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Maximal ϕ-Inequalities for Nonnegative Submartingales

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Maximal ϕ-Inequalities for Nonnegative Submartingales

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Let $(M_n)_{n\geq 0}$ be a nonnegative submartingale and $M_n^* \stackrel{\text{def}}{=} \max_{0\leq k\leq n} M_k$, $n \geq 0$ the associated maximal sequence. For nondecreasing convex functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ (Orlicz functions) we provide various inequalities for $E\phi(M_n^*)$ in terms of $E\Phi_a(M_n)$ where, for $a \geq 0$,

$$\Phi_a(x) \stackrel{\text{def}}{=} \int_a^x \int_a^s \frac{\phi'(r)}{r}\, dr\, ds, \quad x > 0.$$

Of particular interest is the case $\phi(x) = x$ for which a variational argument leads us to

$$EM_n^* \leq \left(1 + \left(E\left(\int_1^{M_n\vee 1} \log x\, dx\right)\right)^{1/2}\right)^2.$$

A further discussion shows that the given bound is better than Doob's classical bound $\frac{e}{e-1}(1+EM_n \log^+ M_n)$ whenever $E(M_n-1)^+ \geq e-2 \approx 0.718$.

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1. Introduction

Let $(M_n)_{n\ge 0}$ be a nonnegative submartingale and $M_n^* \overset{\text{def}}{=} \max_{0\le k\le n} M_k$, $n \ge 0$ the associated maxima. Moment inequalities for the sequence $(M_n^*)_{n\ge 0}$ in terms of $(M_n)_{n\ge 0}$ are usually based on Doob's maximal inequality which states that

$$P(M_n^* > t) \;\le\; \frac{1}{t}\int_{\{M_n^*>t\}} M_n \, dP \tag{1.1}$$

for $n \ge 0$ and $t > 0$. If $p > 1$, a combination of (1.1) with Hölder's inequality shows

$$EM_n^{*p} \;\le\; \left(\frac{p}{p-1}\right)^p EM_n^p \tag{1.2}$$

for $n \ge 0$ [4, p. 255f], the constant being sharp (see [3, p. 14]). In case $p = 1$ one finds with (1.1) that

$$EM_n^* \;\le\; \frac{e}{e-1}\Big(1 + EM_n \log^+ M_n\Big) \tag{1.3}$$

for $n \ge 0$. Clearly, these results apply to $(|M_n|)_{n\ge 0}$ if $(M_n)_{n\ge 0}$ is a martingale.

Orlicz and Young functions. (1.2) and (1.3) are only special cases within a whole class of convex function inequalities based on Doob's inequality which is the main topic of this paper. Let $\mathfrak{C}$ denote the class of *Orlicz functions*, that is unbounded, nondecreasing convex functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. If the right derivative ϕ' is also unbounded then ϕ is called a *Young function* and we denote by $\mathfrak{C}'$ the subclass of such functions. Given any probability space $(\Omega, \mathfrak{A}, P)$, each $\phi \in \mathfrak{C}$ induces the semi-Banach space $(\mathfrak{L}^\phi(P), \|\cdot\|_\phi)$ of ϕ-integrable random variables X on $(\Omega, \mathfrak{A}, P)$, where

$$\|X\|_\phi \;\overset{\text{def}}{=}\; \inf\{\lambda > 0 : \; E\Phi(|X|/\lambda) \le 1\}$$

defines the underlying semi-norm. $(\mathfrak{L}^\phi(P), \|\cdot\|_\phi)$ is called an *Orlicz space* and equals the space of α-times integrable functions $(\mathfrak{L}^\alpha(P), \|\cdot\|_\alpha)$ in case $\phi(x) = x^\alpha$ for some $\alpha \in [1,\infty)$.

Since $\phi(x) = \int_0^x \phi'(s)\, ds \le x\phi'(x)$ by convexity, the numbers

$$p \;= p_\phi \;\overset{\text{def}}{=}\; \inf_{x>0} \frac{x\phi'(x)}{\phi(x)} \quad \text{and} \quad p^* \;=\; p_\phi^* \;\overset{\text{def}}{=}\; \sup_{x>0} \frac{x\phi'(x)}{\phi(x)} \tag{1.4}$$

are both in $[1,\infty]$. ϕ is called *moderate* [5, p. 162] if $p^* < \infty$ or, equivalently [7, Thm. 3.1.1], if for some (and then all) $\lambda > 1$ there exists a finite constant c_λ such that

$$\phi(\lambda x) \;\le\; c_\lambda \phi(x), \quad x \ge 0. \tag{1.5}$$

This property is shared by all $\phi \in \mathfrak{C}$ which are also regularly varying at infinity at order $\alpha \ge 1$ [2], thus including $\phi(x) = x^\alpha$ for $\alpha \in [1,\infty)$. Examples of non-moderate Orlicz functions are $\phi(x) = \exp(x^\alpha) - 1$ for any $\alpha \ge 1$.

Given a Young function ϕ, the right continuous inverse $\psi'(x) \stackrel{\text{def}}{=} \inf\{y : \phi'(y) > x\}$ of ϕ' is also unbounded and thus $\psi(x) \stackrel{\text{def}}{=} \int_0^x \psi'(s)\,ds$ again an element of $\mathfrak{C}'$, called the *conjugate Young function* to ϕ. Obviously, this conjugation is reflexive. A simple geometric argument shows [5, p. 163] that

$$x\phi'(x) \;=\; \phi(x) \;+\; \int_0^{\phi'(x)} \psi'(s)\,ds \;=\; \phi(x) \;+\; \psi(\phi'(x)), \quad x \geq 0. \tag{1.6}$$

and, by reflexivity, the same identity holds true with the roles of ϕ and ψ interchanged. With the help of (1.6), $\psi(x) \geq \frac{1}{p_\psi^*} x\psi'(x)$ and $\psi'(\phi'(x)) \geq x$ we infer as in [5, p. 169] that

$$x\phi'(x) \;=\; \phi(x) + \psi(\phi'(x)) \;\geq\; \phi(x) + \frac{1}{p_\psi^*}\phi'(x)\psi'(\phi'(x)) \;\geq\; \phi(x) + \frac{1}{p_\psi^*}x\phi'(x)$$

for $x \geq 0$ and thus $p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)} \geq \frac{p_\psi^*}{p_\psi^*-1}$. The reverse inequality can also be shown [7, Thm. 3.1.1] so that we have the identity

$$p_\phi \;=\; \frac{p_\psi^*}{p_\psi^* - 1}, \quad \text{or equivalently} \quad \frac{1}{p_\phi} + \frac{1}{p_\psi^*} \;=\; 1. \tag{1.7}$$

As further results stated in [7, Thm. 3.1.1] we mention that for any $\phi \in \mathfrak{C}$ with $p = p_\phi$ the assertions

$$\phi(\lambda x) \;\geq\; \lambda^p \phi(x) \quad \text{for all } \lambda > 1 \text{ and } x > 0; \tag{1.8}$$

$$\frac{\phi(x)}{x^p} \nearrow \tag{1.9}$$

hold true, and that for moderate ϕ with $p^* = p_\phi^*$ furthermore

$$\phi(\lambda x) \;\leq\; \lambda^{p^*} \phi(x) \quad \text{for all } \lambda > 1 \text{ and } x > 0; \tag{1.10}$$

$$\frac{\phi(x)}{x^{p^*}} \searrow . \tag{1.11}$$

The following two inequalities, the first of which may also be found in [5, p. 169], are easily deduced from another inequality stated as (2.12) in the next section. This latter inequality emerges as a special case of one of our results, see Corollary 2.2, but can also be derived by different arguments based on Doob's inequality and the Choquet representation of a function in $\mathfrak{C}$. For the interested reader this is briefly demonstrated in Appendix 1.

PROPOSITION 1.1. *Let ϕ be an Orlicz function with $p = p_\phi > 1$ and $(M_n)_{n\geq 0}$ be a nonnegative submartingale. Then*

$$\|M_n^*\|_\phi \;\leq\; \frac{p}{p-1}\|M_n\|_\phi \tag{1.12}$$

for each $n \geq 0$. If ϕ is also moderate, i.e. $p^* = p^*_\phi < \infty$, then furthermore

$$E\phi(M_n^*) \;\leq\; \left(\frac{p}{p-1}\right)^{p^*} E\phi(M_n) \tag{1.13}$$

for each $n \geq 0$.

2. Main Results

Inequalities of type (1.13) are also the content of our main results to be presented in this section. They are based upon integration of a variational variant of Doob's inequality (1.1) to be proved in Proposition 3.2, namely

$$P(M_n^* > t) \;\leq\; \frac{\lambda}{(1-\lambda)t}\int_t^\infty P(M_n/\lambda > s)\,ds \;=\; \frac{\lambda}{(1-\lambda)t}E\left(\frac{M_n}{\lambda} - t\right)^+ \tag{2.1}$$

for all $n \geq 0$, $t > 0$ and $\lambda \in (0,1)$. Under additional contraints on ϕ we will see that by good choices of λ the constant in (1.13) can be improved considerably. We will also derive an inequality for EM_n^* in terms of $EM_n \log^+ M_n$ which in many situations strictly beats (1.3).

The following two subclasses of $\mathfrak{C}$ will be of interest hereafter. We shall denote by $\mathfrak{C}^*$ the set of all differentiable $\phi \in \mathfrak{C}$ whose derivative is concave or convex, and by $\mathfrak{C}_0$ the set of $\phi \in \mathfrak{C}$ such that $\frac{\phi'(x)}{x}$ is integrable at 0 and thus in particular $\phi'(0) = 0$. Put $\mathfrak{C}_0^* \stackrel{\text{def}}{=} \mathfrak{C}_0 \cap \mathfrak{C}^*$.

Given $\phi \in \mathfrak{C}$ and $a \geq 0$, define

$$\Phi_a(x) \;\stackrel{\text{def}}{=}\; \int_a^x \int_a^s \frac{\phi'(r)}{r}\,dr\;ds, \quad x > 0 \tag{2.2}$$

and note that $\Phi_a \mathbf{1}_{[a,\infty)} \in \mathfrak{C}$ for $a > 0$. If $\phi \in \mathfrak{C}_0$ the same holds true for $\Phi \stackrel{\text{def}}{=} \Phi_0$, whereas $\Phi \equiv \infty$ otherwise. The function Φ will be of great importance in our subsequent analysis. If $\phi \in \mathfrak{C}_0$ then Φ is obviously again an element from this class. If in addition ϕ' is concave or convex the same holds true for $\Phi'(x) = \int_0^x \frac{\phi'(r)}{r}\,dr$, hence $\phi \in \mathfrak{C}_0^*$ implies $\Phi \in \mathfrak{C}_0^*$. Use $\Phi''(x) = \frac{\phi'(x)}{x}$ to see that ϕ and Φ are related through the differential equation

$$x\Phi'(x) - \Phi(x) \;=\; \phi(x), \quad x \geq 0 \tag{2.3}$$

under the initial conditions $\phi(0) = \phi'(0) = \Phi(0) = \Phi'(0) = 0$. If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = \frac{1}{p-1}x^p$, in particular $\Phi = \phi$ in case $\phi(x) = x^2$. If $\phi(x) = x$ then $\Phi \equiv \infty$, but we have $\Phi_1(x) = (x\log x - x + 1)$. Further properties of Φ and its relation to ϕ are collected in Lemma 3.1 at the beginning of Section 3 where it will be seen particularly that Φ and ϕ grow at the same order of magnitude unless ϕ or its conjugate are non-moderate.

Theorem 2.1. *Let $(M_n)_{n\geq 0}$ be a nonnegative submartingale and $\phi \in \mathfrak{C}$. Then*

$$E\phi(M_n^*) \;\leq\; \phi(a) \;+\; \frac{\lambda}{1-\lambda}\,E\Phi_a(M_n/\lambda) \tag{2.4}$$

for all $a \geq 0$, $\lambda \in (0,1)$ and $n \geq 0$, in particular ($\lambda = \frac{1}{2}$)

$$E\phi(M_n^*) \ \leq \ \phi(a) \ + \ E\Phi_a(2M_n) \tag{2.5}$$

for all $a > 0$ and $n \geq 0$. If $(M_n)_{n\geq 0}$ is a positive martingale with $M_{n+1} \leq cM_n$ for some $c > 0$ and all $n \geq 0$, and if $E\Phi_a(M_0) < \infty$, then

$$E\phi(M_n^*) \ \geq \ c^{-1}\Big(E\Phi_a(M_n) - E\Phi_a(M_0)\Big) \tag{2.6}$$

for all $n \geq 0$.

Of course, inequality (2.4) with $a = 0$ is of interest only when $\Phi_0 < \infty$, thus for $\phi \in \mathfrak{C}_0$. The conditions on $(M_n)_{n\geq 0}$ implying (2.6) were given by Gundy [6] to demonstrate that the bound in (1.3) cannot be much improved, see (2.17) below and also [8, p. 71f].

If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = \frac{1}{p-1}x^p$ implies in (2.4) with $a = 0$

$$EM_n^{*p} \ \leq \ \frac{\lambda^{1-p}}{(1-\lambda)(p-1)}EM_n^p \tag{2.7}$$

for all $n \geq 0$ and $\lambda \in (0,1)$. Elementary calculus shows that

$$\lambda^*(p) \ \stackrel{\text{def}}{=} \ \operatorname*{argmin}_{\lambda\in(0,1)} \frac{\lambda^{1-p}}{(1-\lambda)} \ = \ \frac{p-1}{p}$$

With this value of λ in (2.7) Doob's $\mathfrak{L}^p$-inequality (1.2) comes out again. For an extension of it consider nonnegative increasing functions ϕ on $[0,\infty)$ such that $\phi^{1/\gamma}$ is also convex for some $\gamma > 1$. As before let $(M_n)_{n\geq 0}$ be a nonnegative submartingale. Then the same holds true for $(\phi^{1/\gamma}(M_n))_{n\geq 0}$, the associated sequence of maxima being $(\phi^{1/\gamma}(M_n^*))_{n\geq 0}$. Consequently, (1.2) implies

$$E\phi(M_n^*) \ \leq \ \left(\frac{\gamma}{\gamma-1}\right)^{\gamma} E\phi(M_n) \tag{2.8}$$

for each $n \geq 0$. Interesting examples are $\phi(x) = x^p \log^r(1+x)$ for $p > 1$ and $r \geq 0$ (choose $\gamma = p$), as well as $\phi(x) = e^{rx}$ for $r > 0$. For the latter example any $\gamma > 0$ will do and since $(\frac{\gamma}{\gamma-1})^\gamma$ decreases to e as $\gamma \to \infty$, we obtain

$$Ee^{rM_n^*} \ \leq \ e\, Ee^{rM_n} \tag{2.9}$$

for all $n \geq 0$ and $r > 0$.

We will show in the next section that $\Phi(x) \leq \frac{1}{p-1}\phi(x)$ for each $\phi \in \mathfrak{C}$ with $p = p_\phi > 1$ ($\Rightarrow \phi \in \mathfrak{C}_0$). With this at hand the subsequent corollary follows from Theorem 2.1.

COROLLARY 2.2. *Given the situation of Theorem 2.1, let $\phi \in \mathfrak{C}$ be such that $p = p_\phi > 1$. Then*

$$E\phi(M_n^*) \ \leq \ \frac{\lambda}{(1-\lambda)(p-1)}\, E\phi(M_n/\lambda) \tag{2.10}$$

for all $\lambda \in (0,1)$ *and* $n \geq 0$*. If* ϕ *is also moderate* ($p^* < \infty$) *then*

$$E\phi(M_n^*) \;\leq\; \frac{p^*-1}{p-1}\left(\frac{p^*}{p^*-1}\right)^{p^*} E\phi(M_n) \tag{2.11}$$

for each $n \geq 0$.

A comparison of the constants going with $E\phi(M_n)$ in (2.11) and (1.13) shows that the one in (2.11) is strictly better unless $p = p^*$ (see Lemma A.2 in Appendix 2 for a rigorous proof). For large p, p^* and $\beta \stackrel{\text{def}}{=} p^*/p$ we also have that $\frac{p^*-1}{p-1}(\frac{p^*}{p^*-1})^{p^*} \approx \beta e$, while $(\frac{p}{p-1})^{p^*} \approx e^{\beta}$.

Putting $q = \frac{p}{p-1}$ and choosing $\lambda = \frac{1}{q}$ in (2.10), we obtain

$$E\phi(M_n^*) \;\leq\; E\phi(qM_n) \tag{2.12}$$

for each $n \geq 0$ and any $\phi \in \mathfrak{C}$ with $p > 1$. It is this inequality which will easily give the assertions of Proposition 1.1 (see Section 3).

By a similar argument as the one leading to (2.8), Doob's inequality (1.2) can be used to get refinements of (1.13) and (2.11) for functions $\phi \in \mathfrak{C}^*$. However, the main tool for this is not (2.1) but rather a suitable Choquet representation of ϕ exploited in a similar manner as in [1] for the special case of $\phi \in \mathfrak{C}^*$ with concave derivative.

THEOREM 2.3. *Given the situation of Theorem 2.1, let* $\phi \in \mathfrak{C}$*,* $k \geq 1$ *and* $\phi^{(k)}$ *the k-th order derivative of* ϕ*. Then* $\phi^{(k)} \in \mathfrak{C}$ ($\Rightarrow \phi \in \mathfrak{C}^*$) *implies*

$$E\phi(M_n^*) \;\leq\; \left(\frac{k+1}{k}\right)^{k+1} E\phi(M_n) \tag{2.13}$$

as well as

$$\|M_n^*\|_\phi \;\leq\; \frac{k+1}{k}\|M_n\|_\phi \tag{2.14}$$

for each $n \geq 0$.

In case $\phi(x) = x^p$ for integral $p > 1$ we have $\phi^{(p-1)} \in \mathfrak{C}$ and thus that (2.13) coincides with (1.2).

We finally turn to the case $\phi(x) = x$ for which $\Phi \equiv \infty$ but $\Phi_1(x) = x\log x - x + 1$ as mentioned earlier. A more general version of inequality (2.4) proved in Section 3 (Proposition 3.2) will be used to derive the following alternative to Doob's $\mathfrak{L}_1$-inequality (1.3).

THEOREM 2.4. *If* $(M_n)_{n\geq 0}$ *is a nonnegative submartingale, then*

$$EM_n^* \;\leq\; b + \frac{b}{b-1}E\left(\int_1^{M_n\vee 1} \log x \; dx\right) \tag{2.15}$$

for all $b > 1$ and $n \geq 1$. The value of b which minimizes the right hand side equals $b^* \stackrel{\text{def}}{=} 1 + \big(E(\int_1^{M_n \vee 1} \log x \, dx)\big)^{1/2}$ and gives

$$EM_n^* \;\leq\; \left(1 + \left(E\left(\int_1^{M_n \vee 1} \log x \, dx\right)\right)^{1/2}\right)^2 \tag{2.16}$$

If $(M_n)_{n\geq 0}$ is a positive martingale with $M_{n+1} \leq cM_n$ for some $c > 0$ and all $n \geq 0$, and if $EM_0 \log^+ M_0 < \infty$, then

$$EM_n^* \;\geq\; \frac{1}{c}\Big(EM_n \log^+ M_n - EM_0 \log^+ M_0\Big) \tag{2.17}$$

for all $n \geq 0$.

Since $\int_1^x \log y \, dy = x \log^+ x - (x-1)$ for $x \geq 1$, inequality (2.15) may be restated as

$$EM_n^* \;\leq\; b + \frac{b}{b-1}\Big(EM_n \log^+ M_n - E(M_n - 1)^+\Big) \tag{2.18}$$

for all $n \geq 1$ and $b > 1$. For the special choices $b = E(M_n - 1)^+ + 1$ and $b = e$, this yields for each $n \geq 1$

$$EM_n^* \;\leq\; \frac{1 + E(M_n - 1)^+}{E(M_n - 1)^+} EM_n \log^+ M_n \tag{2.19}$$

and

$$EM_n^* \;\leq\; e + \frac{e}{e-1}\Big(EM_n \log^+ M_n - E(M_n - 1)^+\Big), \tag{2.20}$$

respectively, where the right hand side of (2.19) is to be interpreted as 1 if $M_n \leq 1$ a.s. Note that even the choice $b = e$, though only suboptimal, leads to a better bound for EM_n^* than in (1.3) whenever $E(M_n - 1)^+ \geq e - 2 \approx 0.718$.

The previous upper bounds for EM_n may be further improved if $m \stackrel{\text{def}}{=} EM_0 > 1$. To see this note that $(\hat{M}_n)_{n\geq 0} \stackrel{\text{def}}{=} (1, \frac{M_0}{m}, \frac{M_1}{m}, ...)$ forms again a nonnegative submartingale whose maxima $\hat{M}_n^*$ satisfy

$$\hat{M}_n^* \;=\; \frac{M_{n-1}^*}{m} \vee 1$$

for $n \geq 1$. Consequently, (2.15) and (2.18) for $(\hat{M}_n)_{n\geq 0}$ give

Corollary 2.5. *If $(M_n)_{n\geq 0}$ is a nonnegative submartingale with $m = EM_0 > 1$, then*

$$\begin{aligned} EM_n^* \;\leq\; E\hat{M}_{n+1}^* \;&\leq\; b + \frac{b}{b-1} E\left(\int_1^{(M_n/m)\vee 1} \log x \, dx\right) \\ &\leq\; b + \frac{b}{b-1}\left(E\left(\frac{M_n}{m} \log^+\left(\frac{M_n}{m}\right)\right) - E\left(\frac{M_n}{m} - 1\right)^+\right) \end{aligned} \tag{2.21}$$

for all $b > 1$ and $n \geq 1$, in particular

$$EM_n^* \;\leq\; \left(1 + \left(E\left(\int_1^{(M_n/m)\vee 1} \log x \, dx\right)\right)^{1/2}\right)^2 \tag{2.22}$$

when choosing the minimizing b^* (compare (2.16)).

3. Proofs

We begin with a collection of some useful properties of the function $\Phi = \Phi_0$ in (2.2) associated with an element ϕ of $\mathfrak{C}_0$.

Lemma 3.1. *For each $\phi \in \mathfrak{C}_0$ with $p = p_\phi > 1$ the function Φ satisfies*

$$\Phi(x) \;\le\; \frac{1}{p-1}\phi(x), \quad x \ge 0. \tag{3.1}$$

If ϕ is moderate, i.e. $p^* = p^*_\phi < \infty$, then

$$\Phi(x) \;\ge\; \frac{1}{p^*-1}\phi(x), \quad x \ge 0. \tag{3.2}$$

Finally, for each $\phi \in \mathfrak{C}_0$ the inequality

$$\liminf_{x\to\infty} \frac{\Phi(x)}{x\log x} \;>\; 0 \tag{3.3}$$

holds true.

The final inequality shows that $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ whenever $\phi \in \mathfrak{C}_0$ is a function "close to the identity" in the sense that $\phi(x) = o(x\log x)$ as $x \to \infty$. Another class of examples comprises $\phi(x) = rx\log^{r-1}(1+x) - \log^r(1+x)$ for $r \ge 1$ in which case $\Phi(x) = (1+x)\log^r(1+x)$ (use the differential equation (2.3)).

Proof. Using $x\phi'(x) \ge p\phi(x)$, we obtain by partial integration

$$\int_0^s \frac{\phi'(r)}{r}\,dr \;=\; \frac{\phi(s)}{s} \;+\; \int_0^s \frac{\phi(r)}{r^2}\,dr \;\le\; \frac{\phi(s)}{s} \;+\; \frac{1}{p}\int_0^s \frac{\phi'(r)}{r}\,dr$$

and thus $\int_0^s \frac{\phi'(r)}{r}\,dr \le \frac{p}{p-1}\frac{\phi(s)}{s} \le \frac{1}{p-1}\phi'(s)$ for $s \ge 0$. An integration of this inequality with respect to s obviously implies (3.1). The second inequality (3.2) is obtained analogously when utilizing that $p^* < \infty$ implies $x\phi'(x) \le p^*\phi(x)$ for all $x \ge 0$. As to (3.3), choose $a > 0$ such that $\phi'(a) > 0$. Then

$$\Phi(x) \;\ge\; \Phi_a(x) \;=\; \int_a^x\int_a^s \frac{\phi'(1)}{r}\,dr\,ds \;=\; \phi'(a)\Big(x\log(x/a) - x + a\Big)$$

for all $x \ge a$. ◊

The proofs of Theorem 2.1 and 2.4 are based on the following more general proposition.

PROPOSITION 3.2. *Let $(M_n)_{n\ge 0}$ be a nonnegative submartingale and $\phi \in \mathfrak{C}$. Then inequality (2.1) holds true and furthermore*

$$E\phi(M_n^*) \;\le\; \phi(b) + \frac{\lambda}{1-\lambda}\int_{\{M_n/\lambda>b\}}\left(\Phi_a\left(\frac{M_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{M_n}{\lambda}-b\right)\right)dP \qquad (3.4)$$

for all $n \ge 0$, $a, b > 0$ and $\lambda \in (0,1)$. If $\frac{\phi'(x)}{x}$ is integrable at 0, i.e. $\phi \in \mathfrak{C}_0$, then (3.4) remains valid for $b = 0$.

PROOF. Doob's maximal inequality gives

$$\begin{aligned} P(M_n^* > t) \;&\le\; \frac{1}{t}\int_{\{M_n^*>t\}} M_n\; dP \\ &=\; \frac{1}{t}\int_0^\infty P(M_n^* > t, M_n > s)\; ds \\ &=\; \frac{1}{t}\int_0^{\lambda t} P(M_n^* > t)\; ds \;+\; \frac{1}{t}\int_{\lambda t}^\infty P(M_n > s)\; ds \\ &\le\; \lambda P(M_n^* > t) \;+\; \frac{\lambda}{t}\int_t^\infty P(M_n/\lambda > s)\; ds \end{aligned} \qquad (3.5)$$

and thus

$$P(M_n^* > t) \;\le\; \frac{\lambda}{(1-\lambda)t}\int_t^\infty P(M_n/\lambda > s)\; ds$$

for all $n \ge 0$, $t > 0$ and $\lambda \in (0,1)$, which is (2.1). With this result we further infer

$$\begin{aligned} E\phi(M_n^*) \;&\le\; \phi(b) + \int_b^\infty \phi'(t)\, P(M_n^* > t)\; dt \\ &\le\; \phi(b) + \frac{\lambda}{1-\lambda}\int_b^\infty \frac{\phi'(t)}{t}\int_t^\infty P(M_n/\lambda > s)\; ds\; dt \\ &=\; \phi(b) + \frac{\lambda}{1-\lambda}\int_b^\infty \left(\int_b^s \frac{\phi'(t)}{t}\; dt\right) P(M_n/\lambda > s)\; ds \\ &=\; \phi(b) + \frac{\lambda}{1-\lambda}\int_b^\infty \left(\Phi_a'(s) - \Phi_a'(b)\right) P(M_n/\lambda > s)\; ds \\ &=\; \phi(b) + \frac{\lambda}{1-\lambda}\int_{\{M_n/\lambda>b\}}\left(\Phi_a\left(\frac{M_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{M_n}{\lambda}-b\right)\right) dP \end{aligned}$$

for all $n \ge 0$, $b, t > 0$, $\lambda \in (0,1)$ and $a > 0$, where $a = 0$ may also be chosen if $\frac{\phi'(x)}{x}$ is integrable at 0. ♢

PROOF OF THEOREM 2.1. Inequality (2.4) follows directly from Proposition 3.2 if we choose $b = a$ and recall that $\Phi_a(a) = \Phi_a'(a) = 0$. Hence we are left with the proof of (2.6). If $(M_n)_{n\ge 0}$ is a positive martingale satisfying $M_{n+1} \le cM_n$ for all $n \ge 0$ and some $c > 0$, then

$$P(M_n^* > t) \;\ge\; \frac{1}{ct}\int_{\{M_n^*>t\}} M_n\; dP \;-\; \frac{1}{ct}\int_{\{M_0>t\}} M_0\; dP$$

for all $n \geq 0$ and $t > 0$, see [8, p. 72]. Consequently,

$$
\begin{aligned}
P(M_n^* > t) \;&\geq\; \frac{1}{ct}\int_{\{M_n>t\}} M_n\; dP \;-\; \frac{1}{ct}\int_{\{M_0>t\}} M_0\; dP \\
&=\; \frac{1}{ct}\int_t^\infty \Big(P(M_n > s) - P(M_0 > s)\Big)\; ds \\
&\quad +\; \frac{1}{c}\Big(P(M_n > t) - P(M_0 > t)\Big) \qquad (3.6)
\end{aligned}
$$

for all $n \geq 0$ and $t > 0$. Assuming $E\Phi_a(M_0) < \infty$, and also $E\phi(M_n^*) < \infty$ (there is nothing to prove otherwise), (2.6) now follows upon integration over $(0, \infty)$ with respect to t of both sides of (3.6) multiplied with $\phi'(t)$. We must only note that

$$
\int_0^\infty \phi'(t)\Big(P(M_n > t) - P(M_0 > t)\Big)\; dt \;=\; E\phi(M_n) - E\phi(M_0) \;\geq\; 0
$$

because ϕ is convex and $E\phi(M_k) \leq E\phi(M_n) < \infty$ for $0 \leq k \leq n$. ♢

We continue with a short proof of Proposition 1.1 stated in the Introduction.

Proof of Proposition 1.1. Let $\phi \in \mathfrak{C}$ with $p = p_\phi > 1$, put $q \stackrel{\text{def}}{=} \frac{p}{p-1}$ and recall from (2.12) that $E\phi(M_n^*) \leq E\phi(qM_n)$ for each $n \geq 0$. Setting $\gamma_n \stackrel{\text{def}}{=} \|M_n\|_\phi$, this inequality implies

$$
E\phi(M_n^*/q\lambda_n) \;\leq\; E\phi(M_n/\lambda_n) \;=\; 1
$$

as well as (use also (1.10))

$$
E\phi(M_n^*) \;\leq\; E\phi(qM_n) \;\leq\; q^{p^*} E\phi(M_n).
$$

The second inequality proves (1.13) while the first one yields $\|M_n^*\|_\phi \leq q\lambda_n$ and thus assertion (1.12). ♢

Proof of Theorem 2.3. The basic tool for the proof of Theorem 2.3 is to convert the assumption of smooth convexity ($\phi^{(k)} \in \mathfrak{C}$) into a suitable Choquet decomposition of ϕ. Each $\phi \in \mathfrak{C}$ can be written as

$$
\phi(x) \;=\; \int_{[0,\infty)} (x-t)^+\; Q_\phi(dt)
$$

where $Q_\phi(dt) = \phi'(0)\delta_0 + \phi'(dt)$. Hence, if $\phi' \in \mathfrak{C}$, then

$$
\begin{aligned}
\phi(x) \;&=\; \int_0^x \phi'(y)\; dy \\
&=\; \int_0^x \int_{[0,\infty)} (y-t)^+\; Q_{\phi'}(dt)\; dy \\
&=\; \int_{[0,\infty)} \int_0^x (y-t)^+\; dy\; Q_{\phi'}(dt) \\
&=\; \int_{[0,\infty)} \frac{((x-t)^+)^2}{2}\; Q_{\phi'}(dt).
\end{aligned}
$$

An inductive argument now gives that

$$\phi(x) \;=\; \int_{[0,\infty)} \frac{((x-t)^+)^{k+1}}{(k+1)!}\; Q_{\phi^{(k)}}(dt) \tag{3.7}$$

for each $\phi \in \mathfrak{C}$ with $\phi^{(k)} \in \mathfrak{C}$. Put $\varphi_{k,t}(x) \stackrel{\text{def}}{=} \frac{((x-t)^+)^{k+1}}{(k+1)!}$ for $k \ge 1$ and $t \in [0,\infty)$. Note that $\varphi_{k,t}^{1/(k+1)}$ is convex for each t. Thus we infer with the help of (3.7) and the argument which proved (2.8) that

$$\begin{aligned} E\phi(M_n^*) \;&=\; \int_{[0,\infty)} E\varphi_{k,t}(M_n^*)\; Q_{\phi^{(k)}}(dt) \\ &\le\; \left(\frac{k+1}{k}\right)^{k+1} \int_{[0,\infty)} E\varphi_{k,t}(M_n)\; Q_{\phi^{(k)}}(dt) \\ &=\; \left(\frac{k+1}{k}\right)^{k+1} E\phi(M_n) \end{aligned}$$

for each $n \ge 0$. Replacing M_n^* with M_n^*/γ_n in the previous estimation, where $\gamma_n \stackrel{\text{def}}{=} \frac{k+1}{k}\|M_n\|_\phi$, further gives (2.14) by a similar argument as in the proof of Proposition 1.1. ◊

Proof of Theorem 2.4. If $\phi(x) = x$, then $\Phi_1(x) = x\log x - x + 1$ and $\Phi_1'(x) = \log x$ for $x > 0$. Inequality (3.4) with these functions reads

$$\begin{aligned} EM_n^* \;&\le\; b + \frac{\lambda}{1-\lambda} \int_{\{M_n > \lambda b\}} \left(\frac{M_n}{\lambda} \log\left(\frac{M_n}{\lambda}\right) - \frac{M_n}{\lambda} + b - \log b\,\frac{M_n}{\lambda} \right) dP \\ &=\; b + \frac{1}{1-\lambda} \int_{\{M_n > \lambda b\}} \Big(M_n \log M_n - M_n\big(\log\lambda + \log b + 1\big) + \lambda b \Big)\, dP \end{aligned}$$

for all $b > 0$ and $\lambda \in (0,1)$. Choose $b > 1$ and $\lambda = \frac{1}{b}$ to obtain (2.15) in its equivalent form (2.18). (2.16) follows by elementary calculus. Finally, if $(M_n)_{n\ge 0}$ is a positive martingale with $M_{n+1} \le cM_n$ for some $c > 0$ and all $n \ge 0$, and if $EM_0 \log^+ M_0 < \infty$, then a use of the first inequality in (3.6) gives after partial integration

$$\begin{aligned} EM_n^* \;&\ge\; \int_1^\infty \frac{1}{ct}\left(\int_{\{M_n > t\}} M_n\; dP \;-\; \int_{\{M_0 > t\}} M_0\; dP \right) dt \\ &=\; \frac{1}{c} E\left(M_n \int_1^{M_n \vee 1} \frac{1}{t}\; dt \;-\; M_0 \int_1^{M_0 \vee 1} \frac{1}{t}\; dt \right) \\ &=\; \frac{1}{c}\Big(EM_n \log^+ M_n - EM_0 \log^+ M_0 \Big) \end{aligned}$$

for all $n \ge 1$ and hence the asserted inequality (2.17). ◊

Appendix 1

Inequality (2.12) from which Proposition 1.1 was deduced may be viewed as a specialization of inequality (A.2) below to the pair $(X, Y) = (M_n, M_n^*)$. The purpose of this short appendix is to provide a proof of (A.2) based upon a Choquet representation of the involved convex function ϕ.

LEMMA A.1. *Let X, Y be nonnegative random variables satisfying Doob's inequality*

$$t\,P(Y \geq t) \;\leq\; E\mathbf{1}_{\{Y\geq t\}}X \tag{A.1}$$

for all $t \geq 0$. Then

$$E\phi(Y) \;\leq\; E\phi(qX) \tag{A.2}$$

holds for each Orlicz function ϕ, where $q \overset{\text{def}}{=} q_\phi = \frac{p_\phi}{p_\phi - 1}$.

PROOF. If $p = p_\phi = 1$, thus $q = \infty$, there is nothing to prove. So let $p > 1$ and put $V = qX$. Write ϕ in Choquet decomposition, that is

$$\phi(x) \;=\; \int_{[0,\infty)} (x-t)^+ \;\phi'(dt), \quad x \geq 0.$$

Then we obtain

$$\begin{aligned}
E\phi(V) - E\phi(Y) \;&=\; E\Big(\int_{[0,\infty)} (V-t)^+ - (Y-t)^+ \;\phi'(dt)\Big) \\
&=\; E\Big(\int_{[0,V]} (V-t)\;\phi'(dt) \;-\; \int_{[0,Y]} (Y-t)\;\phi'(dt)\Big) \\
&=\; E\Big(\int_{[0,V]} (V-t)\;\phi'(dt) \;-\; \int_{[0,Y]} (V-t)\;\phi'(dt)\Big) \\
&+\; E\Big(\int_{[0,Y]} (V-t)\;\phi'(dt) \;-\; \int_{[0,Y]} (Y-t)\;\phi'(dt)\Big) \\
&=\; E\Big(\mathbf{1}_{\{V\geq Y\}}\int_{(Y,V]} (V-t)\;\phi'(dt) \;+\; \mathbf{1}_{\{V<Y\}}\int_{(V,Y]} (t-V)\;\phi'(dt)\Big) \\
&+\; E\Big(\int_{[0,Y]} (V-Y)\;\phi'(dt)\Big) \\
&=\; I_1 + I_2.
\end{aligned}$$

It is easily seen that $I_1 \geq 0$. So it remains to prove that $I_2 = EV\phi'(V) - EY\phi'(Y) \geq 0$. To this end write

$$\begin{aligned}
EY\phi'(Y) \;\geq\; p\,E\phi(Y) \;&=\; p\,E\Big(\int_{[0,\infty)} (Y-t)^+ \;\phi'(dt)\Big) \\
&=\; p\,EY\phi'(Y) \;-\; p\,E\Big(\int_{\{Y\geq t\}} t\;\phi'(dt)\Big)
\end{aligned}$$

which, after rearranging terms and using (A.1), leads to

$$EY\phi'(Y) \;\leq\; q\,E\Big(\int_{\{Y\geq t\}} t\;\phi'(dt)\Big) \;\leq\; q\,E\Big(\int_{\{Y\geq t\}} X\;\phi'(dt)\Big) \;=\; EV\phi'(Y)$$

and thus the desired conclusion. $\diamondsuit$

Appendix 2

We claimed in Section 2 that

$$\frac{y-1}{x-1}\left(\frac{y}{y-1}\right)^{y} \;\le\; \left(\frac{x}{x-1}\right)^{y} \tag{A.3}$$

holds true for all $y \ge x > 1$ and that the inequality is strict unless $x = y$. (A.3) may be rewritten as

$$\left(\frac{x-1}{y-1}\right)^{y-1} \;\le\; \left(\frac{x}{y}\right)^{y}.$$

Taking logarithms we arrive at

$$(y-1)\Big(\log(x-1)-\log(y-1)\Big) \;\le\; y\Big(\log x-\log y\Big).$$

The desired conclusion is now an immediate consequence of the following lemma.

Lemma A.2. *For each $y > 1$, the function $f_y : (1, y] \to I\!R$,*

$$f_y(x) \;\stackrel{\text{def}}{=}\; (y-1)\Big(\log(x-1)-\log(y-1)\Big) - y\Big(\log x-\log y\Big)$$

is strictly increasing with $f_y(y) = 0$.

Proof. It suffices to note that

$$f_y'(x) \;=\; \frac{y-1}{x-1} - \frac{y}{x} \;>\; 0$$

for all $x \in (1, y)$. ◊

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