof. By (3.6), the triangular scheme $(Y_{k,n})_{n\ge 1}^{0\le k<n}$, defined by

$$Y_{k,n} := \log L(\Psi_{n:k+1}),$$

is subadditive in the sense that $Y_{0,n} \le Y_{0,k} + Y_{k,n}$ a.s. for all $0 \le k < n$. It also satisfies all other conditions of the subadditive ergodic theorem A.5 in the Appendix as the reader can readily check, leading to the conclusion that

$$\lim_{n\to\infty} \frac{1}{n} \log L(\Psi_{n:1}) \;=\; Y_\infty \quad \text{a.s.}$$

for some random variable Y_∞ with mean ℓ if ℓ is finite, and $Y_\infty = -\infty$ otherwise. And in the first case, the convergence is also in L^1. Finally, the fact that Y_∞ actually a.s. equals its mean value ℓ follows by an appeal to the Kolmogorov zero-one law, for Y_∞ is measurable with respect to the terminal σ-field of $\Psi_1, \Psi_2, \ldots$ □

A combination of the previous result with (3.7) and the SLLN further provides us with:

Corollary 3.5. *Let $(X_n)_{n\ge 0}$ be an IFS of iid Lipschitz maps with Lipschitz constants $L_1, L_2, \ldots$ satisfying $\mathbb{E}\log^+ L_1 < \infty$. Then its Liapunov exponent ℓ satisfies*

$$\ell \;\le\; \mathbb{E}\log L_1. \tag{3.8}$$

Proof. It suffices to note that $(\sum_{k=1}^n \log L_k)_{n\ge 0}$ forms a SRW with drift $\mathbb{E}\log L_1$ and that $\log L(\Psi_{n:1}) \le \sum_{k=1}^n \log L_k$ for each $n \ge 1$. □

It should not be surprising that the Lipschitz constants $L(\Psi_{n:1})$ play an important role in the stability analysis of $(X_n)_{n\ge 0}$. This will already become quite clear in the next section when studying *strongly contractive IFS* to be defined below along with other contraction conditions. Recall that a Lipschitz map ψ is called *contractive* or a *contraction* if $L(\psi) < 1$.

Definition 3.6. An IFS $(X_n)_{n\ge 0}$ of iid Lipschitz maps is called

– *strongly contractive*	if $\log L_1 \le -l$ a.s. for some $l \in \overline{\mathbb{R}}_>$.	
– *strongly mean contractive of order p*	if $\log \mathbb{E}L_1^p < 0$.	$(p > 0)$
– *mean contractive*	if $\mathbb{E}\log L_1 < 0$.	
– *contractive*	if it has Liapunov exponent $\ell < 0$.	

It is obvious that strong contraction implies contraction and strong mean contraction of any order, while an application of Jensen's inequality shows that the latter implies mean contraction; for a converse see Problem 3.14. Moreover, strong mean contraction of order p may always be reduced to the case $p = 1$ by switching the metric [☞ Problem 3.15].

3.1.3 Forward versus backward iterations

The recursive character of an IFS naturally entails that its state X_n at any time n is obtained via *forward iteration* or *left multiplication* of the random Lipschitz functions $\Psi_1, \ldots, \Psi_n$. This means, we first apply Ψ_1 to X_0, then Ψ_2 to $\Psi_1(X_0)$, and so on until we finally apply Ψ_n to $\Psi_{n-1} \circ \ldots \circ \Psi_1(X_0)$. On the other hand, since

$$(\Psi_1, \ldots, \Psi_n) \stackrel{d}{=} (\Psi_n, \ldots, \Psi_1),$$

the distribution of the forward iteration X_n is at all times n the same as of the *backward iteration* or *right multiplication* $\widehat{X}_n := \Psi_1 \circ \ldots \circ \Psi_n(X_0)$, that is

$$\widehat{X}_n \stackrel{d}{=} X_n \quad \text{for all } n \geq 0. \tag{3.9}$$

Consequently, we may also study the sequence of backward iterations $(\widehat{X}_n)_{n\geq 0}$ when trying to prove asymptotic stability of $(X_n)_{n\geq 0}$, i.e. $X_n \stackrel{d}{\to} \pi$ for some $\pi \in \mathscr{P}(\mathbb{X})$. The usefulness of this observation relies on the fact that in the stable case the $\widehat{X}_n$ exhibit a stronger pathwise convergence as we will see, which particularly shows that the joint distributions of $(X_n)_{n\geq 0}$ and $(\widehat{X}_n)_{n\geq 0}$ are generally very different. Most notably, $(\widehat{X}_n)_{n\geq 0}$ is not a Markov chain except for trivial cases.

In the following, we put $\Psi_{k:n} := \Psi_k \circ \ldots \circ \Psi_n$ for $1 \leq k \leq n$ and note as direct counterparts of (3.6) and (3.7) that

$$L(\Psi_{1:n}) \leq L(\Psi_{1:k})L(\Psi_{k+1:n}) \quad \text{for any } 1 \leq k < n \tag{3.10}$$

$$\text{and} \quad L(\Psi_{1:n}) \leq \prod_{k=1}^{n} L_k \quad \text{for any } n \geq 1. \tag{3.11}$$

Also Theorem 3.4 and its corollary remain valid when replacing $L(\Psi_{n:1})$ with $L(\Psi_{1:n})$ in (3.8).

3.1.4 Examples

At the end of this section we present a collection of examples some of which we have already encountered before.

Example 3.7 (Random difference equations). Iterations of iid linear functions $\Psi_n(x) = M_n x + Q_n$ on $\mathbb{R}$, with Lipschitz constants $L_n = |M_n|$, constitute one of the basic examples of an IFS and lead back to the one-dimensional random difference equation (RDE)

$$X_n := M_n X_{n-1} + Q_n, \quad n \geq 1, \tag{3.12}$$

discussed in Section 1.5. Recall from there that

$$X_n = \Psi_{n:1}(X_0) = \Pi_n X_0 + \sum_{k=1}^{n} \Pi_{k+1:n} Q_k \tag{3.13}$$

for each $n \in \mathbb{N}$, where $\Pi_{k:n} := M_k \cdot \ldots \cdot M_n$ for $1 \leq k \leq n$, $\Pi_{n+1:n} := 1$ and $\Pi_n := \Pi_{1:n}$, which shows that

$$L(\Psi_{n:1}) \;=\; |\Pi_n| \;=\; \prod_{k=1}^{n} L_k$$

for each $n \in \mathbb{N}$. The distributional equality (3.9) of forward and backward iteration at any time n was also stated there in (1.32), viz.

$$X_n \;\stackrel{d}{=}\; \Pi_n X_0 + \sum_{k=1}^{n} \Pi_{k-1} Q_k \;=\; \Psi_{1:n}(X_0) \;=\; \widehat{X}_n.$$

From these facts we see that $(X_n)_{n\geq 0}$ is

– *strongly contractive*	if $\log|M_1| \leq -l$ a.s. for some $l \in \overline{\mathbb{R}}_>$.	
– *strongly mean contractive of order p*	if $\log \mathbb{E}|M_1|^p < 0$.	$(p > 0)$
– *mean contractive*	if $\mathbb{E}\log|M_1| < 0$.	

In the multivariate case, the RDE (3.12) is defined on $\mathbb{R}^m$ for some $m \geq 2$ with $X_0, X_1, \ldots$ and $Q_1, Q_2, \ldots$ being column vectors and $M_1, M_2, \ldots$ being $m \times m$ real matrices. For $x \in \mathbb{R}^m$ and a $m \times m$ matrix A, let $|x|$ be the usual Euclidean norm and

$$\|A\| \;:=\; \max_{|x|=1} |Ax|$$

the usual operator norm of A. Contraction conditions must now be stated in terms of $\|M_1\|$ but look the same as before otherwise. Hence, $(X_n)_{n\geq 0}$ is

– *strongly contractive*	if $\log\|M_1\| \leq -l$ a.s. for some $l \in \overline{\mathbb{R}}_>$.	
– *strongly mean contractive of order p*	if $\log \mathbb{E}\|M_1\|^p < 0$.	$(p > 0)$
– *mean contractive*	if $\mathbb{E}\log\|M_1\| < 0$.	

Example 3.8 (Lindley processes). Lindley processes were introduced in Section 1.2 in connection with the G/G/1-queue and have the general form

$$X_n \;:=\; (X_{n-1} + \xi_n)^+, \quad n \geq 1, \tag{3.14}$$

for a sequence $(\xi_n)_{n\geq 1}$ of iid real-valued random variables which are not a.s. vanishing. This is an example of an IFS of iid Lipschitz functions on $\mathbb{X} = \mathbb{R}_\geq$ having Lipschitz constants $L_n = 1$, namely $\Psi_n(x) := (x + \xi_n)^+$ for $n \geq 1$. Denote by $(S_n)_{n\geq 0}$ the SRW associated with the ξ_n, thus $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$ for $n \geq 1$. Forward and backward iterations are easily computed as

$$\begin{aligned} X_n \;&=\; \Psi_{n:1}(X_0) \;=\; \max\{0, S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n - S_1, X_0 + S_n\} \\ \text{and} \quad \widehat{X}_n \;&=\; \Psi_{1:n}(X_0) \;=\; \max\{0, S_1, S_2, \ldots, S_{n-1}, X_0 + S_n\}, \end{aligned}$$

and the latter sequence is obviously nondecreasing. Furthermore, it converges a.s. to a finite limit, viz. $\widehat{X}_\infty := \sup_{n\geq 0} S_n$, iff $(S_n)_{n\geq 0}$ is negatively divergent, which particularly holds if $\mathbb{E}\xi < 0$ [☞ Thm. 1.4]. In this case, X_n converges to the same limit in distribution by (3.9). Notice that despite this stability result none of the above contraction conditions is valid, for $L_1 = 1$.

Example 3.9 (AR(1)-model with ARCH(1) errors). As another recurring example of an IFS we mention the AR(1)-model with ARCH(1) errors, defined by

$$X_n \;=\; \alpha X_{n-1} + \left(\beta + \lambda X_{n-1}^2\right)^{1/2} \varepsilon_n, \quad n \geq 1, \tag{3.15}$$

for $(\alpha, \beta, \lambda) \in \mathbb{R} \times \mathbb{R}_>^2$ and a sequence $(\varepsilon_n)_{n\geq 1}$ of iid symmetric random variables. Here $\Psi_n(x) := \alpha x + (\beta + \lambda x^2)^{1/2} \varepsilon_n$ for $n \geq 1$ and therefore

$$\frac{|\Psi_n(x) - \Psi_n(y)|}{|x-y|} \;=\; \left|\alpha + \frac{\lambda^{1/2}(x+y)\varepsilon_n}{(\beta/\lambda + x^2)^{1/2} + (\beta/\lambda + y^2)^{1/2}}\right| \;\leq\; \alpha + \lambda^{1/2}|\varepsilon_n|$$

for all $x, y \in \mathbb{R}$. By combining this with

$$\lim_{x,y\to\pm\infty} \frac{|\Psi_n(x) - \Psi_n(y)|}{|x-y|} \;=\; |\alpha \pm \lambda^{1/2}\varepsilon_n|,$$

it follows easily that $L(\Psi_n) = \alpha + \lambda^{1/2}|\varepsilon_n|$. Hence, the IFS is mean contractive if $\mathbb{E}\log(\alpha + \lambda^{1/2}|\varepsilon|) < 0$. On the other hand, neither forward nor backward iterations are easily computed here so that stability can only be analyzed by more sophisticated tools than in the previous two examples.

Example 3.10 (Random logistic maps). The *logistic map* $x \mapsto \theta x(1-x)$ is a self-map of the unit interval $[0,1]$ if $0 \leq \theta \leq 4$. Therefore, we obtain an IFS of i.i.d. Lipschitz functions on $[0,1]$ by defining

$$X_n \;=\; \xi_n X_{n-1}(1 - X_{n-1}), \quad n \geq 1, \tag{3.16}$$

for a sequence $(\xi_n)_{n\geq 1}$ of iid random variables taking values in $[0,4]$. Hence $\Psi_n(x) = \xi_n x(1-x)$, which has Lipschitz constant $L_n = \xi_n$ as one can easily verify. Contraction conditions as introduced before are thus to be formulated in terms of moments of ξ, but it should be noted that the Markov chain $(X_n)_{n\geq 0}$ possesses a stationary distribution in any case by Lemma 1.23 because the state space is compact which trivially ensures tightness of $(X_n)_{n\geq 0}$. In fact, if the chain is mean contractive, i.e. $\mathbb{E}\log\xi < 0$ holds, then X_n converges a.s. to zero under any initial distribution and geometrically fast [☞ Problem 3.15] which appears to be fairly boring and leaves us with the real challenge to find out what happens if mean contraction fails to hold and to provide conditions under which a stationary distribution $\pi \neq \delta_0$ exists. These questions have been addressed in a number of articles by Dai [20], Steinsaltz [55], Athreya & Dai [7, 8], and Athreya & Schuh [9]. We will return to this question in Subsection **??**.

Problems

Problem 3.11. Let $(X_n)_{n\ge 0}$ be an IFS of iid Lipschitz maps in a canonical model as stated in (3.4). Prove that, under each $\mathbb{P}_\lambda$ and for each $n\in\mathbb{N}_0$, X_n and $(\Psi_k)_{k>n}$ are independent with

$$\mathbb{P}_\lambda(X_n\in\cdot,(\Psi_k)_{k>n}\in\cdot)\ =\ \mathbb{P}_{\lambda_n}(X_0\in\cdot,(\Psi_k)_{k\ge 1}\in\cdot),$$

where $\lambda_n:=\mathbb{P}_\lambda(X_n\in\cdot)$.

Problem 3.12. Prove that the transition kernel P defined in (3.5) is Fellerian, i.e. it maps bounded continuous functions on $\mathbb{X}$ to functions of the same type [☞ Section 1.6].

Problem 3.13. Prove Lemma 3.3.

Problem 3.14. Let $(X_n)_{n\ge 0}$ be a mean contractive IFS of iid Lipschitz maps. Prove that, if $\mathbb{E}L_1^p<\infty$ for some $p>0$, then $(X_n)_{n\ge 0}$ is also strongly mean contractive of some order $q\le p$.

Problem 3.15. Let $(X_n)_{n\ge 0}$ be an IFS of iid Lipschitz maps which is strongly mean contractive of order $p\ne 1$. Prove that there exists a complete separable metric d' on $\mathbb{X}$ generating the same topology as d such that $(X_n)_{n\ge 0}$ is strongly mean contractive of order one under d', that is, when using the Lipschitz constants defined with the help of d'.

Problem 3.16. Consider the IFS $(X_n)_{n\ge 0}$ of random logistic maps introduced in Example 3.10. Prove that, if $\mathbb{E}\log\xi<0$, then $\mu^{-n}X_n\to 0$ a.s. for any $\mu<1$ such that $\log\mu>\mathbb{E}\log\xi$. What happens if $\mathbb{E}\log\xi=0$?

3.2 Geometric ergodicity of strongly contractive IFS

Aiming at an ergodic theorem for mean contractive IFS, we first study the simpler strongly contractive case for which we are going to prove geometric ergodicity, i.e. convergence to a stationary distribution at a geometric rate (under a mild moment condition). The distance between probability distributions on $\mathbb{X}$ is measured by the *Prokhorov metric* associated with d (the metric on $\mathbb{X}$) and denoted by the same letter. Given $\lambda_1,\lambda_2\in\mathscr{P}(\mathbb{X})$, it is defined as the infimum of the $\delta>0$ such that

$$\lambda_1(A)\le\lambda_2(A^\delta)+\delta\quad\text{and}\quad\lambda_2(A)\le\lambda_1(A^\delta)+\delta$$

for all $A\in\mathscr{B}(\mathbb{X})$, where $A^\delta:=\{x\in\mathbb{X}:d(x,y)<\delta\text{ for some }y\in A\}$. We note that $d(\lambda_1,\lambda_2)\le 1$ and without proof that convergence in the Prokhorov metric is equivalent to weak convergence, that is

$$d(\lambda_n,\lambda)\to 0\quad\text{iff}\quad\lambda_n\stackrel{w}{\to}\lambda.$$

The following simple coupling lemma provides a useful tool to derive an estimate for $d(\lambda_1,\lambda_2)$.

Lemma 3.17. *Let X_1,X_2 be two $\mathbb{X}$-valued random elements on the same probability space $(\Omega,\mathfrak{A},\mathbb{P})$ such that $\mathscr{L}(X_1)=\lambda_2$ and $\mathscr{L}(X_2)=\lambda_2$. Then $\mathbb{P}(d(X_1,X_2)\ge\delta)<\delta$ implies $d(\lambda_1,\lambda_2)\le\delta$.*

Proof. The assertion follows from the obvious inequality

$$\max\{\mathbb{P}(X_1\in A,X_2\notin A^\delta),\mathbb{P}(X_1\notin A^\delta,X_2\in A)\}\ \le\ \mathbb{P}(d(X_1,X_2)\ge\delta)$$

for all $A\in\mathscr{B}(\mathbb{X})$ and $\delta>0$. □

After these preliminary remarks we are ready to state the announced ergodic theorem.

Proposition 3.18. *Given a strongly contractive IFS $(X_n)_{n\ge0}$ of iid Lipschitz maps in a standard model such that $\log L_1\le -l$ for some $l\in\mathbb{R}_>$ and*

$$\mathbb{E}\log^+ d(x_0,\Psi_1(x_0))<\infty, \tag{3.17}$$

for some $x_0\in\mathbb{X}$, the following assertions hold true:

(a) For any $x\in\mathbb{X}$, the backward iteration $\widehat{X}_n$ converges $\mathbb{P}_x$-a.s. to a random element $\widehat{X}_\infty$ with distribution π which does not depend on x and satsifies the SFPE

$$\widehat{X}_\infty=\Psi_1(\widehat{X}'_\infty) \tag{3.18}$$

where $\widehat{X}'_\infty$ is a copy of $\widehat{X}_\infty$ independent of Ψ_1.

(b) For any $x\in\mathbb{X}$ and $\gamma\in(1,e^l)$,

$$\lim_{n\to\infty}\gamma^n d(\widehat{X}_n,\widehat{X}_\infty)=0\quad \mathbb{P}_x\text{-a.s.} \tag{3.19}$$

(c) For any $x\in\mathbb{X}$, the forward iteration X_n converges to π in distribution under $\mathbb{P}_x$, and π is the unique stationary distribution of $(X_n)_{n\ge0}$.

(d) Under $\mathbb{P}_\pi$, $(X_n)_{n\ge0}$ forms an ergodic stationary sequence, i.e.

$$\mathbf{1}_B(X_0,X_1,\ldots)=\mathbf{1}_B(X_1,X_2,\ldots)\quad \mathbb{P}_\pi\text{-a.s.}$$

implies $\mathbb{P}_\pi((X_n)_{n\ge0}\in B)\in\{0,1\}$.

(e) If (3.17) is sharpened to

$$\mathbb{E}\,d(x_0,\Psi_1(x_0))^p<\infty, \tag{3.20}$$

for some $p > 0$ and $x_0 \in \mathbb{X}$, then geometric ergodicity in the sense that

$$\lim_{n\to\infty} r^n\, d(\mathbb{P}_x(X_n \in \cdot), \pi) = 0 \tag{3.21}$$

for some $r > 1$ holds true.

Before we turn to the proof of this result, some comments are in order.

Remark 3.19. A fundamental conclusion from this result is that forward and backward iterations, despite having the same one-dimensional marginals, exhibit a drastically different behavior. While backward iterations converge a.s. at a geometric rate to a limit having distribution π, the convergence of the forward iterations, naturally to the same limit, occurs in the distributional sense only. Their trajectories, however, typically oscillate wildly in space due to the ergodicity which ensures that every π-positive subset is visited infinitely often. This is illustrated in Figure 3.1 below.

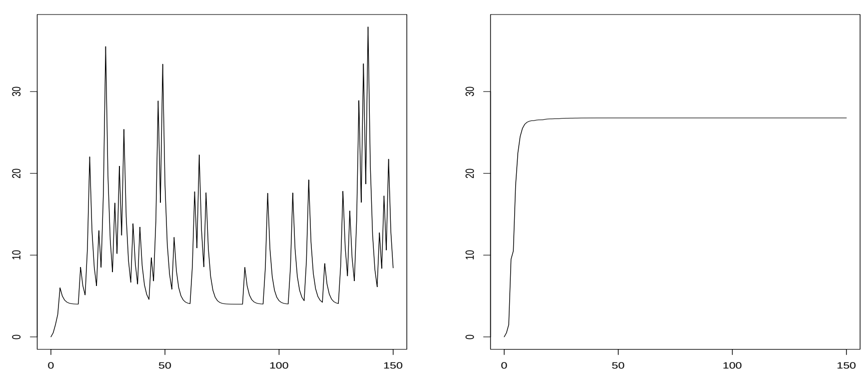

Fig. 3.1 Ergodic behavior of the forward iterations (left panel) versus pathwise convergence of the backward iterations (right panel), illustrated by a simulation of 150 iterations of the IFS which picks the function $\psi_1(x) = 0.5x + 2$ with probability 0.75 and the function $\psi_2(x) = 2x + 0.5$ with probability 0.25 at each step.

Remark 3.20. The extra moment conditions (3.17) – in which $\log^+ x$ may be replaced with the subadditive majorant $\log^* x := \log(1+x)$ – and (3.20), frequently called *jump-size conditions* hereafter, are needed beside strong contraction to ensure that the chain is not carried away too far in one step when moving in space. The reader should realize that this is indeed a property not guaranteed by contraction, which rather ensures forgetfulness of initial conditions. Let us further point out that, if any of these jump-size conditions is valid for one $x_0 \in \mathbb{X}$, then it actually holds for all $x_0 \in \mathbb{X}$. This follows from

$$\begin{aligned} d(x,\Psi_1(x)) &\le d(x,x_0)+d(x_0,\Psi_1(x_0))+d(\Psi_1(x_0),\Psi_1(x)) \\ &\le (1+L_1)\,d(x,x_0)+d(x_0,\Psi_1(x_0)) \\ &\le (1+e^{-l})\,d(x,x_0)+d(x_0,\Psi_1(x_0)) \end{aligned}$$

for all $x\in\mathbb{X}$.

In the following we will use $\mathbb{P}$ for probabilities that do not depend on the initial distribution of $(X_n)_{n\ge 0}$. For instance,

$$\mathbb{P}_x(X_n\in\cdot)\;=\;\mathbb{P}(\Psi_{n:1}(x)\in\cdot)$$

because $\Psi_1,\Psi_2,\ldots$ are independent of X_0. Note also that, for any $x,y\in\mathbb{X}$,

$$(X_n^x,X_n^y):=(\Psi_{n:1}(x),\Psi_{n:1}(y)),\quad n\ge 0$$

provides a canonical coupling of two forward chains starting at x and y. A similar coupling is naturally given by $(\widehat{X}_n^x,\widehat{X}_n^y):=(\Psi_{1:n}(x),\Psi_{1:n}(y))$, $n\ge 0$, for the backward iterations.

Proof (of Prop. 3.18). We leave it as an exercise [☞ Problem 3.21] to verify that

$$\gamma^n d(\widehat{X}_n,\widehat{X}_\infty)\to 0\quad \mathbb{P}_x\text{-a.s.}$$

for $\gamma\ge 1$ and a random variable $\widehat{X}_\infty$ if

$$\sum_{n\ge 0}\gamma^n\,d(\widehat{X}_n,\widehat{X}_{n+1})<\infty\quad \mathbb{P}_x\text{-a.s.}\tag{3.22}$$

Strong contraction implies for any $\gamma\in[1,e^l)$

$$\begin{aligned} \gamma^n\,d(\widehat{X}_n,\widehat{X}_{n+1}) &= \gamma^n\,d(\Psi_{1:n}(X_0),\Psi_{1:n}\circ\Psi_{n+1}(X_0)) \\ &\le \beta^n\,d(X_0,\Psi_{n+1}(X_0)) \\ &= \beta^n\,d(x,\Psi_{n+1}(x))\quad \mathbb{P}_x\text{-a.s.} \end{aligned}$$

for any $x\in\mathbb{X}$, where $\beta:=\gamma e^{-l}\in(0,1]$. Now use (3.17), by Remark 3.20 valid for any $x\in\mathbb{X}$, to infer

$$\begin{aligned} \sum_{n\ge 0}\mathbb{P}_x(\gamma^n\,d(\widehat{X}_n,\widehat{X}_{n+1})>\varepsilon) &\le \sum_{n\ge 0}\mathbb{P}(\beta^n\,d(x,\Psi_{n+1}(x))>\varepsilon) \\ &= \sum_{n\ge 0}\mathbb{P}(\log d(x,\Psi_1(x))>\log\varepsilon+n\log(1/\beta)) \\ &\le \frac{\log(1/\varepsilon)+\mathbb{E}\log^+ d(x,\Psi_1(x))}{\log(1/\beta)}\;<\;\infty \end{aligned}$$

for any $\varepsilon>0$ and thus $Z_n(\gamma):=\gamma^n\,d(\widehat{X}_n,\widehat{X}_{n+1})\to 0$ $\mathbb{P}_x$-a.s. for any $\gamma\in[1,e^l)$ by an appeal to the Borel-Cantelli lemma. But the last conclusion further implies (3.22), because

$$\sum_{n\geq 0}\gamma^n d(\widehat{X}_n,\widehat{X}_{n+1}) \;\leq\; \sum_{n\geq 0}\left(\frac{\gamma}{\beta}\right)^n Z_n(\beta) \;<\; \infty$$

for any $1\leq\gamma<\beta<e^l$. This completes the proof of (b) and the first part of (a). As

$$d(\widehat{X}_n^x,\widehat{X}_n^y) \;=\; d(\Psi_{1:n}(x),\Psi_{1:n}(y)) \;\leq\; e^{-nl}\,d(x,y)\quad \mathbb{P}\text{-a.s.}$$

we see that $\widehat{X}_\infty$ and its distribution are the same under every $\mathbb{P}_x$. Also, $(\Psi_{2:n})_{n\geq 2}$ being a copy of $(\Psi_{1:n})_{n\geq 1}$, we find that $\Psi_{2:n}(x)$ converges a.s. to some $\widehat{X}_\infty'$ not depending on x and with the same law as $\widehat{X}_\infty$. Finally, the asserted SFPE (3.18) follows from

$$\widehat{X}_\infty \;=\; \lim_{n\to\infty}\Psi_1(\Psi_{2:n}(x)) \;=\; \Psi_1\left(\lim_{n\to\infty}\Psi_{2:n}(x)\right) \;=\; \Psi_1(\widehat{X}_\infty')\quad \mathbb{P}\text{-a.s.}$$

where the continuity of Ψ_1 enters in the second equality. This completes the proof of (a).

As for part (c), the first assertion is obvious from (a) because $X_n\stackrel{d}{=}\widehat{X}_n$ under each $\mathbb{P}_x$ and therefore each $\mathbb{P}_\lambda$, $\lambda\in\mathscr{P}(\mathbb{X})$. But this also implies that π must be the unique stationary distribution of $(X_n)_{n\geq 0}$. Indeed, any stationary π' satisfies

$$\int f(x)\;\pi'(dx) \;=\; \mathbb{E}_{\pi'}f(X_n)\;\stackrel{n\to\infty}{\longrightarrow}\;\int f(x)\;\pi(dx)$$

for all $f\in\mathscr{C}_b(\mathbb{X})$ and thus $\pi'=\pi$ because the class $\mathscr{C}_b(\mathbb{X})$ is measure-determining.

The proof of (d) forces us to make an excursion into ergodic theory and follows the argument given by ELTON [25, p. 43]. First of all, we may w.l.o.g. extend $(\Psi_n)_{n\geq 0}$ to a *doubly infinite* sequence $(\Psi_n)_{n\in\mathbb{Z}}$ of iid Lipschitz maps. This sequence is ergodic [☞ Prop. A.1], which in the terminology of ergodic theory means that the shift $\mathscr{S}_1:(\ldots,\psi_{-1},\psi_0,\psi_1,\ldots)\mapsto(\ldots,\psi_0,\psi_1,\psi_2,\ldots)$ consitutes a *measure-preserving ergodic transformation* on $(\mathscr{C}_{Lip}(\mathbb{X})^{\mathbb{Z}},\mathscr{A}^{\mathbb{Z}},\Lambda^{\mathbb{Z}})$. Next, fix any $x\in\mathbb{X}$ and define the doubly infinite stationary sequence

$$Y_n:=\lim_{k\to\infty}\Psi_{-k:n}(x),\quad n\in\mathbb{Z}$$

which is clearly a function φ, say, of $(\Psi_n)_{n\in\mathbb{Z}}$ and does not depend on the choice of x (by part (a)). Let Γ denote its distribution and notice that $Y_1=\widehat{X}_\infty$ as well as $\mathbb{P}((Y_n)_{n\geq 0}\in\cdot)=\mathbb{P}_\pi((X_n)_{n\geq 0}\in\cdot)$. The stationarity of $(Y_n)_{n\in\mathbb{Z}}$ means that the shift $\mathscr{S}_2:(\ldots,x_{-1},x_0,x_1,\ldots)\mapsto(\ldots,x_0,x_1,x_2,\ldots)$ is a measure-preserving transformation on $(\mathbb{X}^{\mathbb{Z}},\mathscr{B}(\mathbb{X})^{\mathbb{Z}},\Gamma)$. Now the ergodicity of $\mathscr{S}_2$, and thus of $(Y_n)_{n\in\mathbb{Z}}$, follows because *$\mathscr{S}_2$ is a factor of $\mathscr{S}_1$*, viz. $\varphi\circ\mathscr{S}_1=\mathscr{S}_2\circ\varphi$ a.s. for the measure-preserving map $\varphi:(\mathscr{C}_{Lip}(\mathbb{X})^{\mathbb{Z}},\mathscr{A}^{\mathbb{Z}},\Lambda^{\mathbb{Z}})\to(\mathbb{X}^{\mathbb{Z}},\mathscr{B}(\mathbb{X})^{\mathbb{Z}},\Gamma)$, defined by $(\psi_n)_{n\in\mathbb{Z}}\mapsto(\limsup_{k\to\infty}\psi_{-k:n}(x))_{n\in\mathbb{Z}}$ [☞ Prop. A.2 in the Appendix and before for further information].

Turning to (e), assume (3.20) for some $p>0$, w.l.o.g. $p\leq 1$. Then it follows with the help of the subadditivity of $x\mapsto x^p$ on $\mathbb{R}_\geq$ that, for any $s>0$,

$$\begin{aligned}
\mathbb{P}_x(d(\widehat{X}_n,\widehat{X}_\infty)\ge s^{-n}) &\le s^{np}\,\mathbb{E}_x\,d(\widehat{X}_n,\widehat{X}_\infty)^p\\
&\le s^{np}\sum_{k\ge n}\mathbb{E}_x\,d(\widehat{X}_k,\widehat{X}_{k+1})^p\\
&= s^{np}\sum_{k\ge n}\mathbb{E}\,d(\Psi_{1:k}(x),\Psi_{1:k+1}(x))^p\\
&\le (se^{-l})^{np}\,\mathbb{E}\,d(x,\Psi_1(x))^p\sum_{k\ge 0}e^{-klp}\\
&= (se^{-l})^{np}\,\frac{\mathbb{E}\,d(x,\Psi_1(x))^p}{1-e^{-lp}}.
\end{aligned}$$

The last expression is ultimately bounded by $o(1)\,s^{-n}$ as $n\to\infty$ if $(se^{-l})^p<s^{-1}$ or, equivalently, $1<s<e^{ql}$ with $q:=p/(p+1)$. Hence, for any such s we have shown that

$$\lim_{n\to\infty}s^n\,\mathbb{P}_x(d(\widehat{X}_n,\widehat{X}_\infty)\ge s^{-n})=0,$$

and this entails (3.21) for $r\in(1,s)$ by an appeal to Lemma 3.17, for $(\widehat{X}_n,\widehat{X}_\infty)$ constitutes a coupling of $\mathbb{P}_x(X_n\in\cdot)$ and π under $\mathbb{P}_x$. □

Problems

Problem 3.21. Given a sequence $(X_n)_{n\ge0}$ of random variables taking values in a complete metric space $(\mathbb{X},d)$, prove that

$$\sum_{n\ge0}d(X_n,X_{n+1})<\infty\quad\mathbb{P}\text{-a.s.}$$

implies the a.s. convergence of X_n to a random variable X_∞. More generally, if

$$\sum_{n\ge0}a_n\,d(X_n,X_{n+1})<\infty\quad\mathbb{P}\text{-a.s.}$$

holds true for a nondecreasing sequence $(a_n)_{n\ge0}$ in $\mathbb{R}_>$, then

$$\lim_{n\to\infty}a_n\,d(X_n,X_\infty)=0.$$

Problem 3.22. The proof of part (e) of Prop. 3.18 has shown that (3.21) holds true for any $r\in(1,e^{pl/(p+1)})$, provided that $p\le1$ in the jump-size condition (3.20). Show that this remains valid in the case $p>1$ as well. [Hint: Show first that

$$\big(\mathbb{E}_x d(\widehat{X}_n,\widehat{X}_\infty)^p\big)^{1/p}\ \le\ e^{-ln}\,\frac{\big(\mathbb{E}\,d(x,\Psi_1(x))^p\big)^{1/p}}{1-e^{-l}}$$

and then argue as in the afore-mentioned proof.]

Problem 3.23. Suppose that $\Psi,\Psi_1,\Psi_2,\ldots$ are iid Lipschitz maps on a complete metric space $(\mathbb{X},d)$ such that, for

$$\alpha := \mathbb{P}(\Psi \equiv x_0) > 0$$

for some $x_0 \in \mathbb{X}$. Define $\sigma := \inf\{n \geq 1 : \Psi_n \equiv x_0\}$ and then the pre-σ-occupation measure

$$\pi(A) \ := \ \mathbb{E}\left(\sum_{n=0}^{\sigma-1} \mathbf{1}_A\big(\Psi_{n:1}(x_0)\big)\right), \quad A \in \mathscr{A},$$

where $\Psi_{0:1}(x) = x$. Show that π is the unique stationary distribution of the strongly contractive IFS generated by $(\Psi_n)_{n\geq 1}$.

3.3 Ergodic theorem for mean contractive IFS

We will now proceed with the main result of this chapter, an ergodic theorem for mean contractive IFS of iid Lipschitz maps. The basic idea for its proof is taken from [3] and combines our previous result for strongly contractive IFS with a renewal argument based on the observation that any weakly contractive IFS contains a strongly contractive one. Before dwelling on this further, let us state the result we are going to prove in this section.

Theorem 3.24. *Given a mean contractive IFS $(X_n)_{n\geq 0}$ of iid Lipschitz maps in a standard model which also satisfies the jump-size condition* (3.17) *for some and thus all $x_0 \in \mathbb{X}$, the following assertions hold true:*

(a) *For any $x \in \mathbb{X}$, the backward iteration $\widehat{X}_n$ converges $\mathbb{P}_x$-a.s. to a random element $\widehat{X}_\infty$ with distribution π which does not depend on x and satisfies the SFPE* (3.18).

(b) *For some $\gamma > 1$ and any $x \in \mathbb{X}$,* (3.19) *holds true, that is*

$$\lim_{n\to\infty} \gamma^n \, d(\widehat{X}_n, \widehat{X}_\infty) = 0 \quad \mathbb{P}_x\text{-a.s.}$$

(c) *For any $x \in \mathbb{X}$, the forward iteration X_n converges to π in distribution under $\mathbb{P}_x$, and π is the unique stationary distribution of $(X_n)_{n\geq 0}$.*

(d) *Under $\mathbb{P}_\pi$, $(X_n)_{n\geq 0}$ forms an ergodic stationary sequence.*

(e) *If, for some $p > 0$, $(X_n)_{n\geq 0}$ is even strongly mean contractive of order p and satisfies the sharpened jump-size condition* (3.20)*, then geometric ergodicity in the sense of* (3.21) *for some $r > 1$ holds true.*

Remark 3.25. In view of Problem 3.14, the assumptions of (e) could be relaxed to

$$\mathbb{E}L_1^p < \infty \quad \text{and} \quad \mathbb{E}d(x_0,\Psi_1(x_0))^p < \infty \tag{3.23}$$

for some $x_0 \in \mathbb{X}$ and $p > 0$.

For the rest of this section, the assumptions of Theorem 3.24 will always be in force without further mention. We embark on the crucial observation that, given a weakly contractive IFS $(X_n)_{n\ge 0}$ of iid Lipschitz maps with Lipschitz constants $L_1, L_2, ...$, the SRW

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{k=1}^{n} \log L_k \quad \text{for } n \ge 1$$

has negative drift. Hence we may fix any $l > 0$ and consider the SRP of a.s. finite level $-l$ ladder epochs, defined by $\sigma_0 := 0$ and

$$\sigma_n := \inf\{k > \sigma_{n-1} : S_k - S_{\sigma_{n-1}} < -l\}$$

for $n \ge 1$. For simplicity, we choose l such that $\mathbb{P}(\sigma_1 = 1) > 0$ and thus $(\sigma_n)_{n\ge 0}$ is 1-arithmetic. The following lemma is basic.

Lemma 3.26. *The embedded sequence $(X_{\sigma_n})_{n\ge 0}$ forms a strongly contractive IFS of iid Lipschitz maps satisfying* (3.17)*, and the same holds true for the sequence $(Y_n)_{n\ge 0}$, defined by $Y_0 := X_0$ and*

$$Y_n := \Psi_{\sigma_{n-1}+1:\sigma_n} \circ \ldots \circ \Psi_{1:\sigma_1}(X_0)$$

for $n \ge 1$.

Proof. Plainly, $X_{\sigma_n} = \Phi_{n:1}(X_0)$ with $\Phi_n := \Psi_{\sigma_n:\sigma_{n-1}+1} \in \mathscr{C}_{Lip}(\mathbb{X})$ for $n \ge 1$. Since $(\sigma_n)_{n\ge 0}$ has iid increments, one can readily check that $\Phi_1, \Phi_2, \ldots$ are iid as well. Moreover, by (3.7),

$$\log L(\Phi_1) \;=\; \log L(\Psi_{\sigma_1} \circ \ldots \circ \Psi_1) \;\le\; S_{\sigma_1} \;<\; -l$$

which a confirms the strong contraction property. In order to verify the jump-size condition (3.17), i.e.

$$\mathbb{E} \log^+ d(x_0, \Psi_{\sigma_1:1}(x_0)) < \infty$$

for some and thus all $x_0 \in \mathbb{X}$, we will use $\log^+ x \le \log^* x := \log(1+x)$ and the subadditivity of the latter function. With $S_k^* := \sum_{j=1}^{k} \log^* L_j$ for $k \ge 1$, we infer

$$\begin{aligned}
\log d(x_0, \Psi_{\sigma_1:1}(x_0)) \;&\le\; \log\left(d(x_0, \Psi_{\sigma_1}(x_0)) + \sum_{n=1}^{\sigma_1 - 1} d(\Psi_{\sigma_1:n+1}(x_0), \Psi_{\sigma_1:n}(x_0)) \right) \\
&\le\; \log\left(\sum_{n=1}^{\sigma_1} e^{S_{\sigma_1} - S_n} d(x_0, \Psi_n(x_0)) \right) \\
&\le\; \log\left(e^{S^*_{\sigma_1}} \sum_{n=1}^{\sigma_1} d(x_0, \Psi_n(x_0)) \right)
\end{aligned}$$

$$\leq\ S^*_{\sigma_1} + \sum_{n=1}^{\sigma_1} \log^* d(x_0, \Psi_n(x_0)), \tag{3.24}$$

whence, by using Wald's equation,

$$\mathbb{E}\log^+ d(x_0, \Psi_{\sigma_1:1}(x_0))\ \leq\ \Big(\mathbb{E}\log^* L_1 + \mathbb{E}\log^* d(x_0, \Psi_1(x_0))\Big)\,\mathbb{E}\sigma_1\ <\ \infty$$

as claimed.

Turning to the sequence $(Y_n)_{n\geq 0}$, it suffices to point out that it is obtained from $(X_{\sigma_n})_{n\geq 0}$ by reversing the iteration order within, and only within, the segments determined by the σ_n [☞ Fig. 3.2]. In other words, the Φ_n are substituted for $\Phi_n^{\leftarrow} := \Psi_{\sigma_{n-1}+1:\sigma_n}$, which are still iid and satisfy $L(\Phi_n^{\leftarrow}) \leq -l$ a.s. Again, the bound (3.24) is obtained, when embarking on

$$\log d(x_0, \Psi_{1:\sigma_1}(x_0))\ \leq\ \log\left(d(x_0, \Psi_1(x_0)) + \sum_{n=2}^{\sigma_1} d(\Psi_{1:n-1}(x_0), \Psi_{1:n}(x_0))\right).$$

The rest is straightforward. ◻

$$\begin{array}{rl}
X_{\sigma_n}: & \left|\Psi_{\sigma_n}\ldots\Psi_{\sigma_{n-1}+1}\right|\ \left|\Psi_{\sigma_{n-1}}\ldots\Psi_{\sigma_{n-2}+1}\right|\ \ldots\ \left|\Psi_{\sigma_1}\ldots\Psi_1\right| \\
Y_n: & \left|\Psi_{\sigma_{n-1}+1}\ldots\Psi_{\sigma_n}\right|\ \left|\Psi_{\sigma_{n-2}+1}\ldots\Psi_{\sigma_{n-1}}\right|\ \ldots\ \left|\Psi_1\ldots\Psi_{\sigma_1}\right| \\
\widehat{Y}_n = \widehat{X}_{\sigma_n}: & \left|\Psi_1\ldots\Psi_{\sigma_1}\right|\ \ldots\ \left|\Psi_{\sigma_{n-2}+1}\ldots\Psi_{\sigma_{n-1}}\right|\ \left|\Psi_{\sigma_{n-1}+1}\ldots\Psi_{\sigma_n}\right|
\end{array}$$

Fig. 3.2 Schematic illustration of how the blocks of Ψ_k's determined by the σ_n are composed in the definition of X_{σ_n}, Y_n and $\widehat{Y}_n = \widehat{X}_{\sigma_n}$.

The reason for introducing $(Y_n)_{n\geq 0}$ becomes apparent when observing the following twist: the backward iterations of $(X_n)_{n\geq 0}$ at the ladder epochs σ_n, i.e. $(\widehat{X}_{\sigma_n})_{n\geq 0}$, are *not* given by the backward iterations of the IFS $(X_{\sigma_n})_{n\geq 0}$ but rather of $(Y_n)_{n\geq 0}$, thus $(\widehat{Y}_n)_{n\geq 0}$. By the previous lemma in combination with Prop. 3.18, we infer the $\mathbb{P}_x$-a.s. convergence of $\widehat{Y}_n$ to a limit not depending on x, which is clearly the candidate for the a.s. limit of $\widehat{X}_n$ and therefore denoted $\widehat{X}_\infty$.

Let us define $\tau(n) := \inf\{k \geq 0 : \sigma_k \geq n\}$ for $n \geq 0$, and

$$\begin{aligned}
C_n\ :=&\ d(X_0, \Phi_n^{\leftarrow}(X_0)) \vee \max_{\sigma_{n-1}<k<\sigma_n} \big\{d(\Psi_{\sigma_{n-1}+1:k}(X_0), \Psi_{\sigma_{n-1}+1:\sigma_n}(X_0))\big\} \\
=&\ \text{distance between } \Phi_n^{\leftarrow}(X_0) = \Psi_{\sigma_{n-1}+1:\sigma_n}(X_0) \text{ and the set} \\
&\quad \{X_0, \Psi_{\sigma_{n-1}+1}(X_0), \Psi_{\sigma_{n-1}+1:\sigma_{n-1}+2}(X_0), \ldots, \Psi_{\sigma_{n-1}+1:\sigma_n-1}(X_0)\}
\end{aligned} \tag{3.25}$$

for $n \geq 1$. By the elementary renewal theorem [☞ Lemma 2.1(e) and (f)],

$$n^{-1}\tau(n) \to \mu^{-1} \quad \text{as well as} \quad n^{-1}\mathbb{E}\tau(n) \to \mu^{-1},$$

where $\mu = \mu(l) := \mathbb{E}\sigma_1$. Under each $\mathbb{P}_x$, the C_n are clearly iid, and a standard renewal argument shows that $C_{\tau(n)}$ converges in distribution to a random variable C_∞ [☞ Problem 3.30]. However, the really needed piece of information about $C_{\tau(n)}$ will be a consequence of the following lemma.

Lemma 3.27. *For any $x \in \mathbb{X}$, $\mathbb{E}_x \log^+ C_1 < \infty$ and hence $n^{-1}\log^+ C_n \to 0$ as well as $e^{-\varepsilon n} C_n \to 0$ $\mathbb{P}_x$-a.s. for each $\varepsilon > 0$.*

Indeed, as $\tau(n) \to \infty$, the last assertion particularly implies

$$\lim_{n\to\infty} e^{-\varepsilon\tau(n)} C_{\tau(n)} = 0 \quad \mathbb{P}_x\text{-a.s.} \tag{3.26}$$

for all $x \in \mathbb{X}$ and $\varepsilon > 0$.

Proof. Fix any $x \in \mathbb{X}$. By proceeding as for (3.24), we find

$$\begin{aligned} \log C_1 \ &\le\ \log\left(d(x_0,\Psi_1(x_0)) + \sum_{n=2}^{\sigma_1} d(\Psi_{1:n-1}(X_0),\Psi_{1:n}(X_0))\right) \\ &\le\ \log\left(\sum_{n=1}^{\sigma_1} e^{S_{n-1}} d(X_0,\Psi_n(X_0))\right) \\ &\le\ S_{\sigma_1}^* + \sum_{n=1}^{\sigma_1} \log^* d(X_0,\Psi_n(X_0)) \quad \mathbb{P}_x\text{-a.s.} \end{aligned}$$

and the last expression has finite expectation under $\mathbb{P}_x$ by an appeal to Wald's equation. As a consequence, $n^{-1}\log^+ C_n \to 0$ $\mathbb{P}_x$-a.s., and this is readily seen to also give $e^{-\varepsilon n} C_n \to 0$ $\mathbb{P}_x$-a.s. for all $\varepsilon > 0$. □

The crucial estimate of $d(\widehat{X}_n,\widehat{X}_\infty)$ in terms of the previously introduced variables is provided in a further lemma.

Lemma 3.28. *For all $n \ge 0$ and $x \in \mathbb{X}$, the inequality*

$$d(\widehat{X}_n,\widehat{X}_\infty)\ \le\ e^{-(\tau(n)-1)l} C_{\tau(n)} + d(\widehat{Y}_{\tau(n)},\widehat{X}_\infty) \quad \mathbb{P}_x\text{-a.s.}$$

holds, where $C_0 := 0$.

Proof. Using the strong contraction property of $(\widehat{Y}_n)_{n\ge 0}$ and $\widehat{Y}_n \to \widehat{X}_\infty$ $\mathbb{P}_x$-a.s., we obtain

$$\begin{aligned} d(\widehat{X}_n,\widehat{X}_\infty)\ &\le\ d(\widehat{X}_n,\widehat{Y}_{\tau(n)}) + d(\widehat{Y}_{\tau(n)},\widehat{X}_\infty) \\ &\le\ e^{-(\tau(n)-1)l} C_{\tau(n)} + d(\widehat{Y}_{\tau(n)},\widehat{X}_\infty) \quad \mathbb{P}_x\text{-a.s.} \end{aligned}$$

and this is the asserted inequality. □

Proof (of Theorem 3.24). (a) Recall that $\widehat{X}_\infty$ equals the $\mathbb{P}_x$-a.s. limit of $\widehat{Y}_n = \widehat{X}_{\sigma_n}$ for each $x \in \mathbb{X}$. By combining this with (3.26) and the previous lemma, the a.s. convergence $\widehat{X}_n \to \widehat{X}_\infty$ under each $\mathbb{P}_x$ follows. The SFPE for $\widehat{X}_\infty$ is obtained in the same manner as in the proof of Prop. 3.18.

(b) First note that, by Prop. 3.18(b),

$$\lim_{n\to\infty} \beta^{\tau(n)} d(\widehat{Y}_{\tau(n)}, \widehat{X}_\infty) = 0 \quad \mathbb{P}_x\text{-a.s.}$$

for some $\beta > 1$. Since $n^{-1}\tau(n) \to \mu^{-1}$ $\mathbb{P}$-a.s., we can pick $\varepsilon > 0$ and $\gamma > 1$ such that $\gamma^{n/\tau(n)} < \beta$ and $\gamma^n e^{-l(\tau(n)-1)} < e^{-\varepsilon\tau(n)}$ $\mathbb{P}$-a.s. for all sufficiently large n (depending on the realization of the $\tau(n)$). By another use of Lemma 3.28, we then infer

$$\begin{aligned} \gamma^n d(\widehat{X}_n, \widehat{X}_\infty) &\le \gamma^n e^{-l(\tau(n)-1)} C_{\tau(n)} + \big(\gamma^{n/\tau(n)}\big)^{\tau(n)} d(\widehat{Y}_{\tau(n)}, \widehat{X}_\infty) \\ &\le e^{-\varepsilon\tau(n)} C_{\tau(n)} + \beta^{\tau(n)} d(\widehat{Y}_{\tau(n)}, \widehat{X}_\infty) \quad \mathbb{P}_x\text{-a.s.} \end{aligned}$$

for all sufficiently large n and any $x \in \mathbb{X}$, and this yields (3.19) upon letting n tend to ∞.

(c) and (d) follow again in the same way as in the proof of Prop. 3.18.

(e) Now assume strong mean contraction of order p, w.l.o.g. $0 < p \le 1$, and (3.20). Put $\rho := (\mathbb{E}L_1^p)^{1/p} = (\mathbb{E}e^{pS_1})^{1/p}$. Using the independence of S_n and $\Psi_{n+1}(x)$ for all $n \in \mathbb{N}_0$ and $x \in \mathbb{X}$, a similar estimation as in the proof of part (e) of Prop. 3.18 leads to

$$\begin{aligned} \mathbb{P}_x(d(\widehat{X}_n, \widehat{X}_\infty) \ge s^{-n}) &\le s^{np}\, \mathbb{E}_x d(\widehat{X}_n, \widehat{X}_\infty)^p \\ &\le s^{np} \sum_{k\ge n} \mathbb{E}_x d(\widehat{X}_k, \widehat{X}_{k+1})^p \\ &= s^{np} \sum_{k\ge n} \mathbb{E}\, d(\Psi_{1:k}(x), \Psi_{1:k+1}(x))^p \\ &\le s^{np} \sum_{k\ge n} \mathbb{E}\, e^{pS_k} d(\Psi_{1:k}(x), \Psi_{1:k+1}(x))^p \\ &\le (s\rho)^{np}\, \mathbb{E}\, d(x, \Psi_1(x))^p \sum_{k\ge 0} \rho^{kp} \\ &= (s\rho)^{np} \frac{\mathbb{E}\, d(x, \Psi_1(x))^p}{1-\rho^p}. \end{aligned}$$

for $s > 0$. It follows that

$$\lim_{n\to\infty} s^n\, \mathbb{P}_x(d(\widehat{X}_n, \widehat{X}_\infty) \ge s^{-n}) = 0$$

for $1 < s < \rho^{-p/(p+1)}$, and once again this entails (3.21) for $r \in (1,s)$ by an appeal to Lemma 3.17. □

We close this section with a result on the existence of moments of π, more precisely of

$$\int_{\mathbb{X}} d(x_0,x)^p\ \pi(dx)\ =\ \mathbb{E}d(x_0,\widehat{X}_\infty)^p$$

for any fixed $x_0 \in \mathbb{X}$ and $p > 0$. In the case of Euclidean space $\mathbb{X} = \mathbb{R}^m$ for some $m \geq 1$ with the usual norm $d(x,y) = |x-y|$ and $x_0 = 0$, this means to consider

$$\int_{\mathbb{R}^m} |x|^p\ \pi(dx)\ =\ \mathbb{E}|\widehat{X}_\infty|^p.$$

The following theorem is due to BENDA [10, Prop. 2.2] for $p \geq 1$; for a weaker version see [3, Theorem 2.3(d)]. It does not only complement the main result of this section, but will also be useful in connection with the implicit renewal theory developed in Chapter 4.

Theorem 3.29. *If $(X_n)_{n\geq 0}$ is strongly mean contractive of order $p > 0$ and satisfies the corresponding sharpened jump-size condition* (3.20)*, i.e. $\mathbb{E}L_1^p < 1$ and $\mathbb{E}d(x_0,\Psi_1(x_0))^p < \infty$ for some/all $x_0 \in \mathbb{X}$, then*

$$\mathbb{E}d(x_0,\widehat{X}_\infty)^p\ =\ \int_{\mathbb{X}} d(x_0,x)^p\ \pi(dx)\ <\ \infty$$

for some and then all $x_0 \in \mathbb{X}$.

Proof. Put $\beta := \mathbb{E}L_1^p = \mathbb{E}e^{pS_1}$. If $p \leq 1$, we infer by using the subadditivity of $x \mapsto x^p$ and the model assumptions that, for any $x_0 \in \mathbb{X}$,

$$\begin{aligned}
\mathbb{E}d(x_0,\widehat{X}_\infty)^p\ &\leq\ \mathbb{E}\left(d(x_0,\Psi_1(x_0)) + \sum_{n\geq 1} d(\Psi_{n:1}(x_0),\Psi_{n+1:1}(x_0))\right)^p\\
&\leq\ \mathbb{E}d(x_0,\Psi_1(x_0))^p + \sum_{n\geq 1}\mathbb{E}e^{pS_n} d(x_0,\Psi_{n+1}(x_0))^p\\
&\leq\ \mathbb{E}d(x_0,\Psi_1(x_0))^p \sum_{n\geq 0}\mathbb{E}e^{pS_n}\\
&=\ \frac{\mathbb{E}d(x_0,\Psi_1(x_0))^p}{1-\beta}\ <\ \infty.
\end{aligned}$$

In the case $p > 1$ we can use Minkowski's inequality, which also holds for infinitely many summands, to obtain in a similar manner

$$\begin{aligned}
\|d(x_0,\widehat{X}_\infty)\|_p\ &\leq\ \left\|d(x_0,\Psi_1(x_0)) + \sum_{n\geq 1} d(\Psi_{n:1}(x_0),\Psi_{n+1:1}(x_0))\right\|_p\\
&\leq\ \|d(x_0,\Psi_1(x_0))\|_p + \sum_{n\geq 1}\|e^{S_n} d(x_0,\Psi_{n+1}(x_0))\|_p
\end{aligned}$$

$$\leq \|d(x_0, \Psi_1(x_0))\|_p \sum_{n \geq 0} \|e^{S_n}\|_p$$
$$= \frac{\mathbb{E}d(x_0, \Psi_1(x_0))^p}{1 - \beta^{1/p}} < \infty.$$

This completes the proof. □

Problems

Problem 3.30. Use a renewal argument to prove that C_n defined in (3.25) converges in distribution to a random variable C_∞ with cdf

$$\mathbb{P}_x(C_\infty \leq t) = \frac{1}{\mu} \sum_{n \geq 0} \mathbb{P}_x(\sigma_1 > n, C_1 \leq t), \quad t \in \mathbb{R}_{\geq}.$$

[Recall that l in the definition of σ_1 was chosen such that σ_1 is 1-arithmetic.]

Chapter 4
Power law behavior of stochastic fixed points and implicit renewal theory

The previous chapter has shown that any mean contractive IFS $(X_n)_{n\geq 0}$ of iid Lipschitz maps $\Psi_1,\Psi_2,...$ converges in distribution to a unique stationary limit π which is characterized as the unique solution to the SFPE

$$X \stackrel{d}{=} \Psi(X) \tag{4.1}$$

where Ψ denotes a generic copy of the Ψ_n independent of X. In the following, we will deal with the problem of gaining information about the tail behavior of $\pi = \mathbb{P}(X\in\cdot)$, more precisely, the behavior of

$$\mathbb{P}(X>t) \quad \text{and/or} \quad \mathbb{P}(X<-t) \quad \text{as } t\to\infty.$$

If they are asymptotically equal to a nonzero constant times a power $|t|^{\vartheta}$ for some $\vartheta>0$, we say that X (or π) exhibits a *power law behavior*.[1] For the case when $\Psi(x)$, for x large, is approximately Mx for a random variable M, GOLDIE [34] developed a method he called *implicit renewal theory*, which allows to establish power law behavior of X under appropriate moment conditions on M. The present chapter is devoted to a presentation of his main results.

4.1 Goldie's implicit renewal theorem

Given an SFPE of type (4.1) with $\Psi(x)\approx Mx$ for a random variable M if x is large, Goldie's basic idea to study the asymptotics of $\mathbb{P}(X>t)$ and $\mathbb{P}(X<-t)$ as $t\to\infty$ is to consider the differences

$$\mathbb{P}(X>t)-\mathbb{P}(MX>t) \quad \text{and} \quad \mathbb{P}(X<-t)-\mathbb{P}(MX<-t)$$

[1] In some papers like [22] it is alternatively said that X has *algebraic tails*.

as t tends to ∞. Additionally assuming that M is nonnegative, the renewal-theoretic character of this approach becomes immediately apparent after a logarithmic transform. Put $Y := \log X^+$, $\xi := \log M$ and $G(t) := \mathbb{P}(Y > t)$ for $t \in \mathbb{R}$. Since M and X are independent, we infer for $\Delta(t) := \mathbb{P}(X > e^t) - \mathbb{P}(MX > e^t)$ that

$$\Delta(t) \ = \ G(t) - \int G(t-x)\, \mathbb{P}(\xi \in dx), \quad t \in \mathbb{R}, \tag{4.2}$$

or, equivalently, the renewal equation $G = \Delta + G * Q$ with $Q := \mathbb{P}(\xi \in \cdot)$ holds. However, unlike the usual situation [☞ Section 2.7], the function Δ is also unknown here and indeed an integral involving G. That renewal-theoretic arguments still work to draw conclusions about G is the key feature of the approach and the following result in particular. It will be made more precise in the next section.

Theorem 4.1. [Implicit renewal theorem] *Let M, X be independent random variables such that, for some $\vartheta > 0$,*

(IRT-1) $\mathbb{E}|M|^{\vartheta} = 1$.
(IRT-2) $\mathbb{E}|M|^{\vartheta} \log^+ |M| < \infty$.
(IRT-3) *The conditional law $\mathbb{P}(\log|M| \in \cdot | M \neq 0)$ of $\log|M|$ given $M \neq 0$ is nonarithmetic, in particular, $\mathbb{P}(|M| = 1) < 1$.*

Then $-\infty \le \mathbb{E} \log|M| < 0$, $0 < \mu_{\vartheta} := \mathbb{E}|M|^{\vartheta} \log|M| < \infty$, and the following assertions hold true:

(a) Suppose M is a.s. nonnegative. If

$$\int_0^{\infty} \big|\mathbb{P}(X > t) - \mathbb{P}(MX > t)\big|\, t^{\vartheta - 1}\, dt \ < \ \infty \tag{4.3}$$

or, respectively,

$$\int_0^{\infty} \big|\mathbb{P}(X < -t) - \mathbb{P}(MX < -t)\big|\, t^{\vartheta - 1}\, dt \ < \ \infty, \tag{4.4}$$

then

$$\lim_{t \to \infty} t^{\vartheta}\, \mathbb{P}(X > t) \ = \ C_+, \tag{4.5}$$

respectively

$$\lim_{t \to \infty} t^{\vartheta}\, \mathbb{P}(X < -t) \ = \ C_-, \tag{4.6}$$

where C_+ and C_- are given by the equations

$$C_+ \ := \ \frac{1}{\mu_{\vartheta}} \int_0^{\infty} \big(\mathbb{P}(X > t) - \mathbb{P}(MX > t)\big)\, t^{\vartheta - 1}\, dt, \tag{4.7}$$

$$C_- \ := \ \frac{1}{\mu_{\vartheta}} \int_0^{\infty} \big(\mathbb{P}(X < -t) - \mathbb{P}(MX < -t)\big)\, t^{\vartheta - 1}\, dt. \tag{4.8}$$

(b) *If* $\mathbb{P}(M<0)>0$ *and* (4.3), (4.4) *are both satisfied, then* (4.5) *and* (4.6) *hold with* $C_+ = C_- = C/2$, *where*

$$C := \frac{1}{\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(|X|>t)-\mathbb{P}(|MX|>t)\big)\,t^{\vartheta-1}\,dt. \tag{4.9}$$

The proof of this result naturally requires some work which will be carried out in the Section 4.3. Let us rather point here as in [34] that the theorem has real content only if $\mathbb{E}|X|^\vartheta=\infty$, because otherwise, by the independence of M and X and condition (IRT-1),

$$C = \frac{1}{\vartheta\mu_\vartheta}\Big(\mathbb{E}|X|^\vartheta-\mathbb{E}|MX|^\vartheta\Big) = \frac{1}{\vartheta\mu_\vartheta}\,\mathbb{E}|X|^\vartheta\Big(1-\mathbb{E}|M|^\vartheta\Big) = 0$$

in which case (4.5) and (4.6) take the form

$$\lim_{t\to\infty} t^\vartheta\,\mathbb{P}(|X|>t) = 0,$$

Naturally, this follows also directly from $t^\vartheta\,\mathbb{P}(|X|>t)\le\mathbb{E}\mathbf{1}_{\{|X|>t\}}|X|^\vartheta\to 0$. We thus see that the "right" choice of M and ϑ is crucial.

The next corollary specializes to the situation where X additionally satisfies the SFPE (4.1) for a Borel-measurable random function Ψ.

Corollary 4.2. *Let* $(\Omega,\mathfrak{A},\mathbb{P})$ *be any probability space,* $\Psi:\Omega\times\mathbb{R}\to\mathbb{R}$ *a* $\mathfrak{A}\otimes\mathscr{B}(\mathbb{R})$*-measurable function and* X,M *further random variables on* Ω *such that* X *solves* (4.1) *and is independent of* (Ψ,M). *Suppose also that* M *satisfies (IRT-1)-(IRT-3). Then, in Theorem 4.1, conditions* (4.3) *and* (4.4) *may be replaced by (the generally stronger)*

$$\mathbb{E}\Big|(\Psi(X)^+)^\vartheta-((MX)^+)^\vartheta\Big|<\infty \tag{4.10}$$

and

$$\mathbb{E}\Big|(\Psi(X)^-)^\vartheta-((MX)^-)^\vartheta\Big|<\infty \tag{4.11}$$

respectively, and the formulae in (4.7), (4.8) *and* (4.9) *by*

$$C_+ = \frac{1}{\vartheta\mu_\vartheta}\,\mathbb{E}\Big((\Psi(X)^+)^\vartheta-((MX)^+)^\vartheta\Big), \tag{4.12}$$

$$C_- = \frac{1}{\vartheta\mu_\vartheta}\,\mathbb{E}\Big((\Psi(X)^-)^\vartheta-((MX)^-)^\vartheta\Big), \tag{4.13}$$

and

$$C \;=\; \frac{1}{\vartheta\mu_\vartheta}\,\mathbb{E}\Big(|\Psi(X)|^\vartheta - |MX|^\vartheta\Big), \tag{4.14}$$

respectively.

The proof with the help of Theorem 4.1 is quite simple and provided after the following lemma.

Lemma 4.3. *Let X,Y be two real-valued random variables and $\vartheta > 0$. Then*

$$\int_0^\infty \big|\mathbb{P}(X>t) - \mathbb{P}(Y>t)\big|\,t^{\vartheta-1}\,dt \;\le\; \frac{1}{\vartheta}\,\mathbb{E}\Big|(X^+)^\vartheta - (Y^+)^\vartheta\Big|, \tag{4.15}$$

finite or infinite. If finite, absolute value signs may be removed to give

$$\int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(Y>t)\big)\,t^{\vartheta-1}\,dt \;=\; \frac{1}{\vartheta}\,\mathbb{E}\Big((X^+)^\vartheta - (Y^+)^\vartheta\Big). \tag{4.16}$$

Proof. Problem 4.6. □

Remark 4.4. The previous lemma bears a subtlety that is easily overlooked at first reading (and has actually been done so also in [34]). If F,G denote the df's of X,Y, then (4.15) may be restated as

$$\int_0^\infty \big|F(t) - G(t)\big|\,t^{\vartheta-1}\,dt \;\le\; \frac{1}{\vartheta}\,\mathbb{E}\Big|(X^+)^\vartheta - (Y^+)^\vartheta\Big| \tag{4.17}$$

and holds true for *every* (F,G)*-coupling* (X,Y) on some probability space $(\Omega,\mathfrak{A},\mathbb{P})$, which means that $\mathscr{L}(X) = F$ and $\mathscr{L}(Y) = G$. Moreover, any such coupling having $\mathbb{E}\big|(X^+)^\vartheta - (Y^+)^\vartheta\big| < \infty$ leads to the same value when removing the absolute value signs, namely the left-hand integral in (4.16), i.e.

$$\int_0^\infty \big(F(t) - G(t)\big)\,t^{\vartheta-1}\,dt.$$

This is trivial if $\mathbb{E}((X^+)^\vartheta$ and $\mathbb{E}((Y^+)^\vartheta$ are finite individually, but requires a proof otherwise.

Concerning inequality (4.17), it is natural to ask whether equality can be achieved by choosing a special (F,G)-coupling (X,Y). This is indeed the case when $X = F^{-1}(U)$ and $Y = G^{-1}(U)$, where U is a *Unif*$(0,1)$ random variable and F^{-1},G^{-1} the *pseudo-inverses* of F,G, defined by $F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \ge u\}$ for $u \in (0,1)$. The reader is asked for a proof in Problem 4.6.

Proof (of Corollary 4.2). By combining the SFPE (4.1) with the previous lemma, we see that (4.3) turns into

$$\begin{aligned}\infty \;>\; &\int_0^\infty \big|\mathbb{P}(\Psi(X)>t)-\mathbb{P}(MX>t)\big|\,t^{\vartheta-1}\,dt\\ =\; &\mathbb{E}\big|(\Psi(X)^+)^\vartheta-((MX)^+)^\vartheta\big|\end{aligned}$$

which is condition (4.10). Since all other asserted replacements follow in the same manner, the result is proved. □

Problems

Problem 4.5. Prove that, if at least one of (4.3) and (4.4) is valid, then these conditions hold together iff

$$\int_0^\infty \big|\mathbb{P}(|X|>t)-\mathbb{P}(|MX|>t)\big|\,t^{\vartheta-1}\,dt \;<\; \infty. \tag{4.18}$$

Problem 4.6. Prove Lemma 4.3 and, furthermore, that

$$\int_0^\infty \big|F(t)-G(t)\big|\,t^{\vartheta-1}\,dt \;=\; \frac{1}{\vartheta}\,\mathbb{E}\Big|(F^{-1}(U)^+)^\vartheta-(G^{-1}(U)^+)^\vartheta\Big| \tag{4.19}$$

where F^{-1},G^{-1} and U are as stated in Remark 4.4.

4.2 Making explicit the implicit

Let us take as a starting point a *two-sided renewal equation* of the form

$$G(t) \;=\; \Delta(t) \;+\; \int G(t-x)\,Q(dx),\quad t\in\mathbb{R},$$

as in (4.2), where $G,\Delta:\mathbb{R}\to\mathbb{R}$ are unknown bounded functions vanishing at ∞, i.e.

$$\lim_{t\to\infty} G(t)=0 \quad\text{and}\quad \lim_{t\to\infty}\Delta(t)=0,$$

and Q is a given probability measure with mean $\mu<0$. Let $\mathbb{U}:=\sum_{n\ge 0}Q^{*n}$ denote its renewal measure. The goal is to determine the precise asymptotic behavior of $G(t)$ as $t\to\infty$. Although we have studied only standard renewal equations in Section 2.7, where G,Δ and Q vanish on $\mathbb{R}_>$, it is reasonable to believe and sustained by an iteration argument that $G=\Delta*\mathbb{U}$. In fact, if $\Delta*\mathbb{U}$ exists, this only takes to verify that $\lim_{n\to\infty}G*Q^{*n}(t)=0$ for all $t\ge 0$, as

$$G(t) \;=\; \sum_{k=0}^{n-1}\Delta*Q^{*k}(t) \;+\; G*Q^{*n}(t),\quad t\in\mathbb{R}$$

for each $n \in \mathbb{N}$. But with $(S_n)_{n\ge 0}$ denoting a SRW with increment distribution Q and thus negative drift, it follows indeed that

$$\lim_{n\to\infty} G*Q^{*n}(t) \;=\; \lim_{n\to\infty} \mathbb{E}G(t-S_n) \;=\; 0$$

by an appeal to the dominated convergence theorem and $G(t)\to 0$ as $t\to\infty$.

So far we have not really gained any new insight because an application of the key renewal theorem 2.67 to $G(t)=\Delta*\mathbb{U}(t)$, if possible, only reconfirms what we already know, namely that $G(t)\to 0$ as $t\to\infty$. On the other hand, defining $G_\theta(t):=e^{\theta t}G(t)$, $\Delta_\theta(t):=e^{\theta t}\Delta(t)$ and $Q_\theta(dx):=e^{\theta x}Q(dx)$ for $\theta\in\mathbb{R}$, we find as in Lemma 2.43 that G_θ solves a renewal equation as well, viz.

$$G_\theta(t) \;=\; \Delta_\theta(t)+\int G_\theta(t-x)\;Q_\theta(dx),\quad t\in\mathbb{R}.$$

Hence, if Q possesses a *characteristic exponent* ϑ, defined by the unique (if it exists) value $\ne 0$ such that $\phi_Q(\vartheta)=\int e^{\vartheta x}\,Q(dx)=1$[2], it appears to be natural to use this renewal equation with $\theta=\vartheta$ which should lead to $G_\vartheta=\Delta_\vartheta*\mathbb{U}_\vartheta$, $\mathbb{U}_\vartheta:=\sum_{n\ge 0}Q_\vartheta^{*n}$, and then to the conclusion that $G_\vartheta(t)$ converges to a constant as $t\to\infty$ by an appeal to the key renewal theorem, thus

$$\lim_{t\to\infty} e^{\vartheta t}G(t)=C$$

for some $C\in\mathbb{R}$ which in the best case is $\ne 0$. Naturally, further conditions must be imposed to make this work for us. They are stated in the following proposition together with the expected conclusion. Let us mention that ϑ, if it exists, is necessarily positive and that Q_ϑ has positive, possibly infinite mean. This follows from the fact that the mgf of Q, i.e. $\phi_Q(\theta)=\int e^{\theta x}\,Q(dx)$, is convex on its natural domain $\mathbb{D}_Q$ and that $\phi'(0)=\int x\,Q(dx)<0$.

Proposition 4.7. *In addition to the assumptions on G,Δ and Q stated at the beginning of this section suppose that Q is nonarithmetic and possesses a characteristic exponent $\vartheta>0$ and let $\mu_\vartheta:=\int xe^{\vartheta x}\,Q(dx)$ denote the (positive) mean of Q_ϑ. Also assume that Δ is dRi and*

$$\int_{-\infty}^{\infty} e^{\vartheta x}\,|\Delta(x)|\;dx \;<\; \infty. \tag{4.20}$$

Then $G=\Delta+G*Q$ implies $G=\Delta*\mathbb{U}$ and

$$\lim_{t\to\infty} e^{\vartheta t}G(t) \;=\; \frac{1}{\mu_\vartheta}\int_{-\infty}^{\infty} e^{\vartheta x}\Delta(x)\;dx, \tag{4.21}$$

which, by our usual convention, equals 0 if $\mu_\vartheta=\infty$.

[2] In Subsection 2.7.1, a slightly different definition has been used for bounded measures on $\mathbb{R}_\ge$

Proof. First note that, since Δ is dRi and (4.20) holds, the function Δ_ϑ is dRi as well [☞ Problem 4.8]. This in combination with the uniform local boundedness of $\mathbb{U}_\vartheta$ [☞ Lemma 2.64] implies that $\Delta_\vartheta * \mathbb{U}_\vartheta$ is everywhere finite. We have already argued above that $G = \Delta * \mathbb{U}$ so that $G_\vartheta = \Delta_\vartheta * \mathbb{U}_\vartheta$. Therefore, assertion (4.21) follows by an appeal to the (nonarithmetic version of the) key renewal theorem 2.67. □

The previous result should be kept in mind as a kind of general version of what is actually derived for special triples G, Δ and Q in the proof of the implicit renewal theorem we are now going to prove.

Problems

Problem 4.8. [☞ also Lemma 2.28] Let $g : \mathbb{R} \to \mathbb{R}$ be a dRi function and $\theta \in \mathbb{R}$ be such that $g_\theta(x) = e^{\theta x} g(x)$ is $\lambda\!\!\lambda_0$-integrable. Prove that g_θ is then dRi as well.

Problem 4.9. [Two-sided renewal equation] Prove that, given a two-sided renewal equation $G = g + G * Q$ with a dRi function g and a probability measure Q on $\mathbb{R}$, the set of solutions equals

$$\big\{a + g * \mathbb{U} : a \in \mathbb{R}\big\},$$

where $\mathbb{U}$ denotes the renewal measure of Q.

4.3 Proof of the implicit renewal theorem

It suffices to show (4.5) and the formula for C_+ in the respective parts (a) and (b) because the other assertions follow by considering $-X$ instead of X. The proof will be carried out for the three cases

$$M \geq 0 \text{ a.s.}, \quad \mathbb{P}(M > 0) \wedge \mathbb{P}(M < 0) > 0 \quad \text{and} \quad M \leq 0 \text{ a.s.}$$

separately and frequently make use of the following notation most of which has already been used earlier. Let $X, M, M_1, M_2, \ldots$ be independent random variables on a common probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ such that $M, M_1, M_2, \ldots$ are further identically distributed. Then

$$\Pi_0 := 1 \quad \text{and} \quad \Pi_n := \prod_{k=1}^n M_k \quad \text{for } n \geq 1,$$

$$\xi := \log|M|, \quad Q := \mathscr{L}(\xi),$$

$$\xi_n := \log|M_n|, \quad S_0 := 0 \quad \text{and} \quad S_n := \log|\Pi_n| = \sum_{k=1}^n \xi_k \quad \text{for } n \geq 1,$$

$$\mathbb{U} := \sum_{n\geq 0} Q^{*n} = \sum_{n \geq 0} \mathbb{P}(S_n \in \cdot), \quad \mathbb{U}_\theta(dx) := e^{\theta x}\, \mathbb{U}(dx),$$

$$G(t) := \mathbb{P}(X > e^t), \quad G_\theta(t) := e^{\theta t} G(t),$$
$$\Delta(t) := \mathbb{P}(X > e^t) - \mathbb{P}(MX > e^t), \quad \Delta_\theta(t) := e^{\theta t}\Delta(t) \quad \text{for } \theta, t \in \mathbb{R},$$
$$\overline{f}(t) := \int_{(-\infty,t]} e^{-(t-x)} f(x)\,\lambda\!\!\lambda_0(dx) = \mathbb{E}f(t-Z) \quad \text{for suitable } f:\mathbb{R}\to\mathbb{R},$$

where Z is a standard exponential random variable. The function $\overline{f}$ has already been introduced in Subsection 2.6.1 and called *exponential smoothing of f*. Recall from Lemma 2.30 there that $\overline{f}$ is dRi whenever $f \in L^1$. The next simple lemma further shows that exponential smoothing is preserved under convolutions with measures.

Lemma 4.10. *Let $f:\mathbb{R}\to\mathbb{R}$ be a measurable function and V a finite measure on $\mathbb{R}$ such that $\overline{f}$ as well as $f*V$ exist as real-valued functions on $\mathbb{R}$. Then*

$$\overline{f}*V = \overline{f*V}. \tag{4.22}$$

and $f = g + f*V$ for a measurable function g with exponential smoothing $\overline{g}$ implies $\overline{f} = \overline{g} + \overline{f}*V$.

Proof. W.l.o.g. suppose that $\|V\| = 1$. Let Y, Z be independent random variables such that $\mathscr{L}(Y) = V$ and $\mathscr{L}(Z) = Exp(1)$. For (4.22), it then suffices to note that

$$\overline{f}*V(t) \;=\; \mathbb{E}f(t-Y-Z) \;=\; \overline{f*V}(t)$$

for all $t \in \mathbb{R}$, while the last assertion then follows from

$$\overline{f} \;=\; \overline{g+f*V} \;=\; \overline{g} + \overline{f*V} \;=\; \overline{g} + \overline{f}*V.$$

having used that exponential smoothing is a linear operation. □

By combing this lemma with next one, we will be able to use exponential smoothing when studying the asymptotic properties of the function G_ϑ in the proof of the implicit renewal theorem.

Lemma 4.11. [Smoothing lemma] *If $\mathbb{P}(X > t)$ satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x^\vartheta\, \mathbb{P}(X > x)\, dx \;=\; C_+$$

for some $\vartheta > 0$ and $C_+ \in \mathbb{R}_\geq$, then (4.5) holds true as well.

In other words, if the Césaro smoothing of $e^{\vartheta t}\,\mathbb{P}(X > t)$ converges to some C_+, then so does the function $e^{\vartheta t}\,\mathbb{P}(X > t)$ itself as $t \to \infty$.

Proof. Fixing any $b > 1$, we infer

$$\begin{aligned} C_+(b-1)t &\simeq \int_0^{bt} x^{\vartheta}\,\mathbb{P}(X>x)\,dx - \int_0^{t} x^{\vartheta}\,\mathbb{P}(X>x)\,dx \\ &= \int_t^{bt} x^{\vartheta}\,\mathbb{P}(X>x)\,dx \\ &\le \mathbb{P}(X>t)\int_t^{bt} x^{\vartheta}\,dx \\ &= \frac{b^{\vartheta+1}-1}{\vartheta+1}t^{\vartheta+1}\,\mathbb{P}(X>t) \end{aligned}$$

and thereby

$$\liminf_{t\to\infty} t^{\vartheta}\,\mathbb{P}(X>t) \ \ge\ C_+(\vartheta+1)\frac{b-1}{b^{\vartheta+1}-1}.$$

Now let b tend to 1 and use

$$\lim_{b\downarrow 1}\frac{b-1}{b^{\vartheta+1}-1} \ =\ \frac{1}{\vartheta+1}$$

to conclude $\liminf_{t\to\infty} t^{\vartheta}\,\mathbb{P}(X>t) \ge C_+$.

By an analogous argument for $0 < b < 1$, one finds that

$$C_+(1-b)t \ \simeq\ \int_{bt}^{t} x^{\vartheta}\,\mathbb{P}(X>x)\,dx \ \ge\ \frac{1-b^{\vartheta+1}}{\vartheta+1}t^{\vartheta+1}\,\mathbb{P}(X>t)$$

which upon letting b again tend to 1 yields $\limsup_{t\to\infty} t^{\vartheta}\,\mathbb{P}(X>t) \le C_+$. □

In view of this lemma, it suffices to verify $t^{-1}\int_0^t x^{\vartheta}\,\mathbb{P}(X>x)\,dx \to C_+$ as $t\to\infty$ instead of (4.5), and since

$$\begin{aligned} \frac{1}{t}\int_0^t x^{\vartheta}\,\mathbb{P}(X>x)\,dx &= \frac{1}{t}\int_{-\infty}^{\log t} e^{(\vartheta+1)x}\,\mathbb{P}(Y>e^x)\,dx \\ &= \frac{1}{t}\int_{-\infty}^{\log t} e^{x}\,G_{\vartheta}(x)\,dx \\ &= \int_{-\infty}^{\log t} e^{-(\log t - x)}\,G_{\vartheta}(x)\,dx \\ &= \overline{G_{\vartheta}}(\log t) \end{aligned}$$

this means to show that

$$\lim_{t\to\infty}\overline{G_{\vartheta}}(t) = C_+.$$

4.3.1 The case when $M \ge 0$ *a.s.*

Rewriting (IRT-1)-(IRT3) in terms of $\xi = \log M$, we have

(IRT-1) $\|Q_\vartheta\| = \mathbb{E}e^{\vartheta\xi} = 1$.
(IRT-2) $\int_{\mathbb{R}_\ge} x\,Q_\vartheta(dx) = \mathbb{E}e^{\vartheta\xi}\xi^+ < \infty$
[thus $0 < \mu_\vartheta = \int x\,Q_\vartheta(dx) = \mathbb{E}e^{\vartheta\xi}\xi < \infty$ as explained before Prop. 4.7].
(IRT-3) Q_ϑ is nonarithmetic.

We have already argued for this case that $G = \Delta + G * Q$, $G = \Delta * \mathbb{U}$ and thus $G_\vartheta = \Delta_\vartheta * \mathbb{U}_\vartheta$. As

$$\begin{aligned}\int_{-\infty}^{\infty} |\Delta_\vartheta(x)|\,dt \;&=\; \int_{-\infty}^{\infty} \big|\mathbb{P}(X > e^x) - \mathbb{P}(MX > e^x)\big|\,e^{\vartheta x}\,dx\\ &=\; \int_{-\infty}^{\infty} \big|\mathbb{P}(X > t) - \mathbb{P}(MX > t)\big|\,t^{\vartheta - 1}\,dt,\end{aligned}$$

we see that $\Delta_\vartheta \in L^1$ by (4.3), and also (when removing absolute values)

$$\frac{1}{\mu_\vartheta}\int_{-\infty}^{\infty} \Delta_\vartheta(x)\,dx \;=\; C_+.$$

By Lemma 2.30 and (2.23), $\overline{\Delta_\vartheta}$ is dRi and $\mu_\vartheta^{-1}\int_{-\infty}^{\infty}\overline{\Delta_\vartheta}(x)\,dx = C_+$ as well. Now use Lemma 4.10, the smoothing lemma 4.11 and the key renewal theorem 2.67 to conclude

$$\lim_{t\to\infty} G_\vartheta(t) \;=\; \lim_{t\to\infty}\overline{G_\vartheta}(t) \;=\; \lim_{t\to\infty}\overline{\Delta_\vartheta} * \mathbb{U}_\vartheta(t) \;=\; \frac{1}{\mu_\vartheta}\int_{-\infty}^{\infty}\overline{\Delta_\vartheta}(x)\,dx \;=\; C_+$$

as claimed.

4.3.2 The case when $\mathbb{P}(M > 0) \wedge \mathbb{P}(M < 0) > 0$

The main idea for the proof of the remaining two cases is to reduce it to the first case by comparison of X with $\Pi_\sigma X$, where

$$\sigma \;:=\; \inf\{n \ge 1 : \Pi_n \ge 0\} \;=\; \begin{cases} 1, & \text{if } M_1 \ge 0,\\ \inf\{n \ge 2 : M_n \le 0\}, & \text{otherwise.}\end{cases}$$

Plainly, σ is a.s. finite, and we may thus hope to be successful in our endeavor if Π_σ satisfies (IRT-1)-(IRT-3).

The reader should keep in mind that, besides (IRT-1)-(IRT-3) for M, we are now *always* assuming (4.3) *and* (4.4), or, equivalently [☞ Problem 4.5],

$$\int_{-\infty}^{\infty} \Delta_\vartheta^*(x)\,dx \;=\; \int_0^{\infty} \big|\mathbb{P}(|X| > t) - \mathbb{P}(|MX| > t)\big|\,t^{\vartheta-1}\,dt \;<\; \infty,$$

where $\Delta^*(x) := e^{\vartheta x}\Delta^*(x)$ as usual and

$$\Delta^*(x) := \mathbb{P}(|X| > e^x) - \mathbb{P}(|MX| > e^x), \quad x \in \mathbb{R}.$$

We begin with a lemma that verifies (IRT-1)-(IRT-3) for Π_σ.

Lemma 4.12. *The stopped product $\Pi_\sigma = e^{S_\sigma}$ satisfies the conditions (IRT-1)-(IRT-3), i.e.*

$$\mathbb{E}\Pi_\sigma^\vartheta = \mathbb{E}e^{\vartheta S_\sigma} = 1, \quad \mathbb{E}\Pi_\sigma^\vartheta \log^+ \Pi_\sigma = \mathbb{E}e^{\vartheta S_\sigma} S_\sigma^+ < \infty,$$

and the law of $\log \Pi_\sigma$ *given* $\Pi_\sigma \neq 0$ *is nonarithmetic. Moreover,*

$$\mathbb{E}\Pi_\sigma^\vartheta \log \Pi_\sigma = \mathbb{E}e^{\vartheta S_\sigma} S_\sigma = 2\mu_\vartheta.$$

Proof. First note that $(|\Pi_n|^\vartheta)_{n\geq 0}$ constitutes a nonnegative mean one product martingale with respect to the filtration $\mathscr{F}_n := \sigma(\Pi_0, M_1, ..., M_n)$ for $n \geq 0$. As usual, put $\mathscr{F}_\infty := \sigma(\bigcup_{n\geq 0} \mathscr{F}_n)$. As in the proof of Theorem 2.68, define a new probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathscr{F}_\infty)$ by

$$\widehat{\mathbb{P}}(A) := \mathbb{E}|\Pi_n|^\vartheta \mathbf{1}_A = \mathbb{E}e^{\vartheta S_n}\mathbf{1}_A \quad \text{for } A \in \mathscr{F}_n \text{ and } n \geq 0.$$

Then $M_1, M_2, ...$ are still iid under $\widehat{\mathbb{P}}$ with common distribution

$$\widehat{\mathbb{P}}(M_1 \in B) = \mathbb{E}|M_1|^\vartheta \mathbf{1}_B(M_1), \quad B \in \mathscr{B}(\mathbb{R}).$$

Equivalently, $(S_n)_{n\geq 0}$ remains a SRW under $\widehat{\mathbb{P}}$ with increment distribution

$$\widehat{\mathbb{P}}(\xi_1 \in B) = \mathbb{E}e^{\vartheta \xi_1}\mathbf{1}_B(\xi_1) \quad , B \in \mathscr{B}(\mathbb{R}),$$

and drift $\widehat{\mathbb{E}}\xi_1 = \mathbb{E}e^{\vartheta\xi_1}\xi_1 = \mu_\vartheta$. It is shown in Problem 4.15 that σ is a.s. finite and has finite moments of any order under $\widehat{\mathbb{P}}$. The almost sure finiteness ensures that, for any $A \in \mathscr{F}_\sigma$,

$$\widehat{\mathbb{P}}(A) = \sum_{n\geq 1} \widehat{\mathbb{P}}(A \cap \{\sigma = n\}) = \sum_{n\geq 1} \mathbb{E}|\Pi_n|^\vartheta \mathbf{1}_{A\cap\{\sigma=n\}} = \mathbb{E}|\Pi_\sigma|^\vartheta \mathbf{1}_A,$$

for $A \cap \{\sigma = n\} \in \mathscr{F}_n$ for each $n \geq 1$. Choosing $A = \Omega$, we particularly find that $\mathbb{E}|\Pi_\sigma|^\vartheta = 1$. Next, use the $\mathscr{F}_\sigma$-measurability of S_σ and Wald's equation to infer

$$\mathbb{E}e^{\vartheta S_\sigma} S_\sigma^+ \;=\; \widehat{\mathbb{E}}S_\sigma^+ \;\leq\; \widehat{\mathbb{E}}\left(\sum_{k=1}^{\sigma} \xi_k^+\right) \;=\; \widehat{\mathbb{E}}\xi_1^+ \,\widehat{\mathbb{E}}\sigma$$

which is finite because $\widehat{\mathbb{E}}\sigma < \infty$ and $\widehat{\mathbb{E}}\xi_1^+ = \mathbb{E}|M|^\vartheta \log^+ |M| < \infty$ by (IRT-2). But then, by another use of Wald's equation,

$$\mathbb{E}e^{\vartheta S_\sigma}S_\sigma \;=\; \widehat{\mathbb{E}}S_\sigma \;=\; \widehat{\mathbb{E}}\xi_1\,\widehat{\mathbb{E}}\sigma \;=\; \mu_\vartheta\,\widehat{\mathbb{E}}\sigma \;=\; 2\mu_\vartheta,$$

where $\widehat{\mathbb{E}}\sigma = 2$ is again shown as a part of Problem 4.15.

Finally, use [and prove as part (c) of Problem 4.15] that

$$\mathbb{P}(S_\sigma \in \cdot, \Pi_\sigma \neq 0) = p\,Q_> + (1-p)^2 \sum_{n\geq 0} p^n\, Q_>^{*n} * Q_<^{*2}, \tag{4.23}$$

where $p := \mathbb{P}(M>0)$, $Q_> := \mathbb{P}(\xi \in \cdot|M>0)$ and $Q_< := \mathbb{P}(\xi \in \cdot|M<0)$. Since, by (IRT-3), at least one of $Q_<$ or $Q_>$ is nonarithmetic, the same must hold for the conditional law $\mathbb{P}(S_\sigma \in \cdot|\Pi_\sigma \neq 0)$ as one may easily deduce with the help of FT's [☞ again Problem 4.15]. □

Lemma 4.13. *If* (4.3), (4.4)*, and thus* (4.18) *are valid, then*

$$\int_0^\infty \big|\mathbb{P}(|X|>t) - \mathbb{P}(|\Pi_\sigma X|>t)\big|\, t^{\vartheta-1}\, dt \;<\; \infty. \tag{4.24}$$

holds true as well and, furthermore,

$$\frac{1}{2\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(|X|>t) - \mathbb{P}(|\Pi_\sigma X|>t)\big)\, t^{\vartheta-1}\, dt \;=\; C \tag{4.25}$$

for C as defined in (4.9).

Proof. First observe that, for all $t \geq 0$,

$$\begin{aligned}
&\big|\mathbb{P}(|X|>t) - \mathbb{P}(|\Pi_\sigma X|>t)\big| \\
&\quad= \lim_{m\to\infty}\big|\mathbb{P}(|X|>t) - \mathbb{P}(|\Pi_{\sigma\wedge m} X|>t)\big| \\
&\quad= \lim_{m\to\infty}\left|\mathbb{E}\left(\sum_{n=1}^{\sigma\wedge m} \mathbf{1}_{(t,\infty)}(|\Pi_{n-1}X|) - \mathbf{1}_{(t,\infty)}(|\Pi_n X|)\right)\right| \\
&\quad= \lim_{m\to\infty}\left|\mathbb{E}\left(\sum_{n=1}^{m} \mathbf{1}_{\{\sigma\geq n\}}\Big(\mathbf{1}_{(t,\infty)}(|\Pi_{n-1}X|) - \mathbf{1}_{(t,\infty)}(|\Pi_n X|)\Big)\right)\right| \\
&\quad\leq \sum_{n\geq 1}\Big|\mathbb{P}(\sigma\geq n, |\Pi_{n-1}X|>t) - \mathbb{P}(\sigma\geq n, |\Pi_n X|>t)\Big|.
\end{aligned}$$

Consequently, defining $P_n(ds) := \mathbb{P}(\sigma \geq n, \Pi_{n-1} \in ds)$ for $n \geq 1$, we obtain

$$\begin{aligned}
&\int_0^\infty \big|\mathbb{P}(|X|>t) - \mathbb{P}(|\Pi_\sigma X|>t)\big|\, t^{\vartheta-1}\, dt \qquad\qquad (4.26)\\
&\quad\leq \sum_{n\geq 1}\int_0^\infty \Big|\mathbb{P}(\sigma\geq n, |\Pi_{n-1}X|>t) - \mathbb{P}(\sigma\geq n, |\Pi_n X|>t)\Big|\, t^{\vartheta-1}\, dt \\
&\quad= \sum_{n\geq 1}\int_0^\infty \left|\int_{\mathbb{R}_>} \mathbb{P}\left(|X|>\frac{t}{s}\right) - \mathbb{P}\left(|MX|>\frac{t}{s}\right)\, P_n(ds)\right|\, t^{\vartheta-1}\, dt
\end{aligned}$$

$$\le \sum_{n\ge 1}\int_{\mathbb{R}_>}\int_0^\infty \left|\mathbb{P}\left(|X|>\frac{t}{s}\right)-\mathbb{P}\left(|MX|>\frac{t}{s}\right)\right|\left(\frac{t}{s}\right)^{\vartheta-1} dt\ s^{\vartheta-1}P_n(ds)$$
$$= \left(\int_0^\infty \big|\mathbb{P}(|X|>t)-\mathbb{P}(|MX|>t)\big|\,t^{\vartheta-1}\,dt\right)\sum_{n\ge 1}\mathbb{E}\mathbf{1}_{\{\sigma\ge n\}}|\Pi_{n-1}|^{\vartheta},$$

where the independence of $(\{\sigma\ge n\},\Pi_{n-1})\in\sigma(\Pi_0,M_1,...,M_{n-1})$ and (M_n,X) has been utilized for the third line and the change of variables $t/s\rightsquigarrow t$ for the last one. In view of the fact that (4.18) holds true, it remains to verify for (4.24) that the last series, which may also be written as $\mathbb{E}(\sum_{n=1}^{\sigma}|\Pi_{n-1}|^{\vartheta})$, is finite. To this end, let $\widehat{\mathbb{P}}$ be defined as in the proof of Lemma 4.12. Then

$$a_n := \mathbb{E}\mathbf{1}_{\{\sigma\ge n\}}|\Pi_{n-1}|^{\vartheta} = \widehat{\mathbb{P}}(\sigma\ge n)$$

for each $n\ge 1$, because $\{\sigma\ge n\}\in\mathscr{F}_{n-1}$ and therefore [☞ Problem 4.15(b)]

$$\sum_{n\ge 1}a_n = \widehat{\mathbb{E}}\sigma = 2,$$

which completes the proof of (4.24). But by now repeating the calculation in (4.26) without absolute value signs, all inequalities turn into equalities, giving

$$\int_0^\infty \big(\mathbb{P}(|X|>t)-\mathbb{P}(|\Pi_\sigma X|>t)\big)\,t^{\vartheta-1}\,dt$$
$$= \left(\int_0^\infty \big(\mathbb{P}(|X|>t)-\mathbb{P}(|MX|>t)\big)\,t^{\vartheta-1}\,du\right)\sum_{n\ge 1}\mathbb{E}\mathbf{1}_{\{\sigma\ge n\}}|\Pi_{n-1}|^{\vartheta}$$
$$= 2\int_0^\infty \big(\mathbb{P}(|X|>t)-\mathbb{P}(|MX|>t)\big)\,t^{\vartheta-1}\,dt$$

and so (4.25) upon multiplication with $(2\mu_\vartheta)^{-1}$. □

Finally, we must verify condition (4.3) when substituting M for Π_σ and provide the formula that replaces (4.7) in this case.

Lemma 4.14. *Under the same assumptions as in the previous lemma,*

$$\int_0^\infty \big|\mathbb{P}(X>t)-\mathbb{P}(\Pi_\sigma X>t)\big|\,t^{\vartheta-1}\,dt \;<\; \infty \tag{4.27}$$

as well as

$$\frac{1}{2\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(X>t)-\mathbb{P}(\Pi_\sigma X>t)\big)\,t^{\vartheta-1}\,dt \;=\; \frac{C}{2}. \tag{4.28}$$

Proof. We leave it as an exercise to first verify (4.27) along similar lines as in (4.26) [☞ Problem 4.16]. Keeping the notation of the proof of the previous lemma, we then obtain (using $\Pi_{n-1} = -|\Pi_{n-1}|$ on $\{\sigma \geq n\}$ for any $n \geq 2$ and $\sum_{n\geq 2} a_n = 1$)

$$
\begin{aligned}
&\int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(\Pi_\sigma X > t)\big)\, t^{\vartheta-1}\, dt \\
&\quad = \sum_{n\geq 1} \int_0^\infty \Big(\mathbb{P}(\sigma \geq n, \Pi_{n-1} X > t) - \mathbb{P}(\sigma \geq n, \Pi_n X > t)\Big)\, t^{\vartheta-1}\, dt \\
&\quad = \int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(MX>t)\big)\, t^{\vartheta-1}\, dt \\
&\quad + \sum_{n\geq 2} \int_0^\infty \Big(\mathbb{P}(\sigma \geq n, |\Pi_{n-1}| X < -t) - \mathbb{P}(\sigma \geq n, |\Pi_{n-1}| M_n X < -t)\Big)\, t^{\vartheta-1}\, dt \\
&\quad = \int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(MX>t)\big)\, t^{\vartheta-1}\, dt \\
&\quad + \sum_{n\geq 2} \int_{\mathbb{R}_>} \int_0^\infty \left(\mathbb{P}\left(X < -\frac{t}{s}\right) - \mathbb{P}\left(MX > -\frac{t}{s}\right)\right) \left(\frac{t}{s}\right)^{\vartheta-1} dt\; s^{\vartheta-1}\, P_n(ds) \\
&\quad = \int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(MX>t)\big)\, t^{\vartheta-1}\, dt \\
&\quad + \left(\int_0^\infty \big(\mathbb{P}(X<-t) - \mathbb{P}(MX<-t)\big)\, t^{\vartheta-1}\, dt\right) \sum_{n\geq 2} a_n \\
&\quad = \int_0^\infty \big(\mathbb{P}(|X|>t) - \mathbb{P}(|MX|>t)\big)\, t^{\vartheta-1}\, dt
\end{aligned}
$$

and therefore

$$
\frac{1}{2\mu_\vartheta} \int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(\Pi_\sigma X > t)\big)\, t^{\vartheta-1}\, dt \;=\; \frac{C}{2}
$$

as claimed. □

Taking a deep breath, we are finally able to settle the present case by using part (a) of the theorem upon replacing M with $\Pi_\sigma \geq 0$. Lemma 4.12 ensures validity of (IRT-1)-(IRT-3) under this replacement and also that $2\mu_\vartheta$ takes the place of μ_ϑ. Condition (4.3) now turns into (4.27), which has been verified as part of Lemma 4.14. Therefore, we conclude

$$
C_+ \;=\; \lim_{t\to\infty} t^\vartheta\, \mathbb{P}(X>t) \;=\; \frac{1}{2\mu_\vartheta} \int_0^\infty \big(\mathbb{P}(X>t) - \mathbb{P}(\Pi_\sigma > t)\big)\, t^{\vartheta-1}\, dt,
$$

and, by (4.28), the last expression equals $C/2$ as asserted.

4.3.3 The case when $M \leq 0$ a.s.

This case is handled by the same reduction argument as the previous one, but is considerably simpler because of the obvious fact that $\sigma \equiv 2$ holds true here. We leave it to the reader to check all necessary conditions as well as to show that (4.25) and (4.28) remain valid [☞ Problem 4.17].

Problems

Problem 4.15. Given the assumptions of Subsection 4.3.2, prove the following assertions:

(a) $\mathbb{P}(\sigma-2\in\cdot|\sigma\ge 2)=Geom(\theta)$ with $\theta=\mathbb{P}(M\le 0)$.
(b) $\widehat{\mathbb{P}}(\sigma-2\in\cdot|\sigma\ge 2)=Geom(\hat\theta)$ with $\hat\theta=\mathbb{E}|M|^{\vartheta}\mathbf{1}_{\{M<0\}}$, and $\widehat{\mathbb{E}}\sigma=2$.
(c) The conditional law under $\mathbb{P}$ of S_σ given $\Pi_\sigma\ne 0$ satisfies (4.23).
(d) Compute the FT of $\mathbb{P}(S_\sigma\in\cdot|\Pi_\sigma\ne 0)$ in terms of those of $Q_<,Q_>$ and use it to show that this law is nonarithmetic.

Problem 4.16. Give a proof of (4.27) under the assumptions of Lemma 4.14.

Problem 4.17. Give a proof of the implicit renewal theorem for the case $M\le 0$ a.s.

Problem 4.18. [Tail behavior at 0] Prove the following version of the implicit renewal theorem:
Let M,X be independent random variables taking values in $\overline{\mathbb{R}}\backslash\{0\}$ such that, for some $\vartheta>0$,

(IRT2-1) $\mathbb{E}|M|^{-\vartheta}=1$.
(IRT2-2) $\mathbb{E}|M|^{-\vartheta}\log^-|M|<\infty$.
(IRT2-3) The conditional law $\mathbb{P}(\log|M|\in\cdot||M|<\infty)$ of $\log|M|$ given $|M|<\infty$ is nonarithmetic, in particular, $\mathbb{P}(|M|=1)<1$.

Then $0<\mathbb{E}\log|M|\le\infty$, $0<\mu_\vartheta:=-\mathbb{E}|M|^{-\vartheta}\log|M|<\infty$, and the following assertions hold true:

(a) Suppose M is a.s. positive. If

$$\int_0^\infty\big|\mathbb{P}(X\le t)-\mathbb{P}(MX\le t)\big|\,t^{-1-\vartheta}\,dt\;<\;\infty \tag{4.29}$$

or, respectively,

$$\int_0^\infty\big|\mathbb{P}(X\ge -t)-\mathbb{P}(MX\ge -t)\big|\,t^{-1-\vartheta}\,dt\;<\;\infty, \tag{4.30}$$

then

$$\lim_{t\to 0+}t^{-\vartheta}\,\mathbb{P}(0<X\le t)\;=\;C_+, \tag{4.31}$$

respectively

$$\lim_{t\to 0+}t^{\vartheta}\,\mathbb{P}(-t\le X<0)\;=\;C_-, \tag{4.32}$$

where C_+ and C_- are given by the equations

$$C_+ \ := \ \frac{1}{\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(X\le t)-\mathbb{P}(MX\le t)\big)\,t^{-1-\vartheta}\,dt, \tag{4.33}$$

$$C_- \ := \ \frac{1}{\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(X\ge -t)-\mathbb{P}(MX\ge -t)\big)\,t^{-1-\vartheta}\,dt. \tag{4.34}$$

(b) If $\mathbb{P}(M<0)>0$ and (4.29), (4.30) are both satisfied, then (4.31) and (4.32) hold with $C_+=C_-=C/2$, where

$$C \ := \ \frac{1}{\mu_\vartheta}\int_0^\infty \big(\mathbb{P}(|X|\le t)-\mathbb{P}(|MX|\le t)\big)\,t^{-1-\vartheta}\,dt. \tag{4.35}$$

4.4 Applications

We will proceed with an application of the previously developed results to a number of examples some of which are also discussed in [34]. In view of the fact that implicit renewal theory deals with the tail behavior of solutions to SFPE's of the form (4.1) and embarks on linear approximation of the random function Ψ involved, the simplest and most natural example that comes to mind is a RDE and therefore studied first.

4.4.1 Random difference equations and perpetuities

Returning to the situation described in Section 1.5, let $(M,Q),(M_1,Q_1),(M_2,Q_2),\ldots$ be iid two-dimensional random variables and $(X_n)_{n\ge0}$ recursively defined by the (one-dimensional) RDE

$$X_n \ := \ M_nX_{n-1}+Q_n, \quad n\ge 1.$$

As usual, let $\Pi_0:=0$ and $\Pi_n:=M_1\cdot\ldots\cdot M_n$ for $n\ge1$. If $\mathbb{E}\log|M|<0$ and $\mathbb{E}\log^+|Q|<\infty$, then $(X_n)_{n\ge0}$ is a mean contractive IFS on $\mathbb{R}$ satisfying the jump-size condition (3.17) (with $x_0=0$ and $d(x,y)=|x-y|$). By Theorem 3.24, it is then convergent in distribution (under any initial distribution) to the unique solution of the SFPE

$$X \stackrel{d}{=} MX+Q, \qquad X \text{ independent of } (M,Q), \tag{4.36}$$

which is (the law of) the perpetuity $X:=\sum_{n\ge1}\Pi_{n-1}Q_n$ and in turn is obtained as the a.s. limit (under any initial distribution) of the backward iterations. By applying the same arguments to the RDE

$$Y_n=|M_n|Y_{n-1}+|Q_n|, \quad n\ge1,$$

we see that $Y:=\sum_{n\ge1}|\Pi_{n-1}Q_n|$ is a.s. finite as well and its law the unique solution to the SFPE

$$Y = |M|Y + |Q|, \qquad Y \text{ independent of } (M,Q).$$

An application of the implicit renewal theorem provides us with the following result about the tail behavior of X under appropriate conditions on M and Q. Its far more difficult extension to the multidimensional situation is a famous result due to KESTEN [42].

Theorem 4.19. *Suppose that M satisfies (IRT-1)-(IRT-3) and that $\mathbb{E}|Q|^{\vartheta} < \infty$. Then there exists a unique solution to the SFPE* (4.36)*, given by the law of the perpetuity $X := \sum_{n\ge 1} \Pi_{n-1} Q_n$. This law satisfies* (4.5) *as well as* (4.6)*, where*

$$C_{\pm} = \frac{\mathbb{E}\big(((MX+Q)^{\pm})^{\vartheta} - ((MX)^{\pm})^{\vartheta}\big)}{\vartheta \mu_{\vartheta}} \tag{4.37}$$

if $M \ge 0$ a.s., while

$$C_{+} = C_{-} = \frac{\mathbb{E}\big(|MX+Q|^{\vartheta} - |MX|^{\vartheta}\big)}{2\vartheta \mu_{\vartheta}} \tag{4.38}$$

if $\mathbb{P}(M<0) > 0$. Furthermore,

$$C_{+} + C_{-} > 0 \quad \text{iff} \quad \mathbb{P}(Q = c(1-M)) < 1 \quad \text{for all } c \in \mathbb{R}. \tag{4.39}$$

A crucial ingredient to the proof of this theorem is the following moment result that will enable us to verify validity of (4.3) and (4.4) of the implicit renewal theorem. For a.s. nonnegative M, Q and $\kappa > 1$, it was obtained by VERVAAT [59]; for a stronger version see [4] and Problem 4.33.

Proposition 4.20. *Suppose that $\mathbb{E}|M|^{\kappa} \le 1$ and $\mathbb{E}|Q|^{\kappa} < \infty$ for some $\kappa > 0$. Then $Y = \sum_{n\ge 1} |\Pi_{n-1} Q_n|$ satisfies $\mathbb{E}Y^p < \infty$ for any $p \in (0,\kappa)$.*

Proof. This is actually a direct consequence of the more general Theorem 3.29, but we repeat the argument for the present situation because it is short and simple.

As argued earlier, $\mathbb{E}|M|^p < 1$ for any $p \in (0,\kappa)$. If $p \le 1$, the subadditivity of $x \mapsto x^p$ implies that

$$\mathbb{E}Y^p \;\le\; \sum_{n\ge 1} \mathbb{E}|\Pi_{n-1} Q_n|^p \;\le\; \mathbb{E}|Q|^p \sum_{n\ge 1} (\mathbb{E}|M|^p)^{n-1} \;\le\; \frac{\mathbb{E}|Q|^p}{1-\mathbb{E}|M|^p} \;<\; \infty,$$

whereas in the case $p > 1$ a similar estimation with the help of Minkowski's inequality yields

$$\|Y\|_p \;\le\; \sum_{n\ge1}\|\Pi_{n-1}Q_n\|_p \;=\; \|Q\|_p\sum_{n\ge1}\|M\|_p^{n-1} \;=\; \frac{\|Q\|_p}{1-\|M\|_p} \;<\; \infty.$$

This completes the proof. □

GRINCEVIČIUS [35] provided the following extension of Lévy's symmetrization inequalities that will be utilized in the proof of (4.39). Under the assumptions of Theorem 4.19, define $m_0 := med(X)$,

$$\begin{aligned}
\Pi_{k:n} &:= \prod_{j=k}^{n} M_j \quad \text{for } 1\le k\le n,\\
\widehat{X}_n &:= \sum_{k=1}^{n}\Pi_{k-1}Q_k, \quad \widehat{X}_{k:n} := \sum_{j=k}^{n}\Pi_{k:j-1}Q_j \quad \text{for } 1\le k\le n,\\
\widehat{X}_0^* &:= m_0, \quad \widehat{X}_n^* := \widehat{X}_n + \Pi_n m_0 \quad \text{for } n\ge 1,\\
R_k &:= \widehat{X}_k + \Pi_k\, med\big(\widehat{X}_{k+1:n} + \Pi_{k+1:n}y\big) \quad \text{for } 1\le k\le n,\ y\in\mathbb{R}\\
U_n &:= \Pi_{n-1}\big(Q_n - m_0(1-M_n)\big) \quad \text{for } n\ge 1.
\end{aligned}$$

where $\widehat{X}_{n+1:n} := 0$ and $\Pi_{n+1:n} := 1$ in the definition of R_n. The $\widehat{X}_n$ are obviously the backward iterations when $X_0 = 0$ and hence a.s. convergent to $X = \sum_{n\ge1}\Pi_{n-1}Q_n$.

Lemma 4.21. [Grincevičius] *With the given notation,*

$$\mathbb{P}\left(\max_{1\le k\le n} R_k > x\right) \;\le\; 2\,\mathbb{P}(\widehat{X}_n + \Pi_n y > x)$$

for all $x,y\in\mathbb{R}$.

Specializing to $y = 0$, we obtain

$$\mathbb{P}\left(\max_{1\le k\le n}\left(\widehat{X}_k + \Pi_k\, med\big(\widehat{X}_{k+1:n}\big)\right) > x\right) \;\le\; 2\,\mathbb{P}(\widehat{X}_n > x)$$

for any $x\in\mathbb{R}$ and then upon letting $n\to\infty$

$$\mathbb{P}\left(\sup_{n\ge1}\widehat{X}_n^* > x\right) \;=\; \mathbb{P}\left(\sup_{n\ge1}\left(\widehat{X}_n + \Pi_n m_0\right) > x\right) \;\le\; 2\,\mathbb{P}(X > x), \tag{4.40}$$

because $\lim_{n\to\infty}\widehat{X}_{k+1:n} \stackrel{d}{=} X$ for any $k\ge 1$. The same inequality holds, of course, with $-\widehat{X}_n, -X$ instead of $\widehat{X}_n, X$ whence

$$\mathbb{P}\left(\sup_{n\ge1}|\widehat{X}_n^*| > x\right) \;\le\; 2\,\mathbb{P}(|X| > x) \tag{4.41}$$

for all $x\in\mathbb{R}_{\ge}$.

Proof. Fixing any $x, y \in \mathbb{R}$, define

$$A_k := \{R_1 \le x, ..., R_{k-1} \le x, R_k > x\},$$
$$B_k := \left\{\widehat{X}_{k+1:n} + \Pi_{k+1:n} y \ge med\big(\widehat{X}_{k+1:n} + \Pi_{k+1:n} y\big)\right\}$$

for $k = 1, ..., n$. Observe that A_k and B_k are independent events with $\mathbb{P}(B_k) \ge 1/2$ for each k and that

$$\left\{\max_{1 \le k \le n} \Big(\widehat{X}_k + \Pi_k\, med\big(\widehat{X}_{k+1:n} + \Pi_{k+1:n} y\big)\Big) > x\right\} \;=\; \sum_{k=1}^{n} A_k,$$

$$\left\{\widehat{X}_n + \Pi_n y > x\right\} \;\supset\; \sum_{k=1}^{n} A_k \cap B_k.$$

For the last inclusion we have used that, on $A_k \cap B_k$,

$$x \;<\; R_k \;\le\; \widehat{X}_k + \Pi_k \left(\widehat{X}_{k+1:n} + \Pi_{k+1:n} y\right) \;=\; \widehat{X}_n + \Pi_n y$$

for each $k = 1, ..., n$. Now

$$\mathbb{P}(\widehat{X}_n + \Pi_n y > x) \;\ge\; \sum_{k=1}^{n} \mathbb{P}(A_k)\,\mathbb{P}(B_k) \;\ge\; \frac{1}{2}\sum_{k=1}^{n} \mathbb{P}(A_k) \;=\; \frac{1}{2}\mathbb{P}\left(\max_{1 \le k \le n} R_k > x\right)$$

proves the assertion. □

Proof (of Theorem 4.19). We first prove (4.5) and (4.6) for which, by Corollary 4.2, it suffices to verify (4.10) and (4.11). But since $-X$ satisfies the same SFPE as X when replacing (M, Q) with $(M, -Q)$, it is further enough to consider only the first of these two conditions, viz. $\mathbb{E}\big|((MX+Q)^+)^\vartheta - ((MX)^+)^\vartheta\big| < \infty$.

By making use of the inequality

$$(x+y)^p \le x^p + p\,2^{p-1}\big(x^{p-1}y + xy^{p-1}\big) + y^p, \tag{4.42}$$

valid for all $x, y \in \mathbb{R}_{\ge}$ and $p > 1$, we find that

$$\begin{aligned} &((MX+Q)^+)^\vartheta - ((MX)^+)^\vartheta \;\le\; ((MX)^+ + Q^+)^\vartheta - ((MX)^+)^\vartheta \\ &\le \begin{cases} (Q^+)^\vartheta, & \text{if } \vartheta \in (0,1], \\ (Q^+)^\vartheta + c_\vartheta\big(((MX)^+)^{\vartheta-1}Q^+ + (MX)^+(Q^+)^{\vartheta-1}\big), & \text{if } \vartheta > 1, \end{cases} \end{aligned}$$

where $c_\vartheta := \vartheta 2^{\vartheta-1}$. A combination of $(MX)^+ \le |MX|$, the independence of X and (M, Q), $\mathbb{E}|M|^\vartheta < \infty$, $\mathbb{E}|Q|^\vartheta < \infty$, and of $\mathbb{E}|X|^{\vartheta-1} < \infty$ if $\vartheta > 1$ [by Prop. 4.20] hence implies that

$$\mathbb{E}\left(((MX+Q)^+)^\vartheta - ((MX)^+)^\vartheta\right)^+ < \infty.$$

Indeed, if $\vartheta > 1$, the obtained bound is

$$\mathbb{E}(Q^+)^\vartheta + c_\vartheta\, \mathbb{E}(|M|^{\vartheta-1}Q^+)\,\mathbb{E}|X|^{\vartheta-1} + c_\vartheta\, \mathbb{E}(|M|(Q^+)^{\vartheta-1})\,\mathbb{E}|X|$$

and the finiteness of $\mathbb{E}(|M|^{\vartheta-1}Q^+)$, $\mathbb{E}(|M|(Q^+)^{\vartheta-1})$ follows by an appeal to Hölder's inequality.

In order to get

$$\mathbb{E}\left(((MX+Q)^+)^\vartheta - ((MX)^+)^\vartheta\right)^- < \infty,$$

one can argue in a similar manner when using the estimate

$$\begin{aligned} &((MX)^+)^\vartheta - ((MX+Q)^+)^\vartheta \ \le\ ((MX+Q)^+ + Q^-)^\vartheta - ((MX+Q)^+)^\vartheta \\ &\le \begin{cases} (Q^-)^\vartheta, & \text{if } \vartheta \in (0,1], \\ (Q^+)^\vartheta + c_\vartheta\big(((MX+Q)^+)^{\vartheta-1}Q^- + (MX+Q)^+(Q^-)^{\vartheta-1}\big), & \text{if } \vartheta > 1. \end{cases} \end{aligned}$$

The straightforward details are again left as an exercise [☞ Problem 4.34].

Turning to the proof of (4.39), suppose that $Q = c(1-M)$ a.s. for some $c \in \mathbb{R}$. Then $X = c$ a.s. forms the unique solution to (4.36) so that $C_+ = C_- = 0$. For the converse, let $\mathbb{P}(Q = c(1-M)) < 1$ for all $c \in \mathbb{R}$. Since

$$C_+ + C_- \ =\ \lim_{t\to\infty} t^\vartheta\, \mathbb{P}(|X| > t),$$

we must verify that the limit on the right-hand side is positive. To this end, we start by noting that, by our assumption, we can pick $\varepsilon > 0$ such that

$$p := \mathbb{P}(|Q - m_0(1-M)| > \varepsilon) > 0.$$

Next, observe that

$$\widehat{X}^*_{n-1} + U_n = \widehat{X}_{n-1} + \Pi_{n-1}m_0 + \Pi_{n-1}(Q_n - m_0(1-M_n)) \ =\ \widehat{X}^*_n$$

for each $n \ge 1$, which implies

$$\sup_{n\ge 0}|\widehat{X}^*_n| \ \ge\ \sup_{n\ge 1}|\widehat{X}^*_n| \ \ge\ \sup_{n\ge 1}|U_n| - \sup_{n\ge 0}|\widehat{X}^*_n|$$

and therefore

$$\sup_{n\ge 0}|\widehat{X}^*_n| \ \ge\ \frac{1}{2}\sup_{n\ge 1}|U_n|.$$

Since $\widehat{X}^*_0 = m_0$, we thus find for any $t > |m_0|$ that

$$\left\{\sup_{n\ge 1}|\widehat{X}^*_n| > t\right\} \ =\ \left\{\sup_{n\ge 0}|\widehat{X}^*_n| > t\right\} \ \supset\ \left\{\sup_{n\ge 1}|U_n| > 2t\right\}$$

and then in combination with Grincevičius' inequality (4.41)

$$
\begin{aligned}
\mathbb{P}(|X| > t) \;&\geq\; \frac{1}{2}\,\mathbb{P}\left(\sup_{n\geq 1}|\widehat{X}_n^*| > t\right) \\
&\geq\; \frac{1}{2}\,\mathbb{P}\left(\sup_{n\geq 1}|U_n| > 2t\right) \\
&\geq\; \frac{1}{2}\sum_{n\geq 1}\mathbb{P}\left(\tau = n-1, |Q_n - m_0(1-M_n)| > \varepsilon\right) \\
&=\; \frac{p}{2}\,\mathbb{P}\left(\sup_{n\geq 0}|\Pi_n| > \frac{2t}{\varepsilon}\right),
\end{aligned}
$$

where $\tau := \inf\{n \geq 0 : |\Pi_n| > 2t/\varepsilon\}$ and

$$|U_n| = |\Pi_{n-1}(Q_n - m_0(1-M_n))| > 2t \quad \text{on } \{\tau = n-1, |Q_n - m_0(1-M_n)| > \varepsilon\}$$

for each $n \geq 1$ has been utilized for the penultimate line. Finally,

$$\lim_{t\to\infty} t^{\vartheta}\,\mathbb{P}\left(\sup_{n\geq 0}|\Pi_n| > t\right) = \lim_{t\to\infty} e^{\vartheta \log t}\,\mathbb{P}\left(\sup_{n\geq 0} S_n > \log t\right) = K_+ > 0,$$

by Theorem 2.68 leads to the desired conclusion. ◻

Formula (4.38) may be used to derive further information on $C_\pm$ or $C_+ + C_-$ like upper bounds or alternative formulae. We refrain from dwelling on this further and refer to Problems 4.36-4.38.

It is usually impossible to determine the law of a perpetuity explicitly, but there are exceptions. One such class is described in Proposition 4.23 below. Recall that a *beta distribution* with parameters $a, b > 0$ has $\lambda\!\!\lambda$-density

$$g_{a,b}(x) \;:=\; \frac{1}{B(a,b)}\,x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x),$$

where the normalizing constant

$$B(a,b) \;:=\; \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

equals the so-called *complete beta integral* at (a,b). The substitution $\frac{y}{1+y}$ for x provides us with the equivalent formula

$$B(a,b) \;=\; \int_0^\infty y^{a-1}(1+y)^{-a-b}\,dx$$

for all $a, b \in \mathbb{R}_>$. As a consequence,

$$g^*_{a,b}(x) \;:=\; \frac{1}{B(a,b)}\,x^{a-1}(1+x)^{-a-b}\,\mathbf{1}_{\mathbb{R}_>}(x)$$

for $a,b>0$ defines the $\lambda\!\!\lambda$-density of another distribution $\beta^*(a,b)$, say, called *beta distribution of the second kind*. Here is a useful (multiplicative) convolution property of these distributions that is crucial for the proof of the announced proposition.

Lemma 4.22. *If X and Y are two independent random variables with $\mathscr{L}(X)=\beta^*(a,b)$ and $\mathscr{L}(Y)=\beta^*(c,a+b)$ for $a,b,c\in\mathbb{R}_>$, then*

$$\mathscr{L}((1+X)Y)=\beta^*(c,b).$$

Proof. For $s\in(-c,b)$, we obtain

$$\begin{aligned}\mathbb{E}Y^s &= \frac{\Gamma(a+b+c)}{\Gamma(c)\Gamma(a+b)}\int_0^\infty y^{c+s-1}(1+y)^{-((c+s)+(a+b-s))}\,dy\\ &= \frac{\Gamma(c+s)\Gamma(a+b-s)}{\Gamma(c)\Gamma(a+b)}\end{aligned}$$

and further in a similar manner

$$\begin{aligned}\mathbb{E}(1+X)^s &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^\infty x^{a-1}(1+x)^{-(a+b-s)}\,dx\\ &= \frac{\Gamma(a+b)\Gamma(b-s)}{\Gamma(b)\Gamma(a+b-s)}.\end{aligned}$$

Consequently, the independence of X,Y implies

$$\mathbb{E}((1+X)Y)^s = \mathbb{E}(1+X)^s\,\mathbb{E}Y^s = \frac{\Gamma(c+s)\Gamma(b-s)}{\Gamma(c)\Gamma(b)}.$$

But the last expression also equals $\phi(s)=\mathbb{E}Z^s$ if $\mathscr{L}(Z)=\beta^*(c,b)$. The function ϕ is called the *Mellin transform* of Z and is the same as the mgf of $\log Z$ (as Z is positive). But the mgf, if not only defined at 0, determines the distribution of $\log Z$ and thus of Z uniquely, giving $\beta^*(c,b)=\mathscr{L}(Z)=\mathscr{L}((1+X)Y)$ as claimed. □

Here is the announced result, again taken from [34]. The cases $m=1,2$ are due to CHAMAYOU & LETAC [18].

Proposition 4.23. *For $m\in\mathbb{N}$ and positive reals $a_1,\ldots,a_m,b$, let $X,Y_1,\ldots,Y_m$ be independent random variables such that $\mathscr{L}(X)=\beta^*(a_1,b)$ and $\mathscr{L}(Y_k)=\beta^*(a_{k+1},a_k+b)$ for $k=1,\ldots,m$, where $a_{m+1}:=a_1$. Then X satisfies the SFPE* (4.36)*, i.e. $X\overset{d}{=}MX+Q$, for the pair (M,Q) defined by*

$$M := \prod_{k=0}^{m-1}Y_{m-k}\quad\text{and}\quad Q := \sum_{k=0}^{m-1}\prod_{j=0}^{k}Y_{m-j}.$$

Furthermore,

$$\lim_{t\to\infty} t^b\,\mathbb{P}(X>t) \;=\; \frac{1}{b\,B(a_1,b)}. \tag{4.43}$$

Proof. We put $X_1 := X$ and $X_n := (1+X_{n-1})Y_{n-1}$ for $n=2,\ldots,m$. A simple induction in combination with the previous lemma shows that $\mathscr{L}(X_n) = \beta^*(a_n,b)$ for $n = 1,\ldots,m$. Moreover,

$$(1+X_n)Y_n \;=\; Y_n + X_nY_n \;=\; (1+X_{n-1})Y_{n-1}Y_n \;=\ldots=\; Q+MX$$

and $\mathscr{L}((1+X_n)Y_n) = \beta^*(a_{m+1},b) = \beta^*(a_1,b)$ by another appeal to Lemma 4.22.

Left with the proof of (4.43) and using $\mathscr{L}(X) = \beta^*(a_1,b)$, we infer

$$\begin{aligned}\mathbb{P}(X>t) \;&=\; \frac{1}{B(a_1,b)}\int_t^\infty x^{a_1-1}(1+x)^{-a_1-b}\,dx\\ &=\; \frac{1}{B(a_1,b)}\int_t^\infty \left(\frac{x}{1+x}\right)^{a_1-1}\left(\frac{x}{1+x}\right)^{b+1}\,dx.\end{aligned}$$

Since the last integral is easily seen to behave like $\int_t^\infty x^{-b-1}\,dx = bt^{-b}$ as $t\to\infty$, we arrive at the desired conclusion. ☐

We leave it as an exercise [☞ Problem 4.39] to verify that (M,Q) satisfies the assumptions of the implicit renewal theorem 4.1 with $\vartheta = b$ and so, by this result,

$$\frac{1}{b\,B(a_1,b)} \;=\; \frac{\mathbb{E}\big((MX+Q)^b-(MX)^b\big)}{b\mu_b}$$

holds true.

4.4.2 Lindley's equation and a related max-type equation

If we replace addition in (4.36) by the max-operation, we get a new SFPE, namely

$$X \;\stackrel{d}{=}\; MX\vee Q, \tag{4.44}$$

where X and (M,Q) are as usual independent. We make the additional assumption that $M\ge 0$ a.s. If $\mathbb{E}\log M<0$ and $\mathbb{E}\log^+Q^+<\infty$, it has a unique solution which is the unique stationary distribution of the mean contractive IFS of iid Lipschitz maps with generic copy $\Psi(x) := MX\vee Q$ [☞ Problem 4.40] and the law of

$$X := \sup_{n\ge 1}\Pi_{n-1}Q_n.$$

Notice that $\Pi_n \to 0$ a.s. in combination with the stationarity of the Q_n entails $\Pi_{n-1}Q_n \to 0$ in probability and thus $X \ge 0$ a.s. In other words, only the right tail of X needs to be studied hereafter. Theorem 4.24 below constitutes the exact counterpart of Theorem 4.19 for (4.44), but before stating it we want to point out the direct relation of this SFPE with Lindley's equation, which is revealed after a transformation. Namely, if $Q \equiv 1$, then taking logarithms in (4.44) yields

$$Y \stackrel{d}{=} (Y+\xi) \vee 0 \;=\; (Y+\xi)^+$$

for $Y = \log X$, where $\xi := \log M$, and the unique solution is given by the law of

$$\log X = \sup_{n\ge 0} \log \Pi_n \;=\; \sup_{n\ge 0} S_n,$$

a fact already known from Problem 1.6.

Theorem 4.24. *Suppose M satisfies (IRT-1)-(IRT-3) and $\mathbb{E}(Q^+)^\vartheta < \infty$. Then there exists a unique solution to the SFPE* (4.44)*, given by the law of $X = \sup_{n\ge 1} \Pi_{n-1}Q_n$. This law satisfies* (4.5) *with*

$$C_+ = \frac{\mathbb{E}\big(((MX\vee Q)^+)^\vartheta - ((MX)^+)^\vartheta\big)}{\vartheta\mu_\vartheta}. \tag{4.45}$$

Moreover, C_+ is positive iff $\mathbb{P}(Q>0)>0$.

Proof. Problem 4.40 shows that (the law of) $X = \sup_{n\ge 1} \Pi_{n-1}Q_n$ provides the unique solution to the SFPE (4.44) under the assumptions stated here. (4.5) with C_+ given by (4.45) is now directly inferred from Corollary 4.2 because

$$\begin{aligned}\mathbb{E}\big|((MX\vee Q)^+)^\vartheta - ((MX)^+)^\vartheta\big| \\ &= \mathbb{E}\big|Q^\vartheta - ((MX)^+)^\vartheta\big|\,\mathbf{1}_{\{MX<Q,Q>0\}} \\ &\le \mathbb{E}(Q^+)^\vartheta \;<\; \infty\end{aligned}$$

[which is (4.10) in that corollary] holds true.

Turning to the asserted equivalence, one implication is trivial, for $Q \le 0$ a.s. entails $X = 0$ a.s. and thus $C_+ = 0$. Hence, suppose $\mathbb{P}(Q>0)>0$ and fix any $c>0$ such that $\mathbb{P}(Q>c)>0$. Defining the predictable first passage time

$$\tau(t) \;:=\; \inf\{n\ge 1 : \Pi_{n-1} > t/c\}, \quad t \ge 0,$$

we note that

$$\mathbb{P}\left(\sup_{n\ge 1} \Pi_{n-1} > \frac{t}{c}\right) \;=\; \mathbb{P}(\tau(t)<\infty) \tag{4.46}$$

and claim that

$$\mathbb{P}\left(\sup_{n\geq 1}\Pi_{n-1}Q_n>t\right)\ \geq\ \mathbb{P}(Q>c)\,\mathbb{P}(\tau(t)<\infty). \tag{4.47}$$

For a proof of the latter claim, just note that

$$\begin{aligned}\mathbb{P}\left(\sup_{n\geq 1}\Pi_{n-1}Q_n>t\right)\ &\geq\ \sum_{n\geq 1}\mathbb{P}\left(\tau(t)=n,\,\Pi_{n-1}Q_n>t\right)\\ &\geq\ \sum_{n\geq 1}\mathbb{P}\left(\tau(t)=n,\,Q_n>c\right)\\ &=\ \mathbb{P}(Q>c)\,\mathbb{P}(\tau(t)<\infty),\end{aligned}$$

where the last line follows by the independence of $\{\tau(t)=n\}\in\sigma(\Pi_0,...,\Pi_{n-1})$ and Q_n. Now we infer upon using (4.46) and (4.47) that

$$\mathbb{P}(X>t)\ =\ \mathbb{P}\left(\sup_{n\geq 1}\Pi_{n-1}Q_n>t\right)\ \geq\ \mathbb{P}(Q>c)\,\mathbb{P}\left(\sup_{n\geq 1}\Pi_{n-1}>\frac{t}{c}\right)$$

and thereby the desired result $C_+>0$, for $\mathbb{P}(Q>c)>0$ and

$$\lim_{t\to\infty}\left(\frac{t}{c}\right)^{\vartheta}\mathbb{P}\left(\sup_{n\geq 1}\Pi_{n-1}>\frac{t}{c}\right)\ =\ \lim_{t\to\infty}e^{\vartheta t}\,\mathbb{P}\left(\sup_{n\geq 0}S_n>t\right)\ >\ 0$$

by invoking once again Theorem 2.68. ☐

4.4.3 Letac's max-type equation $X\overset{d}{=}M(N\vee X)+Q$

A more general example of a max-type SFPE studied by GOLDIE in [34] was first introduced by LETAC [45, Example E], namely

$$X\ \overset{d}{=}\ M(N\vee X)+Q \tag{4.48}$$

for a random triple (M,N,Q) independent of X such that $M\geq 0$ a.s. As usual, this equation characterizes the unique stationary law of the pertinent IFS of iid Lipschitz maps, defined by

$$X_n\ =\ M_n(N_n\vee X_{n-1})+Q_n,\quad n\geq 1,$$

provided that mean contractivity and the jump-size condition (3.17) hold. The (M_n,N_n,Q_n), $n\geq 1$, are of course independent copies of (M,N,Q). We leave it as an exercise [☞ Problem 4.42] to verify that mean contraction holds if $\mathbb{E}\log M<0$ and (3.17) holds if, furthermore, $\mathbb{E}\log^+ N^+<\infty\ \mathbb{E}\log^+|Q|<\infty$. By computing the backward iterations, one then finds as in [34, Prop. 6.1] that

$$X\ :=\ \max\left\{\sum_{n\geq 1}\Pi_{n-1}Q_n,\sup_{n\geq 1}\left(\sum_{k=1}^{n}\Pi_{k-1}Q_k+\Pi_nN_n\right)\right\}$$

is a.s. finite and its law the unique solution to (4.48). But $\Pi_n N_n \to 0$ in probability in combination with

$$X \;\ge\; \lim_{n\to\infty}\left(\sum_{k=1}^{n}\Pi_{k-1}Q_k+\Pi_n N_n\right) \;=\; \sum_{n\ge 1}\Pi_{n-1}Q_n$$

obviously implies that

$$X \;=\; \sup_{n\ge 1}\left(\sum_{k=1}^{n}\Pi_{k-1}Q_k+\Pi_n N_n\right). \tag{4.49}$$

The following result determines the right tail behavior of X with the help the implicit renewal theorem. As for the left tail behavior see Remark 4.27 below.

Theorem 4.25. *Suppose that M satisfies (IRT-1)-(IRT-3) and*

$$\mathbb{E}(N^+)^{\vartheta}<\infty,\quad \mathbb{E}|Q|^{\vartheta}<\infty\quad \textit{and}\quad \mathbb{E}(MN^+)^{\vartheta}<\infty. \tag{4.50}$$

Then the SFPE (4.48) *has a unique solution given by the law of X in* (4.49)*. This law satisfies* (4.5) *with*

$$C_+=\frac{\mathbb{E}\big(((M(N\vee X)+Q)^+)^{\vartheta}-((MX)^+)^{\vartheta}\big)}{\vartheta\mu_{\vartheta}}. \tag{4.51}$$

Furthermore, C_+ is positive iff $\mathbb{P}(Q=c(1-M))<1$ for all $c\in\mathbb{R}$, or $Q=c(1-M)$ a.s. and $\mathbb{P}(M(N-c)>0)>0$ for some $c\in\mathbb{R}$.

Remark 4.26. In [34, Theorem 6.2] only a sufficient condition for $C_+>0$ was given, namely that $Q-c(1-M)\ge 0$ a.s. and

$$\mathbb{P}(Q-c(1-M)>0)+\mathbb{P}(M(N-c)>0)>0 \tag{4.52}$$

for some constant $c\in\mathbb{R}$.

Remark 4.27. Concerning the left tail of X in Theorem 4.25, let us point out the following: Since $M(N\vee X)+Q\ge MN+Q\mathbf{1}_{\{Q<0\}}$, we have

$$\mathbb{E}(X^-)^{\vartheta} \;=\; \mathbb{E}((M(N\vee X)^-)^{\vartheta} \;\le\; \mathbb{E}(MN^-)^{\vartheta}+\mathbb{E}(Q^-)^{\vartheta}$$

and thus $C_-=\lim_{t\to\infty}t^{\vartheta}\,\mathbb{P}(X<-t)=0$ if the last moment assumption in (4.50) is sharpened to $\mathbb{E}M|N|^{\vartheta}<\infty$.

Proof (of Theorem 4.25). By what has been stated before the theorem and is shown in Problem 4.42(a), (4.50) ensures that the IFS pertaining to the SFPE (4.48) is mean contractive and satisfies (3.17). Therefore the law of X, defined in (4.49), forms the unique solution to (4.48). In order to infer (4.5) for its right tails by Corollary 4.2, we must verify

$$\mathbb{E}\big|((M(N\vee X)+Q)^+)^{\vartheta}-((MX)^+)^{\vartheta}\big|<\infty$$

or, a fortiori,

$$\begin{aligned}&\mathbb{E}\big|((M(N\vee X)+Q)^+)^{\vartheta}-((M(N\vee X)^+)^{\vartheta}\big|<\infty\\ \text{and}\quad &\mathbb{E}\big|(M(N\vee X)^+)^{\vartheta}-((MX)^+)^{\vartheta}\big|<\infty.\end{aligned}$$

If $\vartheta\in(0,1]$, the desired conclusion is obtained by the usual subadditivity argument [☞ proof of Theorem 4.19], namely

$$\mathbb{E}\big|((M(N\vee X)+Q)^+)^{\vartheta}-((MX)^+)^{\vartheta}\big|\ \le\ \mathbb{E}|Q|^{\vartheta}\ <\ \infty.$$

Left with the case $\vartheta>1$, we first point out that the sharpened jump-size condition (3.20) (with $p=\vartheta$, $x_0=0$ and $d(x,y)=|x-y|$) holds, namely

$$\mathbb{E}\big(MN^++Q^+\big)^{\vartheta}<\infty.$$

This is an obvious consequence of (4.50). Therefore $\mathbb{E}|X|^p<\infty$ for any $p\in(0,\vartheta)$ by Theorem 3.29. By another use of inquality (4.42), we find that

$$\begin{aligned}&((M(N\vee X)+Q)^+)^{\vartheta}-((M(N\vee X)^+)^{\vartheta}\\ &\quad\le\ (Q^+)^{\vartheta}+\vartheta\,2^{\vartheta-1}\Big((M(N\vee X)^+)^{\vartheta-1}Q^++M(N\vee X)^+(Q^+)^{\vartheta-1}\Big)\\ &\quad\le\ (Q^+)^{\vartheta}+\vartheta\,2^{\vartheta-1}\Big((MN^+\vee X)^+)^{\vartheta-1}Q^++M(N\vee X)^+(Q^+)^{\vartheta-1}\Big)\\ &\quad\le\ (Q^+)^{\vartheta}+\vartheta\,2^{\vartheta-1}\Big(((MN^+)^{\vartheta-1}+(X^+)^{\vartheta-1})Q^++((MN^+)^{\vartheta-1}\\ &\qquad\qquad+(MX^+)^{\vartheta-1})(Q^+)^{\vartheta-1}\Big).\end{aligned}$$

But the last expression has finite expectation as one can see by using $\mathbb{E}|X|^{\vartheta-1}<\infty$, our moment assumptions (4.50) and the independence of X and (M,N,Q). We refer to the proof of Theorem 4.19 for a very similar argument spelled out in greater detail. Having thus shown

$$\mathbb{E}\Big(((M(N\vee X)+Q)^+)^{\vartheta}-((M(N\vee X)^+)^{\vartheta}\Big)^+<\infty$$

we leave it to the reader to show by similar arguments that the corresponding negative part has finite expectation, too, and that $\mathbb{E}\big|(M(N\vee X)^+)^{\vartheta}-((MX)^+)^{\vartheta}\big|<\infty$ [☞ Problem 4.43].

Turning to the equivalence assertion, recall from above that

$$X\ \ge\ Y:=\sum_{n\ge 1}\Pi_{n-1}Q_n$$

so that $\mathbb{P}(X>t)\ge\mathbb{P}(Y>t)$. But Theorem 4.19 tells us that, under the given assumptions, $t^{\vartheta}\,\mathbb{P}(Y>t)$ converges to a positive limit if $\mathbb{P}(Q=c(1-M))<1$ for all $c\in\mathbb{R}$, whence the same must then hold for $t^{\vartheta}\,\mathbb{P}(X>t)$. On the other hand, if $Q=c(1-M)$ a.s. for some c, then a simple calculation shows that

$$X \;=\; \sup_{n\ge1}\big(c(1-\Pi_n)+\Pi_nN_n\big) \;=\; c+\sup_{n\ge1}\big(\Pi_{n-1}M_n(N_n-c)\big).$$

By Theorem 4.24, $X-c$ then forms the unique solution to the SFPE (4.44) when choosing $Q=M(N-c)$ there. Consequently,

$$C_+ \;=\; \lim_{t\to\infty}t^{\vartheta}\,\mathbb{P}(X>t) \;=\; \lim_{t\to\infty}t^{\vartheta}\,\mathbb{P}(X-c>t) \;>\; 0$$

iff $\mathbb{P}(M(N-c)>0)>0$ as claimed. □

Writing (4.48) in the form $X\overset{d}{=}(MN)\vee MX+Q$, we see that an SFPE of type (4.44), which has been discussed in the previous subsection, yields as a special case when choosing $Q=0$. However, as $\mathbb{P}(MN=0)\ge\mathbb{P}(M=0)$, the statement fails to hold for those equations $X\overset{d}{=}MX\vee Q'$ with $\mathbb{P}(Q'=0)<\mathbb{P}(M=0)$ [☞ Problem 4.44 for further information].

Further specializing to the situation when $Q=0$ and $N>0$ a.s., Letac's equation (4.48) after taking logarithms turns into

$$Y\overset{d}{=}\zeta\vee Y+\xi, \tag{4.53}$$

where $Y:=\log X$, $\xi:=\log M$ and $\zeta:=\log N$. Upon choosing the usual notation for the associated IFS $(Y_n)_{n\ge0}$, say, of iid Lipschitz maps with generic copy $\Psi(x):=\zeta\vee x+\xi$, backward iterations can be shown to satisfy [☞ Problem 4.45(a)]

$$\widehat{Y}_n \;=\; \max\left\{S_n+x,\ \max_{1\le k\le n}(S_k+\zeta_k)\right\} \tag{4.54}$$

if $\widehat{Y}_0=x$ and $(S_n)_{n\ge0}$ denotes the SRW associated with the $\xi_1,\xi_2,\ldots$ Provided that ξ has negative mean and thus $(S_n)_{n\ge0}$ negative drift, we infer a.s. convergence of the $\widehat{Y}_n$ to

$$Y \;:=\; \sup_{n\ge1}(S_n+\zeta_n)$$

the law of which then constitutes the unique solution to (4.53). However, $\mathbb{P}(Y=\infty)$ may be positive. In order to rule out this possibility, it is sufficient to additionally assume $\mathbb{E}\log^+\zeta<\infty$ [☞ Problem 4.45(c)].

HELLAND & NILSEN [38] have studied a random recursive equation leading to a special case of (4.53), namely

$$Y_n \;=\; (Y_{n-1}-D_n)\vee U_n \;=\; \big(Y_{n-1}\vee(U_n+D_n)\big)-D_n,\quad n\ge1,$$

for independent sequences $(D_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ of iid random variables which are also independent of Y_0. The model had been suggested earlier by GADE [33] (with constant D_n) and HELLAND [37] in an attempt to describe the deep water exchanges in a sill fjord, i.e., an inlet containing a relatively deep basin with a shallower sill at the mouth. The water exchanges are described by the following simple mechanism: If, in year n, U_n denotes the density of coastal water adjacent to the fjord and Y_n the density of resident water in the basin, then fresh water running into the fjord causes the resident water density to decrease by an amount D_n from year $n-1$ to n. Nothing happens if this water is still heavier than the coastal water, but resident water is completely replaced with water of density U_n otherwise [☞ [33] and [38] for further information]. Obviously, the distributional limit of Y_n, if it exists, satisfies (4.53) with $\zeta := U+D$ and $\xi := -D$, where as usual (D,U) denotes a generic copy of the (D_n,U_n) independent of Y.

4.4.4 The AR(1)-model with ARCH(1) errors

We return to the nonlinear time series model first introduced in Section 1.6 and briefly discussed further in Example 3.9, namely the AR(1)-model with ARCH(1) errors

$$X_n \;=\; \alpha X_{n-1} + (\beta+\lambda X_{n-1}^2)^{1/2}\varepsilon_n, \quad n\geq 1, \tag{4.55}$$

where $\varepsilon,\varepsilon_1,\varepsilon_2,...$, called innovations, are iid symmetric random variables independent of X_0 and $(\alpha,\beta,\lambda)\in\mathbb{R}\times\mathbb{R}^2_>$. This is an IFS of iid Lipschitz maps of generic form $\Psi(x) := \alpha x+(\beta+\lambda x^2)^{1/2}\varepsilon$. As pointed out in 3.9, Ψ has Lipschitz constant $L(\Psi) = |\alpha|+\lambda^{1/2}|\varepsilon|$. The following result is therefore immediate when using Theorem 3.24.

Proposition 4.28. *The IFS $(X_n)_{n\geq 0}$ stated above is mean contractive and satisfies the jump-size condition* (3.17) *if* $\mathbb{E}\log(|\alpha|+\lambda^{1/2}|\varepsilon|)<0$. *In this case it possesses a unique stationary distribution, which is symmetric and the unique solution to the SFPE*

$$X \;\stackrel{d}{=}\; \alpha X + (\beta+\lambda X^2)^{1/2}\varepsilon, \tag{4.56}$$

where X,ε are independent.

Proof. We only mention that, if (4.56) holds, then

$$-X \;\stackrel{d}{=}\; \alpha(-X) + (\beta+\lambda X^2)^{1/2}(-\varepsilon) \;\stackrel{d}{=}\; \alpha(-X) + (\beta+\lambda(-X)^2)^{1/2}\varepsilon,$$

the second equality by the symmetry of ε. Hence the law of $-X$ also solves (4.56) implying $\mathscr{L}(X)=\mathscr{L}(-X)$ because this SFPE has only one solution. □

Remark 4.29. [☞ also [16, Remark 2]] Let us point out that, if X_n is given by (4.55), then $X_n^\star := (-1)^n X_n$ satisfies the same type of random recursive equation, viz.

$$X_n^\star \;=\; -\alpha X_{n-1}^\star + (\beta + \lambda (X_{n-1}^\star)^2)^{1/2} \varepsilon_n^\star$$

with $\varepsilon_n^\star := (-1)^n \varepsilon_n$ for $n \ge 1$. But the $\varepsilon_n^\star$ are again independent copies of ε_1, for ε_1 is symmetric. In terms of distributions, the IFS $(X_n)_{n\ge 0}$ and $(X_n^\star)_{n\ge 0}$ thus differ merely by a sign change for the parameter α, and under the assumptions of the previous result we further have that $X_n^\star$ converges in distribution to the same limit X. It is therefore no loss of generality to assume $\alpha > 0$.

The symmetry of the law of X, giving

$$\mathbb{P}(X > t) \;=\; \mathbb{P}(X < -t) \;=\; \frac{1}{2}\mathbb{P}(|X| > t) \;=\; \frac{1}{2}\mathbb{P}(X^2 > t^2)$$

for all $t \in \mathbb{R}_\ge$, allows us to subsequently focus on $Y = X^2$, which satisfies the distributional equation

$$Y \;\stackrel{d}{=}\; (\alpha^2 + \lambda \varepsilon^2) Y + 2\alpha\varepsilon X (\beta + \lambda Y)^{1/2} + \beta\varepsilon^2.$$

This is *not* a SFPE as X is not a function of Y, but when observing that $\varepsilon X \stackrel{d}{=} \eta |X|$, where η is a copy of ε independent of $|X| = Y^{1/2}$ and satisfying $\eta^2 = \varepsilon^2$, we are led to

$$\begin{aligned} Y \;&\stackrel{d}{=}\; (\alpha + \lambda^{1/2}\eta)^2 Y + 2\alpha\eta Y^{1/2}\Big((\beta + \lambda Y)^{1/2} - (\lambda Y)^{1/2}\Big) + \beta\eta^2 \\ &=\; (\alpha + \lambda^{1/2}\eta)^2 Y + \frac{2\alpha\beta\eta Y^{1/2}}{(\beta+\lambda Y)^{1/2} + (\lambda Y)^{1/2}} + \beta\eta^2 \;=:\; \Phi(Y). \end{aligned} \tag{4.57}$$

with Y and η being independent. Observe that

$$\Phi_*(y) \;:=\; My - c|\eta| + \beta\eta^2 \;\le\; \Phi(y) \;\le\; My + c|\eta| + \beta\eta^2 \;=:\; \Phi^*(y) \tag{4.58}$$

for all $y \in \mathbb{R}_\ge$, where $M := (\alpha + \lambda^{1/2}\eta)^2$ and $c := \alpha\beta\lambda^{-1/2}$. As a consequence, we obtain the following lemma which will be useful to prove our main result, Theorem 4.31 below.

Lemma 4.30. *Suppose that $\mathbb{E}\log M < 0$ and let $(\Phi_{*,n}, \Phi_n^*, \Phi_n)_{n\ge 1}$ be a sequence of iid copies of (Φ_*, Φ^*, Φ), defined on the same probability space as Y. Then the following assertions hold true:*

(a) $\Phi_{*,1:n} \le \Phi_{1:n} \le \Phi_{1:n}^*$ *for all* $n \ge 1$.
(b) $\Phi_{*,1:n}(Y) \to Y_*$ *and* $\Phi_{1:n}^*(Y) \to Y^*$ *a.s. for random variables* $Y_* \le Y^*$ *satisfying*

$$Y_* \stackrel{d}{=} \Phi_*(Y_*) \quad \textit{and} \quad Y^* \stackrel{d}{=} \Phi^*(Y^*). \tag{4.59}$$

(c) $Y_* \leq_{st} Y \leq_{st} Y^*$, *i.e.*

$$\mathbb{P}(Y_* > t) \ \leq \ \mathbb{P}(Y > t) \ \leq \ \mathbb{P}(Y^* > t)$$

for all $t \in \mathbb{R}$.

Proof. Part (a) follows directly from (4.58). Since $\mathbb{E}\log M < 0$ implies $\mathbb{E}\log^+|\eta| < \infty$, we see that the IFS generated by the $(\Phi_{*,n})_{n\geq 1}$ and $(\Phi_n^*)_{n\geq 1}$ are mean contractive and satisfying (3.17). Therefore, their backward iterations are a.s. convergent under any initial condition to limiting variables solving the SFPE's stated in (4.59). This proves (b). Finally, as Y satisfies the SFPE (4.57), we infer

$$\Phi_{1:n}(Y) \stackrel{d}{=} Y$$

and thus $\Phi_{*,1:n}(Y) \leq_{st} Y \leq_{st} \Phi_{1:n}^*(Y)$ for all $n \geq 1$. Taking the limit $n \to \infty$ yields the assertion. □

The quintessential outcome of the previous lemma is that Y can be sandwiched in the sense of stochastic majorization $(\leq_{st})$ by two perpetuities, Y_* and Y^*. The following result is now derived very easily with the help of the implicit renewal theorem.

Theorem 4.31. *Suppose that* $M = (\alpha + \lambda^{1/2}\eta)^2$ *satisfies (IRT-1)-(IRT-3) and let Y be a nonnegative solution to the SFPE* (4.57). *Then its law satisfies* (4.5) *with*

$$C_+ = \frac{\mathbb{E}\big(\Phi(Y)^\vartheta - (MY)^\vartheta\big)}{\vartheta\mu_\vartheta} \tag{4.60}$$

which is positive if $\vartheta \geq 2$.

Remark 4.32. In all previous applications, the random variable M that appeared in the respective tail result happened to be also the Lipschitz constant of the generic Lipschitz function Ψ in the SFPE under consideration, i.e. $M = L(\Psi)$. In the present situation, however, this is no longer true. We have $L(\Psi) = |\alpha| + \lambda^{1/2}|\eta|$ which, after squaring, would suggest $M' = (|\alpha| + \lambda^{1/2}|\eta|)^2$ in the above theorem. But $M' > M = (\alpha + \lambda^{1/2}\eta)^2$ a.s. even if α is positive. What this essentially tells us is that mean contraction with respect to global Lipschitz constants, albeit constituting a sufficient condition for the distributional convergence of a given IFS of iid Lipschitz maps to a unique limit law, fails to be necessary in general. For the AR(1)-model with ARCH(1) errors, BORKOVEC & KLÜPPELBERG [16, Theorem 1] show that $\mathbb{E}\log M < 0$ in combination with some additional conditions on $\mathscr{L}(\eta)$ (beyond symmetry) already ensures convergence to a unique symmetric stationary distribution. Earlier results in this direction under the second moment condition

$\alpha^2 + \lambda\,\mathbb{E}\eta^2 = \mathbb{E}M^2 < 1$ were obtained by GUÉGAN & DIEBOLT [36] and MAERCKER [50].

Proof. Note that (IRT-1) for $(\alpha + \lambda^{1/2}\eta)^2$ ensures $\mathbb{E}|\eta|^{2\vartheta} < \infty$. Under the assumptions of the theorem und with the notation of the previous lemma, the IFS generated by $(\Phi_{*,n})_{n\ge1}$ and $(\Phi^*)_{n\ge1}$ are strongly contractive of order ϑ and satisfy the sharpened jump-size condition (3.20) for $p = \vartheta$. Hence, the perpetuities Y_* and Y^*, have moments of all orders $p \in (0,\vartheta)$ [☞ Prop. 4.20]. Using Lemma 4.30(c), $\mathbb{E}|Y|^p \le \mathbb{E}|Y_*|^p + \mathbb{E}|Y^*|^p < \infty$ for all $p \in (0,\vartheta)$. Now it follows in a meanwhile routine manner that

$$\mathbb{E}\left|\Phi(Y)^\vartheta - (MY)^\vartheta\right| \;\le\; \mathbb{E}\left|c|\eta| + \beta\eta^2\right|^\vartheta \;<\; \infty$$

if $\vartheta \le 1$, and [use again (4.42) and put $c_\vartheta := \vartheta 2^{\vartheta-1}$]

$$\begin{aligned}\mathbb{E}\left|\Phi(Y)^\vartheta - (MY)^\vartheta\right| \;&\le\; \mathbb{E}\left|\Phi^*(Y)^\vartheta - (MY)^\vartheta\right|\\ \le\; &\mathbb{E}\left|c|\eta| + \beta\eta^2\right|^\vartheta\\ &+ c_\vartheta\left(\mathbb{E}Y^{\vartheta-1}\,\mathbb{E}\left|c|\eta| + \beta\eta^2\right| + \mathbb{E}Y\,\mathbb{E}\left|c|\eta| + \beta\eta^2\right|^{\vartheta-1}\right) \;<\; \infty\end{aligned}$$

if $\vartheta > 1$. Hence, by Corollary 4.2, the right tails of Y satisfy (4.5) with C_+ as stated.

Left with the proof of $C_+ > 0$ if $\vartheta \ge 2$, a Taylor expansion of $\Phi(Y)^\vartheta$ about MY yields

$$\begin{aligned}\Phi(Y)^\vartheta \;&=\; (MY)^\vartheta + \vartheta(MY)^{\vartheta-1}h(Y,\eta) + \vartheta(\vartheta-1)Z^{\vartheta-2}h(Y,\eta)^2\\ &\ge\; (MY)^\vartheta + \vartheta(MY)^{\vartheta-1}h(Y,\eta),\end{aligned}$$

where

$$h(Y,\eta) \;:=\; \frac{2\alpha\beta\eta Y^{1/2}}{(\beta+\lambda Y)^{1/2} + (\lambda Y)^{1/2}} + \beta\eta^2$$

and Z is an intermediate (random) point between MY and $\Phi(Y) = MY + h(Y,\eta)$ and thus ≥ 0. As a consequence,

$$\mathbb{E}\Big(\Phi(Y)^\vartheta - (MY)^\vartheta\Big) \;\ge\; \vartheta\,\mathbb{E}(MY)^{\vartheta-1}h(Y,\eta) \;=\; \vartheta\beta\,\mathbb{E}\eta^2 \;>\; 0,$$

having utilized $\mathbb{E}\eta = 0$ and the independence of Y and η. □

Problems

Problem 4.33. Prove the following converse of Proposition 4.20: If $\mathbb{E}Y^p < \infty$ for some $p \ge 1$ and $\mathbb{P}(|Q| > 0) > 0$, then $\mathbb{E}|M|^p < 1$ and $\mathbb{E}|Q|^p < \infty$. [Hint: Use that $\sum_{n\ge1}|\Pi_{n-1}Q_n|^p \le Y^p$.]

Problem 4.34. Complete the proof of Theorem 4.19.

Problem 4.35. Given the assumptions of Theorem 4.19, suppose additionally that $\vartheta > 1$ and $\mathbb{E}|Q|^{\kappa} < \infty$ for some $\kappa \in [1, \vartheta)$. Prove that

$$\mathbb{E}X^n \;=\; \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\big(M^k Q^{n-k}\big)\,\mathbb{E}X^k \tag{4.61}$$

for all integers $n \le \kappa$. [This was first shown by VERVAAT [59].]

Problem 4.36. Given the assumptions of Theorem 4.19 and $M \ge 0$ a.s., prove that, if $0 < \vartheta \le 1$,

$$C_+ + C_- \;\le\; \frac{1}{\vartheta\mu_\vartheta}\,\mathbb{E}|Q|^{\vartheta},$$

while, if $\vartheta > 1$,

$$\begin{aligned} C_+ + C_- \;&\le\; \frac{2^{\vartheta-1}}{\vartheta\mu_\vartheta}\,\Big(\mathbb{E}|Q|^{\vartheta} + \mathbb{E}\big(M^{\vartheta-1}|Q|\big)\,\mathbb{E}|X|^{\vartheta-1}\Big) \\ &\le\; \frac{2^{\vartheta-1}}{\vartheta\mu_\vartheta}\left(\mathbb{E}|Q|^{\vartheta} + \frac{\mathbb{E}\big(M^{\vartheta-1}|Q|\big)\,\mathbb{E}|Q|^{\vartheta-1}}{1-\|M\|_{\vartheta-1})^{\vartheta-1}}\right). \end{aligned}$$

If $\mathbb{P}(M<0) > 0$ and thus $C_+ = C_-$, the same bounds with an additional factor $1/2$ and M replaced by $|M|$ hold for C_+ and C_-.

Problem 4.37. Given the assumptions of Theorem 4.19, $M, Q \ge 0$ a.s. and $\vartheta \in \mathbb{N}$, prove that $C_- = 0$ and

$$C_+ \;=\; \frac{1}{\vartheta\mu_\vartheta}\sum_{k=0}^{\vartheta-1}\binom{\vartheta}{k}\mathbb{E}\big(M^k Q^{\vartheta-k}\big)\,\mathbb{E}X^k \tag{4.62}$$

with $\mathbb{E}X^k$ being determined by (4.61) for $k = 1, \ldots, \vartheta - 1$. Show further that

$$C_+ \;=\; \begin{cases} \dfrac{\mathbb{E}Q}{\mathbb{E}M\log M}, & \text{if } \vartheta = 1, \\[2ex] \dfrac{1}{\mu_\vartheta}\left(\dfrac{1}{2}\mathbb{E}Q^2 + \dfrac{\mathbb{E}Q\,\mathbb{E}MQ}{1-\mathbb{E}M}\right), & \text{if } \vartheta = 2. \end{cases} \tag{4.63}$$

Problem 4.38. Given the assumptions of Theorem 4.19 and $\vartheta \in 2\mathbb{N}$, prove the following assertions:

(a) If $M \ge 0$ a.s., then (4.62) holds for $C_+ + C_-$ instead of C_+.

(b) If $\mathbb{P}(M < 0) > 0$, then

$$C_+ \;=\; C_- \;=\; \frac{1}{2\vartheta\mu_\vartheta}\sum_{k=0}^{\vartheta-1}\binom{\vartheta}{k}\mathbb{E}\big(M^k Q^{\vartheta-k}\big)\,\mathbb{E}X^k$$

with $\mathbb{E}X^k$ being determined by (4.61) for $k = 1, \ldots, \vartheta - 1$.

(c) If $\vartheta = 2$, then the respective formula in (4.63) holds for $C_+ + C_-$ instead of C_+.

Problem 4.39. Prove that (M,Q) defined in Proposition 4.23 satisfies the conditions of the implicit renewal theorem 4.1 with $\vartheta = b$.

Problem 4.40. Let $(X_n)_{n\ge 0}$ be an IFS generated by iid Lipschitz maps of generic form $\Psi(x) := Mx \vee Q$ and suppose that $\mathbb{E}\log|M| < 0$ and $\mathbb{E}\log^+|Q| < \infty$. Prove that $(X_n)_{n\ge 0}$ has a unique stationary distribution π which forms the unique solution to the SFPE (4.44) and is the distribution of $\sup_{n\ge 1}\Pi_{n-1}Q_n$ (in the usual notation).

Problem 4.41. As a variation of (4.44), consider the SFPE

$$X \stackrel{d}{=} MX \curlyvee Q, \tag{4.64}$$

where as usual X and (M,Q) are independent and

$$x \curlyvee y \ := \ \begin{cases} x, & \text{if } |x| > |y|, \\ y, & \text{otherwise.} \end{cases}$$

Prove the following counterpart of Theorem 4.24:
If M satisfies (IRT-1)-(IRT-3) and $\mathbb{E}|Q|^\vartheta < \infty$, then (4.64) has a unique solution X (in terms of its law) and (4.5), (4.6) hold true. If $M \ge 0$ a.s., then

$$C_\pm = \frac{\mathbb{E}\big(((MX \curlyvee Q)^\pm)^\vartheta - ((MX)^\pm)^\vartheta\big)}{\vartheta\mu_\vartheta},$$

and if $\mathbb{P}(M < 0) > 0$, then

$$C_+ = C_- = \frac{\mathbb{E}\big((|Q|^\vartheta - |MX|^\vartheta)^+\big)}{\vartheta\mu_\vartheta}.$$

Moreover, $C_+ + C_-$ is positive iff $\mathbb{P}(Q \ne 0) > 0$.

Problem 4.42. (Letac's example E in [45]) Consider the IFS, defined by the random recursive equation

$$X_n \ = \ M_n(N_n \vee X_{n-1}) + Q_n, \quad n \ge 1,$$

for iid random triples (M_n, N_n, Q_n), $n \ge 1$, in $\mathbb{R}_\ge \times \mathbb{R}^2$ with generic copy (M,N,Q). Show that

(a) $(X_n)_{n\ge 0}$ is mean contractive if $\mathbb{E}\log M < 0$ and satisfies the jump-size condition (3.17) if, furthermore, $\mathbb{E}\log^+ N^+ < \infty$ and $\mathbb{E}\log^+|Q| < \infty$.

(b) If the previous conditions hold, the a.s. limit of the associated backward iterations $\widehat{X}_n$ is given by

$$X \ := \ \max\left\{\sum_{n\ge 1}\Pi_{n-1}Q_n, \sup_{n\ge 1}\left(\sum_{k=1}^n \Pi_{k-1}Q_k + \Pi_n N_n\right)\right\},$$

where Π_n has the usual meaning. [Hint: Use induction over n to verify that

$$\Psi_{n:1}(t) \;=\; \max\left\{\sum_{k=1}^{n}\Pi_{k-1}Q_k+\Pi_n t,\ \max_{1\le m\ge n}\left(\sum_{k=1}^{m}\Pi_{k-1}Q_k+\Pi_m N_m\right)\right\}$$

for any $\in\mathbb{R}$, where $\Psi_n(t) := M_n(N_n\vee t)+Q_n$ for $n\in\mathbb{N}$.]

Problem 4.43. Complete the proof of Theorem 4.25 by showing along similar lines as in the proof of Theorem 4.19 that, for the case $\vartheta > 1$,

$$\mathbb{E}\left(((M(N\vee X)+Q)^+)^\vartheta-((M(N\vee X)^+)^\vartheta\right)^- < \infty$$

as well as

$$\mathbb{E}\left|(M(N\vee X)^+)^\vartheta-((MX)^+)^\vartheta\right| < \infty.$$

Problem 4.44. Prove that, if M,Q are real-valued random variables such that $M\ge 0$ a.s. and $\mathbb{P}(Q=0)\ge\mathbb{P}(M=0)$, then there exist random variables M',N' (on a suitable probability space) such that $(M',M'N')$ forms a copy of (M,Q).

Problem 4.45. Consider an IFS $(Y_n)_{n\ge 0}$ of iid Lipschitz maps with generic copy $\Psi(x)=\zeta\vee x+\xi$ and let $(S_n)_{n\ge 0}$ denote the SRW with increments $\xi_1,\xi_2,\ldots$ (in the usual notation). Prove the following assertions:

(a) The backward iterations $\widehat{Y}_n$, when starting at $\widehat{Y}_0 = x$, are given by (4.54).
(b) If $\mathbb{E}\xi < 0$, then $\widehat{Y}_n \to Y$ a.s., where $Y=\sup_{n\ge 1}(S_n+\zeta_n)$.
(c) If, furthermore, $\mathbb{E}\zeta^+ < \infty$, then Y is a.s. finite.

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Appendix A
A quick look at some ergodic theory and theorems

A.1 Measure-preserving transformations and ergodicity

Given a probability space $(\mathbb{Y},\mathscr{A},\mathbf{P})$, a measurable mapping $T:\mathbb{Y}\to\mathbb{Y}$ is called *measure-preserving transformation of* $(\mathbb{Y},\mathscr{A},\mathbf{P})$ if $\mathbf{P}(T\in\cdot)=\mathbf{P}$. The iterations $(T^n(y))_{n\ge 0}$ for any initial value $y\in\mathbb{Y}$ provide an *orbit* or *trajectory* of the *dynamical system* generated by T. If y is picked according to $\mathbf{P}$ and thus formally replaced with a random element $Y_0:(\Omega,\mathfrak{A},\mathbb{P})\to(\mathbb{Y},\mathscr{A})$ having law $\mathbf{P}$, then

$$Y_n:=T^n(Y_0),\quad n\ge 0$$

forms a *stationary sequence*, the defining property being that

$$(Y_n,\ldots,Y_{n+m})\ \stackrel{d}{=}\ (Y_0,\ldots,Y_m)\quad\text{for all } m,n\in\mathbb{N}_0. \tag{A.1}$$

When studying the statistical properties of an orbit $(T^n(y))_{n\ge 0}$, for instance by looking at absolute or relative frequencies

$$N_{T,n}(y,A):=\sum_{k=0}^{n}\mathbf{1}_A(T^k(y))\quad\text{or}\quad h_{T,n}(y,A):=\frac{1}{n+1}\sum_{k=0}^{n}\mathbf{1}_A(T^k(y)),$$

respectively, for $A\in\mathscr{A}$, the notion of T*-invariance* arises quite naturally. A set $A\in\mathscr{A}$ is called T*-invariant* or *invariant under* T if $T^{-1}(A)=A$. Their collection $\mathscr{I}_T^*$ forms a σ-field, called σ*-field of* T*-invariant sets* or just *invariant* σ*-field of* T. Its completion $\mathscr{I}_T$, say, within $\mathscr{A}$ consists of all sets $A\in\mathscr{A}$ for which T-invariance holds $\mathbf{P}$-a.s., thus

$$\mathscr{I}_T=\{A\in\mathscr{A}:T^{-1}(A)=A\ \mathbf{P}\text{-a.s.}\}$$

Obviously, $y\in A$ for a [$\mathbf{P}$-a.s.] T-invariant set A entails $T^n(y)\in A$ [$\mathbf{P}$-a.s.] for all $n\in\mathbb{N}$. As a consequence , if $\mathscr{I}_T$ contains a set A having $0<\mathbf{P}(A)<1$, then the distribution of $\boldsymbol{Y}=(Y_n)_{n\ge 0}$ under $\mathbf{P}$ may be decomposed as

$$\mathbf{P}(\boldsymbol{Y}\in\cdot)\ =\ \mathbf{P}(A)\,\mathbb{P}(\boldsymbol{Y}\in\cdot|Y_0\in A)+(1-\mathbf{P}(A))\,\mathbb{P}(\boldsymbol{Y}\in\cdot|Y_0\in A^c) \tag{A.2}$$

and $\boldsymbol{Y}$ remains stationary under both, $\mathbb{P}(\cdot|Y_0 \in A)$ and $\mathbb{P}(\cdot|Y_0 \in A^c)$ [☞ Problem **??**]. If no such decomposition exists or, equivalently, $\mathscr{I}_T$ is $\mathbf{P}$-trivial, then the sequence $\boldsymbol{Y}$ as well as the associated transformation T are called *ergodic*.

In probability theory, a stationary sequence $\boldsymbol{Y} = (Y_n)_{n\ge 0}$ of $\mathbb{Y}$-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ is simply defined by property (A.1) and hence does not require the ergodic-theoretic setting of a measure-preserving transformation. On the other hand, when considering the associated coordinate model $(\mathbb{Y}^{\mathbb{N}_0}, \mathscr{A}^{\mathbb{N}_0}, \Lambda)$ with $\Lambda := \mathbb{P}(\boldsymbol{Y} \in \cdot)$ and $\boldsymbol{X} = (X_n)_{n\ge 0}$ denoting the identical mapping on this space, so that $\Lambda(\boldsymbol{X} \in \cdot) = \mathbb{P}(\boldsymbol{Y} \in \cdot)$, the stationarity of $\boldsymbol{Y}$ is equivalent to the property that the *shift* S on $\mathbb{Y}^{\mathbb{N}_0}$, defined by

$$S(y_0, y_1, ...) := (y_1, y_2, ...),$$

is measure-preserving. Therefore it appears to be natural to call $\boldsymbol{Y}$ ergodic if $\mathscr{S}$ is ergodic. Defining the invariant σ-field associated with $\boldsymbol{Y}$ by

$$\mathscr{I}_{\boldsymbol{Y}} := \Big\{ B \in \mathscr{A}^{\mathbb{N}_0} : \mathbf{1}_B(\boldsymbol{Y}) = \mathbf{1}_B(S \circ \boldsymbol{Y}) \ \mathbb{P}\text{-a.s.} \Big\},$$

we further see that $\mathscr{I}_{\boldsymbol{Y}} = \boldsymbol{Y}^{-1}(\mathscr{I}_S)$ and that ergodicity of $\boldsymbol{Y}$ holds iff $\mathscr{I}_{\boldsymbol{Y}}$ is $\mathbb{P}$-trivial.

By Kolmogorov's consistency theorem, any stationary sequence $\boldsymbol{Y} = (Y_n)_{n\ge 0}$ has a *doubly infinite extension* $\boldsymbol{Y}^* = (Y_n)_{n\in\mathbb{Z}}$ with distribution Γ^*, say, which in turn is associated with the measure-preserving shift map S^* on the doubly infinite product space $(\mathbb{Y}^{\mathbb{Z}}, \mathscr{A}^{\mathbb{Z}}, \Gamma^*)$, defined by

$$S^*(..., y_{-1} y_0, y_1, ...) := (... y_0, y_1, y_2, ...).$$

Plainly, S^* is invertible, and the inverse S^{*-1} is also measure-preserving. It should not take one by surprise that both transformations are further ergodic if this is true for S. The following lemma shows that sequences of iid random variables are ergodic.

Proposition A.1. *Any sequence $(Y_n)_{n\in\mathbb{T}}$ of iid random variables, where $\mathbb{T} = \mathbb{N}$ or $= \mathbb{Z}$, is ergodic.*

Proof. This follows from Kolmogorov's zero-one law if $\mathbb{T} = \mathbb{N}$ and extends to $\mathbb{T} = \mathbb{Z}$ by the above remark about the ergodicity of S^* and S^{*-1}. □

As an example of a non-ergodic stationary sequence one can take any stationary positive recurrent (and thus irreducible) periodic discrete Markov chain.

Let us finally introduce the ergodic theoretic notion of a *factor* which appeared in the proof of Prop. 3.18. Given two measure-preserving transformations T_1, T_2 of probability spaces $(\mathbb{Y}_1, \mathscr{A}_1, \mathbf{P}_1)$ and $(\mathbb{Y}_2, \mathscr{A}_2, \mathbf{P}_2)$, respectively, T_2 is called a *factor of* T_1 if there exists a measure-preserving map $\varphi : \mathbb{Y}_1 \to \mathbb{Y}_2$ (i.e. $\mathbf{P}_1(\varphi \in \cdot) = \mathbf{P}_2$) such that $\varphi \circ T_1 = T_2 \circ \varphi$ $\mathbf{P}_1$-a.s.

Proposition A.2. *If T_2 is a factor of an ergodic transformation T_1, then T_2 is ergodic as well.*

Proof. With the notation introduced above, choose any $A_2 \in \mathscr{I}_{T_2}$ and put $A_1 := \varphi^{-1}(A_2)$. Then $\varphi \circ T_1 = T_2 \circ \varphi$ $\mathbf{P}_1$-a.s. implies

$$T_1^{-1}(A_1) = (\varphi \circ T_1)^{-1}(A_2) = (T_2 \circ \varphi)^{-1}(A_2) = \varphi^{-1}(A_2) = A_1 \quad \mathbf{P}_1\text{-a.s.},$$

that is $A_1 \in \mathscr{I}_{T_1}$. Since T_1 is ergodic and $\mathbf{P}_1(\varphi \in \cdot) = \mathbf{P}_2$, we hence infer

$$\mathbf{P}_2(A_2) = \mathbf{P}_2(\varphi^{-1}(A_1)) = \mathbf{P}_1(A_1) \in \{0,1\},$$

which proves that T_2 is ergodic. □

A.2 Birkhoff's ergodic theorem

The following theorem due to BIRKHOFF [13] [☞ also [14] by the same author] is one of the fundamental results in ergodic theory and may also be viewed as the extension of the classical SLLN for sums of iid random variables to stationary sequences. We provide two versions of the result, the first one formulated in terms of a measure-preserving transformation as in [13], the second more probabilistic one in terms of a stationary sequence.

Theorem A.3. [Birkhoff's ergodic theorem for measure-preserving transformations] *Let T be a measure-preserving transformation of a probability space $(\mathbb{Y}, \mathscr{A}, \mathbf{P})$ and $g : \mathbb{Y} \to \mathbb{R}$ be a $\mathbf{P}$-integrable function, i.e. $g \in L^1(\mathbf{P})$. Then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} g \circ T^k = \mathbf{E}(g|\mathscr{I}_T) \quad \mathbf{P}\text{-a.s. and in } L^1(\mathbf{P}), \tag{A.3}$$

and the a.s. convergence ramins valid if g is quasi-$\mathbf{P}$-integrable. As a particular consequence,

$$\lim_{n\to\infty} h_{T,n}(\cdot, A) = \mathbf{P}(A|\mathscr{I}_T) \quad \mathbf{P}\text{-a.s.} \tag{A.4}$$

for any $A \in \mathscr{A}$.

Clearly, the conditional expectations in (A.3) and (A.4) reduce to unconditional ones if T is ergodic and thus $\mathscr{I}_T$ $\mathbf{P}$-trivial.

Theorem A.4. [Birkhoff's ergodic theorem for stationary sequences] *Let* $\boldsymbol{Y} = (Y_n)_{n\ge 0}$ *be a stationary sequence of* $\mathbb{Y}$*-valued random variables on a probability space* $(\Omega, \mathfrak{A}, \mathbb{P})$ *and* $g : \mathbb{Y} \to \mathbb{R}$ *be such that* $g \circ Y_0$ *is integrable. Then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} g \circ Y_k = \mathbb{E}(g \circ Y_0 | \mathscr{I}_{\boldsymbol{Y}}) \quad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}), \tag{A.5}$$

and the a.s. convergence remains valid if g is quasi-$\mathbf{P}$*-integrable. As a particular consequence,*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{1}_A(Y_k) = \mathbb{P}(Y_0 \in A | \mathscr{I}_{\boldsymbol{Y}}) \quad \mathbb{P}\text{-a.s.} \tag{A.6}$$

for any $A \in \mathscr{A}$.

Excellent introductions to the theory of stationary sequences from a probabilist's viewpoint, including a proof of Theorem A.4, may be found in the textbooks by BREIMAN [17] and DURRETT [23].

A.3 Kingman's subadditive ergodic theorem

A sequence of real numbers $(c_n)_{n\ge 1}$ is called *subadditive* if

$$c_{m+n} \le c_m + c_n$$

for all $m, n \in \mathbb{N}$. An old lemma by FEKETE [29] states that every such sequence converges, viz.

$$\lim_{n\to\infty} c_n \;=\; \inf_{n\ge 1} \frac{c_n}{n} \;\in\; \mathbb{R} \cup \{-\infty\}.$$

The subadditive ergodic theorem for triangular schemes $(X_{k,n})_{n\ge 1}^{0\le k<n}$ of real-valued random variables, first obtained by KINGMAN [44] and later improved by LIGGETT [46], builds upon this property together with a certain type of stationarity. Here we present the more general version by LIGGETT.

Theorem A.5. [Subadditive ergodic theorem] *Let* $(X_{k,n})_{n\ge 1}^{0\le k<n}$ *be a family of real-valued random variables which satisfies the following conditions:*

(SA-1) $X_{0,n} \le X_{0,m} + X_{m,n}$ *a.s. for all* $0 \le m < n$.
(SA-2) $(X_{nk,(n+1)k})_{n\ge 1}$ *is a stationary sequence for each* $k \ge 1$.
(SA-3) *The distribution of* $(X_{m,m+n})_{n\ge 1}$ *does not depend on* $m \ge 0$.

(SA-4) $\mathbb{E}X_{0,1}^{+} < \infty$ *and* $\mu := \inf_{n\geq 1} n^{-1}\mathbb{E}X_{0,n} > -\infty$.

Then

(a) $\lim_{n\to\infty} n^{-1}\mathbb{E}X_{0,n} = \mu$.
(b) $n^{-1}X_{0,n}$ *converges a.s. and in* L^1 *to a random variable* X *with mean* μ.
(c) *If all stationary sequences in (SA-2) are ergodic, then* $X = \mu$ *a.s.*
(d) *If* $\mu = -\infty$ *in (SA-4), then* $n^{-1}X_{0,n} \to -\infty$ *a.s.*

We note that Kingman assumed also (SA-4), but instead of (SA-1)-(SA-3) the stronger conditions

(SA-5) $X_{k,n} \leq X_{k,m} + X_{m,n}$ a.s. for all $0 \leq k < m < n$.
(SA-6) The distribution of $(X_{m+k,n+k})_{0\leq m<n}$ does not depend on $k \geq 0$.

A proof of the result may be found in the original article [46] or in the textbook by DURRETT [23, Ch. 6], the latter also containing a good collection of interesting applications including the Furstenberg-Kesten theorem for products of random matrices [☞ Theorem 3.4]. The reader is asked in Problem A.6 to deduce Birkhoff's ergodic theorem A.4 from the result.

Problems

Problem A.6. Give a proof of Birkhoff's ergodic theorem A.4 with the help of the subadditive ergodic theorem.

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