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# Fluctuation Theory of Markov Random Walks and Markov-Modulated Random Difference Equations

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# Fluctuation Theory of Markov Random Walks and Markov-Modulated Random Difference Equations

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# Summary

The thesis at hand is concerned with the study of some Markov-modulated processes, where the underlying Markov chain $(M_n)_{n\geq 0}$ is aperiodic, positive recurrent and has a countable state space $\mathscr{S}$.

Chapter A is devoted to an analysis of Markov random walks (MRWs). A MRW is a process $(S_n)_{n\geq 0}$, $S_n := \sum_{k=1}^n X_k$ for $n \geq 1$ and $S_0 := 0$, where, conditioned on $(M_n)_{n\geq 0}$, the increments $(X_n)_{n\geq 1}$ are independent and $X_m$ only depends on $M_{m-1}, M_m$ for all $m \geq 1$, i.e. $(X_n)_{n\geq 1}$ is Markov-modulated. In the analysis, we dispense with moment assumptions on the increments and focus on extending results from fluctuation theory of ordinary random walks. After verifying a trichotomy for the almost sure asymptotic behaviour of a so-called *non-trivial* MRW, emphasis will be placed on characterising MRWs diverging to $\infty$ almost surely. In particular, equivalent conditions for the existence of power moments of $|\min_{n\geq 0} S_n|$, $\sum_{n\geq 1} \mathbf{1}_{\{S_n \leq 0\}}$ and the last exit time $\sup\{n \geq 0 : S_n \leq 0\}$ will be established, whereas difficulties for finding an equivalent criterion for the existence of power moments of the first passage time will be illustrated. Finally, the well-known arcsine law will be extended to non-trivial MRWs.

Chapter B examines iterations of Markov-modulated random affine functions $\Psi_n(x) := A_n x + B_n$, $x \in \mathbb{R}$, $n \geq 1$, i.e. $(A_n, B_n)_{n\geq 1}$ is Markov-modulated. Let $Z_0$ be a random variable independent of all other occurring random variables given $M_0$ and $\Psi_{k:n} := \Psi_k \circ \ldots \circ \Psi_n$ for all $k, n \geq 1$. On the one hand, we study distributional convergence of the forward iterations $(\Psi_{n:1}(Z_0))_{n\geq 1}$ and relate possible limit distributions to solutions to a stochastic fixed point equation (SFPE). On the other hand, we characterise distributional convergence of the backward iterations $(\Psi_{1:n}(Z_0))_{n\geq 1}$, which does not reduce to the study of distributional convergence of the forward iterations as in the ordinary setup, where $(A_n, B_n)_{n\geq 1}$ are independent and identically distributed (i.i.d.). Moreover, necessary and sufficient conditions will be obtained for $\widehat{Z}_\infty := \sum_{n\geq 1} (\prod_{k=1}^{n-1} A_k)\, B_n$, called *perpetuity*, to exist as the almost sure limit of $(\Psi_{1:n}(0))_{n\geq 1}$ and for its distribution to be a solution to some SFPE.

In both chapters the regenerative structure included in Markov-modulation enables to apply the classical results, i.e. on ordinary random walks and on iterations of i.i.d. affine functions respectively, on some subsequences of the corresponding process to obtain first insights. Nevertheless, great differences appear between the behaviour of this subsequences and the one of the entire process, which arise when $\mathscr{S}$ has countably infinitely many states. Consequently, additional theory will be developed to accomplish the proofs.

# Contents

# A. On Fluctuation Theory of Markov Random Walks

## 1. An Overview of Fluctuation Theory of Random Walks

Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of random variables. Without further mentioning, all random variables in this work are real-valued. Set $S_n := \sum_{k=1}^{n} X_k$ for $n \geq 1$ and $S_0 := 0$. $(S_n)_{n\geq 0}$ is called *random walk*.

A random walk is one of the basic objects in probability theory and is already encountered when dealing with the most known results of probability theory, namely the law of large numbers and the central limit theorem. Naturally, an additive process with increments that are of some regenerative structure arises in many applications. The simplest way of modulation is to assume i.i.d. increments, which leads to a random walk. There are several examples, well-known is queuing theory, where the arrival process, i.e. the sequence of arrivals of customers to the queue, is represented by a random walk with positive increments in the simplest models (e.g., see [7, Chapter III]). Moreover, random walks appear in the analysis of random difference equations (see Section B.1).

Although random walk theory is a relatively old topic (e.g., see [39], dating from 1921), associated research is still ongoing. Results on aspects of the fluctuation behaviour of a random walk are subsumed under the term *fluctuation theory*. This section is devoted to introducing the reader to the topic of fluctuation theory and to state the main results. Simultaneaously, the latter form the set of theorems, which are at least partially generalised in the remaining part of Chapter A.

An introduction to fluctuation theory of random walks and proofs of the basic results can be found in several books (e.g., [20, Chapter XII] and [12, Chapter 8]).

### 1.1. The Fluctuation Type of Random Walks and Finiteness of Fluctuation-Theoretic Quantities

A random walk with increments not degenerate in 0 is called *non-trivial*. It is well-known that such a random walk exhibits one of the following fluctuation types:

$$\text{Positive divergence:} \qquad \lim_{n\to\infty} S_n = \infty \quad \text{a.s.}$$

$$\text{Negative divergence:} \qquad \lim_{n\to\infty} S_n = -\infty \quad \text{a.s.}$$

$$\text{Oscillation:} \qquad \liminf_{n\to\infty} S_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} S_n = \infty \quad \text{a.s.}$$

For the reader's convenience, we dispense occasionally with subindices in the sense that we write $X$ instead of $X_1$. Similarly, we proceed with all other occurring random variables in this thesis. If $\mathbb{E}X$ is well-defined, i.e. $\mathbb{E}X^+ \wedge \mathbb{E}X^- < \infty$ and $\mathbb{E}X := \mathbb{E}X^+ - \mathbb{E}X^-$, the fluctuation type of a non-trivial random walk can be characterised by the mean in the following form:

$$\begin{aligned} \text{Positive divergence} \quad &\Leftrightarrow \quad \mathbb{E}X > 0. \\ \text{Negative divergence} \quad &\Leftrightarrow \quad \mathbb{E}X < 0. \\ \text{Oscillation} \quad &\Leftrightarrow \quad \mathbb{E}X = 0. \end{aligned}$$

Kesten [30] established the following trichotomy for the case $\mathbb{E}X^+ = \mathbb{E}X^- = \infty$:

**Theorem 1.1 ([30], Corollary 3)** *Let $(S_n)_{n\ge 0}$ be a non-trivial random walk with $\mathbb{E}|X| = \infty$. Then, exactly one of the following cases prevails:*

*(i)* $\lim_{n\to\infty} n^{-1} S_n = \infty$ *a.s. and* $(S_n)_{n\ge 0}$ *is positive divergent.*

*(ii)* $\lim_{n\to\infty} n^{-1} S_n = -\infty$ *a.s. and* $(S_n)_{n\ge 0}$ *is negative divergent.*

*(iii)* $\liminf_{n\to\infty} n^{-1} S_n = -\infty$ *a.s.,* $\limsup_{n\to\infty} n^{-1} S_n = \infty$ *a.s. and* $(S_n)_{n\ge 0}$ *oscillates.*

Further characterisations of the fluctuation type will be stated only for positive divergent random walks. These can easily be translated into a negative divergent random walk $(S_n)_{n\ge 0}$, since $(-S_n)_{n\ge 0}$ is positive divergent. Hence, a result on oscillating random walks can be concluded by contraposition.

Let $x \in \mathbb{R}_\ge$. In the study of the fluctuation behaviour of random walks, interesting quantities are the level $x$ first passage times

$$\sigma^>(x) \ := \ \inf\{n \ge 1 : S_n > x\}, \qquad \sigma^\leqslant(-x) \ := \ \inf\{n \ge 1 : S_n \le -x\},$$

the last level $x$ exit time

$$\rho(x) \ := \ \sup\{n \ge 0 : S_n \le x\},$$

the hitting time of the minimum

$$\sigma_{\min} \ := \ \inf\left\{n \ge 1 : S_n = \min_{k\ge 1} S_k\right\}$$

and the renewal counting measure

$$\Lambda(x) \ := \ \sum_{n\ge 1} \mathbf{1}_{\{S_n \le x\}}.$$

Set $\sigma^> := \sigma^>(0)$ and $\sigma^\leqslant := \sigma^\leqslant(0)$. That one can draw a conclusion on the fluctuation type from information on these quantities is part of the next theorem. For $y \in \mathbb{R}_\ge$, define

$$A(y) \ := \ \mathbb{E}(X^+ \wedge y) - \mathbb{E}(X^- \wedge y)$$

and

$$J(y) \ := \ \begin{cases} \frac{y}{\mathbb{E}(X^+ \wedge y)}, & \text{if } \mathbb{P}(X^+ = 0) < 1, \\ y, & \text{if } \mathbb{P}(X^+ = 0) = 1, \end{cases}$$

where $0/\mathbb{E}(X^+ \wedge 0) := 1$. The case $\mathbb{P}(X^+ = 0) = 1$ is actually irrelevant for this work, but $J(y)$ is chosen in correspondence to [19].

**Theorem 1.2** *Let $(S_n)_{n\geq 0}$ be a non-trivial random walk. The following conditions are equivalent:*

*(i)* $(S_n)_{n\geq 0}$ *is positive divergent.*

*(ii)* $A(y) > 0$ *for all sufficiently large* $y$ *and* $\mathbb{E}\,J(X^-) < \infty$.

*(iii)* $\sum_{n\geq 1} n^{-1}\,\mathbb{P}(S_n \leq x) < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(iv)* $\mathbb{E}\,\sigma^{>}(x) < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

Additionally, further equivalences to positive divergence are $\mathbb{P}(\sigma^{\leqslant}(-x) = \infty) > 0$, $\mathbb{P}(\rho(x) < \infty) = 1$, $\mathbb{P}(\sigma_{\min} < \infty) = 1$, $\mathbb{P}(\Lambda(x) < \infty) = 1$ and $\mathbb{P}(|\min_{n\geq 0} S_n| < \infty) = 1$ for some (hence all) $x \in \mathbb{R}_{\geq}$, which follow directly from the definition of positive divergence and the fluctuation type trichotomy.

Positive divergence is equivalent to $\mathbb{E}\,J(X^-) < \infty$ if $\mathbb{E}|X| = \infty$, which is due to Erickson [19, Corollary 1]. In fact, condition (ii) reduces to $\mathbb{E}\,J(X^-) < \infty$ in this case, because $A(y) > 0$ for all sufficiently large $y$ is a consequence of positive divergence (see [33, Lemma 3.2]). If $\mathbb{E}|X| < \infty$, (ii) is equivalent to $0 < \mathbb{E}X = \lim_{y\to\infty} A(y)$, since $\mathbb{E}\,J(X^-)$ is then of magnitude $\mathbb{E}X^-$, which is finite.

Kesten and Maller [33] showed that $\mathbb{E}\,J(X^-) < \infty$ can be replaced with

$$\int \frac{y}{A(y)}\,\mathbb{P}(X^- \in dy) \;<\; \infty.$$

Moreover, they have shown that under the assumption of positive divergence, $J(y)$ and $y/A(y)$ are asymptotically of the same magnitude (see the proof of Lemma 3.1 from [33]), which is why we also state the next theorems with $J(y)$.

The criterion concerning the harmonic renewal series in (iii) goes back to Spitzer [44, Theorem 4.1].

The main outcome of Kesten and Maller's article [33] is a theorem giving equivalent conditions for the finiteness of power moments of the above introduced fluctuation-theoretic quantities. Given $0 < \mathbb{E}X \leq \mathbb{E}|X| < \infty$, this has already been done by many authors, most notably Gut [26] and Janson [29]. For a good overview of the relevant literature, the reader is referred to Chapter III of Gut's monography [27].

**Theorem 1.3 ([33], Theorem 2.1 and p. 27)** *Let $(S_n)_{n\geq 0}$ be a positive divergent random walk and $\alpha > 0$. The following conditions are equivalent:*

*(i)* $\mathbb{E}\,\rho(x)^\alpha < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(ii)* $\mathbb{E}\,J(X^-)^{1+\alpha} < \infty$.

*(iii)* $\mathbb{E}\,\sigma_{\min}^\alpha < \infty$.

*(iv)* $\mathbb{E}\,\sigma^{\leqslant}(-x)^\alpha\,\mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(v)* $\mathbb{E}\,\Lambda(x)^\alpha < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(vi)* $\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}(S_n \leq x) < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(vii)* $\mathbb{E}\,\sigma^{>}(x)^{1+\alpha} < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

Another set of equivalent conditions is formed by the equivalences to finite power moments of $|\min_{n\geq 0} S_n|$.

**Theorem 1.4 ([33], Prop. 4.1)** *Let* $(S_n)_{n\geq 0}$ *be a positive divergent random walk and* $\alpha > 0$. *The following conditions are equivalent:*

*(i)* $\mathbb{E}\,|\min_{n\geq 0} S_n|^{\alpha} < \infty$.

*(ii)* $\mathbb{E}\,[(X^-)^{\alpha}\, J(X^-)] < \infty$.

*(iii)* $\mathbb{E}\,|S_{\sigma^{\leqslant}(-x)}|^{\alpha}\, \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

*(iv)* $\mathbb{E}\,(\max_{0\leq n\leq \rho(x)} |S_n|)^{\alpha} < \infty$ *for some (hence all)* $x \in \mathbb{R}_{\geq}$.

Obviously, Theorem 1.4 (ii) is stronger than Theorem 1.3 (ii), but, if $0 < \mathbb{E}\,X \leq \mathbb{E}\,|X| < \infty$, both conditions are equivalent and reduce to $\mathbb{E}\,(X^-)^{1+\alpha} < \infty$. Hence, as a corollary of the previous results, we obtain Janson's theorem, which has been published years before [33].

**Theorem 1.5 ([29], Theorem 1)** *Let* $(S_n)_{n\geq 0}$ *be a positive divergent random walk with* $0 < \mathbb{E}\,X \leq \mathbb{E}|X| < \infty$ *and* $\alpha > 0$. *The conditions in Theorem 1.3 and 1.4 are equivalent and the respective second condition reduces to* $\mathbb{E}(X^-)^{1+\alpha} < \infty$.

Another result of Kesten and Maller specifies the rate of growth of certain quantities from Theorem 1.3, which will be a basic ingredient for our results. For positive functions on the negative half-line $f$ and $g$ and $y \to \infty$ we denote $f(y) \lesssim g(y)$ if there exists a constant $c \in \mathbb{R}_{>}$ such that

$$f(y) \;\leq\; c\,g(y) \qquad \text{for all sufficiently large } y,$$

and $f(y) \asymp g(y)$, if $f(y) \lesssim g(y)$ and $f(y) \gtrsim g(y)$, i.e.

$$0 \;<\; \liminf_{y\to\infty} \frac{f(y)}{g(y)} \;\leq\; \limsup_{y\to\infty} \frac{f(y)}{g(y)} \;<\; \infty.$$

In particular, this notation is used for constant functions $f$ and $g$, i.e. we abbreviate that both are finite or infinite at the same time by $f \asymp g$.

**Theorem 1.6 ([33], Theorem 2.2 and p. 28)** *Let* $(S_n)_{n\geq 0}$ *be a non-trivial random walk.*

*(i) If* $(S_n)_{n\geq 0}$ *is positive divergent, then*

$$\sum_{n\geq 1} n^{-1}\,\mathbb{P}(S_n \leq y) \;\asymp\; \log J(y) \qquad \textit{as } y \to \infty.$$

*(ii) Suppose* $\mathbb{E}\,\rho(0)^{\alpha} < \infty$ *for some* $\alpha > 0$. *Then,*

$$\mathbb{E}\,\sigma^{>}(y)^{\alpha} \;\asymp\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}(S_n \leq y) \;\asymp\; \mathbb{E}\,\Lambda(y)^{\alpha} \;\asymp\; \mathbb{E}\,\rho(y)^{\alpha} \;\asymp\; J(y)^{\alpha} \qquad \textit{as } y \to \infty.$$

Erickson [19] first showed that $\sum_{n\geq 1} \mathbb{P}(S_n \leq y) \asymp J(y)$ for a random walk $(S_n)_{n\geq 0}$ with non-negative increments. For further remarks on this theorem we refer to those in [33].

## 1.2. Arcsine Law for Random Walks

A result contradicting a person's first intuition is the arcsine law for random walks. Consider the coin tossing game between two players A and B where player A wins a round if a fair coin shows heads, and player B wins otherwise. Let $X_n$ be equal to 1 if player A wins the $n$-th round, and $-1$ otherwise. Moreover, let $(S_n)_{n\geq 0}$ denote the random walk with increments $(X_n)_{n\geq 1}$. By the symmetry of the game, one may expect that it is most likely for large $n$ that

$$\Lambda_n^{>} \ := \ \sum_{k=1}^{n} \mathbf{1}_{\{S_k>0\}},$$

the number of rounds player A has the lead, is approximately $n/2$. In contrast, the arcsine law entails that $n^{-1}\Lambda_n^{>}$ is actually more likely to be close to 0 and 1. In other words, it is more probable that one player is in lead most of the time.

The result for the above setting was first introduced by Lévy [35, Corollaire 2, p. 303], which more detailly states

$$\frac{\Lambda_n^{>}}{n} \ \xrightarrow{d} \ AR(1/2),$$

where $AR(1/2)$ is the classical arcsine distribution given by the distribution function

$$AR(1/2)((-\infty,x]) \ := \ \frac{2}{\pi}\arcsin(\sqrt{x}), \qquad x\in[0,1].$$

More generally, we can introduce the family of arcsine distributions $(AR(\theta))_{\theta\in[0,1]}$. Set $AR(0):=\delta_0$ and $AR(1):=\delta_1$. For $\theta\in(0,1)$, $AR(\theta)$ is defined by having the Lebesgue-density

$$\frac{\sin(\pi\theta)}{\pi}\,\frac{1}{x^{1-\theta}\,(1-x)^{\theta}}\,\mathbf{1}_{(0,1)}(x).$$

Lévy's result is followed by several generalisations. The most important generalisations have been given by Sparre Andersen and Spitzer. Sparre Anderson (see [43, Theorem 3]) showed that a non-trivial random walk $(S_n)_{n\geq 0}$ with

$$\exists\theta\in(0,1): \qquad \lim_{n\to\infty}\mathbb{P}(S_n>0) \ = \ \theta$$

entails $n^{-1}\Lambda_n^{>}\xrightarrow{d}AR(\theta)$. Spitzer established an arcsine law under a weaker assumption. We state his theorem including the trivial cases $\theta\in\{0,1\}$.

**Theorem 1.7 ([44], Theorem 7.1)** *Let $(S_n)_{n\geq 0}$ be a non-trivial random walk, which fulfils*

$$\exists\theta\in[0,1]: \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}(S_k>0) \ = \ \theta.$$

*Then,*

$$\frac{\Lambda_n^{>}}{n} \ \xrightarrow{d} \ AR(\theta) \qquad \text{and} \qquad \frac{\Lambda_n^{\leqslant}}{n} \ \xrightarrow{d} \ AR(1-\theta),$$

*where* $\Lambda_n^{\leqslant}:=n-\Lambda_n^{>}=\sum_{k=1}^{n}\mathbf{1}_{\{S_k\leq 0\}}$.

# 2. Introduction of Markov Random Walks

It is self-evident that a random walk is not always the best choice for modelling sums of increments of a regenerative structure. In the context of queuing theory one may think about temporal bursts or seasonal fluctuations. This asks for a weakening of the i.i.d. assumption. In order to implement dependency, the natural choice is Markovian dependence, i.e. the increments are governed by a Markov chain. This motivation leads to the study of the subsequent defined object.

Let $(M_n)_{n\geq 0}$ be a positive recurrent, aperiodic Markov chain on some countable set $\mathscr{S}$ with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in\mathscr{S}}$ and unique stationary distribution $\pi := (\pi_i)_{i\in\mathscr{S}}$. Additionally, consider a stochastic kernel $K : (\mathscr{S}^2 \times \mathfrak{B}) \to [0,1]$ to define a bivariate Markov chain $(M_n, X_n)_{n\geq 1}$ by

$$\begin{aligned}\mathbb{P}((M_n, X_n) \in (j,E)|(M_k, X_k)_{0\leq k\leq n-1}) &:= \mathbb{P}((M_n, X_n) = (j,E)|M_{n-1}) \\ &:= p_{M_{n-1}\,j}\, K(M_{n-1}, j, E),\end{aligned}$$

for all $j \in \mathscr{S}$, $n \geq 1$ and $E \in \mathfrak{B}$, where $\mathfrak{B}$ denotes the Borel $\sigma$-field on $\mathbb{R}$. Hence,

$$K_{M_{n-1}M_n} \;:=\; K(M_{n-1}, M_n, \cdot\,) \;=\; \mathbb{P}(X_n \in \cdot\,|M_{n-1}, M_n).$$

Furthermore, $(X_1,\ldots,X_n)$ are conditionally independent, i.e.

$$\mathbb{P}((X_1,\ldots,X_n) \in \cdot\,|M_0 = i_0,\ldots,M_n = i_n) \;=\; \bigotimes_{k=1}^{n} K_{i_{k-1}i_k}$$

for all $n \geq 1$ and $i_0,\ldots,i_n \in \mathscr{S}$. The dependence structure allows to say that $(X_n)_{n\geq 1}$ is governed by the *driving chain* $(M_n)_{n\geq 0}$. We also call $(X_n)_{n\geq 1}$ a *Markov-modulated sequence*. Set $S_0 := 0$ and $S_n := \sum_{k=1}^{n} X_k$. $(M_n, S_n)_{n\geq 0}$ and often also $(S_n)_{n\geq 0}$ are called *Markov random walk* (MRW) or *Markov additive process*.

Fluctuation theory of MRWs has been examined by several authors. A classification of MRWs in terms of the almost sure asymptotic behaviour is studied in the articles of Prabhu et al. [40] and Newbould [36]. The latter particularly focused on MRWs with $(M_n)_{n\geq 0}$ having a finite state space. Although we focus on the case of $\mathscr{S}$ being countably infinite, Section 9.1 deals with the simpler case of a finite state space.

Further contributions are due to Alsmeyer. [3] deals with recurrence of MRWs given an ergodic driving chain with general state space. [2] examines ladder times and ladder chains under the assumption of a positive stationary mean in the case of an underlying Harris chain. [4] has been worked out during this thesis and studies ladder chains in our setup. Fuh and Lai [21] as well study ladder variables, but the underlying Markov chain is assumed to be uniformly ergodic.

A Wiener-Hopf factorisation for MRWs has been established by Asmussen (see [8] or [7, Chapter XI]). The latter reference also contains applications of MRWs in queuing theory. [13] provides further references for applications and early developments in Markov renewal theory.

## 2.1. Basic Results and Further Organisation

In the remainder of this chapter, $(M_n,S_n)_{n\geq 0}$ is always assumed to be a MRW. For a first result, we make use of the following considerations.

For $i\in\mathscr{S}$, let $(\tau_n(i))_{n\geq 1}$ be the successive return times of $(M_n)_{n\geq 0}$ to state $i$. Moreover, set $\chi_n(i):=\tau_n(i)-\tau_{n-1}(i)$, $n\geq 1$ and $\tau_0(i):=0$. Due to its particular Markovian structure, $(M_n,X_n)_{n\geq 1}$ splits into i.i.d. cycles

$$(\chi_{n+1}(i),(M_{\tau_n(i)+k},X_{\tau_n(i)+k})_{1\leq k\leq \chi_{n+1}(i)})_{n\geq 0}$$

under $\mathbb{P}_i:=\mathbb{P}(\cdot|M_0=i)$ and is stationary under $\mathbb{P}_\pi:=\sum_{i\in\mathscr{S}}\pi_i\mathbb{P}_i$. In particular, $(S_{\tau_n(i)})_{n\geq 0}$ forms an ordinary random walk under $\mathbb{P}_i$.

Consider a measurable function $f:\mathscr{S}\times\mathbb{R}\to\mathbb{R}$ satisfying $\mathbb{E}_\pi f^+(X_1)\wedge\mathbb{E}_\pi f^-(X_1)<\infty$. It is well-known (e.g., see [37, Lemma 5.2]) that

$$\mathbb{E}_\pi f(X_1) \;=\; \pi_i\,\mathbb{E}_i\left(\sum_{k=1}^{\tau(i)} f(X_k)\right). \tag{2.1}$$

Given $\mathbb{E}_\pi X_1^+\wedge\mathbb{E}_\pi X_1^-<\infty$, we obtain

$$\mathbb{E}_\pi X_1 \;=\; \pi_i\,\mathbb{E}_i S_{\tau(i)}. \tag{2.2}$$

It is a simple result to show that $\mathbb{P}_i(S_{\tau(i)}=0)=1$ holds either for all $i\in\mathscr{S}$ or none (e.g., see Lemma 2.2). Excluding this degenerate case, we are able to state a first result on the behaviour of MRWs. Since "$\mathbb{P}_\pi$-a.s." means the same as "$\mathbb{P}_i$-a.s." for all $i\in\mathscr{S}$, we use "a.s." synonymous for both.

**Theorem 2.1** *Let $(M_n,S_n)_{n\geq 0}$ be a MRW with $\mathbb{E}_\pi X_1^+\wedge\mathbb{E}_\pi X_1^-<\infty$ and $\mathbb{P}_i(S_{\tau(i)}=0)<1$ for all $i\in\mathscr{S}$. The following assertions are true:*

*(i)* $(S_n)_{n\geq 0}$ *is either positive divergent, negative divergent or oscillating.*

*(ii)* $(S_n)_{n\geq 0}$ *and* $(S_{\tau_n(i)})_{n\geq 0}$ *have the same fluctuation type for all $i\in\mathscr{S}$.*

*(iii)* $(S_n)_{n\geq 0}$ *is positive divergent if and only if* $\mathbb{E}_\pi X_1>0$.

*(iv)* $\lim_{n\to\infty} n^{-1}S_n=\mathbb{E}_\pi X_1$ *a.s.*

**Proof.** If $\mathbb{E}_i S_{\tau(i)}=0$, $(S_{\tau_n(i)})_{n\geq 0}$ oscillates, which naturally implies oscillation of $(S_n)_{n\geq 0}$. Consequently, assertion (i) to (iii) follow directly from the occupation measure formula (2.2) if we prove (iv). Moreover, Birkhoff's ergodic theorem entails (iv) if we show that $(X_n)_{n\geq 1}$ is ergodic under $\mathbb{P}_\pi$.

The stationarity is clear. Hence, let us consider an invariant set $E\in\mathfrak{B}^\infty$, i.e.

$$E \;=\; \{(X_n,X_{n+1},\dots)\in B\}\quad\text{for all } n\geq 1 \text{ and some } B\in\mathfrak{B}^\infty.$$

Define the $\mathbb{P}_\pi$-independent $\sigma$-algebras

$$\mathfrak{A}_n \;:=\; \sigma\Big(\chi_{n+1}(i),(M_{\tau_n(i)+k},X_{\tau_n(i)+k})_{1\leq k\leq\chi_{n+1}(i)}\Big),\qquad n\geq 0.$$

By Kolmogorov's zero-one law $\bigcap_{k\ge 1}\sigma(\bigcup_{n\ge k}\mathfrak{A}_n)$ is $\mathbb{P}_\pi$-trivial. Hence, it suffices to show $E\in\sigma(\bigcup_{n\ge k}\mathfrak{A}_n)$ for all $k\ge 1$. This follows from

$$\begin{aligned} E &= \bigcup_{n\ge k}\{\tau_k(i)=n\}\cap E = \bigcup_{n\ge k}\{\tau_k(i)=n,(X_{n+1},X_{n+2},\dots)\in B\} \\ &= \bigcup_{n\ge k}\{\tau_k(i)=n, X_{\tau_k(i)+1},X_{\tau_k(i)+2},\dots)\in B\} \\ &= \{X_{\tau_k(i)+1},X_{\tau_k(i)+2},\dots)\in B\} \ \in\sigma\Big(\bigcup_{n\ge k}\mathfrak{A}_n\Big). \end{aligned}$$

□

This theorem sums up most of the current knowledge of the fluctuation behaviour of a MRW before this thesis. Besides, Prabhu et al. [40] examined the introduced degeneracy condition and showed that the trichotomy in Theorem 2.1 (i) remains true without any moment assumptions (see their Theorem 7). Nevertheless, we prove this fluctuation type trichotomy ourselves and widen the degeneracy discussion by the aspect of *null-homology* in Section 2.2.

The study of a MRW does not reduce to the study of its embedded random walks. In fact, $(S_n)_{n\ge 0}$ may be regarded as the countable union of $(S_{\tau_n(i)})_{n\ge 0}$, $i\in\mathscr{S}$, but the way these are intertwined cause several complications. For example, we will reveal the intriguing fact that the assertion of Theorem 2.1 (ii) does not necessarily hold if $\mathbb{E}_\pi X_1^+ = \mathbb{E}_\pi X_1^- = \infty$. As it turns out throughout this chapter, one always has to take the behaviour of the excursions between successive return epochs into account.

Section 3 establishes solidarity results. Particularly, we show that all embedded random walks have the same fluctuation type. Auxiliary results are also given in Section 5, which deals with ladder chains. The task of finding equivalent conditions for finiteness of power moments of the first level $x$ passage time will remain unsolved, a discussion can be found in Section 6.5. The main results are contained in the Sections 4 and 6, which prove partial generalisations of Theorems 1.1–1.6. Not all assertions can be generalised to MRWs, Section 7 gathers some counterexamples.

Just recently, Alsmeyer, Iksanov and Meiners published an article [5] that studies fluctuation behaviour of *perturbed random walks*. Some of their results can be translated to MRWs and vice versa. Hence, we will extend some of their results and give different proofs. See Section 8 for a comparison of both results.

Section 9 aims at finding stronger versions of our main results for some special cases (e.g. $|\mathscr{S}|<\infty$). Finally, Section 10 generalises the arcsine law for ordinary random walks to MRWs.

A useful object will be the *dual of* $(M_n,S_n)_{n\ge 0}$, denoted $({}^\# M_n,{}^\# S_n)_{n\ge 0}$ herafter, which is again a MRW with $({}^\# M_n)_{n\ge 0}$ being the time reversal of $(M_n)_{n\ge 0}$ under $\mathbb{P}_\pi$ with transition matrix

$${}^\#\mathbf{P} := \left(\frac{\pi_j\, p_{ji}}{\pi_i}\right)_{i,j\in\mathscr{S}}.$$

Moreover,

$${}^\# K_{{}^\# M_{n-1}{}^\# M_n} := \mathbb{P}({}^\# X_n\in\cdot\,|{}^\# M_{n-1},{}^\# M_n) := K_{{}^\# M_n{}^\# M_{n-1}}$$

for all $n \geq 1$. Then, one easily infers

$$\mathbb{P}_{i_0}(M_1 = i_1, \ldots, M_n = i_n) \;=\; \frac{\pi_{i_n}}{\pi_{i_0}} \mathbb{P}_{i_n}({}^{\#}M_1 = i_{n-1}, \ldots, {}^{\#}M_n = i_0) \tag{2.3}$$

and

$$\begin{aligned} &\mathbb{P}((X_1, \ldots, X_n) \in \cdot \,|M_0 = i_0, \ldots, M_n = i_n) \\ &= \bigotimes_{k=1}^{n} K_{i_{k-1} i_k} \;=\; \bigotimes_{k=1}^{n} {}^{\#}K_{i_k i_{k-1}} \\ &= \mathbb{P}\Big(({}^{\#}X_n, \ldots, {}^{\#}X_1) \in \cdot \,|{}^{\#}M_0 = i_n, \ldots, {}^{\#}M_n = i_0\Big) \end{aligned} \tag{2.4}$$

for all $n \geq 1$ and $i_0, \ldots, i_n \in \mathscr{S}$.

Considering a doubly infinite extension $(M_n, X_n)_{n \in \mathbb{Z}}$ of the stationary chain $(M_n, X_n)_{n \geq 1}$ under $\mathbb{P}_\pi$ and putting $S_0 := 0$ and $S_n := S_{n-1} + X_n$ for $n \neq 0$, thus

$$S_n \;:=\; \begin{cases} X_1 + \ldots + X_n, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \\ -X_0 - \ldots - X_{n+1}, & \text{if } n \leq -1, \end{cases}$$

one can easily verify that the $\mathbb{P}_\pi$-laws of the dual and of $(M_{-n}, -S_{-n})_{n \geq 0}$ are the same. Equivalently, $({}^{\#}M_n, {}^{\#}X_n)_{n \geq 1}$ has the law of $(M_{-n}, X_{-n+1})_{n \geq 1}$ under $\mathbb{P}_\pi$.

For $x \in \mathbb{R}_{\geq}$, we define $\sigma^{>}(x)$, $\sigma^{\leqslant}(-x)$ etc. for MRWs in the same way as for ordinary random walks. $({}^{\#}\tau_n(i))_{n \geq 1}$, ${}^{\#}\sigma^{>}(x)$, ${}^{\#}\sigma^{\leqslant}(-x)$ etc. denote the corresponding quantities for the dual MRW $({}^{\#}M_n, {}^{\#}S_n)_{n \geq 0}$. Moreover, in the context of the dual MRW, we also write $\mathbb{P}_i$ for $\mathbb{P}(\cdot\,|{}^{\#}M_0 = i)$.

## 2.2. Null-Homologous Markov Random Walks

In Theorem 2.1 we have already seen that excluding the property

$$\mathbb{P}_i(S_{\tau(i)} = 0) \;=\; 1 \qquad \text{for all } i \in \mathscr{S} \tag{2.5}$$

provides the usual fluctuation type trichotomy for MRWs with $\mathbb{E}_\pi X_1$ well-defined. Our first aim is to validate this assertion even when the stationary mean is undefined. This will take place in the next section, but we study MRWs fulfilling (2.5) before. These MRWs have already been studied in [40] to some extent. Due to Lalley [34], we call a MRW *null-homologous* if there exists a function $g : \mathscr{S} \to \mathbb{R}$ such that

$$X_n \;=\; g(M_n) - g(M_{n-1}) \qquad \text{a.s.} \tag{2.6}$$

Notice that such a function is not uniquely determined, because if $g$ satisfies (2.6), then the same holds for $g'(i) := g(i) + c$, $c \in \mathbb{R}$. Our first result is that the class of null-homologous MRWs and the class of MRWs fulfilling (2.5) coincide.

**Lemma 2.2** *The following conditions are equivalent:*

*(i)* $(M_n, S_n)_{n \geq 0}$ *is null-homologous.*

*(ii)* $\mathbb{P}_i(S_{\tau(i)} = 0) = 1$ *for some* $i \in \mathscr{S}$.

*(iii)* $\mathbb{P}_i(S_{\tau(i)} = 0) = 1$ *for all* $i \in \mathscr{S}$.

**Proof.** "(i)⇒(ii)" is trivial.

"(ii)⇒(iii)" Let $\psi_{sj}$ be the characteristic function of $S_{\tau(j)}$ under $\mathbb{P}_s$ for $s, j \in \mathscr{S}$. Then, with $i \in \mathscr{S}$ such that $(S_{\tau_n(i)})_{n \geq 0}$ has zero increments, we easily find that

$$\psi_{is_1}(t)\,\psi_{s_1 s_2}(t) \cdot \ldots \cdot \psi_{s_{n-1}s_n}(t)\,\psi_{s_n i}(t) \;=\; 1 \tag{2.7}$$

for all $t \in \mathbb{R}$, $n \geq 1$ and $s_1, \ldots, s_n \in \mathscr{S}$. In particular, $\psi_{is}\,\psi_{ss}^n\,\psi_{si} \equiv 1$ for all $s \in \mathscr{S}$ and $n \geq 0$. Consequently, $\psi_{ss} \equiv 1$ for all $s \in \mathscr{S}$, which proves (iii).

"(iii)⇒(i)" (2.7) implies $|\psi_{sj}| \equiv 1$ and $\psi_{sj}\,\psi_{js} \equiv 1$ for all $s, j \in \mathscr{S}$: Hence, $\psi_{sj}(t) = e^{\mathbf{i}h(s,j)t}$ for some function $h : \mathscr{S}^2 \to \mathbb{R}$ and

$$e^{\mathbf{i}h(j,s)t} \;=\; \psi_{js}(t) \;=\; \overline{\psi_{sj}(t)} \;=\; e^{-\mathbf{i}h(s,j)t}.$$

This further yields $h(j,s) = -h(s,j)$ and particularly $h(s,s) = 0$. Fix some $i \in \mathscr{S}$, define $g(s) := h(i,s)$ for $s \in \mathscr{S}$ and use $\psi_{is}\,\psi_{sj}\,\psi_{ji} \equiv 1$ to infer

$$0 \;=\; h(i,s) + h(s,j) + h(j,i) \;=\; h(i,s) + h(s,j) - h(i,j) \;=\; g(s) + h(s,j) - g(j),$$

i.e. $h(s,j) = g(j) - g(s)$ for all $s, j \in \mathscr{S}$. But the latter means that $S_{\tau(j)} = g(j) - g(s)$ $\mathbb{P}_s$-a.s. and therefore

$$\begin{aligned}
\mathbb{P}_s(X_1 = g(M_1) - g(M_0)) \;&=\; \sum_{j \in \mathscr{S}} \mathbb{P}_s(X_1 = g(j) - g(s),\, M_1 = j) \\
&=\; \sum_{j \in \mathscr{S}} \mathbb{P}_s(S_{\tau(j)} = g(j) - g(s),\, \tau(j) = 1) \\
&=\; \sum_{j \in \mathscr{S}} \mathbb{P}_s(\tau(j) = 1) \;=\; 1
\end{aligned}$$

for all $s, j \in \mathscr{S}$ which shows that $(M_n, S_n)_{n \geq 0}$ is indeed null-homologous. □

The following result uses of the regenerative structure of a null-homologous MRW.

**Lemma 2.3** *Let* $(M_n, S_n)_{n \geq 0}$ *be a null-homologous MRW. Then,* $(S_n)_{n \geq 0}$ *converges in distribution.*

**Proof.** Obviously, $(M_n, S_n)_{n \geq 0}$ is a Markov chain with

$$\mathbb{P}_i((M_n, S_n) = (i, 0) \text{ i.o.}) \;=\; \mathbb{P}_i(M_n = i \text{ i.o.}) \;=\; 1.$$

Since $\mathbb{P}_i(\tau(i) \in \cdot)$ is aperiodic,

$$\zeta(\cdot) \;:=\; \pi_i\,\mathbb{E}_i\Bigg(\sum_{k=0}^{\tau(i)-1} \mathbf{1}_{\{S_k \in \cdot\}}\Bigg),$$

is the unique stationary measure of $(S_n)_{n \geq 0}$ (e.g., see [7, Cor. VI.1.5 (i)]). □

Here is a classification result for null-homologous MRWs, the straightforward proof is omitted.

**Proposition 2.4** *If $(M_n,S_n)_{n\geq 0}$ is null-homologous with $g$ as in (2.6), then exactly one of the following alternatives is true:*

*(i)* $g\equiv 0$ *and* $S_n=0$ *a.s. for all* $n\geq 0$*.*

*(ii)* $0\neq\sup_{i\in\mathscr{S}}|g(i)|<\infty$ *and*

$$-\infty\ <\ \liminf_{n\to\infty} S_n\ \leq\ \limsup_{n\to\infty} S_n\ <\ \infty\qquad a.s.$$

*(iii)* $-\infty=\inf_{i\in\mathscr{S}} g(i)<\sup_{i\in\mathscr{S}} g(i)<\infty$ *and*

$$\liminf_{n\to\infty} S_n\ =\ -\infty\qquad and\qquad \limsup_{n\to\infty} S_n\ <\ \infty\qquad a.s.$$

*(iv)* $-\infty<\inf_{i\in\mathscr{S}} g(i)<\sup_{i\in\mathscr{S}} g(i)=\infty$ *and*

$$-\infty<\ \liminf_{n\to\infty} S_n\qquad and\qquad \limsup_{n\to\infty} S_n\ =\ \infty\qquad a.s.$$

*(v)* $\inf_{i\in\mathscr{S}} g(i)=-\infty$, $\sup_{i\in\mathscr{S}} g(i)=\infty$ *and*

$$\liminf_{n\to\infty} S_n\ =\ -\infty\qquad and\qquad \limsup_{n\to\infty} S_n\ =\ \infty\qquad a.s.$$

Notice that alternatives (iii)–(v) are only possible if $\mathscr{S}$ has infinitely many states.

In correspondence to ordinary random walks, we call a MRW *non-trivial* if (2.5) is not satisfied or equivalently if the MRW is not null-homologous. The following lemma is a main ingredient for the proof of the fluctuation type trichotomy for non-trivial MRWs in the next section.

**Lemma 2.5** *Suppose* $\liminf_{n\to\infty} S_n\in\mathbb{R}$ *or* $\limsup_{n\to\infty} S_n\in\mathbb{R}$ $\mathbb{P}_i$*-a.s. for some* $i\in\mathscr{S}$*, then* $(M_n,S_n)_{n\geq 0}$ *is null-homologous.*

**Proof.** Since $\liminf_{n\to\infty} S_n=-\limsup_{n\to\infty}(-S_n)$, suppose w.l.o.g. $\mathbb{P}_i(\limsup_{n\to\infty} S_n\in\mathbb{R})=1$. Then, let $Y$ be a copy of $\limsup_{n\to\infty} S_n$ under $\mathbb{P}_i$, which is independent of $S_{\tau(i)}$. Since $\limsup_{n\to\infty}(S_n-S_{\tau(i)})\overset{d}{=}Y$ under $\mathbb{P}_i$, we obtain the stochastic fixed point equation

$$Y\ \overset{d}{=}\ S_{\tau(i)}+Y$$

under $\mathbb{P}_i$. The use of characterstic functions easily yields $\mathbb{P}_i(S_{\tau(i)}=0)=1$, i.e. the MRW is null-homologous. □

## 2.3. The Fluctuation Type Trichotomy for Non-Trivial Markov Random Walks

The exclusion of null-homologous MRWs may raise the hope to have found a class of MRWs whose behaviour is close to that of ordinary random walks. In fact, the same fluctuation type trichotomy is satisfied for non-trivial MRWs and for non-trivial ordinary random walks.

**Theorem 2.6** *A non-trivial MRW* $(M_n, S_n)_{n\geq 0}$ *is either positive divergent, negative divergent or oscillating.*

**Proof.** It suffices to show that $\liminf_{n\to\infty} S_n$ and $\limsup_{n\to\infty} S_n$ are a.s. constant and take values in $\{\pm\infty\}$. Using the argumentation from above, we can even restrict the proof on examining $\limsup_{n\to\infty} S_n$ . Fix some $i \in \mathscr{S}$ and suppose $p := \mathbb{P}_i(\limsup_{n\to\infty} S_n = \infty) > 0$. Since Lemma 2.5 entails $\mathbb{P}_i(\limsup_{n\to\infty} S_n = \pm\infty) = 1$, the proof is complete if we show $p = 1$. Notice that

$$\left\{\limsup_{n\to\infty} S_n = \infty\right\} \;=\; \bigcup_{m\geq 1} \left\{\max_{1\leq k\leq m} S_k > x, \limsup_{n\to\infty}(S_n - S_{\tau_m(i)}) = \infty\right\}$$

for all $x \in \mathbb{R}_\geq$. We obtain

$$\begin{aligned} p &= \lim_{m\to\infty} \mathbb{P}_i\left(\max_{1\leq k\leq m} S_k > x, \limsup_{n\to\infty}(S_n - S_{\tau_m(i)}) = \infty\right) \\ &= p \lim_{m\to\infty} \mathbb{P}_i\left(\max_{1\leq k\leq m} S_k > x\right) \\ &= p\,\mathbb{P}_i\left(\sup_{n\geq 1} S_n > x\right) \end{aligned}$$

for all $x \in \mathbb{R}_\geq$. Since we assumed $p > 0$, we derive $\mathbb{P}_i(\sup_{n\geq 1} S_n > x) = 1$ for all $x \in \mathbb{R}_\geq$ and thus $p = 1$. □

As a matter of fact, non-trivial MRWs and non-trivial ordinary random walks are different in other aspects when the stationary mean is undefined. The following example illustrates several of these. For the reader's convenience, we dispense with modelling aperiodicity in the examples.

**Example 2.7** Let $(M_n)_{n\geq 0}$ be a Markov chain on $\mathbb{N}_0$ which, when in state 0, picks an arbitrary $i \in \mathbb{N}$ with positive probability $p_{0i}$ and jumps back to 0 otherwise, thus $p_{i0} = 1$. With all $p_{0i}$ being positive, the chain is clearly irreducible and positive recurrent with stationary probabilities $\pi_0 = \frac{1}{2}$ and

$$\pi_i \;=\; \frac{1}{2}\mathbb{E}_0\left(\sum_{k=0}^{\tau(0)-1} \mathbf{1}_{\{M_k=i\}}\right) \;=\; \frac{1}{2}\mathbb{P}_0(M_1 = i) \;=\; \frac{p_{0i}}{2}.$$

Turning to the additive component, we define $X_n$ by

$$X_n \;:=\; \begin{cases} -p_{0i}^{-1}, & \text{if } M_{n-1}=0, M_n=i,\\ 2+p_{0i}^{-1}, & \text{if } M_{n-1}=i, M_n=0.\end{cases}$$

In other words, $K_{0i}=\delta_{-p_{0i}^{-1}}$ and $K_{i0}=\delta_{2+p_{0i}^{-1}}$ for all $i\in\mathbb{N}$. Notice that $\mathbb{E}_\pi X_1^+=\mathbb{E}_\pi X_1^-=\infty$. By definition, we have

$$S_n \;=\; \begin{cases} n, & \text{if } n \text{ even},\\ n-1-p_{0M_n}^{-1}, & \text{if } n \text{ is odd},\end{cases}$$

under $\mathbb{P}_0$. Hence, $S_n$ tends almost surely to $\infty$ for $n$ even, but since

$$\infty \;=\; \sum_{i\in\mathscr{S}} p_{0i}\,p_{0i}^{-1} \;=\; \mathbb{E}_0 X_1^- \;\asymp\; \sum_{n\geq 0}\mathbb{P}_0(X_{\tau_n(0)+1}^- > 2n)$$

and

$$\{M_0=0,\, X_{\tau_n(0)+1}^- > 2n \text{ infinitely often}\} \;=\; \{M_0=0,\, S_{\tau_n(0)+1}<0 \text{ infinitely often}\},$$

the Borel-Cantelli lemma entails $S_n<0$ infinitely often $\mathbb{P}_0$-a.s. Consequently, the MRW is oscillating, while $(S_{\tau_n(0)})_{n\geq 0}$ is positive divergent.

This example does also reveal further interesting properties. First of all, we note that the MRW is oscillating and $\mathbb{E}_i\sigma^>(x)<\infty$ for all $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$. Secondly,

$$\liminf_{n\to\infty} n^{-1}S_n \;\leq\; 0 \qquad\text{and}\qquad \limsup_{n\to\infty} n^{-1}S_n \;=\; 1 \qquad \text{a.s.}$$

Moreover, $\mathbb{E}_0 S_{\tau(0)}=2$ exists, but the stationary mean of $X_1$ does not. In addition, the fluctuation type of the MRW is different from the one of the dual MRW as it can be seen by

$$^{\#}S_n \;=\; \begin{cases} n, & \text{if } n \text{ even},\\ n+1+p_{0\,^{\#}M_n}^{-1}, & \text{if } n \text{ odd},\end{cases}$$

under $\mathbb{P}_0$.

The phenomenon that $\mathbb{E}_i S_{\tau(i)}$, $i\in\mathscr{S}$, exists, but the stationary mean of $X_1$ does not, has already been pointed out by Ney and Nummelin in a similar example (see [37, Example 6.3]). Our example already showed that Theorem 1.1, Kesten's trichotomy, can not be generalised to MRW with $\mathbb{E}_\pi|X_1|=\infty$. Under an additional assumption the trichotomy is true again (see Section 4.3).

The mentioned phenomena in Example 2.7 arise from an embedded null-homologous MRW whose extreme values push the MRW into the opposite direction infinitely often. Therefore, a study of a more restrictive class seems reasonable (see Section 9.2). Nevertheless, we will be able to generalise the results from fluctuation theory from ordinary random walks to non-trivial MRW to a great extent.

If we figure all $i \in \mathbb{N}$ being placed on a circle around 0, the transition diagram of the Markov chain looks like a *flower with infinitely many petals*, each of the petals representing a transition from 0 to some $i$ and back.

In most of our counterexamples the underlying Markov chain is a more general *infinite petal flower chain*, where starting in 0 a deterministic cycle of length $n$ is chosen with probability $\mathbb{P}(\varGamma = n)$, where $\varGamma$ will be adjusted for specific examples. We define this general chain here for future use. Again, we dispense with modelling aperiodicity.

**Example 2.8** Let $\varGamma$ be a positive, integer-valued random variable with finite mean and $\mathbb{P}(\varGamma \geq 2) = 1$. We construct a positive recurrent Markov chain $(M_n)_{n\geq 0}$ on $\mathscr{S} \subset \{0\} \cup \mathbb{N}^2$ by

$$\begin{aligned}\mathbb{P}_0(M_1 = (n,1), M_2 = (n,2), \ldots, M_{n-1} = (n,n-1), M_n = 0) &:= \mathbb{P}_0(M_1 = (n,1)) \\ &:= \mathbb{P}(\varGamma = n)\end{aligned}$$

for all $n \geq 2$. Hence, we have $\mathbb{P}_0(\tau(0) \in \cdot\,) = \mathbb{P}(\varGamma \in \cdot\,)$.

# 3. Solidarity Results

In the previous section, we have seen that the fluctuation type of a MRW $(S_n)_{n\geq 0}$ can be different from the one of $(S_{\tau_n(i)})_{n\geq 0}$ for some $i \in \mathscr{S}$. Our first result will be that the embedded random walks share the same fluctuation type. Moreover, further solidarity results will be given which are fundamental in future proofs.

For $i \in \mathscr{S}$, define

$$\nu(x) \;:=\; \nu(x,i) \;:=\; \inf\{n \geq 1 : S_{\tau_n(i)} > x\}, \qquad x \in \mathbb{R}_\geq,$$

the first level $x$ passage time for $(S_{\tau_n(i)})_{n\geq 0}$, $\nu := \nu(0)$,

$$\tau_1^>(i) \;:=\; \tau_\nu(i) \;=\; \inf\{k \geq 1 : S_k > 0, M_k = i\}$$

and inductively

$$\tau_n^>(i) \;:=\; \inf\{k \geq \tau_{n-1}^>(i) + 1 : S_k - S_{\tau_{n-1}^>(i)} > 0, M_k = i\}$$

for $n \geq 2$.

**Lemma 3.1** *If $(M_n, S_n)_{n\geq 0}$ is non-trivial, then all $(S_{\tau_n(i)})_{n\geq 0}$, $i \in \mathscr{S}$, are of the same fluctuation type.*

**Proof.** Non-triviality of the MRW implies $\liminf_{n\to\infty} S_{\tau_n(i)}$ and $\limsup_{n\to\infty} S_{\tau_n(i)}$ to be almost surely equal to $\infty$ or $-\infty$ for all $i \in \mathscr{S}$. Fix some distinct $i, j \in \mathscr{S}$ and assume w.l.o.g. $\limsup_{n\to\infty} S_{\tau_n(i)} = \infty$ a.s. Hence, $\lim_{n\to\infty} S_{\tau_n^>(i)} = \infty$ a.s. Now, pick $m \in \mathbb{N}$ and $x > 0$ such that $\mathbb{P}_i(M_m = j, |S_m| \leq x) > 0$. We infer from a geometric trials argument that

$$\mathbb{P}_i\left(M_{\tau_n^>(i)+m} = j,\, |S_{\tau_n^>(i)+m} - S_{\tau_n^>(i)}| \leq x \text{ infinitely often}\right) \;=\; 1$$

and thus $\limsup_{n\to\infty} S_{\tau_n(j)} = \limsup_{n\to\infty} S_{\tau_n(i)} = \infty$ a.s. $\square$

Additionally, one can show that the positive divergent embedded random walks share the same finite power moments of fluctuation-theoretic quantities. The easiest way would be to prove that the tails are of the same magnitude, i.e. $\mathbb{P}_i(S^+_{\tau(i)} > y) \asymp \mathbb{P}_i(S^+_{\tau(j)} > y)$ and $\mathbb{P}_i(S^-_{\tau(i)} > y) \asymp \mathbb{P}_j(S^-_{\tau(j)} > y)$ as $y \to \infty$, and make use of the integral criteria, but this is generally not possible. Referring to Example 2.7, we see that $\mathbb{P}_0(S_{\tau(0)} > y) = 0$ for $y > 2$ while $\mathbb{P}_i(S_{\tau(i)} > y) > 0$ for all $y \in \mathbb{R}_\geq$ for any $i \in \mathbb{N}$. However, we use will use a similar approach.

For distinct $i, j \in \mathscr{S}$, define

$$\upsilon \;:=\; \upsilon(i,j) \;:=\; \inf\{n \geq 1 : \tau_n(i) > \tau(j)\}$$

and notice that $\mathbb{E}_i \upsilon^{1+\alpha} < \infty$ for any $\alpha \geq 0$ due to $\mathbb{P}_i(\upsilon > n) = \mathbb{P}_i(\tau(i) < \tau(j))^n$ for all $n \geq 1$.

**Lemma 3.2** *Let $i, j \in \mathscr{S}$ be some distinct states.*

*(i) There exists $x \in \mathbb{R}_\geq$ such that*

$$\mathbb{P}_j(S_{\tau(j)} > y) \;\lesssim\; \mathbb{P}_i(S_{\tau_\upsilon(i)} > y - x) \qquad \textit{as } y \to \infty.$$

*(ii) $\mathbb{P}_j(S_{\tau(j)} > 0) > 0$ implies $\mathbb{P}_i(S_{\tau_\upsilon(i)} > 0) > 0$.*

**Proof.** (i) We begin with

$$\mathbb{P}_j(S_{\tau(j)} > y) \;=\; \mathbb{P}_j(S_{\tau(j)} > y, \tau(i) < \tau(j)) + \mathbb{P}_j(S_{\tau(j)} > y, \tau(i) > \tau(j)).$$

On the one hand, we obtain

$$\begin{aligned}
\mathbb{P}_j(S_{\tau(j)} > y, \tau(i) < \tau(j)) \;&=\; \mathbb{P}_j(S_{\tau(i)} + (S_{\tau(j)} - S_{\tau(i)}) > y, \tau(i) < \tau(j)) \\
&=\; \int \mathbb{P}_j(S_{\tau(i)} > y - x, \tau(i) < \tau(j))\; \mathbb{P}_i(S_{\tau(j)} \in dx) \\
&\leq\; \int \mathbb{P}_i(S_{\tau_\upsilon(i)} - S_{\tau(j)} > y - x)\; \mathbb{P}_i(S_{\tau(j)} \in dx) \\
&=\; \mathbb{P}_i(S_{\tau_\upsilon(i)} > y).
\end{aligned}$$

This proves the assertion if $\mathbb{P}_j(\tau(i) > \tau(j)) = 0$. Otherwise, there exist $x_1, x_2 \in \mathbb{R}_\geq$ with

$$p_1 \;:=\; \mathbb{P}_i(S_{\tau(j)} \geq -x_1, \tau(i) > \tau(j)) \;>\; 0 \qquad \text{and} \qquad p_2 \;:=\; \mathbb{P}_j(S_{\tau(i)} \geq -x_2) \;>\; 0.$$

Set $x := 2\,(x_1 + x_2)$. We infer

$$\begin{aligned}
&\mathbb{P}_j(S_{\tau(j)} > y, \tau(i) > \tau(j)) \\
&\leq [p_1 \cdot p_2]^{-1}\, \mathbb{P}_i(S_{\tau(i)} > y - x/2, \tau(i) > \tau_2(j), S_{\tau(j)} \geq -x_1, S_{\tau(i)} - S_{\tau_2(j)} \geq -x_2) \\
&\lesssim \mathbb{P}_i(S_{\tau(i)} > y - x/2).
\end{aligned}$$

Furthermore, in this case,

$$\mathbb{P}_i(S_{\tau_\upsilon(i)} > y - x)$$

$$
\begin{aligned}
&= \mathbb{P}_i(S_{\tau(i)} > y-x, \upsilon = 1) + \mathbb{P}_i(S_{\tau(i)} + (S_{\tau_\upsilon(i)} - S_{\tau(i)}) > y-x, \upsilon > 1) \\
&\geq \mathbb{P}_i(S_{\tau(i)} > y-x/2, \upsilon = 1) \\
&\qquad + \mathbb{P}_i(S_{\tau(i)} + (S_{\tau_\upsilon(i)} - S_{\tau(i)}) > y-x, \upsilon > 1, (S_{\tau_\upsilon(i)} - S_{\tau(i)}) \geq -x_1 - x_2) \\
&\geq \mathbb{P}_i(S_{\tau(i)} > y-x/2, \upsilon = 1) + p_1 \cdot p_2 \cdot \mathbb{P}_i(S_{\tau(i)} > y-x/2, \upsilon > 1) \\
&\gtrsim \mathbb{P}_i(S_{\tau(i)} > y-x/2).
\end{aligned}
$$

In conclusion, we obtain the assertion with $x$ chosen as above if $\mathbb{P}_j(\tau(i) > \tau(j)) > 0$.

(ii) If $\mathbb{P}_j(S_{\tau(j)} > 0, \tau(i) < \tau(j)) > 0$, the assertion follows immediately from the proof in (i). Otherwise,

$$\mathbb{P}_j(S_{\tau(j)} > \varepsilon, \tau(i) > \tau(j)) \; > \; 0$$

for some $\varepsilon > 0$. Then, choosing $x_2 \in \mathbb{R}_{\geq}$ and $p_2 > 0$ as in (i), the assertion follows from

$$
\begin{aligned}
\mathbb{P}_i(S_{\tau_\upsilon(i)} > 0) \; &\geq \; p_2 \cdot \mathbb{P}_j(S_{\tau_n(j)} > x_2, \tau(i) > \tau_n(j)) \\
&\geq \; p_2 \cdot \mathbb{P}_j(S_{\tau(j)} > \varepsilon, \tau(i) > \tau(j))^n \; > \; 0
\end{aligned}
$$

for $n = \lceil x_2/\varepsilon \rceil$.

□

We need further definitions. For $i \in \mathscr{S}$, $\gamma \in [0,1]$ and $y \in \mathbb{R}_{\geq}$, introduce

$$A_i(y) \; := \; \mathbb{E}_i(S^+_{\tau(i)} \wedge y) - \mathbb{E}_i(S^-_{\tau(i)} \wedge y)$$

and

$$J_{i,\gamma}(y) \; := \; \begin{cases} \frac{y}{[\mathbb{E}_i(S^+_{\tau(i)} \wedge y)]^\gamma}, & \text{if } \mathbb{P}_i(S^+_{\tau(i)} = 0) < 1, \\ y, & \text{if } \mathbb{P}_i(S^+_{\tau(i)} = 0) = 1, \end{cases}$$

where $0/[\mathbb{E}_i(S^+_{\tau(i)} \wedge 0)]^\gamma := 1$ if $\gamma > 0$. In addition, set $J_i := J_{i,1}$. Our main theorems will contain integral criteria with powers of $J_{i,\gamma}$ as the integrand. The next lemma gathers properties of $J_{i,\gamma}$ and $A_i$.

**Lemma 3.3** *The following assertions are true for any $\gamma \in [0,1]$:*

*(i) $J_{i,\gamma}$ is subadditive and non-decreasing for all $i \in \mathscr{S}$.*

*(ii) $J_{i,\gamma}(y) \asymp J_{i,\gamma}(x+y)$ as $y \to \infty$ for all $(x,i) \in \mathbb{R} \times \mathscr{S}$.*

*(iii) $J_{i,\gamma}(y) \asymp J_{j,\gamma}(y)$ as $y \to \infty$ for all $i,j \in \mathscr{S}$.*

*If $(S_{\tau_n(i)})_{n \geq 0}$ is positive divergent for some (hence all) $i \in \mathscr{S}$, then furthermore:*

*(iv) $A_i(y) > 0$ for all sufficiently large $y$ for all $i \in \mathscr{S}$.*

*(v) $A_i(y) \asymp \mathbb{E}_i(S^+_{\tau(i)} \wedge y) \asymp \mathbb{E}_i(S_{\tau^>(i)} \wedge y)$ as $y \to \infty$ for all $i \in \mathscr{S}$.*

**Proof.** (i) Subadditivity is trivial and $J_{i,\gamma}$ being non-decreasing follows from the identity

$$J_{i,\gamma}(y) \;=\; \frac{y^{1-\gamma}}{[y^{-1}\int_0^y \mathbb{P}_i(S_{\tau(i)}^+ > x)\;dx]^\gamma} \;=\; \frac{y^{1-\gamma}}{[\int_0^1 \mathbb{P}_i(S_{\tau(i)} > yx)\;dx]^\gamma}, \qquad y \in \mathbb{R}_\geq$$

(cf. the proof of [5, Lemma 5.4 (a)]).

(ii) Follows immediately from the properties claimed in (i).

(iii) Fix some $i,j \in \mathscr{S}$ and let $x \in \mathbb{R}_\geq$ be the constant provided by Lemma 3.2 (i). The assertion is obvious if we verify

$$\mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \;\asymp\; \mathbb{E}_j(S_{\tau(j)}^+ \wedge y) \qquad \text{as } y \to \infty.$$

We obtain

$$\begin{aligned}
\mathbb{E}_j(S_{\tau(j)}^+ \wedge y) \;&=\; \int_0^y \mathbb{P}_j(S_{\tau(j)} > z)\;dz \\
&\lesssim\; \int_0^y \mathbb{P}_i(S_{\tau_\upsilon(i)} > z - x)\;dz \\
&\lesssim\; \int_0^{y-x} \mathbb{P}_i(S_{\tau_\upsilon(i)} > z)\;dz \\
&=\; \mathbb{E}_i[S_{\tau_\upsilon(i)}^+ \wedge (y-x)] \\
&=\; \sum_{n\geq 1} \mathbb{P}_i(\upsilon = n)\,\mathbb{E}_i[S_{\tau_n(i)}^+ \wedge (y-x) | \upsilon = n] \\
&\leq\; \sum_{n\geq 1} \mathbb{P}_i(\upsilon = n) \sum_{k=1}^n \mathbb{E}_i\Big[(S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ \wedge (y-x)\Big|\upsilon = n\Big],
\end{aligned}$$

where we used Lemma 3.2 (ii) in the third step. Notice that assertion (ii) particularly yields

$$\mathbb{E}_i(S_{\tau(i)}^+ \wedge (y-x)) \;\asymp\; \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \qquad \text{as } y \to \infty.$$

Now, observe

$$\begin{aligned}
\mathbb{E}_i\Big[(S_{\tau_n(i)} - S_{\tau_{n-1}(i)})^+ \wedge (y-x)\Big|\upsilon = n\Big] \;&=\; \mathbb{E}_i[S_{\tau(i)}^+ \wedge (y-x) | \tau(i) > \tau(j)] \\
&\lesssim\; \mathbb{E}_i[S_{\tau(i)}^+ \wedge (y-x)] \\
&\asymp\; \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \qquad \text{as } y \to \infty
\end{aligned}$$

and, given $\mathbb{P}_i(\tau(i) < \tau(j)) > 0$,

$$\begin{aligned}
\mathbb{E}_i\Big[(S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ \wedge (y-x)\Big|\upsilon = n\Big] \;&=\; \mathbb{E}_i[S_{\tau(i)}^+ \wedge (y-x) | \tau(i) < \tau(j)] \\
&\lesssim\; \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \qquad \text{as } y \to \infty
\end{aligned}$$

for $1 \leq k < n$. Consequently,

$$\mathbb{E}_j(S_{\tau(j)}^+ \wedge y) \;\lesssim\; \sum_{n\geq 1} \mathbb{P}_i(\upsilon = n)\, n \cdot \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \;=\; \mathbb{E}_i\upsilon \cdot \mathbb{E}_i(S_{\tau(i)}^+ \wedge y),$$

which shows one part of the assertion. The other part follows by symmetry of the argument.

(iv) and (v) can be extracted from [33, Lemma 3.2, the proof of Lemma 3.1 and (4.5)]. □

The following solidarity lemma contains the announced result that positive divergent embedded random walks are of the same magnitude in terms of finiteness of power moments of fluctuation-theoretic quantities (cf. Theorems 1.3–1.4).

**Lemma 3.4** *The following assertions hold either for all $i \in \mathscr{S}$ or none:*

*(i)* $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$.

*(ii)* $\mathbb{E}_i[(S^-_{\tau(i)})^\alpha J_i(S^-_{\tau(i)})] < \infty$.

*(iii)* $\mathbb{E}_i|S_{\tau(i)}| < \infty$.

**Proof.** We will prove that $\mathbb{E}_i J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha} < \infty$, $\gamma \in [0,1]$, holds either for all $i \in \mathscr{S}$ or none. Since $J_{i,1} = J_i$ and

$$\mathbb{E}_i J_{i,1/(1+\alpha)}(S^-_{\tau(i)})^{1+\alpha} \;=\; \mathbb{E}_i[(S^-_{\tau(i)})^\alpha J_i(S^-_{\tau(i)})],$$

this already shows (i) and (ii). Furthermore, as the proof works also in terms of

$$\int \left(\frac{y}{[\mathbb{E}_i(S^-_{\tau(i)} \wedge y)]^\gamma}\right)^{1+\alpha} \mathbb{P}_i(S^+_{\tau(i)} \in dy),$$

choosing $\gamma = 0$ yields (iii).

Suppose $\mathbb{E}_i J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha} < \infty$. Pick some arbitrary $j \in \mathscr{S} \setminus \{i\}$. An application of Lemma 3.2 (i) on $(M_n, -S_n)_{n\geq 0}$ ensures the existence of $x \in \mathbb{R}_\geq$ with

$$\mathbb{P}_j(S^-_{\tau(j)} > y) \;\lesssim\; \mathbb{P}_i(S^-_{\tau_\upsilon(i)} > y - x) \qquad \text{as } y \to \infty.$$

Hence, by an appeal to Lemma 3.3 (ii)–(iii), it suffices to prove

$$\mathbb{E}_i J_{i,\gamma}(S^-_{\tau_\upsilon(i)})^{1+\alpha} \;<\; \infty.$$

Subadditivity of $J_{i,\gamma}$ yields

$$\begin{aligned}
\mathbb{E}_i J_{i,\gamma}(S^-_{\tau_\upsilon(i)})^{1+\alpha} \;&\leq\; \mathbb{E}_i\left[\sum_{k=1}^{\upsilon} J_{i,\gamma}\Big(\big(S_{\tau_k(i)} - S_{\tau_{k-1}(i)}\big)^-\Big)\right]^{1+\alpha} \\
&\leq\; \mathbb{E}_i\left[\upsilon^\alpha \sum_{k=1}^{\upsilon} J_{i,\gamma}\Big(\big(S_{\tau_k(i)} - S_{\tau_{k-1}(i)}\big)^-\Big)^{1+\alpha}\right] \\
&=\; \sum_{n\geq 1} \mathbb{P}_i(\upsilon = n)\, n^\alpha \sum_{k=1}^{n} \mathbb{E}_i\Big[J_{i,\gamma}\Big(\big(S_{\tau_k(i)} - S_{\tau_{k-1}(i)}\big)^-\Big)^{1+\alpha}\Big|\upsilon = n\Big].
\end{aligned}$$

Now, we use

$$\mathbb{E}_i\Big[J_{i,\gamma}\Big(\Big(S_{\tau_n(i)}-S_{\tau_{n-1}(i)}\Big)^-\Big)^{1+\alpha}\Big|\upsilon=n\Big] \;=\; \mathbb{E}_i\Big(J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha}\Big|\tau(i)>\tau(j)\Big) \;=:\; c_1 \;<\; \infty$$

and, given $\mathbb{P}_i(\tau(i)<\tau(j))>0$,

$$\mathbb{E}_i\Big[J_{i,\gamma}\Big(\Big(S_{\tau_k(i)}-S_{\tau_{k-1}(i)}\Big)^-\Big)^{1+\alpha}\Big|\upsilon=n\Big] \;=\; \mathbb{E}_i\Big(J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha}\Big|\tau(i)<\tau(j)\Big) \;=:\; c_1 \;<\; \infty$$

for $1\le k<n$ to infer

$$\mathbb{E}_iJ_{i,\gamma}(S^-_{\tau_\upsilon(i)})^{1+\alpha} \;\le\; (c_1\vee c_2)\,\mathbb{E}_i\upsilon^{1+\alpha} \;<\; \infty.$$

□

The next solidarity lemma provides sufficient conditions for the existence of power moments of $\sigma^>$. For a further discussion of these quantities we refer to Section 6.5.

Note that a geometric number of cycles marked by successive visits to a state $i$ contains a visit to $j\in\mathscr{S}$, from which one easily concludes that $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$ is satisfied either for all $i\in\mathscr{S}$ or none.

**Lemma 3.5** *Let $\alpha\ge 0$ and suppose $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$ for some (hence all) $i\in\mathscr{S}$. The following conditions are equivalent:*

*(i) $\mathbb{E}_i\tau_{\nu(x)}(i)^{1+\alpha}<\infty$ for some (hence all) $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$.*

*(ii) $A_i(y)>0$ for all sufficiently large $y$ and $\mathbb{E}_iJ_i(S^-_{\tau(i)})^{1+\alpha}<\infty$ for some (hence all) $i\in\mathscr{S}$.*

*In particular, these conditions imply*

$$\mathbb{E}_i\sigma^>(x)^{1+\alpha} \;<\; \infty \qquad \textit{for all } (x,i)\in\mathbb{R}_\ge\times\mathscr{S}. \tag{3.1}$$

**Proof.** By Lemma 3.1, Lemma 3.4 and Theorem 1.2, (ii) holds either for all $i\in\mathscr{S}$ or none. Moreover, the implication of (3.1) follows from $\tau_{\nu(x)}(i)\ge\sigma^>(x)$.

Suppose $\mathbb{E}_i\tau_{\nu(x)}(i)^{1+\alpha}<\infty$ for some $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$. Since $\tau_{\nu(x)}(i)\ge\nu(x)$, we obtain $\mathbb{E}_i\nu(x)^{1+\alpha}<\infty$, which is equivalent to (ii) by Theorems 1.2–1.3.

As we assumed $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$, the reverse implication follows directly from $\mathbb{E}_i\nu(x)^{1+\alpha}<\infty$ and Theorem 1.5.4 from [27]. □

For $i\in\mathscr{S}$, define

$$D^i_n \;:=\; \max_{\tau_{n-1}(i)<k\le\tau_n(i)}\Big(S_k-S_{\tau_{n-1}(i)}\Big)^-, \qquad n\ge 1,$$

as the maximal downward excursion between $\tau_{n-1}(i)+1$ and $\tau_n(i)$. Our main theorems will contain integral criteria in terms of $D^i$ and the next lemma facilitates future proofs by showing that these criteria hold either for all $i\in\mathscr{S}$ or none. Notice that $D^i\ge S^-_{\tau(i)}\ge 0$.

**Lemma 3.6** *The following assertions hold either for all $i \in \mathscr{S}$ or none:*

*(i)* $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$.

*(ii)* $\mathbb{E}_i[(D^i)^\alpha J_i(D^i)] < \infty$.

*(iii)* $\mathbb{E}_i(D^i)^{1+\alpha} < \infty$.

**Proof.** As seen in the proof of Lemma 3.4, it suffices to prove that $\mathbb{E}_i J_{i,\gamma}(D^i)^{1+\alpha} < \infty$ holds either for all $i \in \mathscr{S}$ or none. Suppose $\mathbb{E}_i J_{i,\gamma}(D^i)^{1+\alpha} < \infty$ for some $i \in \mathscr{S}$ and $\gamma \in [0,1]$. Define

$$D_\upsilon \;:=\; \max_{1\le k\le \tau_\upsilon(i)} S_k^-.$$

Using $D_\upsilon \le \sum_{k=1}^{\upsilon} D_k^i$, one obtains $\mathbb{E}_i J_{i,\gamma}(D_\upsilon)^{1+\alpha} < \infty$ analogous to the finiteness of $\mathbb{E}_i J_{i,\gamma}(S^-_{\tau_\upsilon(i)})^{1+\alpha}$ in the proof of Lemma 3.4.

Then, define

$$\upsilon_2 \;:=\; \inf\{n \ge 1 : \tau_n(i) > \tau_2(j)\}$$

and

$$D_{\upsilon_2} \;:=\; \max_{1\le k\le \tau_{\upsilon_2}(i)} S_k^-.$$

Now, notice

$$D_{\upsilon_2} \;\le\; D_\upsilon + \max_{\tau_\upsilon(i)<k\le \tau_{\upsilon_2}(i)} (S_k - S_{\tau_\upsilon(i)})^-.$$

Either $\tau_\upsilon(i) < \tau_{\upsilon_2}(i)$ and the latter summand is an independent copy of $D_\upsilon$ or $\tau_\upsilon(i) = \tau_{\upsilon_2}(i)$ and the latter summand summand is equal to 0. Hence, in both cases one easily obtains $\mathbb{E}_i J_{i,\gamma}(D_{\upsilon_2})^{1+\alpha} < \infty$. Pick $x \in \mathbb{R}_\ge$ with $\mathbb{P}_i(S_{\tau(j)} \le x) > 0$. By the use of Lemma 3.3, we finally derive

$$\begin{aligned}
\infty \;&>\; \mathbb{E}_i J_{i,\gamma}(D_{\upsilon_2})^{1+\alpha} \;\gtrsim\; \int_{(0,\infty)} J_{i,\gamma}(y)^{1+\alpha}\, \mathbb{P}_i(D_{\upsilon_2} \in dy \,|\, S_{\tau(j)} \le x) \\
&\ge \int_{(0,\infty)} J_{i,\gamma}(y)^{1+\alpha}\, \mathbb{P}_i\Big(\max_{\tau_1(j)<k\le\tau_2(j)} S_k^- \in dy \,\Big|\, S_{\tau_1(j)} \le x\Big) \\
&\ge \int_{(0,\infty)} J_{i,\gamma}(y)^{1+\alpha}\, \mathbb{P}_i\Big(\max_{\tau_1(j)<k\le\tau_2(j)} (S_k - S_{\tau_1(j)})^- - x \in dy \,\Big|\, S_{\tau_1(j)} \le x\Big) \\
&= \int_{(0,\infty)} J_{i,\gamma}(y)^{1+\alpha}\, \mathbb{P}_j(D_1^j - x \in dy) \\
&\asymp \mathbb{E}_j J_{j,\gamma}(D^j)^{1+\alpha}.
\end{aligned}$$

□

# 4. Characterisation of the Fluctuation Type of Non-Trivial Markov Random Walks

## 4.1. The General Case

This section is devoted to investigating equivalences for a non-trivial MRW to be positive divergent. As argued for ordinary random walks, these results entail criteria for negative divergent and oscillating MRWs.

Theorem 1.2 can partially be generalised:

**Theorem 4.1** *Let $(M_n,S_n)_{n\geq 0}$ be a non-trivial MRW. Consider the following assertions:*

*(i)* $(S_n)_{n\geq 0}$ *is positive divergent.*

*(ii)* $A_i(y)>0$ *for all sufficiently large* $y$ *and* $\mathbb{E}_i J_i(D^i)<\infty$ *for some (hence all)* $i\in\mathscr{S}$*.*

*(iii)* $\sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n\leq x)<\infty$ *for all* $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$*.*

*(iv)* $\mathbb{E}_i\sigma^{>}(x)<\infty$ *for all* $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$*.*

*Then, (i) ⇔ (ii) ⇒ (iii) ⇒ (iv).*

We can consider an ordinary random walk as a MRW modulated by a constant Markov chain, i.e. $M_n=i$ for all $n\geq 0$ and some $i$. Then, $S^+_{\tau(i)}=X^+$ and $D^i=X^-$, which justifies calling (ii) a generalisation of Theorem 1.2 (ii).

Further equivalent conditions are $\mathbb{P}_i(\rho(x)<\infty)=1$, $\mathbb{P}_i(\sigma_{\min}<\infty)=1$, $\mathbb{P}_i(\Lambda(x)<\infty)=1$ and $\mathbb{P}_i(|\min_{n\geq 0} S_n|<\infty)=1$ for some (hence all) $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$, which follow, as in the ordinary random walk case, directly from the definition of positive divergence and the fluctuation type trichotomy for non-trivial MRWs. In addition, $\mathbb{P}_i(\sigma^{\leqslant}(-x)=\infty)>0$ for some $x\in\mathbb{R}_{\geq}$ is another equivalent condition. Notice that for a positive divergent MRW $\mathbb{P}_i(\sigma^{\leqslant}(-x)=\infty)>0$ is always satisfied for large $x$, but does not need to be for small $x$ (e.g. $\mathbb{P}_i(X_1\leq -x)=1$).

On the one hand, one can dispense with the condition $A_i(y)>0$ for all sufficiently large $y$, when $\mathbb{E}_i|S_{\tau(i)}|=\infty$ and hence $\mathbb{E}_i S^+_{\tau(i)}+\mathbb{E}_i D^i=\infty$, as the proof will reveal. On the other hand, if $\mathbb{E}_i|S_{\tau(i)}|<\infty$, assertion (ii) does not reduce to $A_i(y)>0$ for all sufficiently large $y$, since $\mathbb{E}_i D^i=\infty$ is not excluded.

Naturally, one may conjecture that all necessary information is incorporated in the stationary increment distribution. Hence, one would rather like to have $\mathbb{E}_i J_i(D^i)<\infty$ replaced with

$$\int \frac{y}{\mathbb{E}_\pi(X_1^+\wedge y)}\ \mathbb{P}_\pi(X_1^-\in dy)\ <\ \infty,$$

but Example 7.2 and the remark before will explain its falsity.

The missing reverse implications in Theorem 4.1 are generally not true. We pointed out in Example 2.7 that $\mathbb{E}_i\sigma^{>}(x)<\infty$ for all $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$ is satisfied, but $(S_n)_{n\geq 0}$ oscillates. In Section 6.4 we will give an equivalent criterion for (iii), which reduces to $A_i(y)>0$ for all sufficiently large $y$, $\mathbb{E}_i J_i(S^-_{\tau(i)})<\infty$ and $\mathbb{E}_i\log J_i(D^i)<\infty$, when $\mathbb{P}_i(\tau(i)\leq c)=1$ for some $c \in \mathbb{R}_>$. This can be seen to be weaker than positive divergence by comparison with Theorem 4.1 (ii).

For ordinary random walks, the equivalence to (iii) follows from [44, Corollary 2], often subsumed under Spitzer's formulas. In particular, the cited result entails the equivalence to (iv), too. In the MRW context, we do not have a generalised version of [44, Corollary 2] at hand and thus we will use a different approach for the implication of (iii).

We need a straightforward improvement of [19, Lemma 4] for the proof of Theorem 4.1 (e.g., see the proof of [23, Lemma 5.2]).

**Lemma 4.2** *Let $(X_n, Y_n)_{n\geq 1}$ be an i.i.d. sequence of non-negative random variables with $\mathbb{E}X + \mathbb{E}Y = \infty$. Then*

$$\limsup_{n\to\infty} \frac{Y_{n+1}}{\sum_{k=1}^n X_k} \;=\; 0 \qquad \textit{or} \qquad =\; \infty \qquad \textit{a.s.}$$

*according to*

$$\int \frac{y}{\mathbb{E}(X\wedge y)}\,\mathbb{P}(Y\in dy) \;<\; \infty \qquad \textit{or} \qquad =\; \infty.$$

This enables us to prove a part of Theorem 4.1. Notice that the implication "(i)⇒(ii)" is adapted from the proof of Lemma 5.2 from [23]. Moreover, whenever $i \in \mathscr{S}$ is fixed, we use the notation

$$N(n) \;:=\; N(i,n) \;:=\; \sup\{k\geq 0 : \tau_k(i)\leq n\}.$$

**Proof of Theorem 4.1 "(i)⇔(ii)" and "(iii)⇒(iv)".** Pick some arbitrary $i \in \mathscr{S}$.
"(ii)⇒(i)" We distinguish between two cases.

Case 1. $\mathbb{E}_i S_{\tau(i)}^+ + \mathbb{E}_i D^i < \infty$. Then, $0 < \lim_{y\to\infty} A_i(y) = \mathbb{E}_i S_{\tau(i)} < \infty$ ensures positive divergence of $(S_{\tau_n(i)})_{n\geq 0}$ and $\lim_{n\to\infty} n^{-1} S_{\tau_n(i)} = \mathbb{E}_i S_{\tau(i)}$ a.s. Moreover, $\mathbb{E}_i D^i < \infty$ implies

$$\limsup_{n\to\infty} n^{-1} D_{n+1}^i \;=\; 0 \qquad \text{a.s.}$$

by the Borel-Cantelli lemma. Consequently,

$$\begin{aligned} S_n \;&\geq\; S_{\tau_{N(n)}(i)} - D^i_{N(n)+1} \\ &=\; S_{\tau_{N(n)}(i)}\left(1 - \frac{N(n)}{S_{\tau_{N(n)}(i)}}\frac{D^i_{N(n)+1}}{N(n)}\right) \xrightarrow{n\to\infty} \infty \qquad \text{a.s.} \end{aligned}$$

Case 2. $\mathbb{E}_i S_{\tau(i)}^+ + \mathbb{E}_i D^i = \infty$. Due to Lemma 4.2, the finiteness of $\mathbb{E}_i J_i(D^i)$ is equivalent to

$$\lim_{n\to\infty} \frac{D^i_{n+1}}{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+} \;=\; 0 \qquad \text{a.s.} \tag{4.1}$$

Further, [41, Lemma 8.1] yields

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^-}{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+} \;=\; 0 \qquad \text{a.s.} \tag{4.2}$$

Then, we obtain $\lim_{n\to\infty} S_n = \infty$ a.s. from

$$\begin{aligned} S_n &\ge \sum_{k=1}^{N(n)} (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ - \sum_{k=1}^{N(n)} (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^- - D^i_{N(n)+1} \\ &= \Big(\sum_{k=1}^{N(n)} (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+\Big)(1-o(1)-o(1)) \xrightarrow{n\to\infty} \infty \qquad \text{a.s.,} \end{aligned}$$

"(i)⇒(ii)" As $(S_{\tau_n(i)})_{n\ge0}$ must also be positive divergent, Lemma 3.3 (iv) yields $A_i(y) > 0$ for all sufficiently large $y$. Since $S_{\tau_n(i)} - D^i_{n+1} = S_{\kappa_n}$ for an increasing sequence $(\kappa_n)_{n\ge0}$, we conclude $\lim_{n\to\infty}(S_{\tau_n(i)} - D^i_{n+1}) = \infty$ a.s. As a consequence,

$$\limsup_{n\to\infty} \frac{D^i_{n+1}}{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+} \le \limsup_{n\to\infty} \frac{D^i_{n+1}}{S_{\tau_n(i)}} < 1 \qquad \text{a.s.}$$

and thus $\mathbb{E}_i\, J_i(D^i) < \infty$ by Lemma 4.2.

"(iii)⇒(iv)" The assumption implies

$$\mathbb{E}_i\Big(\sum_{n\ge1} \tau_n(i)^{-1}\, \mathbf{1}_{\{S_{\tau_n(i)} \le x\}}\Big) < \infty$$

and due to Lemma C.3 this term is of magnitude $\sum_{n\ge1} n^{-1}\, \mathbb{P}_i(S_{\tau_n(i)} \le x)$. Theorem 1.2 yields $A_i(y) > 0$ for all sufficiently large $y$ and $\mathbb{E}_i J_i(S_{\tau(i)}) < \infty$. Therefore, Lemma 3.5 entails (iv). □

We are only left with the proof of "(i)⇒(iii)". We embark on the use of

$$D_n^{>,i} := \max_{\tau^>_{n-1}(i) < k \le \tau^>_n(i)} \Big(S_k - S_{\tau^>_{n-1}(i)}\Big)^-, \qquad n \ge 1, i \in \mathscr{S},$$

the maximal downward excursion between $\tau^>_{n-1}(i) + 1$ and $\tau^>_n(i)$.

**Lemma 4.3** $(S_n)_{n\ge0}$ *is positive divergent if and only if* $A_i(y) > 0$ *for all sufficiently large* $y$ *and*

$$\sum_{n\ge1} \mathbb{P}_i(S_{\tau^>_n(i)} - D^{>,i}_{n+1} \le x) < \infty$$

*for some (hence all)* $(x,i) \in \mathbb{R}_\ge \times \mathscr{S}$.

**Proof.** Fix some arbitrary $i \in \mathscr{S}$. Analogous to Theorem 4.1 "(i)⇔(ii)" one proves the equivalence to $A_i(y) > 0$ for all sufficiently large $y$, which yields $\mathbb{P}_i(\tau^>(i) < \infty) = 1$, and

$$\int \frac{y}{\mathbb{E}_i(S_{\tau^>(i)} \wedge y)}\, \mathbb{P}_i(D^{>,i} \in dy) < \infty. \tag{4.3}$$

Lemma 3.3 yields

$$\frac{y}{\mathbb{E}_i(S_{\tau^>(i)} \wedge y)} \asymp \frac{x+y}{\mathbb{E}_i(S_{\tau^>(i)} \wedge (x+y))} \qquad \text{as } y \to \infty$$

for all $x \in \mathbb{R}_{\geq}$. Since $\mathbb{E}_i \sup\{n \geq 0 : S_{\tau_n^>(i)} \leq 0\} = 0 < \infty$, Theorem 1.6 (ii) entails the equivalence of (4.3) and

$$\begin{aligned}\infty \;>\; & \int \sum_{n\geq 1} \mathbb{P}_i(S_{\tau_n^>(i)} \leq x+y)\; \mathbb{P}_i(D^{>,i} \in dy) \\ = \; & \sum_{n\geq 1} \mathbb{P}_i(S_{\tau_n^>(i)} - D_{n+1}^{>,i} \leq x).\end{aligned}$$

□

Now, we are able to complete the proof of Theorem 4.1.

**Proof of Theorem 4.1 "(i)⇒(iii)".** Pick some arbitrary $(x,i) \in \mathbb{R}_{\geq} \times \mathscr{S}$. We estimate

$$\begin{aligned}& \sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n \leq x) \\ = \; & \sum_{n\geq 1} \mathbb{E}_i \left( \sum_{k=\tau_{n-1}^>(i)+1}^{\tau_n^>(i)} k^{-1}\, \mathbf{1}_{\left\{S_{\tau_{n-1}^>(i)} + \left(S_k - S_{\tau_{n-1}^>(i)}\right) \leq x\right\}} \right) \\ \leq \; & \sum_{n\geq 1} \mathbb{E}_i \left( (\tau_{n-1}^>(i)+1)^{-1} \sum_{k=\tau_{n-1}^>(i)+1}^{2\tau_{n-1}^>(i) \wedge \tau_n^>(i)} \mathbf{1}_{\left\{S_{\tau_{n-1}^>(i)} - D_n^{>,i} \leq x\right\}} \right) \\ & + \sum_{n\geq 1} \mathbb{E}_i \left( \sum_{k=2\tau_{n-1}^>(i)+1}^{\tau_n^>(i)} k^{-1}\, \mathbf{1}_{\{\tau_n^>(i) - \tau_{n-1}^>(i) > \tau_{n-1}^>(i)\}} \right) \\ \leq \; & \sum_{n\geq 1} \mathbb{P}_i(S_{\tau_{n-1}^>(i)} - D_n^{>,i} \leq x) + \sum_{n\geq 1} \mathbb{E}_i \left( \sum_{k=2\tau_{n-1}^>(i)+1}^{\tau_n^>(i)} k^{-1}\, \mathbf{1}_{\{\tau_n^>(i) - \tau_{n-1}^>(i) > \tau_{n-1}^>(i)\}} \right),\end{aligned}$$

where due to Lemma 4.3, we only need to show the finiteness of the last summand. Notice that the proviso entails $\lim_{n\to\infty} S_{\tau_n(i)} = \infty$ a.s. and hence $\mathbb{E}_i \tau^>(i) < \infty$. Therefore,

$$\begin{aligned}& \sum_{n\geq 1} \mathbb{E}_i \left( \sum_{k=2\tau_{n-1}^>(i)+1}^{\tau_n^>(i)} k^{-1}\, \mathbf{1}_{\{\tau_n^>(i) - \tau_{n-1}^>(i) > \tau_{n-1}^>(i)\}} \right) \\ \leq \; & \sum_{n\geq 1} \sum_{k\geq 1} \mathbb{E}_i \Big[ (2\tau_{n-1}^>(i) + k)^{-1}\, \mathbf{1}_{\{\tau_n^>(i) - \tau_{n-1}^>(i) \geq n-1+k\}} \Big] \\ \leq \; & \sum_{n\geq 1} \sum_{k\geq n} k^{-1}\, \mathbb{P}_i(\tau^>(i) \geq k) \\ = \; & \sum_{n\geq 1} \mathbb{P}_i(\tau^>(i) \geq n) \;=\; \mathbb{E}_i \tau^>(i) \;<\; \infty\end{aligned}$$

as claimed. □

## 4.2. The Case $\mathbb{E}_\pi X_1^+ \wedge \mathbb{E}_\pi X_1^- < \infty$

We revisit the case $\mathbb{E}_\pi X_1^+ \wedge \mathbb{E}_\pi X_1^- < \infty$, where Theorem 2.1 presented first insights. Under this assumption, the property that the MRW and the embedded random walks share the same fluctuation type, enables us to strengthen the assertion of Theorem 4.1.

**Theorem 4.4** *Let $(M_n, S_n)_{n\geq 0}$ be a non-trivial MRW with $\mathbb{E}_\pi X_1^+ \wedge \mathbb{E}_\pi X_1^- < \infty$. Then, positive divergence is equivalent to*

$$\sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n \leq x) \;<\; \infty \qquad \textit{for some (hence all) } (x,i) \in \mathbb{R}_\geq \times \mathscr{S}. \tag{4.4}$$

**Proof.** Assume $(S_n)_{n\geq 0}$ is not positive divergent. Theorem 2.1 entails that $(S_{\tau_n(i)})_{n\geq 0}$, $i \in \mathscr{S}$, is not positive divergent either. As a consequence, Theorem 1.2 and Lemma C.3 yield

$$\sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n \leq x) \;\geq\; \mathbb{E}_i\Bigg(\sum_{n\geq 1} \tau_n(i)^{-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\leq x\}}\Bigg) \;\asymp\; \sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_{\tau_n(i)} \leq x) \;=\; \infty$$

for all $x \in \mathbb{R}_\geq$, which finishes the proof. □

Unfortunately, we can neither prove nor disprove the sufficiency of condition Theorem 4.1 (iv) in the given context, since we can not rule out $\mathbb{E}_i\sigma^>(x) < \infty$ for all $i \in \mathscr{S}$, when $\mathbb{E}_\pi X = 0$ and $|\mathscr{S}| = \infty$. See Section 9.1 for a discussion of the case $|\mathscr{S}| < \infty$.

## 4.3. The Case $\mathbb{E}_i|S_{\tau(i)}| = \infty$

Example 2.7 revealed that $\mathbb{E}_\pi|X_1| = \infty$ is the wrong assumption to generalise Kesten's trichotomy, i.e. Theorem 1.1. In this section, we prove that the trichotomy is true under the assumption of $\mathbb{E}_i|S_{\tau(i)}| = \infty$ for some (hence all) $i \in \mathscr{S}$, which is stronger than $\mathbb{E}_\pi|X_1| = \infty$.

**Theorem 4.5** *Suppose $\mathbb{E}_i|S_{\tau(i)}| = \infty$ for some $i \in \mathscr{S}$. Then, exactly one of the following cases prevails:*

*(i)* $\lim_{n\to\infty} n^{-1} S_n = \infty$ *and* $(S_n)_{n\geq 0}$ *is positive divergent.*

*(ii)* $\lim_{n\to\infty} n^{-1} S_n = -\infty$ *and* $(S_n)_{n\geq 0}$ *is negative divergent.*

*(iii)* $\liminf_{n\to\infty} n^{-1} S_n = -\infty$ *a.s.,* $\limsup_{n\to\infty} n^{-1} S_n = \infty$ *a.s. and* $(S_n)_{n\geq 0}$ *oscillates.*

**Proof.** Since $(X_n)_{n\geq 1}$ is an ergodic stationary sequence under $\mathbb{P}_\pi$, $\liminf_{n\to\infty} n^{-1} S_n$ and $\limsup_{n\to\infty} n^{-1} S_n$ are $\mathbb{P}_\pi$-a.s. constant. Therefore, it suffices to prove only $\mathbb{P}_i$-a.s. equality in (i)–(iii) for some fixed $i \in \mathscr{S}$. Moreover, we can restrict our effort on proving (i) and (iii) as usual. The assumption of $\mathbb{E}_i|S_{\tau(i)}| = \infty$ guarantees non-trivial MRWs and by the fluctuation type trichtomy we only have to deal with the first part of the assertions.

(i) Suppose $\lim_{n\to\infty} S_n = \infty$ a.s. and thus $\mathbb{E}_i S_{\tau(i)}^+ = \infty$. Then, (4.1), (4.2) and Theorem 1.1 yield

$$\liminf_{n\to\infty} \frac{S_n}{n} \;\geq\; \liminf_{n\to\infty} \frac{S_{\tau_n(i)} - D_{n+1}^i}{\tau_{n+1}(i)}$$

$$= \liminf_{n\to\infty} \frac{n}{\tau_{n+1}(i)} \frac{\sum_{k=1}^{n}(S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+}{n}(1-o(1))$$
$$= \infty \qquad \text{a.s.}$$

(iii): Referring to the fluctuations of the embedded random walks, we obtain at least either $\liminf_{n\to\infty} n^{-1} S_n = -\infty$ or $\limsup_{n\to\infty} n^{-1} S_n = \infty$ a.s. W.l.o.g., we assume $\limsup_{n\to\infty} n^{-1} S_n = \infty$ a.s. If $\liminf_{n\to\infty} n^{-1} S_n = -c$ a.s. for some $c \in \mathbb{R}_\geq$, we derive $\liminf_{n\to\infty} n^{-1}[S_n + n(c+1)] = 1 > 0$. Consequently, $S_n + n(c+1) \xrightarrow{n\to\infty} \infty$ a.s., but, as shown in (i), this already implies

$$\infty = \liminf_{n\to\infty} n^{-1}[S_n + n(c+1)] = \liminf_{n\to\infty} n^{-1} S_n + c + 1 \qquad \text{a.s.},$$

which is clearly not true. □

Example 2.7 also showed that $\limsup S_n \in \mathbb{R}$ a.s. is possible, when $\mathbb{E}_\pi|X_1| = \infty$. Hence, it may appear dubious whether $\lim_{n\to\infty} n^{-1} S_n = \mu \in \mathbb{R}$ if and only if $\mathbb{E}_\pi X_1 = \mu$ is true, which would be a more general law of large numbers (e.g., [12, Theorem 5.4.2]). The affirmation that this assertion is not true can be extracted from Section 7.

# 5. On the Ladder Chain $(M_n^>)_{n\geq 0}$

Set $\sigma_1^> := \sigma^>$ and $\sigma_1^\leqslant := \sigma^\leqslant$. Inductively define the $n$-th strictly ascending ladder epochs by

$$\sigma_n^> := \inf\left\{k \geq \sigma_{n-1}^> + 1 : S_k > S_{\sigma_{n-1}^>}\right\}, \qquad n \geq 2.$$

In addition, $\sigma_0^> := 0$ and $\sigma_0^\leqslant := 0$. Set $M_n^> := M_{\sigma_n^>} \mathbf{1}_{\{\sigma_n^> < \infty\}} + M_{\sigma_*^>} \mathbf{1}_{\{\sigma_n^> = \infty\}}$, where $\sigma_*^> := \sup\{n \geq 1 : \sigma_n^> < \infty\}$. Define ${}^\#\sigma_n^>$, ${}^\# M_n^>$, $\sigma_n^\leqslant$, $M_n^\leqslant$ etc. analogously. Given a non-trivial MRW, $(M_n^>)_{n\geq 0}$ forms a Markov chain on $\mathscr{S}^> \subset \mathscr{S}$ if $\sigma_n^> < \infty$ for all $n \geq 1$. i.e. if $(S_n)_{n\geq 0}$ is either positive divergent or oscillating. In this case, we examine the existence of a stationary distribution of $(M_n^>)_{n\geq 0}$. The main result of this section, Theorem 5.3, will be crucial for future proofs.

**Lemma 5.1**

$$\pi_i \mathbb{E}_i \sigma^\leqslant = \sum_{n\geq 0} \mathbb{P}_\pi({}^\# M_n^> = i, {}^\#\sigma_n^> < \infty) \tag{5.1}$$

*is true for all* $i \in \mathscr{S}$. *The assertion remains true, when replacing* $(\sigma^\leqslant, {}^\# M_n^>, {}^\#\sigma^>)$ *with* $(\sigma^>, {}^\# M_n^\leqslant, {}^\#\sigma^\leqslant)$.

**Proof.** We begin with the obvious equation of

$$\pi_i \frac{1 - \mathbb{E}_i t^{\sigma^\leqslant}}{1-t} = \mathbb{E}_\pi\left(\mathbf{1}_{\{M_0 = i\}} \sum_{n=0}^{\sigma^\leqslant - 1} t^n\right)$$

for all $t \in (0,1)$ and $i \in \mathscr{S}$. Using (2.3) and (2.4), we obtain

$$\mathbb{E}_\pi\left(\mathbf{1}_{\{M_0=i\}} \sum_{n=0}^{\sigma^\leqslant - 1} t^n\right) - \mathbb{P}_\pi({}^\# M_0^> = i)$$

$$
\begin{aligned}
&= \mathbb{E}_\pi\Big(\sum_{n\geq 1} t^n \mathbf{1}_{\{M_0=i, \sigma^\leqslant > n\}}\Big) \\
&= \mathbb{E}_\pi\Big(\sum_{n\geq 1} t^n \mathbf{1}_{\{M_0=i, S_k>0 \text{ for } 1\leq k\leq n\}}\Big) \\
&= \sum_{n\geq 1} t^n \sum_{j\in\mathscr{S}} \mathbb{P}_\pi(M_0=i,\, M_n=j,\, S_k>0 \text{ for } 1\leq k\leq n) \\
&= \sum_{n\geq 1} t^n \sum_{j\in\mathscr{S}} \mathbb{P}_\pi({}^\# M_0=j,\, {}^\# M_n=i,\, {}^\# S_n - {}^\# S_k>0 \text{ for } 0\leq k< n) \\
&= \sum_{n\geq 1} t^n \sum_{\ell\geq 1} \mathbb{P}_\pi({}^\#\sigma_\ell^> = n,\, {}^\# M_n=i) \\
&= \mathbb{E}_\pi\Big(\sum_{n\geq 1} t^n \mathbf{1}_{\{{}^\# M_n^>=i, {}^\#\sigma_n^> <\infty\}}\Big).
\end{aligned}
$$

Finally, let $t$ tend to 1 to obtain the assertion. □

**Proposition 5.2** *Let $(M_n, S_n)_{n\geq 0}$ be a non-trivial MRW. The following assertions are true:*

*(i) $(M_n^>)_{n\geq 0}$ is a recurrent Markov chain on $\mathscr{S}^> \subset \mathscr{S}$ if and only if $\mathbb{E}_i {}^\#\sigma^\leqslant = \infty$ for some $i\in\mathscr{S}$.*

*(ii) Suppose $\mathscr{S}^> := \{i\in\mathscr{S} : \mathbb{E}_i {}^\#\sigma^\leqslant = \infty\} \neq \emptyset$. Then, $(M_n^>)_{n\geq 0}$ is irreducible on $\mathscr{S}^>$ and $\mathbb{P}_i(M_n^> \in \mathscr{S}^>$ eventually$) = 1$ for all $i\in\mathscr{S}$.*

**Proof.** (i) At the beginning, we point out that $\mathbb{E}_i {}^\#\sigma^\leqslant = \infty$ excludes negative divergence of $(S_n)_{n\geq 0}$, because otherwise $S_{\tau(i)} \stackrel{d}{=} {}^\# S_{{}^\#\tau(i)}$ under $\mathbb{P}_i$ yields negative divergence of $({}^\# S_{{}^\#\tau_n(i)})_{n\geq 0}$, which implies $\mathbb{E}_i {}^\#\sigma^\leqslant < \infty$ (cf. Lemma 3.5 for $\alpha = 0$). Therefore, $(M_n^>)_{n\geq 0}$ forms a Markov chain on $\mathscr{S}^> \subset \mathscr{S}$. $(M_n^>)_{n\geq 0}$ has a recurrent state $i\in\mathscr{S}$ if and only if $\sum_{n\geq 0}\mathbb{P}_\pi(M_n^> = i) = \infty$, because

$$
\begin{aligned}
\pi_i \sum_{n\geq 0} \mathbb{P}_i(M_n^> = i) &\leq \sum_{n\geq 0} \mathbb{P}_\pi(M_n^> = i) \\
&= \sum_{j\in\mathscr{S}} \pi_j \, \mathbb{E}_j\Big(\sum_{n\geq 0} \mathbf{1}_{\{M_n^> = i\}}\Big) \\
&\leq \Big[\pi_i + \sum_{j\in\mathscr{S}\setminus\{i\}} \pi_j \, \mathbb{P}_j(\tau^>(i)<\infty)\Big] \mathbb{E}_i\Big(\sum_{n\geq 0} \mathbf{1}_{\{M_n^> = i\}}\Big) \\
&= \sum_{n\geq 0} \mathbb{P}_i(M_n^> = i).
\end{aligned}
$$

Consequently, an appeal to Lemma 5.1 proves (i).

(ii) Let $\mathscr{S}^{>,i}$ be the communicating class of $i\in\mathscr{S}^>$ corresponding to the transition mechanism of $(M_n^>)_{n\geq 0}$. We employ a coupling argument in order to show that $\mathbb{P}_j(M_n^> \in \mathscr{S}^{>,i}$ eventually$) = 1$ for all $j\in\mathscr{S}$, which proves our claim. Pick some $j\in\mathscr{S}$ and let $(M_n', S_n')_{n\geq 0}$, $(M_n'', S_n'')_{n\geq 0}$ be two independent MRWs with the same transition kernel as $(M_n, S_n)_{n\ge 0}$ and initial values $M'_0 = j$, $M''_0 = i \in \mathscr{S}^{>,i}$. Note that $\mathbb{P}(M''^{>}_n \in \mathscr{S}^{>,i}$ for all $n \ge 0) = 1$. Let $T$ be the a.s. finite coupling time for the two driving chains, i.e. $T = \inf\{n \ge 0 : M'_n = M''_n\}$, and define the coupling process

$$(\widehat{M}_n, \widehat{X}_n) \ := \ \begin{cases} (M'_n, X'_n), & \text{if } n \le T, \\ (M''_n, X''_n), & \text{if } n > T, \end{cases}$$

which is a copy of $(M'_n, X'_n)_{n\ge 0}$. It follows that $\widehat{S}_{T+n} - S''_{T+n} = S'_T - S''_T$ for all $n \ge 0$, and from this it is easily inferred that the ladder epochs $\widehat{\sigma}^{>}_m$ and $\sigma'^{>}_n$ will eventually be synchronised in the sense that

$$\widehat{\sigma}^{>}_{\kappa+n} \ = \ \sigma'^{>}_{\nu+n}$$

for suitable finite random times $\kappa, \nu$ and all $n \ge 0$. As a consequence,

$$\begin{aligned} \mathbb{P}_j(M^{>}_n \in \mathscr{S}^{>,i} \text{ eventually}) \ &= \ \mathbb{P}(M'^{>}_n \in \mathscr{S}^{>,i} \text{ eventually}) \\ &= \ \mathbb{P}(\widehat{M}^{>}_n \in \mathscr{S}^{>,i} \text{ eventually}) \\ &= \ \mathbb{P}(M''^{>}_n \in \mathscr{S}^{>,i} \text{ eventually}) \ = \ 1. \end{aligned}$$

□

We will point out in Section 6.5 that $\mathbb{E}_i{}^{\#}\sigma^{\leqslant} < \infty$ and $\mathbb{E}_i{}^{\#}\sigma^{\geqslant} < \infty$ for all $i \in \mathscr{S}$ is possible for a MRW. Consequently, there does not need to be any recurrent ladder chain. It is not clear whether $(M^{>}_n)_{n\ge 0}$ can be null recurrent. There is one important case, when we can rule out null-recurrency.

Let

$$\begin{aligned} \mathbf{P}^{>} \ &:= \ \left(\mathbb{P}_i(M^{>}_1 = j, \sigma^{>} < \infty)\right)_{i,j\in\mathscr{S}}, \\ {}^{\star}\mathbf{P}^{\leqslant} \ &:= \ \left(\frac{\pi_j}{\pi_i}\,\mathbb{P}_j({}^{\#}M^{\leqslant}_1 = i, {}^{\#}\sigma^{\leqslant} < \infty)\right)_{i,j\in\mathscr{S}} \end{aligned}$$

and $\mathbf{I}$ be the associated unity matrix. Asmussen's Wiener-Hopf factorisation for MRWs [8, Theorem 4.1] entails the useful identity

$$\mathbf{I} - \mathbf{P} \ = \ (\mathbf{I} - {}^{\star}\mathbf{P}^{\leqslant})(\mathbf{I} - \mathbf{P}^{>}). \tag{5.2}$$

**Theorem 5.3** *Let $(M_n, S_n)_{n\ge 0}$ be a MRW with positive divergent dual MRW. Then, $(M^{>}_n)_{n\ge 0}$ is a Markov chain with unique stationary distribution $\pi^{>} := (\pi^{>}_i)_{i\in\mathscr{S}}$ given by*

$$\pi^{>}_i \ = \ \frac{\pi_i\,\mathbb{P}_i({}^{\#}\sigma^{\leqslant} = \infty)}{\mathbb{P}_\pi({}^{\#}\sigma^{\leqslant} = \infty)}.$$

*Moreover, $\mathscr{S}^{>} = \{i \in \mathscr{S} : \mathbb{P}_i({}^{\#}\sigma^{\leqslant} = \infty) > 0\}$, $\mathbb{P}_i(M^{>}_n \in \mathscr{S}^{>}$ eventually$) = 1$ for all $i \in \mathscr{S}$ and*

$$\mathbb{E}_{\pi^{>}}\sigma^{>} \ = \ \frac{1}{\mathbb{P}_\pi({}^{\#}\sigma^{\leqslant} = \infty)} \ < \ \infty. \tag{5.3}$$

**Proof.** Positive divergence of the dual MRW provides $\mathbb{P}_\pi(^\#\sigma^\leqslant = \infty) > 0$. Hence,

$$\pi^> \;:=\; \frac{1}{\mathbb{P}_\pi(^\#\sigma^\leqslant = \infty)}\,\pi\,(\mathbf{I} - {}^\star\mathbf{P}^\leqslant) \;=\; \left(\frac{\pi_i\,\mathbb{P}_i(^\#\sigma^\leqslant = \infty)}{\mathbb{P}_\pi(^\#\sigma^\leqslant = \infty)}\right)_{i\in\mathscr{S}}$$

defines a probability vector. Now, $\pi\,\mathbf{P} = \mathbf{P}$ and (5.2) yield

$$\pi^>\,(\mathbf{I} - \mathbf{P}^>) \;=\; \frac{1}{\mathbb{P}_\pi(^\#\sigma^\leqslant = \infty)}\,\pi\,(\mathbf{I} - {}^\star\mathbf{P}^\leqslant)\,(\mathbf{I} - \mathbf{P}^>) \;=\; \frac{1}{\mathbb{P}_\pi(^\#\sigma^\leqslant = \infty)}\,\pi\,(\mathbf{I} - \mathbf{P}) \;=\; 0.$$

(5.3) takes more effort to prove so that we refer to [4, Corollary 2.5]. The remaining assertions follow from Proposition 5.2. ☐

# 6. On Fluctuation-Theoretic Quantities

We are well-prepared to study in how far we can generalise Theorems 1.3–1.6 to MRWs. In fact, we will find equivalent conditions for the finiteness of $\mathbb{E}_i|\min_{n\geq 0} S_n|^\alpha$ for some $i \in \mathscr{S}$. Moreover, we will show that the translated statements of Theorem 1.3 are not equivalent anymore and separate into four sets of equivalent conditions. Firstly, the set of equivalences of $\mathbb{E}_i\rho(0)^\alpha < \infty$ for some $i \in \mathscr{S}$, secondly, those of

$$\mathbb{E}_i\Lambda(0)^\alpha \;<\; \infty \qquad \text{for some } i \in \mathscr{S}, \tag{6.1}$$

then equivalences of

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0) \;<\; \infty \qquad \text{for some } i \in \mathscr{S} \tag{6.2}$$

and lastly, $\mathbb{E}_i\sigma^>(x)^{1+\alpha} < \infty$ for all $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$. In Section 6.5, a discussion of the latter condition will take place and an intriguing example will reveal the difficulty of finding an equivalent condition. Furthermore, we will not be able to find an equivalent integral criterion for (6.1) in the case $\alpha \in (0,1)$.

In the study of the power moments of the above-mentioned quantities, one can generally not dispense with the condition $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some $i \in \mathscr{S}$ given $\alpha > 0$. For example, consider a MRW with $K_{sj}$ equal $\delta_0$ if $j \neq i$ and unequal $\delta_0$ otherwise for all $s \in \mathscr{S}$. In addition, $K_{si}$, $s \in \mathscr{S}$, is chosen such that $\mathbb{E}_i S_{\tau(i)} > 0$. Then, $\rho(0)+1, \Lambda(0)+1, \sigma^>(0) \in \{\tau_n(i) : n \geq 1\}$ $\mathbb{P}_i$-a.s., which entails that we need a moment assumption on $\tau(i)$.

Under the assumption of $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$, $\alpha > 0$, for some $i \in \mathscr{S}$ the following implications are true:

$$\mathbb{E}_i\Big|\min_{n\geq 0} S_n\Big|^\alpha \;<\; \infty \quad\Rightarrow\quad \mathbb{E}_i\rho(0)^\alpha \;<\; \infty \quad \begin{array}{l} \nearrow\!\!\!\nearrow \;\; \mathbb{E}_i\Lambda(0)^\alpha \;<\; \infty \\[2ex] \searrow\!\!\!\searrow \;\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0) \;<\; \infty. \end{array}$$

The first implication will become clear by comparing the integral criteria in Theorem 6.1 and Theorem 6.6, while another follows directly from $\rho(0) \geq \Lambda(0)$ a.s. The remaining implication can be verified by

$$\mathbb{E}_i\rho(0)^\alpha \asymp \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\rho(0) > n) \geq \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0).$$

In contrast to ordinary random walks, (6.1) and (6.2) do not generally imply $\mathbb{E}_i\rho(0)^\alpha < \infty$. In particular, (6.2) does not even require positive divergence for $\alpha < 1$ as it will be explained in Section 6.4. The dependencies between (6.1) and (6.2) will be discussed in Proposition 6.15.

Moreover, given $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ and $\alpha > 0$, the following assertions are true:

$$\begin{array}{ccc} \mathbb{E}_i\Lambda(0)^\alpha < \infty & & \\ & \searrow & \\ & & \mathbb{E}_i\sigma^{>}(x)^{1+\alpha} < \infty \quad \text{for all } (x,i) \in \mathbb{R}_{\geq} \times \mathscr{S}. \\ & \nearrow & \\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0) < \infty & & \end{array}$$

Both implications can be verified by the use of Theorem 1.3, Lemma 3.5 and the inequalities

$$\mathbb{E}_i\Lambda(0)^\alpha \geq \mathbb{E}_i\Big(\sum_{n\geq 1}\mathbf{1}_{\{S_{\tau_n(i)}\leq 0\}}\Big)^\alpha$$

and

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0) \gtrsim \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \leq 0)$$

(see Lemma C.3).

## 6.1. Finiteness of Power Moments of $|\min_{n\geq 0} S_n|$

The counterpart of Theorem 1.4 does not require a moment assumption on the return times.

**Theorem 6.1** *Let $(M_n, S_n)_{n\geq 0}$ be a positive divergent MRW and $\alpha > 0$. The following conditions are equivalent:*

*(i)* $\mathbb{E}_i|\min_{n\geq 0} S_n|^\alpha < \infty$ *for some (hence all)* $i \in \mathscr{S}$.

*(ii)* $\mathbb{E}_i[(D^i)^\alpha\, J_i(D^i)] < \infty$ *for some (hence all)* $i \in \mathscr{S}$.

In comparison to Theorem 1.4, condition (i) of the latter theorem does still trivially entail

$$\mathbb{E}_i|S_{\sigma^{\leqslant}(-x)}|^\alpha\,\mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty \qquad \text{for all } (x,i) \in \mathbb{R}_{\geq} \times \mathscr{S},$$

but the reverse implication is generally not true (see Example 6.4). Furthermore, (i) does not imply

$$\mathbb{E}_i\Big(\max_{0\leq n\leq\rho(x)} |S_n|\Big)^\alpha < \infty \qquad \text{for some } (x,i) \in \mathbb{R}_{\geq} \times \mathscr{S}$$

(see Example 6.5).

Preliminarily, we give two auxiliary lemmata. Given the ladder chain $(M_n^>)_{n\geq 0}$ being recurrent on some set $\mathscr{S}^> \subset \mathscr{S}$, define $\sigma_0^{>,i} := 0$ and

$$\sigma_n^{>,i} \; := \; \inf\{k \geq \sigma_{n-1}^{>,i} + 1 : M_k^> = i\}, \qquad n \geq 1,$$

for $i \in \mathscr{S}^>$.

**Lemma 6.2** *Consider a MRW $(M_n, S_n)_{n\geq 0}$ with positive divergent dual MRW. Let $i \in \mathscr{S}^>$, i.e. with $\mathbb{P}_i(^{\#}\sigma^{\leqslant} = \infty) > 0$, then*

$$\mathbb{E}_i(S_{\sigma^{>,i}} \wedge y) \; \asymp \; \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \qquad \textit{as } y \to \infty.$$

**Proof.** On the one hand, use $S_{\sigma^{>,i}} \geq S_{\tau^>(i)} \geq S_{\tau(i)}^+$ to infer

$$\mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \; \lesssim \; \mathbb{E}_i(S_{\sigma^{>,i}} \wedge y).$$

For the other side, set $\kappa := \inf\{n \geq 1 : \tau_n^>(i) = \sigma_1^{>,i}\}$. $\kappa$ is a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$,

$$\mathcal{F}_n \; := \; \sigma\Big(\tau_1^>(i), \ldots, \tau_n^>(i), (M_k, X_k)_{1\leq k\leq \tau_n^>(i)}\Big).$$

Wald's equation yields

$$\begin{aligned} \mathbb{E}_i(S_{\sigma^{>,i}} \wedge y) \; &= \; \mathbb{E}_i\left[\left(\sum_{k=1}^{\kappa}(S_{\tau_k^>(i)} - S_{\tau_{k-1}^>(i)})\right) \wedge y\right] \leq \; \mathbb{E}_i\left[\sum_{k=1}^{\kappa}(S_{\tau_k^>(i)} - S_{\tau_{k-1}^>(i)}) \wedge y\right] \\ &= \; \mathbb{E}_i\kappa \cdot \mathbb{E}_i(S_{\tau^>(i)} \wedge y) \end{aligned}$$

and, by an appeal to Lemma 3.3 (v), it remains to show $\mathbb{E}_i\kappa < \infty$ to conclude

$$\mathbb{E}_i(S_{\sigma^{>,i}} \wedge y) \; \lesssim \; \mathbb{E}_i(S_{\tau(i)}^+ \wedge y) \qquad \text{as } y \to \infty.$$

By Theorem 5.3, $(M_n^>)_{n\geq 0}$ is positive recurrent on $\mathscr{S}^>$ with stationary distribution $\pi^>$ and $\mathbb{E}_{\pi^>}\sigma^> < \infty$. Notice that $(M_n^>, \sigma_n^>)_{n\geq 0}$ forms a MRW with embedded random walk $(\sigma_n^{>,i})_{n\geq 0}$.. Therefore, the identity (2.2) yields

$$\mathbb{E}_i\kappa \; \leq \; \mathbb{E}_i\sigma^{>,i} \; \leq \; (\pi_i^>)^{-1} \cdot \mathbb{E}_{\pi^>}\sigma^> \; < \; \infty.$$

□

In this section, we need the following lemma only in the case $\mathbb{P}_i(\sigma^{\leqslant} = \infty) > 0$ and $\alpha = 0$, which is much easier to prove. The general version is needed for the proof of Theorem 6.6 "(iv)⇒(ii)".

**Lemma 6.3** *Consider a positive divergent MRW with* $\mathbb{P}_i(\sigma^{\leqslant}(-x)=\infty)>0$ *for some* $x\in\mathbb{R}_{\geq}$. *Then,*

$$J_i(y)^{1+\alpha} \ \lesssim\ \mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha}\,\mathbf{1}_{\{S_{\tau_n(i)}\leq y,\,\min_{1\leq k\leq\tau_n(i)}S_k>-x\}}\Big)\qquad \text{as } y\to\infty$$

*for any* $\alpha\geq 0$.

**Proof.** Choose $j\in\mathscr{S}$ with $\mathbb{P}_j(\sigma^{\leqslant}=\infty)>0$. Observe that positive divergence ensures the existence of such state, since otherwise $\mathbb{P}_\pi(\sigma^{\leqslant}=\infty)=0$ and $\lim_{n\to\infty}S_{\sigma_n^{\leqslant}}=-\infty$ a.s. Hence, $j\in{}^{\#}\mathscr{S}^{>}$ and we can define $({}^{\#}\sigma_n^{>,j})_{n\geq 0}$ as the sequence of ladder epochs in $j$ in terms of the dual MRW. Since $\mathbb{P}_i(\sigma^{\leqslant}(-x)=\infty)>0$, there exists $x_1\in\mathbb{R}_{\geq}$ and $n_1,n_2\in\mathbb{N}$ such that

$$E\ :=\ \Big\{\min_{1\leq k\leq n_1}S_k>-x,S_{n_1}\leq x_1,M_{n_1}=j,\tau_{n_2}(i)\leq n_1<\tau_{n_2+1}(i)\Big\}$$

has positive probability under $\mathbb{P}_i$. We can estimate

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha}\,\mathbf{1}_{\{S_{\tau_n(i)}\leq y,\,\min_{1\leq k\leq\tau_n(i)}S_k>-x\}}\Big)\\
&\geq \mathbb{E}_i\Big(\sum_{n\geq n_2+1}\tau_n(i)^{\alpha}\,\mathbf{1}_{E\cap\{S_{\tau_n(i)}-S_{n_1}\leq y-x_1,\,\min_{n_1<k\leq\tau_n(i)}(S_k-S_{n_1})>0\}}\Big)\\
&\geq \mathbb{P}_i(E)\,\mathbb{E}_j\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha}\,\mathbf{1}_{\{S_{\tau_n(i)}\leq y-x_1,\,\min_{1\leq k\leq\tau_n(i)}S_k>0\}}\Big)\\
&\gtrsim \sum_{n\geq 1}\sum_{m\geq 1}m^{\alpha}\,\mathbb{P}_j\Big(S_{\tau_n(i)}\leq y-x_1,\,S_k>0\text{ for all }1\leq k\leq\tau_n(i),\,\tau_m(j)\leq\tau_n(i)<\tau_{m+1}(j)\Big).
\end{aligned}$$

Using (2.3) and (2.4), we derive

$$\begin{aligned}
&\mathbb{P}_j\Big(S_{\tau_n(i)}\leq y-x_1,\,S_k>0\text{ for all }1\leq k\leq\tau_n(i),\,\tau_m(j)\leq\tau_n(i)<\tau_{m+1}(j)\Big)\\
&=\frac{\pi_i}{\pi_j}\,\mathbb{P}_i\Big(y-x_1\geq{}^{\#}S_{{}^{\#}\tau_{m+1}(j)}>{}^{\#}S_k\text{ for all }0\leq k<{}^{\#}\tau_{m+1}(j),\\
&\qquad\qquad {}^{\#}\tau_{n-1}(i)\leq{}^{\#}\tau_{m+1}(j)<{}^{\#}\tau_n(i)\Big)\\
&=\frac{\pi_i}{\pi_j}\sum_{\ell=1}^{m+1}\mathbb{P}_i\Big({}^{\#}S_{{}^{\#}\sigma_\ell^{>,j}}\leq y-x_1,\,{}^{\#}\tau_{n-1}(i)\leq{}^{\#}\sigma_\ell^{>,j}<{}^{\#}\tau_n(i),\,{}^{\#}\tau_{m+1}(j)={}^{\#}\sigma_\ell^{>,j}\Big).
\end{aligned}$$

Insertion into the above term yields

$$\begin{aligned}
&\sum_{n\geq 1}\sum_{m\geq 1}m^{\alpha}\,\mathbb{P}_j\Big(S_{\tau_n(i)}\leq y-x_1,\,S_k>0\text{ for all }1\leq k\leq\tau_n(i),\,\tau_m(j)\leq\tau_n(i)<\tau_{m+1}(j)\Big)\\
&\geq\frac{\pi_i}{\pi_j}\sum_{n\geq 1}\sum_{m\geq 1}\sum_{\ell=1}^{m+1}(\ell-1)^{\alpha}\,\mathbb{P}_i\Big({}^{\#}S_{{}^{\#}\sigma_\ell^{>,j}}\leq y-x_1,\,{}^{\#}\tau_{n-1}(i)\leq{}^{\#}\sigma_\ell^{>,j}<{}^{\#}\tau_n(i),\\
&\qquad {}^{\#}\tau_{m+1}(j)={}^{\#}\sigma_\ell^{>,j}\Big)
\end{aligned}$$

$$\asymp \sum_{\ell\geq 1} \ell^{\alpha}\,\mathbb{P}_i({}^{\#}S_{\#\sigma_\ell^{>,j}} \leq y - x_1).$$

Now, choose $x_2 > 0$ with $\mathbb{P}_i({}^{\#}S_{\#\sigma^{>,j}} \leq x_2) =: p > 0$ and use Theorem 1.6, Lemma 3.3, Lemma 6.2 and $S_{\tau(j)} \overset{d}{=} {}^{\#}S_{\#\tau(j)}$ under $\mathbb{P}_j$ to infer

$$\begin{aligned}
&\sum_{\ell\geq 1} \ell^{\alpha}\,\mathbb{P}_i({}^{\#}S_{\#\sigma_\ell^{>,j}} \leq y - x_1) \;\geq\; p\sum_{\ell\geq 1} \ell^{\alpha}\,\mathbb{P}_j({}^{\#}S_{\#\sigma_\ell^{>,j}} \leq y - x_1 - x_2)\\
&\asymp \left(\frac{y}{\mathbb{E}_j({}^{\#}S_{\#\sigma^{>,j}} \wedge y)}\right)^{1+\alpha} \asymp \left(\frac{y}{\mathbb{E}_j({}^{\#}S^{+}_{\#\tau(j)} \wedge y)}\right)^{1+\alpha}\\
&\asymp J_i(y)^{1+\alpha},
\end{aligned}$$

which finishes the proof. □

**Proof of Theorem 6.1.** At the beginning, we point out that due to Lemma 3.4 condition (ii) holds either for all $i \in \mathscr{S}$ or none. Simultaneously, (i) holds either for all $i \in \mathscr{S}$ or none, because

$$\begin{aligned}
\mathbb{E}_i\Big|\min_{n\geq 0} S_n\Big|^{\alpha} \;&=\; \mathbb{E}_i\Big|\min_{n\geq 1} S_n \wedge 0\Big|^{\alpha}\\
&\geq \mathbb{E}_i\Big|\Big(\min_{n\geq \tau(j)+1}(S_n - S_{\tau(j)}) + S_{\tau(j)}\Big)\wedge 0\Big|^{\alpha}\mathbf{1}_{\{S_{\tau(j)}\leq x\}}\\
&\geq\; p\cdot \mathbb{E}_j\Big|\Big(\min_{n\geq 1} S_n + x\Big)\wedge 0\Big|^{\alpha}\\
&\asymp\; \mathbb{E}_j\Big|\min_{n\geq 0} S_n\Big|^{\alpha},
\end{aligned}$$

where $x \in \mathbb{R}_{\geq}$ is chosen such that $p := \mathbb{P}_i(S_{\tau(j)} \leq x) > 0$. Consequently, it suffices to prove the equivalence of (i) and (ii) for some fixed $i \in \mathscr{S}$ with $\mathbb{P}_i(\sigma^{\leqslant} = \infty) > 0$. Define

$$\begin{aligned}
\eta_1 \;&:=\; \inf\{k \geq 1 : S_{\tau_{k-1}(i)} - D_k^i < 0\},\\
\eta_n \;&:=\; \inf\{k \geq \eta_{n-1}+1 : S_{\tau_{k-1}(i)} - S_{\tau_{\eta_{n-1}}(i)} - D_k^i < 0\}, \qquad n \geq 2,
\end{aligned}$$

and

$$\kappa \;:=\; \inf\{n \geq 1 : \eta_n = \infty\}.$$

Since

$$\Big|\min_{n\geq 0} S_n\Big| \;\leq\; \sum_{k=1}^{\kappa-1}\Big|S_{\tau_{\eta_k-1}(i)} - S_{\tau_{\eta_{k-1}}(i)} - D^i_{\eta_k}\Big|$$

and $(S_{\tau_{\eta_k-1}(i)} - S_{\tau_{\eta_{k-1}}(i)} - D^i_{\eta_k})_{1\leq k<\kappa}$ are i.i.d. under $\mathbb{P}_i$, it follows as in [29, (v)$_0$ ⇔(vii)] that (i) holds if and only if

$$\mathbb{E}_i|S_{\tau_{\eta-1}(i)} - D^i_{\eta}|^{\alpha}\,\mathbf{1}_{\{\eta<\infty\}} \;<\; \infty. \tag{6.3}$$

We finish the proof by showing equivalence of (6.3) and (ii). Use Lemma C.6 to obtain

$$F_i(x) \;:=\; \mathbb{P}_i(-S_{\tau_{\eta-1}(i)} + D^i_{\eta} \geq x, \eta < \infty)$$

$$
\begin{aligned}
&= \sum_{n\ge 1} \mathbb{P}_i(-S_{\tau_{n-1}(i)} + D_n^i \ge x, \eta = n)\\
&\le \sum_{n\ge 1} \mathbb{P}_i(S_{\tau_{n-1}(i)} \ge 0, -S_{\tau_{n-1}(i)} + D_n^i \ge x)\\
&= \int_{[x,\infty)} \sum_{n\ge 1} \mathbb{P}_i(0 \le S_{\tau_{n-1}(i)} \le y - x)\ \mathbb{P}_i(D^i \in dy)\\
&\asymp \int_{[x,\infty)} J_i(y-x)\ \mathbb{P}_i(D^i \in dy)
\end{aligned}
$$

for $x \in \mathbb{R}_\ge$. Since $J_i$ is non-decreasing, we infer

$$
\begin{aligned}
\mathbb{E}_i|S_{\tau_{\eta-1}(i)} - D_\eta^i|^\alpha \mathbf{1}_{\{\eta<\infty\}} &\asymp \int_0^\infty x^{\alpha-1} F_i(x)\ dx\\
&\lesssim \int_0^\infty \left( x^{\alpha-1} \int_{[x,\infty)} J_i(y-x)\ \mathbb{P}_i(D^i \in dy) \right) dx\\
&\le \int \left( \int_0^y x^{\alpha-1} J_i(y)\ dx \right) \mathbb{P}_i(D^i \in dy)\\
&\asymp \int y^\alpha J_i(y)\ \mathbb{P}_i(D^i \in dy)\\
&= \mathbb{E}_i[(D^i)^\alpha J_i(D^i)].
\end{aligned}
$$

On the other hand, Lemma 6.3 delivers

$$
\begin{aligned}
F_i(x) &= \int_{[x,\infty)} \sum_{n\ge 1} \mathbb{P}_i\Big( S_{\tau_{n-1}(i)} \le y - x, \min_{1\le k\le \tau_{n-1}(i)} S_k \ge 0 \Big)\ \mathbb{P}_i(D^i \in dy)\\
&\gtrsim \int_{(x,\infty)} J_i(y-x)\ \mathbb{P}_i(D^i \in dy)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathbb{E}_i|S_{\tau_{\eta-1}(i)} - D_\eta^i|^\alpha \mathbf{1}_{\{\eta<\infty\}} &\gtrsim \int_0^\infty \left( x^{\alpha-1} \int_{[x,\infty)} J_i(y-x)\ \mathbb{P}_i(D^i \in dy) \right) dx\\
&\ge \int \left( \int_0^{y/2} x^{\alpha-1} J_i(y-x)\ dx \right) \mathbb{P}_i(D^i \in dy)\\
&\ge \int \left( \int_0^{y/2} x^{\alpha-1} J_i(y/2)\ dx \right) \mathbb{P}_i(D^i \in dy)\\
&\asymp \mathbb{E}_i[(D^i)^\alpha J_i(D^i)].
\end{aligned}
$$

□

We close this section with the announced examples concerning

$$\mathbb{E}_i|S_{\sigma^\leqslant(-x)}|^\alpha \mathbf{1}_{\{\sigma^\leqslant(-x)<\infty\}} < \infty \qquad \text{for all } (x,i) \in \mathbb{R}_\ge \times \mathscr{S}$$

being not sufficient and

$$\mathbb{E}_i\Big( \max_{0\le n\le \rho(x)} |S_n| \Big)^\alpha < \infty \qquad \text{for some } (x,i) \in \mathbb{R}_\ge \times \mathscr{S}$$

being not necessary for the conditions of Theorem 6.1 to be true.

**Example 6.4** Let $(M_n)_{n\ge 0}$ be the general infinite petal flower chain introduced in Example 2.8 with

$$\mathbb{E}\Gamma^{2(1+1/\alpha)} < \infty, \qquad \mathbb{E}\Gamma^{1+\alpha} < \infty \qquad \text{and} \qquad \mathbb{E}\Gamma^{(1+\alpha)(1+1/\alpha)} = \infty$$

for some $\alpha > 1$. In particular, we have $\mathbb{E}_0\tau(0)^{1+\alpha} < \infty$. Define the increments $(X_n)_{n\ge 1}$ by

$$X_n := \begin{cases} -k^{1/\alpha}, & \text{if } M_n = (\ell, k) \text{ for } k, \ell \in \mathbb{N}, \\ 1 + \sum_{k=1}^{\ell-1} k^{1/\alpha}, & \text{if } M_{n-1} = (\ell, \ell-1),\, M_n = 0 \text{ for } \ell \in \mathbb{N}, \end{cases}$$

and hence $S_{\tau(0)} = 1$ $\mathbb{P}_0$-a.s. For verifying positive divergence of $(S_n)_{n\ge 0}$, it suffices to prove $\mathbb{E}_0 D^0 < \infty$, because $J_i(y) \asymp y$ as $y \to \infty$. Since

$$\sum_{k=1}^{\ell} k^{1/\alpha} \asymp \int_1^{\ell} x^{1/\alpha}\, dx,$$

we infer

$$\sum_{k=1}^{\ell} k^{1/\alpha} \asymp \ell^{1+1/\alpha} \qquad \text{as } \ell \to \infty$$

and thus $D^0 = \sum_{k=1}^{\tau(0)-1} k^{1/\alpha} \approx \tau(0)^{1+1/\alpha}$ $\mathbb{P}_0$-a.s. By construction, we have

$$\mathbb{E}_0 D^0 \le \mathbb{E}_0 (D^0)^2 \lesssim \mathbb{E}_0\tau(0)^{2(1+1/\alpha)} < \infty = \mathbb{E}_0\tau(0)^{(1+\alpha)(1+1/\alpha)} \asymp \mathbb{E}_0(D^0)^{1+\alpha}.$$

Consequently, Theorem 6.1 entails

$$\mathbb{E}_i \Big| \min_{n\ge 0} S_n \Big|^{\alpha} = \infty.$$

Now, we prove $\mathbb{E}_0 |S_{\sigma^{\leqslant}(-x)}|^{\alpha} \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty$ for all $x \in \mathbb{R}_{\ge}$. This does clearly imply

$$\mathbb{E}_i |S_{\sigma^{\leqslant}(-x)}|^{\alpha} \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty \qquad \text{for all } (x,i) \in \mathbb{R}_{\ge} \times \mathscr{S}.$$

Define

$$\kappa(x) := \inf\{n \ge 1 : \sigma^{\leqslant}(-x) \le \tau_n(i)\}$$

and notice that given $\sigma^{\leqslant}(-x) < \infty$

$$|S_{\sigma^{\leqslant}(-x)}| \le \tau_{\kappa(x)}(0)^{1/\alpha} \qquad \mathbb{P}_0\text{-a.s.}$$

Consequently, Wald's equation yields

$$\begin{aligned} \mathbb{E}_0 |S_{\sigma^{\leqslant}(-x)}|^{\alpha} \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} &\le \mathbb{E}_0 \tau_{\kappa(x)}(0)\, \mathbf{1}_{\{\kappa(x)<\infty\}} \le \mathbb{E}_0\tau(0) \cdot \mathbb{E}_0 \kappa(x)\, \mathbf{1}_{\{\kappa(x)<\infty\}} \\ &\lesssim \mathbb{E}_0 \sigma^{\leqslant}(-x)\, \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}}. \end{aligned}$$

Applying Theorem 6.6 from the next section, we obtain that the latter upper bound is finite, since $\mathbb{E}_0(D^0)^2 < \infty$.

**Example 6.5** Consider an i.i.d. sequence $(Z_n)_{n\ge1}$ such that $\mathbb{E}|\min_{n\ge0}\sum_{k=1}^n Z_k|^\alpha<\infty$ for some $\alpha>0$ and $\mathbb{P}(Z=0)>0$. Then, let $(M_n)_{n\ge0}$ be as in Example 2.7 and the distribution of the increments $(X_n)_{n\ge1}$ be defined by

$$X_n \overset{d}{=} \begin{cases} p_{0i}^{-1}, & \text{if } M_{n-1}=0, M_n=i, \\ Z_1-p_{0i}^{-1}, & \text{if } M_{n-1}=i, M_n=0. \end{cases}$$

Notice that $\mathbb{E}_0X_1^\alpha=\sum_{i\ge1}p_{0i}^{1-\alpha}=\infty$ for $\alpha\ge1$ regardless of the choice of $(p_{0i})_{i\in\mathscr{S}}$. For $\alpha\in(0,1)$, let $c:=\sum_{i\ge1}i^{-1/(1-\alpha)}<\infty$ and put $p_{0i}=c^{-1}i^{-1/(1-\alpha)}$ so that

$$\sum_{i\ge1}p_{0i}^{1-\alpha} \asymp \sum_{i\ge1}\frac{1}{i} = \infty.$$

By construction, it holds that

$$\mathbb{E}_0\Big|\min_{n\ge0}S_n\Big|^\alpha = \mathbb{E}\Big|\min_{n\ge0}\sum_{k=1}^n Z_k\Big|^\alpha < \infty.$$

Nevertheless, we have

$$\mathbb{E}_0\Big(\max_{0\le n\le\rho(x)}|S_n|\Big)^\alpha \ge \mathbb{P}(Z=0)\cdot\mathbb{E}_0X_1^\alpha = \infty.$$

Moreover,

$$\mathbb{E}_i\Big(\max_{0\le n\le\rho(x)}|S_n|\Big)^\alpha \ge \mathbb{P}(Z=0)\cdot\mathbb{E}_0\Big(\max_{0\le n\le\rho(x)}|-p_{0i}^{-1}+S_n|\Big)^\alpha \asymp \mathbb{E}_0\Big(\max_{0\le n\le\rho(x)}|S_n|\Big)^\alpha$$

for all $i\in\mathbb{N}$ entails

$$\mathbb{E}_i\Big(\max_{0\le n\le\rho(x)}|S_n|\Big)^\alpha = \infty \qquad \text{for all } (x,i)\in\mathbb{R}_\ge\times\mathscr{S}.$$

## 6.2. Finiteness of Power Moments of $\rho(0)$

As mentioned at the beginning of this section, not all equivalences of Theorem 1.3 remain true, when the conditions are translated to the MRW context. The equivalences of $\mathbb{E}_i\rho(0)^\alpha<\infty$ for some $i\in\mathscr{S}$ and $\alpha>0$ are gathered in the following theorem.

**Theorem 6.6** *Let $(M_n,S_n)_{n\ge0}$ be a positive divergent MRW with $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$, $\alpha>0$, for some (hence all) $i\in\mathscr{S}$. The following conditions are equivalent:*

*(i) $\mathbb{E}_i\rho(x)^\alpha<\infty$ for some (hence all) $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$.*

*(ii) $\mathbb{E}_iJ_i(D^i)^{1+\alpha}<\infty$ for some (hence all) $i\in\mathscr{S}$.*

*(iii) $\mathbb{E}_i\sigma_{\min}^\alpha<\infty$ for some (hence all) $i\in\mathscr{S}$.*

*(iv) $\mathbb{E}_i\sigma^\leqslant(-x)^\alpha\mathbf{1}_{\{\sigma^\leqslant(-x)<\infty\}}<\infty$ for some (hence all) $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$ satisfying $\mathbb{P}_i(\sigma^\leqslant(-x)=\infty)>0$.*

*Furthermore, if* $\mathbb{E}_i|S_{\tau(i)}| < \infty$ *for some (hence all)* $i \in \mathscr{S}$*, (i)–(iv) are equivalent to* $\mathbb{E}_i(D^i)^{1+\alpha} < \infty$ *and to the conditions of Theorem 6.1.*

Given the truth of the first part of the theorem, the last assertion is trivial, since then

$$y^\alpha J_i(y) \asymp y^{1+\alpha} \asymp J_i(y)^{1+\alpha} \qquad \text{as } y \to \infty.$$

Some parts of the proof are accomplished in the subsequent two lemmata.

**Lemma 6.7** *Suppose* $\mathbb{E}_i\rho(x)^\alpha < \infty$ *for some* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$*, then* $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$ *for all* $i \in \mathscr{S}$*.*

**Proof.** Positive divergence ensures the existence of $x_0 \in \mathbb{R}_\geq$ with $p := \mathbb{P}_i(\sigma^\leqslant(-x_0) = \infty) > 0$. In particular, we have $\mathbb{E}_i\,\rho(-x_0)^\alpha \leq \mathbb{E}_i\rho(x)^\alpha < \infty$. Now, use

$$\begin{aligned}\{\rho(-x_0) > n/2\} \;&\supseteq \bigcup_{n/2<k\leq n} \left\{S_{\tau_k^>(i)} \leq D_{k+1}^{>,i} - x_0\right\}\\ &\supseteq \bigcup_{n/2<k\leq n} \left\{S_{\tau_k^>(i)} \leq D_{k+1}^{>,i} - x_0, (S_\ell - S_{\tau_{k+1}^>(i)}) > -x_0 \text{ for all } \ell > \tau_{k+1}^>(i)\right\}\end{aligned}$$

to conclude

$$\mathbb{P}_i(\rho(-x_0) > n/2) \;\geq\; p \cdot (n/3) \cdot \mathbb{P}_i(S_{\tau_n^>(i)} \leq D_{n+1}^{>,i} - x_0).$$

Hence, $\mathbb{E}_i\rho(-x_0)^\alpha < \infty$ implies

$$\sum_{n\geq 1} n^\alpha\, \mathbb{P}_i(S_{\tau_n^>(i)} \leq D_{n+1}^{>,i} - x_0) \;<\; \infty,$$

which is equivalent to

$$\int_{[x_0,\infty)} \left(\frac{y - x_0}{\mathbb{E}_i(S_{\tau^>(i)} \wedge (y - x_0))}\right)^{1+\alpha} \mathbb{P}_i(D^{>,i} \in dy) \;<\; \infty$$

by Theorem 1.6 (ii). Use Lemma 3.3 and $D^{>,i} \geq D^i$ to derive $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$. Finally, by appeal to Lemma 3.6 the proof is complete. □

**Lemma 6.8** *Given the situation of Theorem 6.6, suppose* $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$ *for some (hence all)* $i \in \mathscr{S}$*. Then,* $\mathbb{E}_i\rho(x)^\alpha < \infty$ *for all* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$*.*

**Proof.** Fix some $i \in \mathscr{S}$. We distinguish between two cases. Under the stated assumptions, either

$$\mathbb{E}_i S_{\tau(i)}^+ \;<\; \infty \qquad \text{and} \qquad \mathbb{E}_i J_i(D^i)^{1+\alpha} \;\asymp\; \mathbb{E}_i(D^i)^{1+\alpha} \;<\; \infty \tag{6.4}$$

or

$$\mathbb{E}_i J_i(D^i)^{1+\alpha} \;<\; \infty \;=\; \mathbb{E}_i S_{\tau(i)}^+. \tag{6.5}$$

At first, suppose (6.4) is true. In particular, we have

$$0 \;<\; \lim_{y\to\infty} A_i(y) \;=\; \mathbb{E}_i S_{\tau(i)} \;<\; \infty,$$

because $(S_{\tau_n(i)})_{n\ge 0}$ is positive divergent. Set $\mu := \mathbb{E}_i S_{\tau(i)}/[2\,\mathbb{E}_i\tau(i)]$. Then,

$$0 \;<\; \mathbb{E}_i\Big(S_{\tau(i)} - \tau(i)\,\mu\Big) \;<\; \infty.$$

Moreover, since $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$, we derive

$$\mathbb{E}_i\Big(D^i + \tau(i)\,\mu\Big)^{1+\alpha} \;<\; \infty.$$

Consequently, $(M_n, S_n - n\,\mu)_{n\ge 0}$ forms a positive divergent MRW and an appeal to Theorem 6.1 yields

$$\mathbb{E}_i\Big|\min_{n\ge 0}\Big(S_n - n\,\mu\Big)\Big|^{\alpha} \;<\; \infty.$$

Finally,

$$\rho(x)\,\mu \;\le\; x - \Big(S_{\rho(x)} - \rho(x)\,\mu\Big) \;\le\; x - \min_{n\ge 0}\Big(S_n - n\,\mu\Big)$$

shows $\mathbb{E}_i\rho(x)^{\alpha} < \infty$ for every $x \in \mathbb{R}_{\ge}$ (cf. [29, (i)⇒(ii)]).

Now, suppose (6.5) is true. Lemma C.5 yields $\mathbb{E}_i\rho(x)^{\alpha} < \infty$ if and only if $\mathbb{E}_i\widehat{\rho}(x)^{\alpha} < \infty$, where

$$\widehat{\rho}(x) \;:=\; \sup\{n \ge 0 : S_{\tau_n(i)} - D^i_{n+1} \le x\}$$

and

$$\{\rho(x) > \tau_n(i)\} \;=\; \{\widehat{\rho}(x) \ge n\}$$

for all $n \in \mathbb{N}$. Let $(Y_n)_{n\ge 1}$ be i.i.d. random variables, which are independent of all other occurring random variables and fulfil $\mathbb{P}(Y_1 = 0) = \mathbb{P}(Y_1 = 1) = 1/2$. Set

$$\zeta_{\theta,n} \;:=\; (S_{\tau_n(i)} - S_{\tau_{n-1}(i)})^+\,\mathbf{1}_{\{Y_n=\theta\}} - D^i_n\,\mathbf{1}_{\{Y_n=1-\theta\}}$$

for $\theta \in \{0,1\}$ and $n \ge 1$. Notice that $(\sum_{k=1}^n \zeta_{\theta,k})_{n\ge 0}$ forms an ordinary random walk with last level $x$ exit time $\rho_\theta(x)$, which can be written as

$$\rho_\theta(x) \;=\; \sup\Big\{n \ge 0 : \sum_{k=1}^n \zeta_{\theta,k} - \zeta^-_{\theta,n+1} \le x\Big\}.$$

Since

$$\begin{aligned} S_{\tau_n(i)} - D^i_{n+1} \;&=\; \sum_{\theta\in\{0,1\}}\Big(\sum_{k=1}^n\Big(\zeta^+_{\theta,k} - (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^-\,\mathbf{1}_{\{Y_k=1-\theta\}}\Big) - D^i_{n+1}\,\mathbf{1}_{\{Y_{n+1}=1-\theta\}}\Big) \\ &\ge\; \sum_{\theta\in\{0,1\}}\Big(\sum_{k=1}^n \zeta_{\theta,k} - \zeta^-_{\theta,n+1}\Big), \end{aligned}$$

we infer

$$\mathbb{E}_i\widehat{\rho}(x)^{\alpha} \;\le\; \mathbb{E}_i[\rho_0(x)\vee\rho_1(x)]^{\alpha} \;\lesssim\; \mathbb{E}_i\rho_0(x)^{\alpha}.$$

Obviously, $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$ implies

$$C(\beta) \ := \ \int \Big( \frac{y}{\mathbb{E}_i(\zeta_{0,1}^+ \wedge y)} \Big)^{1+\beta} \ \mathbb{P}_i(\zeta_{0,1}^- \in dy)$$

to be finite for all $\beta \in [0, \alpha]$. Since (6.5) implies $\mathbb{E}_i|\zeta_{0,1}| = \infty$, the finiteness of $C(0)$ is sufficient for positive divergence of $(\sum_{k=1}^n \zeta_{0,k})_{n\geq 0}$ (cf. remark after Theorem 1.2). Consequently, we derive from Theorem 1.3 that $C(\alpha) < \infty$ implies $\mathbb{E}_i \rho_0(x)^\alpha < \infty$. □

The remaining part of the proof is relatively short, so we put it together:

**Proof of Theorem 6.6.** Lemma 6.7 and Lemma 6.8 have shown the equivalence of (i) and (ii). Since these conditions hold either for all $i \in \mathscr{S}$ or none, it suffices to prove the remaining implications for some fixed $i \in \mathscr{S}$.

"(i)⇒(iii)" follows directly from $\rho(S_{\tau(i)}) - \tau(i) \overset{d}{=} \rho(0)$ under $\mathbb{P}_i$ and $\rho(S_{\tau(i)}) \geq \sigma_{\min}$ $\mathbb{P}_i$-a.s.

"(iii)⇒(iv)" follows directly from the inequality

$$\sigma_{\min} \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} \ \geq \ \sigma^{\leqslant}(-x)\, \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}}$$

for all $x \in \mathbb{R}_{\geq}$.

"(iv)⇒(ii)" Observe that

$$\begin{aligned}
\mathbb{E}_i \sigma^{\leqslant}(-x)^\alpha \mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} \ &= \ \mathbb{E}_i \Big( \sum_{n\geq 1} n^\alpha \mathbf{1}_{\{\sigma^{\leqslant}(-x)=n\}} \Big) \\
&\geq \ \mathbb{E}_i \Big( \sum_{n\geq 1} \sum_{k=\tau_n(i)+1}^{\tau_{n+1}(i)} k^\alpha \mathbf{1}_{\{\sigma^{\leqslant}(-x)=k\}} \Big) \\
&\geq \ \mathbb{E}_i \Big( \sum_{n\geq 1} \tau_n(i)^\alpha \mathbf{1}_{\{\tau_n(i)<\sigma^{\leqslant}(-x)\leq\tau_{n+1}(i)\}} \Big) \\
&= \ \int \mathbb{E}_i \Big( \sum_{n\geq 1} \tau_n(i)^\alpha \mathbf{1}_{\{S_{\tau_n(i)}\leq y-x,\, \min_{1\leq k\leq \tau_n(i)} S_k > -x\}} \Big) \ \mathbb{P}_i(D^i \in dy).
\end{aligned}$$

Finally, an application of Lemma 6.3 yields (ii). □

## 6.3. Finiteness of Power Moments of $\Lambda(0)$

In this section, we search for an equivalent criterion for finite power moments of $\Lambda(0)$. For $i \in \mathscr{S}$ and $\alpha > 0$, define

$$\mathbb{U}_i \ := \ \sum_{n\geq 0} \mathbb{P}_i(S_{\tau_n(i)} \in \cdot)$$

and let $\mathbb{V}_i^\alpha$ be the measure induced by

$$\mathbb{V}_i^\alpha((x,\infty)) \ := \ \mathbb{E}_i \Big( \sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- > x\}} \Big)^\alpha, \qquad x \in \mathbb{R}_{\geq}.$$

**Theorem 6.9** *Let $(M_n, S_n)_{n\ge 0}$ be a positive divergent MRW and $\mathbb{E}_i\,\tau(i)^{1+\alpha} < \infty$ for some (hence all) $i \in \mathscr{S}$, $\alpha > 0$. Consider the following conditions:*

*(i)* $\mathbb{E}_i\,\Lambda(x)^\alpha < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\ge \times \mathscr{S}$.

*(ii)* $\mathbb{E}_i\,J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ *and*

$$\int J_i(y)\ \mathbb{V}_i^\alpha(dy) \ < \ \infty \tag{6.6}$$

*for some (hence all)* $i \in \mathscr{S}$.

*Then, (i) and (ii) are equivalent if $\alpha \ge 1$ and (ii) implies (i) otherwise.*

It is plausible to believe that the missing implication is also true, but the problem is mainly a missing counterpart to the inequality [5, Lemma 5.6] for $\alpha < 1$. As another indication, $\int J_i(y)\ \mathbb{V}_i^\alpha(dy) < \infty$ is weaker than $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$, which will be verified after the proof of the theorem.

In preparation, we prove the following lemmata.

**Lemma 6.10** *Suppose $\mathbb{E}_i\,\tau(i)^\alpha < \infty$ for some (hence all) $i \in \mathscr{S}$, $\alpha > 0$. $\mathbb{E}_i\,\Lambda(x)^\alpha < \infty$ is either true for all $(x,i) \in \mathbb{R}_\ge \times \mathscr{S}$ or none.*

**Proof.** Suppose $\mathbb{E}_i\,\Lambda(0)^\alpha < \infty$ for some $i \in \mathscr{S}$. In particular, we have

$$\mathbb{E}_i\Big(\sum_{n\ge 1} \mathbf{1}_{\{S_{\tau_n(i)}\le 0\}}\Big)^\alpha \ < \ \infty,$$

which is equivalent to $\mathbb{E}_i\nu(x)^{1+\alpha} < \infty$ for all $x \in \mathbb{R}_\ge$. Using [27, Theorem 1.5.1] for $\alpha \in (0,1)$ and [27, Theorem 1.5.2] for $\alpha \ge 1$, we obtain

$$\mathbb{E}_i\,\tau_{\nu(x)}(i)^\alpha \ \lesssim \ \mathbb{E}_i\tau(i)^\alpha \cdot \mathbb{E}_i\nu(x)^{1\vee\alpha} \ < \ \infty$$

for all $x \in \mathbb{R}_\ge$. For arbitrary $j \in \mathscr{S}$, choose $x_1 \in \mathbb{R}_\ge$ such that

$$0 \ < \ p \ := \ \mathbb{P}_i(S_{\tau(j)} \le x_1,\, \tau(i) \ge \tau(j)).$$

Then, we obtain

$$\infty \ > \ \mathbb{E}_i\tau_{\nu(x+x_1)}(i)^\alpha \ \ge \ \mathbb{E}_i\tau_{\nu(x+x_1)}(i)^\alpha\,\mathbf{1}_{\{S_{\tau(j)}\le x_1,\tau(i)\ge\tau(j)\}} \ \ge \ p\mathbb{E}_j\tau_{\nu(x)}(i)^\alpha$$

for all $x \in \mathbb{R}_\ge$. Set

$$\widetilde{\Lambda}(0) \ := \sum_{n\ge\tau_{\nu(x)}(i)+1} \mathbf{1}_{\{S_n - S_{\tau_{\nu(x)}(i)}\le 0\}}.$$

Now, the assertion follows from the arbitrariness of $j \in \mathscr{S}$,

$$\Lambda(x) \ \le \ \tau_{\nu(x)}(i) + \widetilde{\Lambda}(0) \qquad \mathbb{P}_j\text{-a.s.}$$

and $\mathbb{E}_j\,\widetilde{\Lambda}(0)^\alpha = \mathbb{E}_i\,\Lambda(0)^\alpha$. □

**Lemma 6.11** *Given the situation of Theorem 6.9,* $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ *implies*

$$\mathbb{E}_i\Big(\sum_{n\ge 1}\chi_n(i)\,\mathbf{1}_{\{S_{\tau_{n-1}(i)}\le 0\}}\Big)^{\alpha} \;<\; \infty.$$

**Proof.** The assertions can directly be concluded from an application of Theorem 6.6 on an auxiliary MRW. Let $(M'_n)_{n\ge 0}$ be the Markov chain introduced in Example 2.8 with $\mathbb{P}(\Gamma\in\cdot\,) := \mathbb{P}_i(\tau(i)\in\cdot\,)$. Moreover, we define the increments $(X'_n)_{n\ge 1}$ by

$$\mathbb{P}(X'_1\in\cdot\,|M'_0=j,\,M'_1=s) \;=\; \begin{cases}\mathbb{P}_i(S_{\tau(i)}\in\cdot\,|\tau(i)=n), & \text{if } j=(n,n-1),\, s=0,\\ \delta_0, & \text{else,}\end{cases}$$

for all $n\in\mathbb{N}$ and $j,s\in\mathscr{S}'$. Set $S'_n := \sum_{k=1}^n X'_k$ for $n\ge 1$, $\tau' := \inf\{n\ge 1 : M'_n = 0\}$ and

$$D' \;:=\; \max_{0<k\le\tau'}(S'_k)^-.$$

Since $\mathbb{P}_i(S_{\tau(i)}\in\cdot\,) = \mathbb{P}(S'_{\tau'}\in\cdot\,|M'_0=0)$ and $(S'_{\tau'})^- \stackrel{d}{=} D'$ under $\mathbb{P}(\cdot\,|M'_0=0)$, Theorem 6.6 yields

$$\begin{aligned}\infty \;&>\; \mathbb{E}\Big(\sup\{n\ge 0 : S'_n\le 0\}^{\alpha}\Big|M'_0=0\Big)\\ &\gtrsim\; \mathbb{E}\Big[\Big(\sum_{n\ge 0}\mathbf{1}_{\{S'_n\le 0\}}\Big)^{\alpha}\Big|M'_0=0\Big]\\ &=\; \mathbb{E}_i\Big(\sum_{n\ge 1}\chi_n(i)\,\mathbf{1}_{\{S_{\tau_{n-1}(i)}\le 0\}}\Big)^{\alpha}\end{aligned}$$

as claimed. □

**Proof of Theorem 6.9.** Due to Lemma 6.10, it suffices to prove the equivalence of $\mathbb{E}_i\Lambda(0)^\alpha<\infty$ and (ii) for some fixed $i\in\mathscr{S}$. Lemma C.6 entails $\mathbb{U}_i((0,y])\asymp J_i(y)$ as $y\to\infty$ and therefore

$$\begin{aligned}\int J_i(y)\;\mathbb{V}_i^{\alpha}(dy) \;\asymp\; \int \mathbb{U}_i((0,y])\;\mathbb{V}_i^{\alpha}(dy) \;&=\; \int_{\mathbb{R}_>}\mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)}\mathbf{1}_{\{S_k^-\ge y\}}\Big)^{\alpha}\;\mathbb{U}_i(dy)\\ &=\; \int_{\mathbb{R}_>}\mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)}\mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\;\mathbb{U}_i(dy).\end{aligned}$$

As it is easier to work with, we use

$$\int_{\mathbb{R}_>}\mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)}\mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\;\mathbb{U}_i(dy) \tag{6.7}$$

instead of (6.6).

CASE $\alpha \le 1$. "(ii)⇒(i)" Use Lemma 6.11 to derive that we only need to show

$$\mathbb{E}_i\Big(\sum_{n\ge 1}\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)}>0,\, S_k\le 0\}}\Big)^{\alpha} \;<\; \infty.$$

The subadditivity of $x \mapsto x^\alpha$ entails

$$\begin{aligned}
\mathbb{E}_i\left(\sum_{n\ge 1}\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)}>0,\, S_k\le 0\}}\right)^{\alpha} \;&\le \mathbb{E}_i\left[\sum_{n\ge 1}\left(\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)}>0,\, S_k\le 0\}}\right)^{\alpha}\right]\\
&= \int_{\mathbb{R}_>} \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\, \mathbb{U}_i(dy)
\end{aligned}$$

CASE $\alpha \ge 1$. "(i)⇒(ii)" On the one hand, we can obviously derive $\mathbb{E}_i(\sum_{n\ge 1}\mathbf{1}_{\{S_{\tau_n(i)}\le 0\}})^\alpha < \infty$, which is equivalent to $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$. On the other hand, superadditivity of $x \mapsto x^\alpha$ yields

$$\begin{aligned}
\infty \;>\; \mathbb{E}_i\Lambda(0)^\alpha \;&= \mathbb{E}_i\left(\sum_{n\ge 1}\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_k\le 0\}}\right)^{\alpha}\\
&\ge \mathbb{E}_i\left[\sum_{n\ge 1}\Big(\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_k\le 0\}}\Big)^{\alpha}\right]\\
&\ge \int_{\mathbb{R}_>} \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\, \mathbb{U}_i(dy).
\end{aligned}$$

"(ii)⇒(i)" Preliminarily, notice that $\mathbb{E}_i J_i(S^-_{\tau(i)})^2 < \infty$ implies $\mathbb{U}_i((-\infty,0]) < \infty$ (cf. Theorem 1.3 for $\alpha = 1$) and hence

$$\int_{\mathbb{R}_\le} \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\, \mathbb{U}_i(dy) \;\le\; \mathbb{E}_i\tau(i)^\alpha \cdot \mathbb{U}_i((-\infty,0]) \;<\; \infty.$$

Consequently, we have

$$\begin{aligned}
&\sup_{\beta\in(0,\alpha]} \int_{\mathbb{R}} \mathbb{E}_i\Big(\sum_{n=1}^{\tau(i)} \mathbf{1}_{\{S_k\le -y\}}\Big)^{\beta}\, \mathbb{U}_i(dy)\\
&= \int_{\mathbb{R}} \mathbb{E}_i\Big(\sum_{n=1}^{\tau(i)} \mathbf{1}_{\{S_k\le -y\}}\Big)^{\alpha}\, \mathbb{U}_i(dy) \;=:\; c \;<\; \infty.
\end{aligned}$$

In what follows, we use an induction argument (in $m$) from [5] (see their Theorem 3.7). Suppose $\alpha = m + \delta$, $m \in \mathbb{N}$, $\delta \in (0,1]$, and that (ii) does imply

$$\mathbb{E}_i\Lambda(0)^\beta \;<\; \infty \qquad \text{for all } 0 \le \beta \le m. \tag{6.8}$$

Notice that we have already verified this assertion for $\alpha \le 1$. For $n \in \mathbb{N}_0$, define

$$\Lambda_n(x) \;:=\; \sum_{k \ge \tau_n(i)+1} \mathbf{1}_{\{S_k - S_{\tau_n(i)} \le x\}}, \qquad x \in \mathbb{R},$$

and

$$L_n \;:=\; \sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_k \le 0\}}.$$

Hence,

$$\mathbb{E}_i\Big(\sum_{n\ge 1} L_n^{\beta}\Big) \;=\; \mathbb{E}_i\Big[\sum_{n \ge 1}\Big(\sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \mathbf{1}_{\{S_k \le 0\}}\Big)^{\beta}\Big] \;<\; c$$

for all $0 \le \beta \le m$. Observe that $\Lambda_n(0)$ and $L_n$ are independent and $\mathbb{P}_i(\Lambda_n(0) \in \cdot\,) = \mathbb{P}_i(\Lambda(0) \in \cdot\,)$ for all $n \ge 1$. In addition, notice that

$$\Lambda_n(-S_{\tau_n(i)}) \;=\; L_{n+1} + \Lambda_{n+1}(-S_{\tau_{n+1}(i)})$$

for all $n \in \mathbb{N}_0$. Making use of the inequality [5, Lemma 5.6], namely

$$(x+y)^{\alpha} \;\le\; x^{\alpha} + y^{\alpha} + \alpha\, 2^{\alpha-1}(x\, y^{\alpha-1} + x^m\, y^{\delta})$$

for all $x, y \in \mathbb{R}_{\ge}$, we obtain

$$\begin{aligned}
\Lambda(0)^{\alpha} \;&=\; \Big(L_1 + \Lambda_1(-S_{\tau(i)})\Big)^{\alpha} \\
&\le\; L_1^{\alpha} + \Lambda_1(-S_{\tau(i)})^{\alpha} + \alpha\, 2^{\alpha-1}\Big[L_1 \cdot \Lambda_1(-S_{\tau(i)})^{\alpha-1} + L_1^m \cdot \Lambda_1(-S_{\tau(i)})^{\delta}\Big].
\end{aligned}$$

$\mathbb{E}_i\Lambda(0) < \infty$ and $\lim_{n\to\infty} S_{\tau_n(i)} = \infty$ a.s. yields $\Lambda_n(-S_{\tau_n(i)}) \to 0$ a.s. and thus an iteration of the previous inequality shows

$$\Lambda(0)^{\alpha} \;\le\; \sum_{n\ge 1} L_n^{\alpha} + \alpha\, 2^{\alpha-1} \sum_{n \ge 1}\Big[L_n \cdot \Lambda_n(-S_{\tau_n(i)})^{\alpha-1} + L_n^m \cdot \Lambda_n(-S_{\tau_n(i)})^{\delta}\Big]$$

Furthermore, using

$$\begin{aligned}
&\Lambda_n(-S_{\tau_n(i)}) \\
&\le \sum_{\ell \ge n} \sum_{k=1}^{\chi_{\ell+1}(i)} \mathbf{1}_{\{S_{\tau_\ell(i)+k} - S_{\tau_n(i)} \le -S_{\tau_n(i)},\ S_{\tau_n(i)} > 0\}} + \sum_{\ell \ge n} \sum_{k=1}^{\chi_{\ell+1}(i)} \mathbf{1}_{\{S_{\tau_\ell(i)+k} - S_{\tau_n(i)} \le -S_{\tau_n(i)},\ S_{\tau_n(i)} \le 0\}} \\
&\le \Lambda_n(0) + \sum_{\ell \ge 1} \chi_\ell(i)\, \mathbf{1}_{\{S_{\tau_{\ell-1}(i)} \le 0\}},
\end{aligned}$$

Lemma 6.11, (6.8) and taking means yields

$$\mathbb{E}_i\Lambda_0(0)^{\alpha} \;\lesssim\; c\Big(1 + \alpha\, 2^{\alpha-1}\Big[\mathbb{E}_i\Lambda(0)^{\alpha-1} + \mathbb{E}_i\Lambda(0)^{\delta}\Big]\Big) \;<\; \infty.$$

□

It has already been noted that $\mathbb{E}_i\rho(0)^\alpha < \infty$ implies $\mathbb{E}_i\Lambda(0)^\alpha$ for any $\alpha > 0$, since $\rho(0) \geq \Lambda(0)$ a.s. This can also be verified on the basis of condition Theorem 6.9 (ii). Obviously, we only have to care about (6.6), since $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} \leq \mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$. At first, use

$$\sum_{n\geq 1}\mathbb{P}_i(0 < S_{\tau_n(i)} \leq y) \asymp J_i(y) \asymp \mathbb{U}_i^>((-\infty, y]) := \sum_{n\geq 0}\mathbb{P}_i(S_{\tau_n^>(i)} \leq y) \qquad \text{as } y \to \infty$$

to derive that it makes no difference if we integrate with respect to $\mathbb{U}_i^>$ instead of $\mathbb{U}_i$ in (6.7). Furthermore, let $(T_n)_{n\geq 1}$ be a distributional copy of $(S_{\tau_n^>(i)})_{n\geq 1}$, which is independent of all other occurring random variables. Then, an appeal to Lemma C.4 and Theorem 1.6 yields

$$\begin{aligned}
\int_{\mathbb{R}_>} \mathbb{E}_i\Bigg(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k \leq -y\}}\Bigg)^\alpha \,\mathbb{U}_i^>(dy) &\leq \int \mathbb{E}_i\tau(i)^\alpha\,\mathbf{1}_{\{D^i \geq y\}}\,\mathbb{U}_i^>(dy) \\
&= \mathbb{E}_i\Bigg(\sum_{n\geq 0}\chi_{n+1}(i)^\alpha\,\mathbf{1}_{\{T_n \leq D^i_{n+1}\}}\Bigg) \\
&\lesssim \sum_{n\geq 1} n^\alpha\,\mathbb{P}_i(T_n \leq D^i_{n+1}) \\
&\asymp \mathbb{E}_i J_i(D^i)^{1+\alpha}.
\end{aligned}$$

The following example illustrates that a MRW can behave so improperly within a cycle that $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha}$ is not sufficient for $\mathbb{E}_i\Lambda(0)^\alpha < \infty$ for $\alpha \geq 1$.

**Example 6.12** Let $(M_n)_{n\geq 0}$ be defined as in Example 2.8 with state space $\mathscr{S} \subset \{0\} \cup \mathbb{N}^2$. We define the increments $(X_n)_{n\geq 1}$ by

$$X_n := \begin{cases} -\ell^\beta, & \text{if } M_{n-1} = 0,\, M_n = (\ell, 1) \text{ for some } \ell \in \mathbb{N}, \\ 1 + \ell^\beta, & \text{if } M_{n-1} = (\ell, \ell-1),\, M_n = 0 \text{ for some } \ell \in \mathbb{N}, \\ 0, & \text{else} \end{cases}$$

for some $\beta > 1$. By construction, we have $S_{\tau(0)} = 1$ and $D^0 = \tau(0)^\beta$ $\mathbb{P}_0$-a.s. and hence

$$\mathbb{E}_i\Bigg(\sum_{n\geq 1}\mathbf{1}_{\{S_{\tau_n(i)} \leq 0\}}\Bigg)^\alpha < \infty$$

for any $\alpha > 0$. Furthermore, we suppose $\mathbb{E}\varGamma^\beta = \mathbb{E}_0 D^0 = \mathbb{E}_0\tau(0)^\beta < \infty$, which guarantees positive divergence of the MRW. Now, pick $\alpha \geq 1$ and use superadditivity of $x \mapsto x^\alpha$ to estimate

$$\begin{aligned}
\mathbb{E}_0\Lambda(0)^\alpha &= \mathbb{E}_0\Bigg(\sum_{n\geq 1}(\chi_n(0) - 1)\,\mathbf{1}_{\{S_{\tau_{n-1}(0)} - D^0_n \leq 0\}}\Bigg)^\alpha \\
&\geq \mathbb{E}_0\Bigg(\sum_{n\geq 1}(\chi_n(0) - 1)^\alpha\,\mathbf{1}_{\{S_{\tau_{n-1}(0)} - D^0_n \leq 0\}}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\geq \mathbb{E}_0\Big(\sum_{n\geq 1}(\chi_n(0)-1)^\alpha\,\mathbf{1}_{\{D_n^0\geq n\}}\Big)\\
&\gtrsim \sum_{n\geq 1} n^{\alpha/\beta}\,\mathbb{P}_0(\tau(0)^\beta\geq n)\\
&\asymp \mathbb{E}_0\tau(0)^{\alpha+\beta}
\end{aligned}$$

and we can assume the latter to be infinite.

## 6.4. Finiteness of Certain Weighted Renewal Measures

In this section, we establish equivalent conditions for

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n\leq 0)\ <\ \infty. \tag{6.9}$$

Moreover, we will point out that this does actually require positive divergence only for $\alpha\geq 1$. Set $\mathbb{V}_i:=\mathbb{V}_i^1$.

**Theorem 6.13** *Let $(M_n,S_n)_{n\geq 0}$ be a non-trivial MRW and $\mathbb{E}_i\tau(i)^{1+\alpha}\vee\mathbb{E}_i[\tau(i)\cdot\log\tau(i)]<\infty$ for some $i\in\mathscr{S}$ and $\alpha\geq 0$. The following conditions are equivalent:*

*(i)* $\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n\leq x)<\infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$.

*(ii)* $A_i(y)>0$ *for sufficiently large* $y$, $\mathbb{E}_iJ_i(S_{\tau(i)}^-)^{1+\alpha}<\infty$ *and*

$$\begin{aligned}
\alpha=0:&\qquad \int\log J_i(y)\ \mathbb{V}_i(dy)\ <\ \infty\\
\alpha>0:&\qquad \int J_i(y)^\alpha\ \mathbb{V}_i(dy)\ <\ \infty
\end{aligned}$$

*for some (hence all)* $i\in\mathscr{S}$.

**Proof.** The necessity of $A_i(y)>0$ for all sufficiently large $y$ and $\mathbb{E}_iJ_i(S_{\tau(i)}^-)^{1+\alpha}<\infty$ follows from

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n\leq x)\ \geq\ \mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\leq x\}}\Big)\ \asymp\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x)$$

(see Lemma C.3). Hence, by Theorem 1.6 , it suffices to prove

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n\leq x)\ \asymp\ \int\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy). \tag{6.10}$$

Moreover, we remark that

$$\int\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy)\ \asymp\ \mathbb{E}_i\Big(\sum_{n\geq 1} n^{\alpha-1}\sum_{k=1}^{\chi_n(i)}\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big),$$

which can be concluded from

$$\begin{aligned}
&\int \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dy)\\
&= \sum_{n\ge 1} n^{\alpha-1} \int \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{-S_k^- \le x-y\}}\Big)\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&\le \sum_{n\ge 1} n^{\alpha-1} \int \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- = 0\}}\Big)\mathbf{1}_{\{y=x\}}\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&\qquad + \sum_{n\ge 1} n^{\alpha-1} \int \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k \le x-y\}}\Big)\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&\le\ \mathbb{E}_i\tau(i) \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x) + \sum_{n\ge 1} n^{\alpha-1} \int \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k \le x-y\}}\Big)\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&\asymp \mathbb{E}_i\Big(\sum_{n\ge 1} n^{\alpha-1} \sum_{k=1}^{\chi_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)+k} \le x\}}\Big)\\
&\lesssim \sum_{n\ge 1} n^{\alpha-1} \int \Big[\mathbb{E}_i\tau(i)\,\mathbf{1}_{\{y\le x\}} + \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k \le x-y, S_k<0\}}\Big)\Big]\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&\lesssim \int \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- \ge y-x\}}\Big)\ \mathbb{P}_i(S_{\tau_n(i)} \in dy)\\
&= \int \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dy),
\end{aligned}$$

where we used that $\sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x) < \infty$.

The proof is separated into two cases.

Case $0 \le \alpha \le 1$. Using $(\tau_{n-1}(i)+k)^{\alpha-1} \le n^{\alpha-1}$ for $1 \le k \le \chi_n(i)$ and $n \ge 1$, we obtain

$$\begin{aligned}
\sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \le x)\ &= \mathbb{E}_i\Big(\sum_{n\ge 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\mathbf{1}_{\{S_{\tau_{n-1}(i)+k} \le x\}}\Big)\\
&\le \mathbb{E}_i\Big(\sum_{n\ge 1} n^{\alpha-1} \sum_{k=1}^{\chi_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)+k} \le x\}}\Big)\\
&\asymp \int \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dy)
\end{aligned}$$

For the other side, note first that

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\ge 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{\tau_{n-1}(i)>2n\mathbb{E}_i\tau(i)\}}\Big)\\
&\le \mathbb{E}_i\Big(\sum_{n\ge 1} \chi_n(i)\, n^{\alpha-1}\,\mathbf{1}_{\{\tau_{n-1}(i)>2n\mathbb{E}_i\tau(i)\}}\Big)
\end{aligned}$$

$$= \mathbb{E}_i\tau(i) \sum_{n\geq 1} n^{\alpha-1}\, \mathbb{P}_i\Big(\tau_{n-1}(i) > 2\,n\,\mathbb{E}_i\tau(i)\Big),$$

which is finite by Lemma C.1. In addition, we need

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} n^{\alpha-1}\,\mathbf{1}_{\{\chi_n(i)>n\}}\Big)\\
&= \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{n} n^{\alpha-1}\,\mathbf{1}_{\{\chi_n(i)>n\}}\Big) + \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k\geq n} n^{\alpha-1}\,\mathbf{1}_{\{\chi_n(i)>k\}}\Big)\\
&\leq \sum_{n\geq 1} n^{\alpha}\,\mathbb{P}_i(\tau(i)>n) + \sum_{k\geq 1}\sum_{n=1}^{k} n^{\alpha-1}\,\mathbb{P}_i(\tau(i)>k)\\
&\lesssim \mathbb{E}_i\tau(i)^{1+\alpha} \vee \mathbb{E}_i[\tau(i)\cdot\log\tau(i)] \;<\; \infty,
\end{aligned}$$

where we used that

$$\sum_{n=1}^{k} n^{\alpha-1} \;\asymp\; \int_1^k x^{\alpha-1}\,dx \;\lesssim\; \begin{cases}\log k, & \text{if } \alpha=0,\\ k^{\alpha}, & \text{else},\end{cases}$$

and

$$\sum_{k\geq 1}\log k\cdot\mathbb{P}_i(\tau(i)>k) \;\asymp\; \mathbb{E}_i[\tau(i)\,(\log\tau(i)-1)] \;\asymp\; \mathbb{E}_i[\tau(i)\cdot\log\tau(i)].$$

This enables us to make the following estimation

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\geq \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\,\mathbf{1}_{\{\tau_{n-1}(i)\leq 2\,n\,\mathbb{E}_i\tau(i)\}}\Big)\\
&\gtrsim \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\geq \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\,\mathbf{1}_{\{\chi_n(i)\leq n\}}\Big)\\
&\gtrsim \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} n^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\asymp \int\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\;\mathbb{V}_i(dy).
\end{aligned}$$

Case $\alpha>1$. Using $(\tau_{n-1}(i)+k)^{\alpha-1}\geq n^{\alpha-1}$ for $1\leq k\leq\chi_n(i)$ and $n\geq 1$, we infer

$$\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)$$

$$
\begin{aligned}
&\geq \mathbb{E}_i\Big(\sum_{n\geq 1} n^{\alpha-1}\sum_{k=1}^{\chi_n(i)} \mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\asymp \int \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy).
\end{aligned}
$$

For the reverse estimation, we begin with

$$
\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\lesssim \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\qquad + \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\,\mathbf{1}_{\{\tau_{n-1}(i)>2n\,\mathbb{E}_i\tau(i)\}}\Big).
\end{aligned}
$$

The first summand can be estimated by

$$
\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\Big)\\
&\leq 2^{\alpha-1}\,\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} n^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\,\mathbf{1}_{\{\chi_n(i)\leq n\}}\Big)\\
&\qquad + \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (n+k)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_{n-1}(i)+k}\leq x\}}\mathbf{1}_{\{\chi_n(i)>n\}}\Big)\\
&\lesssim \int \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy) + \mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (2\,\chi_n(i))^{\alpha-1}\,\mathbf{1}_{\{\chi_n(i)>n\}}\Big)\\
&\asymp \int \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy) + \mathbb{E}_i\tau(i)^{1+\alpha}\\
&\asymp \int \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)}\leq x+y)\ \mathbb{V}_i(dy).
\end{aligned}
$$

Concerning the second summand, we use

$$
\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)+k)^{\alpha-1}\,\mathbf{1}_{\{\tau_{n-1}(i)>2n\,\mathbb{E}_i\tau(i)\}}\Big)\\
&\leq (2^{\alpha-2}\vee 1)\,\mathbb{E}_i\Big(\sum_{n\geq 1}\sum_{k=1}^{\chi_n(i)} (\tau_{n-1}(i)^{\alpha-1}+\chi_n(i)^{\alpha-1})\,\mathbf{1}_{\{\tau_{n-1}(i)>2n\,\mathbb{E}_i\tau(i)\}}\Big)\\
&\lesssim 2\,\mathbb{E}_i\tau(i)^{\alpha}\,\mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{\tau_n(i)>2n\,\mathbb{E}_i\tau(i)\}}\Big),
\end{aligned}
$$

which is finite by Lemma C.2. □

Notice that

$$\mathbb{P}_i(D^i > y) \ \le\ \mathbb{V}_i((y,\infty)) \ \le\ \mathbb{E}_i\tau(i)\,\mathbf{1}_{\{D^i>y\}}$$

for all $y \in \mathbb{R}_\geq$. In general, $\mathbb{V}_i((y,\infty)) \lesssim \mathbb{P}_i(D^i > y)$ fails, but this changes if we assume $\mathbb{P}_i(\tau(i) \le c) = 1$ for some $c \in \mathbb{R}_>$. Then, the third condition in (ii) is equivalent to $\mathbb{E}_i \log J_i(D^i) < \infty$ or $\mathbb{E}_i J_i(D^i)^\alpha < \infty$ respectively. Therefore, for $0 \le \alpha < 1$, positive divergence is generally not required for (6.9).

In fact, $\mathbb{E}_i[\tau(i)\cdot\log\tau(i)] < \infty$ is required for the necessity of condition (ii) in the case $\alpha = 0$. For explanation, we give an example.

**Example 6.14** Let $(M_n)_{n\ge0}$ be the Markov chain introduced in Example 2.8 with $\mathbb{E}[\Gamma\cdot\log\Gamma] = \infty$. The increment distribution is given by

$$K(0,(n,1),\cdot\,) \ :=\ \delta_{-(n-1)} \qquad \text{and} \qquad K((n,n-1),0,\cdot\,) \ :=\ \delta_n$$

for $n \ge 2$ and $K(i,j,\cdot\,) := \delta_0$ for any other $i,j \in \mathscr{S}$.

Then, $S_{\tau(0)} = 1$ $\mathbb{P}_0$-a.s. and $\mathbb{E}_\pi|X_1| < \infty$, which suffices for

$$\sum_{n\ge1} n^{-1}\,\mathbb{P}_0(S_n \le 0) \ <\ \infty$$

(cf. Theorem 2.1 and Theorem 4.4). In contrast,

$$\begin{aligned}
\int \log J_0(y)\ \mathbb{V}_0(dy) \ &\asymp\ \mathbb{E}_0\Big(\sum_{n\ge1} n^{-1}\sum_{k=1}^{\chi_n(0)} \mathbf{1}_{\{S_{\tau_{n-1}(0)+k}\le 0\}}\Big)\\
&=\ \mathbb{E}_0\Big(\sum_{n\ge1} n^{-1}\sum_{k=1}^{\chi_n(0)} \mathbf{1}_{\{S_{\tau_{n-1}(0)+k}-S_{\tau_{n-1}(0)}\le -(n-1)\}}\Big)\\
&=\ \mathbb{E}_0\Big(\sum_{n\ge1} n^{-1}\sum_{k=1}^{\chi_n(0)-1} \mathbf{1}_{\{\chi_n(0)\ge n\}}\Big)\\
&=\ \mathbb{E}_0\Big(\sum_{n\ge1}\sum_{k=1}^{n-1} n^{-1}\,\mathbf{1}_{\{\chi_n(0)\ge n\}}\Big) + \mathbb{E}_0\Big(\sum_{n\ge1}\sum_{k\ge n} n^{-1}\,\mathbf{1}_{\{\chi_n(0)>k\}}\Big)\\
&\asymp\ \mathbb{E}_0\tau(0) + \sum_{k\ge1}\sum_{n=1}^{k} n^{-1}\,\mathbb{P}_0(\tau(0) > k)\\
&\asymp\ \mathbb{E}_0\tau(0) + \mathbb{E}_0[\tau(0)\cdot\log\tau(0)] \ =\ \infty.
\end{aligned}$$

As $\mathbb{P}_i(\Lambda(0) < \infty) = 1$ is equivalent to positive divergence, Theorem 4.1 yields that (6.9) for $\alpha = 0$ is weaker. The next result relates (6.9) with $\mathbb{E}_i\Lambda(0)^\alpha < \infty$ for $\alpha > 1$. For $0 < \alpha < 1$, we prove that (6.9) is weaker than the conjectured equivalent integral criterion for $\mathbb{E}_i\Lambda(0)^\alpha < \infty$. For $\alpha = 1$, the conditions are the same.

**Proposition 6.15** *Suppose $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$, $(S_{\tau_n(i)})_{n\ge0}$ being positive divergent and $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ for some $i \in \mathscr{S}$ and some $\alpha \ge 0$. Then, for all $i \in \mathscr{S}$:*

$$0 < \alpha < 1: \qquad \int J_i(y)\ \mathbb{V}_i^\alpha(dy) \ <\ \infty \qquad \Rightarrow \qquad \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_n \le 0) \ <\ \infty.$$

$$\alpha > 1: \qquad \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq 0) \;<\; \infty \qquad \Rightarrow \qquad \mathbb{E}_i\Lambda(0)^\alpha \;<\; \infty.$$

**Proof.** Both assertions are proved in terms of the integral criteria from Theorem 6.9 and Theorem 6.13.

We can assume that $y \mapsto \mathbb{P}_i(S_{\tau(i)} > y)$ is a continuous function. For explanation, let

$$X'_n \;:=\; X_n + U_n\,\mathbf{1}_{\{M_{n-1}=i\}},$$

where $(U_n)_{n\geq 1}$ is an i.i.d. sequence of random variables, which are uniformly distributed on $(0,1)$ and are independent of all other occurring random variables. Clearly, $y \mapsto \mathbb{P}_i(S'_{\tau(i)} > y)$ is continuous. Furthermore,

$$\mathbb{P}_i(S_{\tau(i)} > y) \;\leq\; \mathbb{P}_i(S'_{\tau(i)} > y) \;\leq\; \mathbb{P}_i(S_{\tau(i)} > y-1)$$

for all $y \in \mathbb{R}_\geq$. Analogous arguments relate $\mathbb{P}_i(D^i > y)$ and $\mathbb{V}_i^\alpha((y,\infty))$ to the corresponding ones in terms of $(M_n, X'_n)_{n\geq 1}$. For a MRW with positive divergent embedded random walks, these tails entirely contain the information of finiteness of the quantities examined in this section so far. It is simple to see that the integral criteria are satisfied for $(M_n, S'_n)_{n\geq 0}$ if and only if the same is true for $(M_n, S'_n)_{n\geq 0}$.

Hence, $y \mapsto \mathbb{E}_i(S^+_{\tau(i)} \wedge y) = \int_0^y \mathbb{P}_i(S^+_{\tau(i)} > x)\,dx$, $y \in \mathbb{R}_>$, possesses a continuous derivative. In particular, $\frac{\partial}{\partial y} J_i(y)$ is a continuous function. For any $\beta > 0$,

$$f_\beta(y) \;:=\; [J_i(y) - 1]^\beta, \qquad y \in \mathbb{R}_\geq,$$

is non-decreasing, has a continuous derivative on $\mathbb{R}_>$ and fulfils $f_\beta(0) = 0$. Consequently,

$$\int f_\beta(y)\;\mathbb{V}_i^\gamma(dy) \;\asymp\; \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y)\cdot (J_i(y)-1)^{\beta-1}\cdot \mathbb{V}_i^\gamma((y,\infty))\;dy$$

for any $\beta, \gamma > 0$.

Case $0 < \alpha < 1$. We start with the observation of

$$\begin{aligned}
&\mathbb{V}_i((y,\infty))\\
&\leq \mathbb{E}_i\Bigg[\Bigg(\sum_{k=1}^{J_i(y)} \mathbf{1}_{\{S_k^- > y\}}\Bigg)^{1-\alpha}\cdot\Bigg(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- > y\}}\Bigg)^{\alpha}\mathbf{1}_{\{\tau(i)\leq J_i(y)\}}\Bigg] + \mathbb{E}_i[\tau(i)\,\mathbf{1}_{\{\tau(i)>J_i(y)\}}]\\
&\leq J_i(y)^{1-\alpha}\cdot \mathbb{V}_i^\alpha((y,\infty)) + \mathbb{E}_i[\tau(i)\,\mathbf{1}_{\{\tau(i)>J_i(y)\}}].
\end{aligned}$$

Using the preliminary considerations and integration by substitution, we obtain

$$\begin{aligned}
&\int J_i(y)^\alpha\;\mathbb{V}_i(dy)\\
&\asymp \int (J_i(y)-1)^\alpha\;\mathbb{V}_i(dy)\\
&\asymp \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y)\cdot J_i(y)^{\alpha-1}\cdot \mathbb{V}_i((y,\infty))\;dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y) \cdot \mathbb{V}_i^\alpha((y,\infty))\ dy + \mathbb{E}_i\Big(\tau(i) \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y) \cdot J_i(y)^{\alpha-1} \mathbf{1}_{\{\tau(i)>J_i(y)\}}\ dy\Big) \\
&\lesssim \int J_i(y)\ \mathbb{V}_i^\alpha(dy) + \mathbb{E}_i\Big(\tau(i) \int_1^\infty y^{\alpha-1} \mathbf{1}_{\{\tau(i)>y\}}\ dy\Big) \\
&\asymp \int J_i(y)\ \mathbb{V}_i^\alpha(dy) + \mathbb{E}_i \tau(i)^{1+\alpha}
\end{aligned}
$$

Case $\alpha > 1$. Similar to above, we begin with

$$
\begin{aligned}
&\mathbb{V}_i^\alpha((y,\infty)) \\
&\leq \mathbb{E}_i\Big[\Big(\sum_{k=1}^{J_i(y)} \mathbf{1}_{\{S_k^- > y\}}\Big)^{\alpha-1} \cdot \Big(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- > y\}}\Big) \mathbf{1}_{\{\tau(i) \leq J_i(y)\}}\Big] + \mathbb{E}_i[\tau(i)^\alpha \mathbf{1}_{\{\tau(i)>J_i(y)\}}] \\
&\leq J_i(y)^{\alpha-1} \cdot \mathbb{V}_i((y,\infty)) + \mathbb{E}_i[\tau(i)^\alpha \mathbf{1}_{\{\tau(i)>J_i(y)\}}]
\end{aligned}
$$

to derive

$$
\begin{aligned}
&\int J_i(y)\ \mathbb{V}_i^\alpha(dy) \\
&\asymp \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y) \cdot \mathbb{V}_i^\alpha((y,\infty))\ dy \\
&\leq \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y) \cdot J_i(y)^{\alpha-1} \cdot \mathbb{V}_i((y,\infty))\ dy + \mathbb{E}_i\Big(\tau(i)^\alpha \int_{\mathbb{R}_>} \frac{\partial}{\partial y} J_i(y) \mathbf{1}_{\{\tau(i)>J_i(y)\}}\ dy\Big) \\
&\lesssim \int J_i(y)^\alpha\ \mathbb{V}_i(dy) + \mathbb{E}_i\Big(\tau(i)^\alpha \int_1^\infty \mathbf{1}_{\{\tau(i)>y\}}\ dy\Big) \\
&\asymp \int J_i(y)^\alpha\ \mathbb{V}_i(dy) + \mathbb{E}_i \tau(i)^{1+\alpha}.
\end{aligned}
$$

□

## 6.5. Finiteness of Power Moments of $\sigma^>$

In order to find an integral criterion for power moments of $\sigma^>$, we have already made the observation in Lemma 3.5 that

$$
A_i(y) > 0 \text{ for all sufficiently large } y \qquad \text{and} \qquad \mathbb{E}_i J_i(S_{\tau(i)}^-)^{1+\alpha} \ <\ \infty,
$$

$\alpha \geq 0$, is sufficient for

$$
\mathbb{E}_i \sigma^>(x)^{1+\alpha} \ <\ \infty \qquad \text{for all } (x,i) \in \mathbb{R}_\geq \times \mathscr{S}. \tag{6.11}
$$

In particular, the MRW can be oscillating, but (6.11) is true (see Example 2.7). It turns out that it is a difficult task to find an equivalent criterion. In fact, the embedded random walks can be of arbitrary fluctuation type, but (6.11) is true, as the following example reveals. The last part of the example is close to the proof of Proposition 2.13 from [5].

**Example 6.16** For $\alpha \geq 0$, let $(\sum_{k=1}^{n} Y_k)_{n\geq 0}$ be an ordinary integer-valued random walk with

$$\mathbb{P}(Y^- > n) \;=\; \frac{1}{n^{1+\alpha}} \qquad \text{for all sufficiently large } n \in \mathbb{N},$$

$\mathbb{P}(Y=0)>0$ and $\mathbb{P}(Y^+ \in \cdot)$ such that $\mathbb{E}\inf\{n\geq 1 : \sum_{k=1}^{n} Y_k > 0\}^{1+\alpha} = \infty$. In particular, one can choose $\mathbb{P}(Y^+ \in \cdot) = \delta_0$ for any $\alpha \geq 0$. W.l.o.g., we suppose $\mathbb{P}(Y^- = n) > 0$ for all $n \in \mathbb{N}$.

Define $f : \mathbb{R}_> \to \mathbb{R}_>$ by $f(x) := 2^{\theta x^{1+\alpha}}$ for some $\theta > 1+\alpha$, hence $f(x) \geq x$ for all $x \geq 1$. By construction, we have

$$\begin{aligned}\lim_{n\to\infty} n\,\mathbb{P}(f(Y^-) > 2^n c) \;&=\; \lim_{n\to\infty} n\,\mathbb{P}\Big(Y^- > f^{-1}(2^n c)\Big)\\ &=\; \lim_{n\to\infty} n\,\mathbb{P}\left(Y^- > \left(\frac{n+\log_2 c}{\theta}\right)^{1/(1+\alpha)}\right)\\ &=\; \theta\end{aligned} \tag{6.12}$$

for all $c \in \mathbb{R}_>$.

Let $(M_n)_{n\geq 0}$ be a Markov chain on $\mathbb{N}_0$ with $p_{00} := \mathbb{P}(Y \geq 0)$, $p_{0i} := \mathbb{P}(Y = -i)$ and $p_{i0} := 1$ for all $i \in \mathbb{N}$. Furthermore, set

$$K_{00} \;:=\; \mathbb{P}(Y \in \cdot\,|Y\geq 0), \qquad K_{0i} \;:=\; \delta_{f(i)} \qquad \text{and} \qquad K_{i0} \;:=\; \delta_{-f(i)-i}$$

for all $i \in \mathbb{N}$. Consequently, $\mathbb{P}_0(S_{\tau(0)} \in \cdot\,) = \mathbb{P}(Y \in \cdot\,)$ and $\mathbb{E}_0 \inf\{n\geq 1 : S_{\tau_n(0)} > 0\}^{1+\alpha} = \infty$. Fixing any $x \in \mathbb{R}_\geq$, the following property of the MRW under $\mathbb{P}_0$ is essential for our considerations, namely

$$X_{\tau_n(0)+1} \;\leq\; x \quad\Rightarrow\quad S_{\tau_{n+1}(0)} - S_{\tau_n(0)} \;\geq\; -x.$$

As a consequence of this property, we infer that

$$\begin{aligned}\{\sigma^>(x) > \tau(0), M_0 = 0\} \;&\subset\; \{S_{\tau(0)} \geq -x, M_0 = 0\},\\ \{\sigma^>(x) > \tau_2(0), M_0 = 0\} \;&\subset\; \{S_{\tau(0)} \geq -x, X_{\tau(0)+1} \leq 2x, M_0 = 0\}\\ &\subset\; \{S_{\tau_2(0)} \geq -3x, M_0 = 0\}\end{aligned}$$

and then inductively

$$\{\sigma^>(x) > \tau_n(0), M_0 = 0\} \;\subset\; \Big\{S_{\tau_n(0)} \geq -(2^n-1)x, M_0 = 0\Big\}$$

for all $n \in \mathbb{N}$.

Define $\kappa(x) := \inf\{n\geq 1 : X_{\tau_n(0)+1} > 2^n x\}$. Note that $\sigma^>(x) \leq \kappa(x)$ $\mathbb{P}_0$-a.s. and

$$\mathbb{E}_i \sigma^>(x)^{1+\alpha} \;\leq\; \mathbb{E}_0[1+\sigma^>(x+f(i)+i)]^{1+\alpha}$$

for all $i \in \mathbb{N}$. Hence, we will show $\mathbb{E}_0\kappa(x)^{1+\alpha} < \infty$ for all $x > f(1)$, which easily yields (6.11).

We start with

$$\mathbb{E}_0\,\kappa(x)^{1+\alpha} \;\asymp\; \sum_{n\geq 1} n^\alpha\,\mathbb{P}_0(\kappa(x) > n) \;=\; \sum_{n\geq 1} n^\alpha \prod_{k=1}^{n} F(2^k x),$$

where $F(\cdot) := \mathbb{P}_0(X_1 \in \cdot\,)$. Then, notice that (6.12) yields

$$n\,[1-F(2^{n+1}x)] \xrightarrow{n\to\infty} \theta. \tag{6.13}$$

Set $a_n := n^\alpha \prod_{k=1}^n F(2^k x)$, which is positive, because $x > f(1)$. Due to Raabe's criterion, $\mathbb{E}_0\,\kappa(x)^{1+\alpha} < \infty$ if

$$\liminf_{n\to\infty} \Big(\frac{a_n}{a_{n+1}} - 1\Big) \;>\; 1.$$

Therefore, we use the Taylor approximation

$$(1-n^{-1})^\alpha \;=\; 1+\alpha n^{-1} + o(n^{-1})$$

and (6.13) to conclude

$$\begin{aligned}
\liminf_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) &= \liminf_{n\to\infty} n\left[\frac{n^\alpha - (n+1)^\alpha F(2^{n+1}x)}{(n+1)^\alpha F(2^{n+1}x)}\right] \\
&= \liminf_{n\to\infty} n\,[1-(1+n^{-1})^\alpha F(2^{n+1}x)] \\
&= \liminf_{n\to\infty} n\Big[1-[1+\alpha n^{-1} + o(n^{-1})]\,F(2^{n+1}x)\Big] \\
&= \liminf_{n\to\infty} \Big[n\,[1-F(2^{n+1}x)] - \alpha\,F(2^{n+1}x)\Big] \\
&= \theta - \alpha \;>\; 1,
\end{aligned}$$

thus $\mathbb{E}_0\,\kappa(x)^{1+\alpha} < \infty$ for all $x > f(1)$.

Obviously, the phenomenon of Example 6.16 arises from $S^-_{\tau(i)}$ and $H^i$ being coupled in a problematic way, where

$$H^i_n \;:=\; \max_{\tau_{n-1}(i) < k \le \tau_n(i)} (S_k - S_{\tau_{n-1}(i)})^+, \qquad n \ge 1,$$

is the maximal upward excursion between $\tau_{n-1}(i)+1$ and $\tau_n(i)$. It indicates that an equivalent criterion for (6.11) must include the dependency of $S_{\tau(i)}$ and $H^i$.

Moreover, when $\mathbb{P}(Y^+ \in \cdot\,)$ is chosen as $\delta_0$, we obtain $\mathbb{E}_i\sigma^{\leqslant} < \infty$ and $\mathbb{E}_i\sigma^{\geqslant} < \infty$ for all $i \in \mathscr{S}$. Hence, there does not need to be any recurrent ladder chain (cf. Proposition 5.2).

Just $\mathbb{E}_i\sigma^>(x) < \infty$ for some $(x,i) \in \mathbb{R}_\ge \times \mathscr{S}$, does not imply (6.11), since $\mathbb{P}_i(\sigma^>(x) = n) = 1$ for $n \in \mathbb{N}$ is possible and even regardless of the fluctuation type of the MRW. This changes if one does additionally assume

$$q(i,x) \;:=\; \mathbb{P}_i(\sigma^>(x) > \tau(i), S_{\tau(i)} < 0) \;>\; 0.$$

Notice that $q(i,x)$ is non-decreasing in $x$ and, if $\mathbb{P}_i(S_{\tau(i)} < 0) > 0$, $q(i,x) > 0$ for all sufficiently large $x$. The following lemma provides sufficient conditions for (6.11).

**Lemma 6.17** *Let $(M_n,S_n)_{n\ge 0}$ be a non-trivial MRW and $\alpha \ge 0$. The following conditions are sufficient for* (6.11)*:*

*(i)* $\mathbb{E}_i\sigma^>(x)^{1+\alpha} < \infty$ *for all* $x \in \mathbb{R}_\ge$ *for some* $i \in \mathscr{S}$.

*(ii)* $\mathbb{E}_i\sigma^>(x)^{1+\alpha}<\infty$ *and* $q(i,x)>0$ *for some* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$.

*(iii)* $\mathbb{E}_i\sigma^>(0)^{1+\alpha}<\infty$ *and* $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$ *for all* $i\in\mathscr{S}$.

**Proof.** (i) For any $j\in\mathscr{S}$, there exists $x_0\in\mathbb{R}_\geq$ such that

$$p\ :=\ \mathbb{P}_i(\sigma^>(2x_0)>\tau(j),S_{\tau(j)}\leq x_0)\ >\ 0$$

and thus

$$\mathbb{P}_i(\sigma^>(2x)>\tau(j),S_{\tau(j)}\leq x)\ \geq\ p$$

for all $x\geq x_0$. For such $x$, we infer

$$\infty\ >\ \mathbb{E}_i\sigma^>(2x)^{1+\alpha}\ \geq\ \mathbb{E}_i\sigma^>(2x)^{1+\alpha}\mathbf{1}_{\{\sigma^>(2x)>\tau(j),S_{\tau(j)}\leq x\}}\ \geq\ p\cdot\mathbb{E}_j\sigma^>(x)^{1+\alpha}.$$

Then, use $\mathbb{E}_j\sigma^>(x_0)^{1+\alpha}\geq\mathbb{E}_j\sigma^>(x)^{1+\alpha}$ for all $0\leq x\leq x_0$ to infer (6.11).

(ii) By assumption, we can find $\varepsilon>0$ such that

$$p(x)\ :=\ \mathbb{P}_i(\sigma^>(x)>\tau(i),S_{\tau(i)}\leq-\varepsilon)\ >\ 0.$$

Consequently,

$$\infty\ >\ \mathbb{E}_i\sigma^>(x)^{1+\alpha}\mathbf{1}_{\{\sigma^>(x)>\tau(i),S_{\tau(i)}\leq-\varepsilon\}}\ \geq\ p(x)\cdot\mathbb{E}_i\sigma^>(x+\varepsilon)^{1+\alpha}.$$

Since $p(x+n\varepsilon)\geq p(x+(n-1)\varepsilon)$ for all $n\in\mathbb{N}$, we obtain $\mathbb{E}_i\sigma^>(x+n\varepsilon)^{1+\alpha}<\infty$ for all $n\in\mathbb{N}$ by induction and then (6.11) by an appeal to (i).

(iii) If $\mathbb{P}_i(S_{\tau(i)}\geq 0)=1$ holds for some $i\in\mathscr{S}$ and thus $\mathbb{P}_i(S_{\tau(i)}>0)>0$ by non-triviality, the assertion follows easily from Lemma 3.5.

Now, suppose $\mathbb{P}_i(S_{\tau(i)}<0)>0$ for all $i\in\mathscr{S}$. Fix some $i\in\mathscr{S}$. If $q(i,0)>0$ the assertion follows from (ii). Assume $q(i,0)=0$ and hence $\mathbb{P}_i(\sigma^>(0)<\tau(i),S_{\tau(i)}<0)>0$. We will show that $q_n(j,0):=\mathbb{P}_j(\sigma^>(0)>\tau_n(j),S_{\tau_n(j)}<0)>0$ for some $n\in\mathbb{N}$ and $j\in\mathscr{S}$. Then, analogous to (ii), (6.11) can be concluded.

Define

$$\kappa\ :=\ \inf\Big\{0\leq n\leq\tau(i):S_n=\max_{0\leq k\leq\tau(i)}S_k\Big\}.$$

By assumption, there is $j\in\mathscr{S}\setminus\{i\}$ such that

$$p'\ :=\ \mathbb{P}_i(M_\kappa=j,\kappa=\tau_m(j)\geq\sigma^>(0),S_{\tau(i)}<0,\tau_{m+\ell}(j)<\tau(i)<\tau_{m+\ell+1}(j))\ >\ 0$$

for some $m,\ell\in\mathbb{N}_0$. Then,

$$\begin{aligned}
p'\ &=\mathbb{P}_i\Big(S_{\tau_m(j)}>S_k\text{ for }0\leq k<\tau_m(j),\,S_{\tau(i)}-S_{\tau_m(j)}<-S_{\tau_m(j)},\\
&\qquad S_{\tau_m(j)+k}-S_{\tau_m(j)}\leq 0\text{ for }0<k\leq\tau(i)-\tau_m(j),\,\tau_{m+\ell}(j)<\tau(i)<\tau_{m+\ell+1}(j)\Big)\\
&=\int\mathbb{P}_j\Big(S_{\tau(i)}<-x,\,S_k\leq 0\text{ for }0<k\leq\tau(i),\,\tau_\ell(j)<\tau(i)<\tau_{\ell+1}(j)\Big)\\
&\qquad \mathbb{P}_i(S_{\tau_m(j)}\in dx,\,S_{\tau_m(j)}>S_k\text{ for }0\leq k<\tau_m(j)<\tau(i))\\
&=\mathbb{P}_j\Big(S_{\tau_{m+\ell}(j)}<0,\,S_k\leq 0\text{ for }0<k\leq\tau(i),\,\tau_\ell(j)<\tau(i)<\tau_{\ell+1}(j),
\end{aligned}$$

$$S_{\tau_{m+\ell}(j)} > S_{\tau(i)+k} \text{ for } 0 \le k \le \tau_{m+\ell}(j) - \tau(i)\Big)$$
$$\le \mathbb{P}_j(\sigma^> > \tau_{m+\ell}(j), S_{\tau_{m+\ell}(j)} < 0) \;=\; q_{m+\ell}(j,0).$$

□

Suprisingly, one can not dispense with $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ in (iii) as the following simple example reveals. Consider the Markov chain as in Example 2.8 with

$$K((n,n-1),0,\cdot) \;:=\; K((n,k),(n,k+1),\cdot) \;:=\; \delta_{1/n}$$

for all $1 \le k \le n-2$ and $n \ge 2$. Thus, $\sigma^>(0) = 1$ a.s. and $\sigma^>(1) = \tau(0)$ $\mathbb{P}_0$-a.s.

Example 6.16 and Lemma 6.17 (ii) motivate to study the post-$\tau(i)$ level $x$ first passage time

$$\overline{\sigma}^>(x) \;:=\; \inf\{n \ge \tau(i) : S_n > x\}, \qquad x \in \mathbb{R}_\ge.$$

We will establish an equivalent criterion for $\overline{\sigma}^>(0)$ to possess power moments of order greater than or equal to 1 given an embedded random walk tends stochastically to $\infty$.

Preliminarily, we prove an adapted extension of Lemma 3.5 from [33] to MRWs, where we use $J_i(y)$ instead of $y/A_i(y)$. Moreover, we assume $S_{\tau_n(i)} \xrightarrow{\mathbb{P}} \infty$ for some $i \in \mathscr{S}$. Whether the latter is a solidarity property or if it is equivalent to $S_n \xrightarrow{\mathbb{P}_\pi} \infty$ is not known. As proofs seem not to be simple and the benefit for our results is marginal, we refrained from studying this aspect.

**Lemma 6.18** *Suppose $S_{\tau_n(i)} \xrightarrow{\mathbb{P}_i} \infty$ for some $i \in \mathscr{S}$. Then, for all $j \in \mathscr{S}$ and $y \to \infty$:*

$$\alpha = 0: \qquad \log J_j(y) \;\lesssim\; \sum_{n\ge1} n^{-1}\,\mathbb{P}_j\Big(\max_{1\le k\le n} S_k \le y\Big).$$
$$\alpha > 0: \qquad J_j(y)^\alpha \;\lesssim\; \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_j\Big(\max_{1\le k\le n} S_k \le y\Big) \;\asymp\; \mathbb{E}_j\sigma^>(y)^\alpha.$$

**Proof.** We begin with showing that the assertion follows for all $j \in \mathscr{S}$ if it is true for $i$. Pick some arbitrary $j \in \mathscr{S}$. By Lemma 3.3, it suffices to find some $x \in \mathbb{R}_\ge$ such that

$$\sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i\Big(\max_{1\le k\le n} S_k \le y-x\Big) \;\lesssim\; \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_j\Big(\max_{1\le k\le n} S_k \le y+x\Big).$$

Let $x \in \mathbb{R}_\ge$ be such that

$$p \;:=\; \mathbb{P}_j\Big(\max_{1\le k\le \tau(i)} S_k \le x\Big) \;>\; 0.$$

Then, we infer

$$p\sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i\Big(\max_{1\le k\le n} S_k \le y-x\Big)$$
$$\le \sum_{n\ge1} n^{\alpha-1}\int_{(-\infty,x]} \mathbb{P}_i\Big(\max_{1\le k\le n} S_k \le y-z\Big)\;\mathbb{P}_j\Big(\max_{1\le k\le \tau(i)} S_k \in dz\Big)$$

$$
\begin{aligned}
&\leq \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_j\Big(S_{\tau(i)} + \max_{\tau(i)+1\leq k\leq \tau(i)+n}(S_k - S_{\tau(i)}) \leq y,\ \max_{1\leq k\leq \tau(i)} S_k \leq x\Big)\\
&\leq \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_j\Big(\max_{\tau(i)+1\leq k\leq \tau(i)+n} S_k \leq y,\ \max_{1\leq k\leq \tau(i)} S_k \leq x\Big)\\
&\leq \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_j\Big(\max_{1\leq k\leq \tau(i)+n} S_k \leq y+x\Big)\\
&\leq \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_j\Big(\max_{1\leq k\leq n} S_k \leq y+x\Big)
\end{aligned}
$$

for all $y \in \mathbb{R}_{\geq}$.

For $i \in \mathscr{S}$, $y \in \mathbb{R}_{>}$ and $\delta \in (0,1)$ define

$$
m_\delta(y) \ := \ \inf\Big\{n \geq 1 : \mathbb{P}_i\Big(\max_{1\leq k\leq n} S_k \leq y\Big) < 1-\delta\Big\}
$$

with $m_\delta(y) \uparrow \infty$ as $y \to \infty$. Moreover, $S_{\tau_n(i)} \xrightarrow{\mathbb{P}_i} \infty$ implies $(S_{\tau_n(i)})_{n\geq 0}$ being positive divergent or oscillating and thus $m_\delta(y) < \infty$ for all $(y,\delta) \in \mathbb{R}_> \times (0,1)$. Simple estimations yield

$$
\begin{aligned}
\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i\Big(\max_{1\leq k\leq n} S_k \leq y\Big) \ &\geq \ \sum_{n=1}^{m_\delta(y)} n^{\alpha-1}\,\mathbb{P}_i\Big(\max_{1\leq k\leq n} S_k \leq y\Big)\\
&\gtrsim \ \begin{cases} (1-\delta)\log m_\delta(y), & \text{if } \alpha = 0,\\ (1-\delta)\, m_\delta(y)^\alpha, & \text{if } \alpha > 0. \end{cases}
\end{aligned}
$$

It remains to show

$$
J_i(y) \ \lesssim \ m_\delta(y) \qquad \text{as } y \to \infty. \tag{6.14}
$$

Put $A_i^+(y) := \mathbb{E}_i(S_{\tau(i)}^+ \wedge y)$. We assume that (6.14) fails and infer that we find for all $\varepsilon \in (0,1)$ an increasing, non-negative sequence $(y_\ell)_{\ell\geq 1}$, depending on $\varepsilon$, such that

$$
\sup_{\ell\geq 1}\ 2\,m_\delta(y_\ell)\,\frac{A_i^+(y_\ell)}{y_\ell} \ \leq \ \varepsilon, \tag{6.15}
$$

which is equivalent to

$$
(1-\varepsilon)\,y_\ell + 2\,m_\delta(y_\ell)\,A_i^+(y_\ell) \ \leq \ y_\ell \qquad \text{for all } \ell \geq 1. \tag{6.16}
$$

For the sake of brevity, we set $m_\ell := m_\delta(y_\ell)$. It follows that

$$
\begin{aligned}
&\mathbb{P}_i\Big(\max_{1\leq k\leq \tau_{m_\ell}(i)} S_k > y_\ell\Big)\\
&= \sum_{n=1}^{m_\ell} \mathbb{P}_i\Big(\max_{1\leq k\leq n-1}(S_{\tau_{k-1}(i)} + H_k^i) \leq y_\ell < S_{\tau_{n-1}(i)} + H_n^i\Big)\\
&= \sum_{n=1}^{m_\ell} \mathbb{P}_i\left(\max_{1\leq k\leq n-1}(W^i_{2m_\ell-n+1,k} + H^i_{2m_\ell-n+k+1}) \leq x_\ell < W^i_{2m_\ell-n+1,n} + H^i_{2m_\ell+1}\right),
\end{aligned}
$$

where $W_{m,k}^i := S_{\tau_{m+k-1}(i)} - S_{\tau_m(i)}$. We have used that $(S_{\tau_n(i)} - S_{\tau_{n-1}(i)}, H_n^i)_{n\ge 1}$ forms a sequence of i.i.d. random variables under $\mathbb{P}_i$. $S_{\tau_n(i)} \xrightarrow{\mathbb{P}_i} \infty$ ensures that we can pick $h>0$ and $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}_i(H^i > h) \;\le\; \frac{\delta}{4} \quad\text{and}\quad c \;:=\; \inf_{n\ge n_0} \mathbb{P}_i(S_{\tau_n(i)} > h) \;\ge\; \frac{1}{2}.$$

Choosing $\ell$ such that $m_\ell > n_0$, we further estimate

$$\begin{aligned}
&\mathbb{P}_i\Big(\max_{1\le k\le \tau_{m_\ell}(i)} S_k > y_\ell\Big)\\
&\le \sum_{n=1}^{m_\ell} \mathbb{P}_i\bigg[\Big(\max_{1\le k\le n-1}(W^i_{2m_\ell-n+1,k} + H^i_{2m_\ell-n+k+1}) \le y_\ell < W^i_{2m_\ell-n+1,n} + H^i_{2m_\ell+1}\Big)\\
&\qquad\qquad \cdot \frac{1}{c}\,\mathbb{P}_i(S_{\tau_{2m_\ell-n+1}(i)} > h)\bigg]\\
&\le \frac{1}{c}\sum_{n=1}^{m_\ell} \mathbb{P}_i\bigg(\max_{1\le k\le n-1}(W^i_{2m_\ell-n+1,k} + H^i_{2m_\ell-n+k+1}) \le y_\ell < W^i_{2m_\ell-n+1,n} + H^i_{2m_\ell+1},\\
&\qquad\qquad S_{\tau_{2m_\ell}(i)} + H^i_{2m_\ell+1} > y_\ell + h\bigg)\\
&\le \frac{1}{c}\,\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} + H^i_{2m_\ell+1} > y_\ell + h)\\
&\le \frac{1}{c}\Big[\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} > y_\ell) \;+\; \mathbb{P}_i(H^i > h)\Big]\\
&\le 2\,\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} > y_\ell) \;+\; \frac{\delta}{2}.
\end{aligned}$$

Set $\zeta_{k,\ell} := (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ \wedge y_\ell$, so that $A_i^+(y_\ell) = \mathbb{E}_i\,\zeta_{1,\ell}$. We obtain

$$\begin{aligned}
\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} > y_\ell) \;&\le\; \mathbb{P}_i\Big(\sum_{k=1}^{2m_\ell}(S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ > y_\ell\Big)\\
&\le\; \mathbb{P}_i\Big(\sum_{k=1}^{2m_\ell}\zeta_{k,\ell} > y_\ell\Big) + 2\,m_\ell\,\mathbb{P}_i(S_{\tau(i)} > y_\ell).
\end{aligned}$$

Applying (6.16), Tschebychev's inequality and $y_\ell^2\,\mathbb{P}_i(S_{\tau(i)} > y_\ell) \le \mathbb{E}_i\,\zeta_{1,\ell}^2$ yields

$$\begin{aligned}
\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} > y_\ell) \;&\le \mathbb{P}_i\Big(\sum_{k=1}^{2m_\ell}\zeta_{k,\ell} - 2\,m_\ell\,A_i^+(y_\ell) > (1-\varepsilon)y_\ell\Big) + 2\,m_\ell\,\mathbb{P}_i(S_{\tau(i)} > y_\ell)\\
&\le \frac{2\,m_\ell\,\mathbb{E}_i\zeta_{1,\ell}^2}{(1-\varepsilon)^2\,y_\ell^2} + 2\,m_\ell\,\mathbb{P}_i(S_{\tau(i)} > y_\ell)\\
&\le \frac{4\,m_\ell\,\mathbb{E}_i\,\zeta_{1,\ell}^2}{(1-\varepsilon)^2\,y_\ell^2}
\end{aligned}$$

for all $\ell \ge 1$. For sufficiently large $\ell$, we have $\mathbb{E}_i\zeta_{1,\ell}^2 \le 3\,y_\ell\,A_i^+(y_\ell)$ by Lemma 3.2 from [33]. In combination with (6.15), we obtain for such $\ell$

$$\mathbb{P}_i(S_{\tau_{2m_\ell}(i)} > y_\ell) \;\le\; \frac{12\,m_\ell\,A_i^+(y_\ell)}{(1-\varepsilon)^2\,y_\ell} \;\le\; \frac{6\,\varepsilon}{(1-\varepsilon)^2}.$$

Thus, we have shown

$$\mathbb{P}_i\Big(\max_{1\le k\le \tau_{m_\ell}(i)} S_k > y_\ell\Big) \;\le\; \frac{12\,\varepsilon}{(1-\varepsilon)^2} + \frac{\delta}{2}$$

for all $\varepsilon \in (0,1)$ and sufficiently large $\ell$ not depending on $\varepsilon$. Finally, choose $\varepsilon$ with

$$\frac{12\,\varepsilon}{(1-\varepsilon)^2} \;<\; \frac{\delta}{2}$$

to obtain a contradiction of the definition of $m_\ell$ from

$$\delta \;>\; \mathbb{P}_i\Big(\max_{1\le k\le \tau_{m_\ell}(i)} S_k > y_\ell\Big) \;\ge\; \mathbb{P}_i\Big(\max_{1\le k\le m_\ell} S_k > y_\ell\Big)$$

for large $\ell$. □

Now, we give the announced result on $\overline{\sigma}^>(x)$.

**Proposition 6.19** *Let $(M_n,S_n)_{n\ge 0}$ be a non-trivial MRW such that $S_{\tau_n(i)} \xrightarrow{\mathbb{P}_i} \infty$ and $\mathbb{E}_i\tau(i)^{1+\alpha}<\infty$, $\alpha\ge 0$, for some $i\in\mathscr{S}$. The following conditions are equivalent:*

*(i) $\mathbb{E}_iJ_i(S^-_{\tau(i)})^{1+\alpha}<\infty$ for some (hence all) $i\in\mathscr{S}$.*

*(ii) $\mathbb{E}_i\overline{\sigma}^>(x)^{1+\alpha}<\infty$ for some (hence all) $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$.*

*and imply $\mathbb{E}_i\sigma^>(x)^{1+\alpha}<\infty$ for all $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$.*

**Proof.** First of all, notice that $\mathbb{E}_iJ_i(S^-_{\tau(i)})^{1+\alpha}<\infty$ is either true for all $i\in\mathscr{S}$ or none due to Lemma 3.4.

"(i)⇒(ii)" By Lemma 3.3 (iv), $A_i(y)>0$ for all sufficiently large $y$ is satisfied. Hence, Lemma 3.5 yields $\mathbb{E}_i\tau_{\nu(x)}(i)^{1+\alpha}<\infty$ for all $i\in\mathscr{S}$ and we obtain (ii) from $\tau_{\nu(x)}(i)\ge\overline{\sigma}^>(x)$.

"(ii)⇒(i)" Suppose $\mathbb{E}_i\overline{\sigma}^>(x)^{1+\alpha}<\infty$ for some $i\in\mathscr{S}$. The assertion follows from Lemma 3.5 if $\mathbb{P}_i(\overline{\sigma}^>(0)=\tau(i))=1$. Conversely, if $\mathbb{P}_i(\overline{\sigma}^>(0)>\tau(i))=:p>0$, Lemma 6.18 yields

$$\begin{aligned}
\infty \;&>\; \mathbb{E}_i\overline{\sigma}^>(x)^{1+\alpha} \;\ge\; \mathbb{E}_i\overline{\sigma}^>(0)^{1+\alpha}\,\mathbf{1}_{\{\overline{\sigma}^>(0)>\tau(i)\}}\\
&= \mathbb{E}_i\Big[\tau(i)+(\overline{\sigma}^>(0)-\tau(i))\Big]^{1+\alpha}\,\mathbf{1}_{\{\overline{\sigma}^>(0)>\tau(i)\}}\\
&\ge \int \mathbb{E}_i\sigma^>(y)^{1+\alpha}\;\mathbb{P}_i(S^-_{\tau(i)}\in dy, S_{\tau(i)}\le 0)\\
&\gtrsim \int J_i(y)^{1+\alpha}\;\mathbb{P}_i(S^-_{\tau(i)}\in dy, S_{\tau(i)}\le 0)\\
&\asymp \mathbb{E}_iJ_i(S^-_{\tau(i)})^{1+\alpha}.
\end{aligned}$$

The implication of $\mathbb{E}_i\sigma^>(x)^{1+\alpha}<\infty$ for all $(x,i)\in\mathbb{R}_\ge\times\mathscr{S}$ was already shown in Lemma 3.5. □

Given $\mathbb{E}_i\sigma^>(x)^{1+\alpha} < \infty$ and $q(i,x) > 0$, following the steps of the proof of "(ii)⇒(i)" yields

$$\int J_i(y)^{1+\alpha}\ \mathbb{P}_i(S^-_{\tau(i)} \in dy, H^i \le x)\ <\ \infty$$

for all large $x \in \mathbb{R}_\ge$. As Example 6.16 illustrated, $\mathbb{P}_i(S^-_{\tau(i)} > y, H^i \le x)$ and $\mathbb{P}_i(S^-_{\tau(i)} > y)$ can be of a different magnitude.

## 6.6. Asymptotic Growth

We aim at a counterpart of Theorem 1.6. An important step is the verification of the following lemma.

**Lemma 6.20** *Let $(M_n, S_n)_{n\ge 0}$ be a non-trivial MRW and $\alpha \ge 0$. Suppose $\mathbb{E}_i\tau(i)^{1+\alpha} \vee \mathbb{E}_i[\tau(i)\log\tau(i)] < \infty$ and $\sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_n \le 0) < \infty$ are satisfied for some $i \in \mathscr{S}$. Then,*

$$\sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_n \le y)\ \asymp\ \begin{cases} \log J_i(y), & \text{if } \alpha = 0,\\ J_i(y)^\alpha, & \text{if } \alpha > 0,\end{cases}$$

*as $y \to \infty$ for all $i \in \mathscr{S}$.*

**Proof.** Due to (6.10) it suffices to show

$$\int \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dx)\ \asymp\ f_\alpha(y) \qquad \text{as } y \to \infty,$$

where

$$f_\alpha(y)\ :=\ \begin{cases} \log J_i(y), & \text{if } \alpha = 0,\\ J_i(y)^\alpha, & \text{if } \alpha > 0.\end{cases}$$

The assumption induces $A_i(y) > 0$ for all sufficiently large $y$ and $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ by Theorem 6.13, which implies

$$\sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \asymp\ f_\alpha(x+y) \qquad \text{as } y \to \infty$$

for all $x \in \mathbb{R}_\ge$ (see Theorem 1.6). Hence, one part follows easily from

$$\int \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dx)\ \ge\ \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le y).$$

For the other part, notice first that there exists a constant $c_\alpha > 0$ such that

$$f_\alpha(x+y)\ \le\ c_\alpha\,[f_\alpha(x) + f_\alpha(y)] \qquad \text{for all } x,y \text{ large enough.} \tag{6.17}$$

This follows from subadditivity of $y \mapsto J_i(y)$ and of $y \mapsto \log(y)$ for large values combined with $(x+y)^\alpha \le c'_\alpha(x^\alpha + y^\alpha)$ for all $x, y \in \mathbb{R}_>$ and some constant $c'_\alpha > 0$. Consequently, an appeal to (6.17) yields

$$\int \sum_{n\ge1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le x+y)\ \mathbb{V}_i(dx)\ \asymp\ \int f_\alpha(x+y)\ \mathbb{V}_i(dx)$$

$$\begin{aligned} &\lesssim \int f_\alpha(x) + f_\alpha(y)\ \mathbb{V}_i(dx) \\ &\asymp \int f_\alpha(x)\ \mathbb{V}_i(dx) + f_\alpha(y)\ \asymp\ f_\alpha(y), \end{aligned}$$

which finishes the proof. □

**Theorem 6.21** *Let $(M_n, S_n)_{n\geq 0}$ be a non-trivial MRW. The following assertions are true:*

*(i) If $(S_n)_{n\geq 0}$ is positive divergent and $\mathbb{E}_i[\tau(i)\log\tau(i)] < \infty$, then*

$$\sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n \leq y)\ \asymp\ \log J_i(y) \qquad \text{as } y \to \infty$$

*for all $i \in \mathscr{S}$.*

*(ii) Suppose $\mathbb{E}_i\,\rho(0)^\alpha < \infty$ and $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some $i \in \mathscr{S}$ and $\alpha > 0$. Then,*

$$\mathbb{E}_i\,\sigma^>(y)^\alpha\ \asymp\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq y)\ \asymp\ \mathbb{E}_i\Lambda(y)^\alpha\ \asymp\ \mathbb{E}_i\rho(y)^\alpha\ \asymp\ J_i(y)^\alpha$$

*as $y \to \infty$ for all $i \in \mathscr{S}$.*

**Proof.** (i) Since positive divergence implies the finiteness of the harmonic renewal measures, the assertion results immeditately from Lemma 6.20.

(ii) Fix some arbitrary $i \in \mathscr{S}$. The lower approximation follows from Lemma 6.18, $\sigma^>(y) \leq \Lambda(y) + 1$ and

$$\mathbb{E}_i\,\sigma^>(y)^\alpha\ \asymp\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\max_{1\leq k\leq n} S_k \leq y)\ \leq\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_n \leq y)\ \lesssim\ \mathbb{E}_i\,\rho(y)^\alpha.$$

It remains to prove $\mathbb{E}_i\,\rho(y)^\alpha \lesssim J_i(y)^\alpha$ as $y \to \infty$, because $\Lambda(y) \leq \rho(y)$. Define

$$\widetilde{\rho}(y)\ :=\ \sup\{n \geq 0 : S_{\tau_{\nu(y)}(i)+n} - S_{\tau_{\nu(y)}(i)} \leq 0\}.$$

Observe that $\mathbb{E}_i\,\widetilde{\rho}(y)^\alpha = \mathbb{E}_i\rho(0)^\alpha < \infty$ and

$$\rho(y)\ \leq\ \tau_{\nu(y)}(i) + \widetilde{\rho}(y)$$

entails $\mathbb{E}_i\,\rho(y)^\alpha \lesssim \mathbb{E}_i\tau_{\nu(y)}(i)^\alpha$ as $y \to \infty$. An appeal to Lemma C.5 entails

$$\mathbb{E}_i\tau_{\nu(y)}(i)^\alpha\ \asymp\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_{\nu(y)}(i) > \tau_n(i))\ \leq\ \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \leq y).$$

Furthermore, since $\mathbb{E}_i\rho(0)^\alpha < \infty$ implies $\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \leq 0) < \infty$, the assertion follows from Theorem 1.6. □

Besides, one can generalise the assertion of Lemma C.6 to MRWs.

**Lemma 6.22** *Suppose* $(S_{\tau_n(i)})_{n\ge 0}$ *is positive divergent for some (hence all)* $i\in\mathscr{S}$. *Then*

$$\sum_{n\ge 1}\mathbb{P}_i(0\le S_n\le y)\ \asymp\ J_i(y)\qquad \textit{as } y\to\infty$$

*for all* $i\in\mathscr{S}$.

**Proof.** One part follows directly from Lemma C.6 and

$$\sum_{n\ge 1}\mathbb{P}_i(0\le S_n\le y)\ \ge\ \sum_{n\ge 1}\mathbb{P}_i(0\le S_{\tau_n(i)}\le y).$$

For the other one, set $\mathbb{U}_i^{>}((-\infty,y]):=\mathbb{U}_i^{>}([0,y]):=\sum_{n\ge 0}\mathbb{P}_i(S_{\tau_n^{>}(i)}\le y)$ for $y\in\mathbb{R}_{\ge}$. Then, standard renewal techniques yield

$$\begin{aligned}\sum_{n\ge 1}\mathbb{P}_i(0\le S_n\le y)\ &=\ \sum_{n\ge 1}\mathbb{E}_i\Bigg(\sum_{k=\tau_{n-1}^{>}(i)+1}^{\tau_n^{>}(i)}\mathbf{1}_{\{0\le S_k\le y\}}\Bigg)\\ &=\int\mathbb{E}_i\Bigg(\sum_{k=1}^{\tau^{>}(i)}\mathbf{1}_{\{0\le S_k+x\le y\}}\Bigg)\,\mathbb{U}_i^{>}(dx)\\ &=\mathbb{E}_i\Bigg[\sum_{k=1}^{\tau^{>}(i)}\mathbb{U}_i^{>}([-S_k,y-S_k])\Bigg]\\ &\le\mathbb{E}_i\Bigg[\sum_{k=1}^{\tau^{>}(i)}\mathbb{U}_i^{>}([0,y])\Bigg]\\ &=\mathbb{E}_i\tau^{>}(i)\cdot\mathbb{U}_i^{>}([0,y]),\end{aligned}$$

whereupon an application of Theorem 1.6 entails the assertion. □

# 7. Further Counterexamples

In this section, we discuss briefly three assertions, which arise from seeking generalisations of results from fluctuation theory of random walks.

We begin with a remark on a strong law of large numbers for MRWs. As noted in Theorem 2.1 (iv), $\mathbb{E}_\pi X_1^+\wedge\mathbb{E}_\pi X_1^-<\infty$ implies $\lim_{n\to\infty}n^{-1}S_n=\mathbb{E}_\pi X_1$ a.s. Additionally, this assumption ensures

$$\mathbb{E}_\pi X_1\ =\ \mathbb{E}_\pi X_1^+-\mathbb{E}_\pi X_1^-\ =\ \pi_i\left[\mathbb{E}_i\Bigg(\sum_{k=1}^{\tau(i)}X_k^+\Bigg)-\mathbb{E}_i\Bigg(\sum_{k=1}^{\tau(i)}X_k^-\Bigg)\right]$$

and

$$\pi_i\,\mathbb{E}_iS_{\tau(i)}\ =\ \pi_i\,(\mathbb{E}_iS_{\tau(i)}^+-\mathbb{E}_iS_{\tau(i)}^-)\tag{7.1}$$

to be well-defined and equal. Given $\mathbb{E}_\pi X_1^+ \wedge \mathbb{E}_\pi X_1^- = \infty$, (7.1) can still be well-defined and thus $\lim_{n\to\infty} \tau_n(i)^{-1} S_{\tau_n(i)} = \mu \in \mathbb{R}$ a.s., but $\mathbb{E}_i(\sum_{k=1}^{\tau(i)} X_k^+) - \mathbb{E}_i(\sum_{k=1}^{\tau(i)} X_k^-)$ is not well-defined. Hence, it appears dubious, whether it is true that $\lim_{n\to\infty} n^{-1} S_n = \mu \in \mathbb{R}$ a.s. implies $\mathbb{E}_\pi X_1 = \mu$. It is generally known that the latter implication is true for ordinary random walks. However, the subsequent example shows that it is wrong for MRWs.

**Example 7.1** Let $(M_n)_{n\geq 0}$ be a birth-death chain on $\mathbb{N}_0$ with transition probabilities

$$p_{01} := 1 \qquad \text{and} \qquad p_{i\,i-1} := 1 - p_{i\,i+1} := \frac{i+2}{2\,(i+1)}, \qquad i \in \mathbb{N}.$$

Using that the stationary distribution for birth-death chains is given by

$$\pi_i = \frac{\prod_{k=1}^{i} p_{k-1\,k}}{\prod_{k=1}^{i} p_{k\,k-1}}\,\pi_0, \qquad i \in \mathbb{N},$$

and $\pi_0$ such that $\sum_{i\geq 0} \pi_i = 1$, we infer

$$\pi_i \asymp \frac{1}{(i+1)(i+2)} \asymp \frac{1}{i^2}.$$

Define a null-homologous MRW by setting

$$X_n := g(M_n) - g(M_{n-1}), \qquad n \geq 1,$$

where $g(0) := 0$ and

$$g(2i) := -g(2i-1) := i$$

for $i \in \mathbb{N}$. Then, since $p_{2i-1\,2i} \asymp \frac{1}{2}$ as $i \to \infty$, we obtain

$$\begin{aligned}\mathbb{E}_\pi X_1^+ &\geq \sum_{i\geq 1} \pi_{2i-1} \cdot p_{2i-1\,2i} \cdot \mathbb{E}(X_1 | M_0 = 2i-1, M_1 = 2i) \\ &\asymp \sum_{i\geq 1} \frac{1}{(2i)^2} \cdot \frac{1}{2} \cdot 2i = \infty\end{aligned}$$

and $\mathbb{E}_\pi X_1^- = \infty$ follows analogously. Notice that

$$|S_{\tau_n(0)+k}| \leq |X_{\tau_n(0)+k}| \leq k \quad \mathbb{P}_0\text{-a.s.}$$

for all $n \in \mathbb{N}_0$ and $1 \leq k < \chi_{n+1}(0)$. Since $\mathbb{E}_0 \tau(0) < \infty$, we infer

$$\left|\frac{S_n}{n}\right| \leq \frac{\chi_{N(n)+1}(0)}{n} \stackrel{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}_0\text{-a.s.}$$

and then the same under $\mathbb{P}_\pi$, where $N(n) := \sup\{0 \leq k \leq n : \tau_k(0) \leq n\}$.

Choose any sequence $(Z_n)_{n\geq 1}$ of i.i.d. random variables independent of $(M_n)_{n\geq 0}$ with $\mathbb{E}Z_1 =: \mu \in \mathbb{R} \setminus \{0\}$ and set

$$X_n' := Z_n + g(M_n) - g(M_{n-1}), \qquad n \geq 1.$$

The associated MRW $(M_n, S_n')_{n\geq 0}$ is easily seen to be non-trivial with $\lim_{n\to\infty} n^{-1} S_n' = \mu$ $\mathbb{P}_\pi$-a.s., although $\mathbb{E}_\pi X_1'$ does not exist.

As mentioned before, one would rather like to have $\mathbb{E}_i J_i(D^i)^{1+\alpha} < \infty$, $\alpha \geq 0$, replaced with

$$\int \left(\frac{y}{\mathbb{E}_\pi(X_1^+ \wedge y)}\right)^{1+\alpha} \mathbb{P}_\pi(X_1^- \in dy) \;<\; \infty. \tag{7.2}$$

Our example will show that

$$\mathbb{P}_\pi(X^+ > y) \;\lesssim\; \mathbb{P}_i(S^+_{\tau(i)} > y) \qquad \text{as } y \to \infty \tag{7.3}$$

and particularly

$$\frac{y}{\mathbb{E}_\pi(X^+ \wedge y)} \;\gtrsim\; J_i(y) \qquad \text{as } y \to \infty$$

are generally not true. Since the corresponding MRW is non-negative, the example can be adjusted by adding a 1-step path from 0 to 0 with $\mathbb{P}_0(X_1 \in \cdot | M_1 = 0) = \mathbb{P}_0(X_1 \in \cdot | M_1 = 0, X_1 < 0)$ such that $\mathbb{E}_i J_0(D^0)^{1+\alpha} < \infty$ and (7.2) are not equivalent. Additionally, notice that the example does again contain a dominating null-homologous MRW.

**Example 7.2** Let $\beta > 1$ and $(M_n)_{n\geq 0}$ be a Markov chain on $\mathbb{N}_0$ with transition probabilities

$$p_{01} \;:=\; 1 \qquad \text{and} \qquad p_{i\,i+1} \;:=\; 1 - p_{i0} \;:=\; \left(\frac{i}{i+1}\right)^\beta \qquad \text{for } i \in \mathbb{N}.$$

The Markov chain's stationary distribution is given by

$$\begin{aligned}\pi_i \;&=\; c\mathbb{E}_0\Big(\sum_{k=1}^{\tau(0)} \mathbf{1}_{\{M_k = i\}}\Big) \;=\; c\mathbb{P}_0(\tau(0) > i) \\ &=\; c\prod_{k=1}^{i} p_{k-1\,k} \;=\; \frac{c}{i^\beta}\end{aligned}$$

with $c = 1/\mathbb{E}_0\tau(0) < \infty$. The increments are defined by $X_n := M_n$, $n \geq 1$, so that $S^+_{\tau(0)} = \sum_{k=1}^{\tau(0)} X_k^+ = \tau(0)\,(\tau(0)-1)/2$ $\mathbb{P}_0$-a.s.

We obtain

$$\mathbb{P}_0(S^+_{\tau(0)} > n) \;=\; \mathbb{P}_0\Big(\tau(0)\,(\tau(0)-1) > 2n\Big) \;\asymp\; \mathbb{P}_0(\tau(0) > \sqrt{n}) \;\asymp\; \frac{1}{n^{\beta/2}}$$

and

$$\mathbb{P}_\pi(X_1^+ > n) \;=\; \mathbb{P}_\pi(M_1 > n) \;=\; \sum_{k>n} \pi_k \;=\; \sum_{k>n} \frac{c}{k^\beta} \;\asymp\; \frac{1}{n^{\beta-1}}$$

as $n \to \infty$, which shows (7.3). In particular, $\beta$ can be chosen to be in $(1,2)$ and thus $\mathbb{E}_\pi|X_1| = \infty$.

There is a well-known result from fluctuation theory we have not dealt with so far. [27, Theorem 3.3.1 (ii)] states that for a random walk with positive drift

$$\mathbb{E}(X^+)^{1+\alpha} \;<\; \infty \qquad \Leftrightarrow \qquad \mathbb{E}(S_{\sigma^>(x)})^{1+\alpha} \;<\; \infty \qquad \text{for some (hence all) } x \in \mathbb{R}_\geq$$

is satisfied for any $\alpha \geq 0$. Translated to MRWs, given $0 < \mathbb{E}_\pi X_1 < \infty$, one would conjecture an equivalence between

$$\mathbb{E}_i(S_{\sigma^>(x)})^{1+\alpha} \ < \ \infty \qquad \text{for all } (x,i) \in \mathbb{R}_\geq \times \mathscr{S}$$

and

$$\mathbb{E}_i\Big(\max_{1\leq k\leq\tau(i)} X_k^+\Big)^{1+\alpha} \ < \ \infty \qquad \text{for all } i \in \mathscr{S}.$$

In fact, the latter assertion can be stronger, as the subsequent example reveals.

**Example 7.3** Suppose $\alpha > 0$. Let $(M_n)_{n\geq 0}$ be the Markov chain with state space $\mathscr{S} \subset \{0\} \cup \mathbb{N}^2$ from Example 2.8 with $\Gamma$ such that

$$\mathbb{P}(\Gamma = n) \ \asymp \ \frac{1}{n^{3+\alpha+1/\alpha}} \qquad \text{as } n \to \infty.$$

We derive

$$\mathbb{E}\Gamma^{2+1/\alpha} \ < \ \infty \qquad \text{and} \qquad \mathbb{E}\Gamma^{2+\alpha+1/\alpha} \ = \ \infty. \tag{7.4}$$

Define the increments $(X_n)_{n\geq 1}$ by

$$X_n \ := \ k^{1+1/\alpha}, \qquad \text{if } M_n = (\ell,k), \text{ or } M_{n-1} = (k,k-1) \text{ and } M_n = 0 \text{ for } k,\ell \in \mathbb{N}.$$

The MRW is non-negative and fulfils

$$0 \ < \ \mathbb{E}_\pi X_1 \ = \ \pi_0\,\mathbb{E}_0 S_{\tau(0)} \ \leq \ \pi_0\,\mathbb{E}\Gamma^{2+1/\alpha} \ < \ \infty$$

by (7.4). As $X_1 \geq 1$ $\mathbb{P}_\pi$-a.s., we infer $\sigma^>(x) \leq \lceil x \rceil$ a.s. for all $i \in \mathscr{S}$ and thus

$$\begin{aligned} \mathbb{E}_i(S_{\sigma^>(x)})^{1+\alpha} \ &\leq \ \mathbb{E}_i\Big(\sum_{k=1}^{\lceil x\rceil} X_k\Big)^{1+\alpha} \ \leq \ \lceil x\rceil^{1+\alpha} \max_{1\leq k\leq\lceil x\rceil} \mathbb{E}_i X_k^{1+\alpha} \\ &\leq \ \lceil x\rceil^{1+\alpha} \cdot \lceil i+x\rceil^{2+\alpha+1/\alpha} \ < \ \infty. \end{aligned}$$

However, (7.4) yields

$$\mathbb{E}_0\Big(\max_{1\leq k\leq\tau(0)} X_k^+\Big)^{1+\alpha} \ = \ \mathbb{E}_0 X_{\tau(0)}^{1+\alpha} \ = \ \mathbb{E}\Gamma^{2+\alpha+1/\alpha} \ = \ \infty.$$

# 8. Comparison with Perturbed Random Walks

Given an i.i.d. sequence $(Z_n,\eta_n)_{n\geq 1}$, $(\sum_{k=1}^{n-1} Z_k + \eta_n)_{n\geq 1}$ is called *perturbed random walk* (PRW). Naturally, MRWs and PRWs are different stochastic processes, but in some regards and under additional assumptions their study reduces to the same object. Positive divergence of MRWs is equivalent to positive divergence of $(S_{\tau_{n-1}(i)} - D_n^i)_{n\geq 1}$. Moreover, under the assumption of $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$, an application of Lemma C.5 yields that $\mathbb{E}_i\rho(0)^\alpha < \infty$ if and only if the $\alpha$-moment of the last exit time of $(S_{\tau_{n-1}(i)} - D_n^i)_{n\geq 1}$ is finite. Analogously, under an additional moment assumption on $\tau(i)$, a study of power moments of $\sigma^>$ under $\mathbb{P}_i$ reduces to a study of power moments of the first passage time of $(S_{\tau_{n-1}(i)} + H_n^i)_{n\geq 1}$.

As these two processes are PRWs, we can translate corresponding results on PRWs into the context of MRWs. In fact, the translation works also vice versa. W.l.o.g., one can assume that $\mathbb{P}(\eta \in \mathbb{Z}) = 1$, since the distribution of $\eta$ has only an impact on the fluctuations through its tails. Therefore, the approach in Example 6.16 demonstrated that one can construct a MRW such that $S_{\tau(i)} \stackrel{d}{=} Z$, $H^i \stackrel{d}{=} \eta^+$ and $D^i \stackrel{d}{=} \eta^-$.

Alsmeyer, Iksanov and Meiners have studied fluctuation theory of PRWs. The corresponding article [5] has been published during this work. Their results on power moments of the last level $x$ exit time can be translated into the MRW context and therefore coincides partially with Theorem 6.6. We have added two more equivalences and have used different proofs, where we explicitly used the structure of MRWs, namely the ladder chain $(M_n^>)_{n\geq 0}$ and the reduction to the study of $(S_{\tau_{n-1}^>(i)} - D_n^{>,i})_{n\geq 1}$. Another benefit of the use of MRWs is an integral criterion for

$$\left|\min_{n\geq 0} S_n\right| \;=\; \left|\min_{n\geq 1}(S_{\tau_{n-1}(i)} - D_n^i)\right|$$

to possess power moments. However, we also have not found an equivalent condition for the existence of power moments of the first level $x$ passage times, but improved the insights provided by their example (see [5, Proposition 2.13]) by another example (see Example 6.16). Furthermore, for a PRW with $\sum_{k=1}^n Z_k \xrightarrow{\mathbb{P}} \infty$ and $(Z, \eta)$ being an independent pair of random variables, the proof of Proposition 6.19 and the remark thereafter entails that the existence of $(1+\alpha)$-moments of the first level $x$ passage time for the PRW is equivalent to same finiteness in terms of the random walk $(\sum_{k=1}^n Z_k)_{n\geq 0}$. Unfortunately, $Z$ and $\eta$ are dependent in most examples.

In contrast to the above-mentioned fluctuation-theoretic quantities, the criteria for the finiteness of power moments of $\Lambda(0)$ and of the weighted renewal measures from Section 6.4 differ from the corresponding ones for PRWs, since these quantities are influenced by the entire excursion within a cycle and the cycle length can be unbounded. Nevertheless, our approach for dealing with power moments of $\Lambda(0)$ is close to the one from [5].

# 9. Special Cases

The initial aim was to find a not too restrictive class of MRWs whose fluctuation behaviour is close to that of random walks. In contrast to the canonical class of non-trivial MRWs, one would rather like to have the fluctuation results expressed in terms of the embedded random walks or the stationary increments.

To consider MRWs with a finite state space is very restrictive, but an important example. Since such a MRW can be regarded as a finite union of its embedded random walks, we can easily relate fluctuation-theoretic results on the MRW to those on the embedded random walks. Moreover, it is possible to give integral criteria in terms of the stationary increments (see Section 9.1).

In Section 9.2, we will proceed with a study of certain stochastically bounded MRWs, whose assumptions are motivated by our counterexamples for non-trivial MRWs. Afterwards, our focus lies on fluctuation theory for tail-homogeneous MRWs.

## 9.1. The Case $|\mathscr{S}| < \infty$

Under the assumption $|\mathscr{S}| < \infty$, one intuitively supposes that all assertions from fluctuation theory of random walks can be generalised to MRWs. We will show the truth of this intuition. In particular, even power moments of $\sigma^>$ are manageable. Furthermore, we give integral criteria in terms of the stationary increments.

A first lemma including the stationary increment distribution is the following one.

**Lemma 9.1** *Let $(M_n, S_n)_{n\geq 0}$ be a MRW with finite state space. The following assertions are true for all $i \in \mathscr{S}$:*

*(i) There exists $x \in \mathbb{R}_\geq$ such that $\mathbb{P}_i(S^-_{\tau(i)} > y) \gtrsim \mathbb{P}_\pi(X_1^- > y+x)$ as $y \to \infty$.*

*(ii) $\mathbb{E}_\pi|X_1| < \infty$ if and only if $\mathbb{E}_i|S_{\tau(i)}| < \infty$.*

*(iii) Suppose $\mathbb{E}_\pi X^+ = \infty$. Then, $\mathbb{E}_i(S^+_{\tau(i)} \wedge y) \asymp \mathbb{E}_\pi(X_1^+ \wedge y)$ as $y \to \infty$.*

**Proof.** (i) Fix some arbitrary $i \in \mathscr{S}$. Our aim is to show

$$\mathbb{P}_i(S^-_{\tau(i)} > y) \ \gtrsim \ \mathbb{P}_j(X_1^- > y + x_{js}|M_1 = s) \qquad \text{as } y \to \infty$$

for some $x_{js} \in \mathbb{R}_\geq$ for all $j, s \in \mathscr{S}$ with $p_{js} > 0$. Then, by finiteness of $|\mathscr{S}|$, we can choose $x := \max_{j,s\in\mathscr{S}} x_{js} < \infty$ and obtain

$$\mathbb{P}_i(S^-_{\tau(i)} > y) \ \gtrsim \ \sum_{j\in\mathscr{S}} \pi_j p_{js} \mathbb{P}_j(X_1^- > y+x|M_1 = s) \ = \ \mathbb{P}_\pi(X_1^- > y+x)$$

as $y \to \infty$.

For $u \in \mathscr{S}$, define

$$\tau^0(u) \ := \ \inf\{n \geq 0 : M_n = u\}.$$

Pick arbitrary $j, s \in \mathscr{S}$ with $p_{js} > 0$. There exist $m_1, m_2 \in \mathbb{N}_0$, $z \in \mathbb{R}_\geq$ such that

$$p_1 \ := \ \mathbb{P}_i(\tau^0(j) = m_1 < \tau(i), |S_{m_1}| \leq z) \ > \ 0$$

and

$$p_2 \ := \ \mathbb{P}_s(\tau^0(i) = m_2, |S_{m_2}| \leq z) \ > \ 0.$$

It follows that

$$\begin{aligned}
\mathbb{P}_i(S^-_{\tau(i)} > y) \ &\geq \mathbb{P}_i(\tau^0(j) = m_1, |S_{m_1}| \leq z, M_{m_1+1} = s, \tau(i) = m_1+m_2+1, \\
&\qquad |S_{m_1+m_2+1} - S_{m_1+1}| \leq z, S^-_{\tau(i)} > y) \\
&\geq p_1 \cdot p_2 \cdot \mathbb{P}_j(X_1^- > y+2z|M_1 = s) \\
&\asymp \mathbb{P}_j(X_1^- > y+2z|M_1 = s).
\end{aligned}$$

(ii) Notice that the assertion (i) is also valid for $(S^+_{\tau(i)}, X_1^+)$. Consequently, using assertion (i) for the positive and the negative part, we derive that $\mathbb{E}_i|S_{\tau(i)}| < \infty$ implies $\mathbb{E}_\pi|X_1| < \infty$. The reverse implication follows from (2.2).

(iii) Let $x \in \mathbb{R}_\geq$ be the constant provided by assertion (i) for $(S^+_{\tau(i)}, X^+_1)$. We infer

$$\mathbb{E}_i(S^+_{\tau(i)} \wedge y) \;=\; \int_0^y \mathbb{P}_i(S^+_{\tau(i)} > z)\, dz \;\gtrsim\; \int_0^y \mathbb{P}_\pi(X^+_1 > z + x)\, dz \qquad \text{as } y \to \infty.$$

$\mathbb{E}_\pi X^+_1 = \infty$ ensures that the latter term is positive and thus

$$\int_0^y \mathbb{P}_\pi(X^+_1 > z + x)\, dz \;\asymp\; \int_0^{y+x} \mathbb{P}_\pi(X^+_1 > z)\, dz \;=\; \mathbb{E}_\pi[X^+_1 \wedge (y + x)] \qquad \text{as } y \to \infty.$$

Then, one side follows upon using $\mathbb{E}_\pi[X^+_1 \wedge (y+x)] \asymp \mathbb{E}_\pi(X^+_1 \wedge y)$ (cf. Lemma 3.3 (ii)).

For the other side, we use the occupation measure formula (2.1) to infer

$$\mathbb{E}_\pi(X^+_1 \wedge y) \;=\; \pi_i \, \mathbb{E}_i\left[\sum_{k=1}^{\tau(i)} (X^+_k \wedge y)\right] \;\geq\; \pi_i \, \mathbb{E}_i\left[\left(\sum_{k=1}^{\tau(i)} X^+_k\right) \wedge y\right] \;\geq\; \pi_i \, \mathbb{E}_i(S^+_{\tau(i)} \wedge y)$$

for all $y \in \mathbb{R}_\geq$. □

The assumption $\mathbb{E}_\pi X^+_1 = \infty$ is needed, since $\mathbb{E}_i S^+_{\tau(i)} = 0$ and $0 < \mathbb{E}_\pi X^+_1 < \infty$ is possible otherwise. In particular, a MRW with $\mathbb{E}_\pi X^+_1 = \infty$ and $|\mathscr{S}| < \infty$ is non-trivial.

For $y \in \mathbb{R}_\geq$, define

$$A_\pi(y) \;:=\; \mathbb{E}_\pi(X^+_1 \wedge y) - \mathbb{E}_\pi(X^-_1 \wedge y)$$

and

$$J_{\pi,\gamma}(y) \;:=\; \begin{cases} \frac{y}{[\mathbb{E}_\pi(X^+_1 \wedge y)]^\gamma}, & \text{if } \mathbb{P}_\pi(X^+_1 = 0) < 1, \\ y, & \text{if } \mathbb{P}_\pi(X^+_1 = 0) = 1, \end{cases}$$

where $0/[\mathbb{E}_\pi(X^+_1 \wedge 0)]^\gamma := 1$ if $\gamma > 0$. In particular, set $J_\pi := J_{\pi,1}$.

The next lemma puts the corresponding integral criteria in relation. We will use the well-known result that $\mathbb{E}_i \tau(i)^{1+\alpha} < \infty$, $i \in \mathscr{S}$, for all $\alpha \geq 0$ if $|\mathscr{S}| < \infty$.

**Lemma 9.2** *Let $(M_n, S_n)_{n \geq 0}$ be a non-trivial MRW with $|\mathscr{S}| < \infty$. Then, for all $\alpha \geq 0$ and $\gamma \in [0,1]$, the following conditions are equivalent:*

*(i)* $\mathbb{E}_i J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha} < \infty$ *for some (hence all) $i \in \mathscr{S}$.*

*(ii)* $\mathbb{E}_i J_{i,\gamma}(D^i)^{1+\alpha} < \infty$ *for some (hence all) $i \in \mathscr{S}$.*

*(iii)* $\mathbb{E}_\pi J_{\pi,\gamma}(X^-_1)^{1+\alpha} < \infty$.

**Proof.** If $\mathbb{E}_\pi X^+_1 < \infty$, an appeal to the occupation measure formula (2.1) shows

$$\mathbb{E}_\pi X^+_1 \;=\; \pi_i \, \mathbb{E}_i\left(\sum_{k=1}^{\tau(i)} X^+_k\right) \;\geq\; \pi_i \, \mathbb{E}_i S^+_{\tau(i)}$$

and therefore $J_{\pi,\gamma}(y) \asymp y \asymp J_{i,\gamma}(y)$ as $y \to \infty$. Otherwise, $\mathbb{E}_\pi X_1^+ = \infty$ and Lemma 9.1 (iii) entails $J_{\pi,\gamma}(y) \asymp J_{i,\gamma}(y)$ as $y \to \infty$. Hence, it suffices to just compare the corresponding integrals with integrand $J_{\pi,\gamma}^{1+\alpha}$.

"(ii)⇒(i)" follows directly from $D^i \geq S_{\tau(i)}^-$.

"(i)⇒(iii)" Let $x \in \mathbb{R}_\geq$ be the constant provided by Lemma 9.1 (i). $J_{\pi,\gamma}(y-x) \asymp J_{\pi,\gamma}(y)$ as $y \to \infty$ yields

$$\infty \;>\; \mathbb{E}_i J_{\pi,\gamma}(S_{\tau(i)}^-)^{1+\alpha} \;\gtrsim\; \mathbb{E}_\pi J_{\pi,\gamma}((X_1^- - x)^+)^{1+\alpha} \;\asymp\; \mathbb{E}_\pi J_{\pi,\gamma}(X_1^-)^{1+\alpha}.$$

"(iii)⇒(ii)" Let $F_{j,s}$ be the distribution function of $X_1^-$ given $M_0 = j$ and $M_1 = s$ and denote by $F_{j,s}^{-1}$ its pseudo-inverse. Now, given an i.i.d. sequence $(U_n)_{n\geq 1}$, which is independent of all other occurring random variables and with $U_1$ uniformly distributed on $(0,1)$, $(M_n, F_{M_{n-1},M_n}^{-1}(U_n))_{n\geq 1}$ forms a distributional copy of $(M_n, X_n^-)_{n\geq 1}$. Define $\widehat{X}_n := F_{M_{n-1},M_n}^{-1}(U_n)$ and $\widehat{S}_n = \sum_{k=1}^n \widehat{X}_k$ for $n \geq 1$. Moreover, set

$$1 - G(y) \;:=\; \max_{j,s\in\mathscr{S}} \mathbb{P}_j(X_1^- > y | M_1 = s),$$

where $G$ is a distribution function, since its right-continuity follows from the finiteness of $|\mathscr{S}|$. $(W_n)_{n\geq 1} := (G^{-1}(U_n))_{n\geq 1}$ is an i.i.d. sequence, which is independent of $(M_n)_{n\geq 0}$ and $\widehat{X}_n \leq W_n$ for all $n \geq 1$.

Back to the actual assertion, we use that $J_{\pi,\gamma}$ is a subadditive, non-decreasing function to derive

$$\begin{aligned}
\mathbb{E}_i J_{\pi,\gamma}(D^i)^{1+\alpha} &\leq \mathbb{E}_i J_{\pi,\gamma}\Big(\sum_{k=1}^{\tau(i)} X_k^-\Big)^{1+\alpha} \;=\; \mathbb{E}_i J_{\pi,\gamma}(\widehat{S}_{\tau(i)})^{1+\alpha} \\
&\leq \mathbb{E}_i\Big(\sum_{k=1}^{\tau(i)} J_{\pi,\gamma}(\widehat{X}_k)\Big)^{1+\alpha} \;\leq\; \mathbb{E}\Big(\sum_{k=1}^{\tau(i)} J_{\pi,\gamma}(W_k)\Big)^{1+\alpha}
\end{aligned}$$

and according to Theorem 1.5.4 from [27] the upper bound is finite if and only if $\mathbb{E}_i \tau(i)^{1+\alpha} < \infty$ and $\mathbb{E} J_{\pi,\gamma}(W)^{1+\alpha} < \infty$. As remarked before, we only have to verify the latter finiteness, which follows from

$$\begin{aligned}
\infty \;>\; \mathbb{E}_\pi J_{\pi,\gamma}(X_1^-)^{1+\alpha} \;&=\; \sum_{j,s\in\mathscr{S}} \pi_j\, p_{js}\, \mathbb{E}_j(J_{\pi,\gamma}(X_1^-)^{1+\alpha} | M_1 = s) \\
&\geq c \sum_{j,s\in\mathscr{S}: p_{js}>0} \int J_{\pi,\gamma}(y)^{1+\alpha}\; \mathbb{P}_j(X_1^- \in dy | M_1 = s) \\
&\geq c\,\mathbb{E} J_{\pi,\gamma}(W)^{1+\alpha},
\end{aligned}$$

where $c := \min\{\pi_j\, p_{js} : j,s \in \mathscr{S} \text{ and } p_{js} > 0\}$. □

The following two theorems form generalisations of Theorems 1.2–1.4 for the prevailing case.

**Theorem 9.3** *Let $(M_n, S_n)_{n\geq 0}$ be a non-trivial MRW with $|\mathscr{S}| < \infty$. The following conditions are equivalent:*

*(i)* $(S_n)_{n\geq 0}$ *is positive divergent.*

*(ii)* $A_i(y)>0$ *for all sufficiently large* $y$ *and* $\mathbb{E}_i J_i(S^-_{\tau(i)})<\infty$ *for some (hence all)* $i\in\mathscr{S}$*.*

*(iii)* $A_\pi(y)>0$ *for sufficiently large* $y$ *and* $\mathbb{E}_\pi J_\pi(X_1^-)<\infty$*.*

*(iv)* $\sum_{n\geq 1} n^{-1}\,\mathbb{P}_i(S_n\leq x)<\infty$ *for all* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$*.*

*(v)* $\mathbb{E}_i\sigma^>(x)<\infty$ *for all* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$*.*

**Proof.** "(i)⇒(iv)⇒(v)" can be concluded from Theorem 4.1 and "(ii)⇒(v)" is also already known from Lemma 3.5.

"(v)⇒(ii)" The assumption entails that $(M_n^>)_{n\geq 0}$ forms a Markov chain on a finite state space $\mathscr{S}^>$. Hence, $(M_n^>)_{n\geq 0}$ possesses a unique stationary distribution $\pi^>$ and $\mathbb{E}_i\kappa<\infty$, where $\kappa:=\inf\{n\geq 1: M_n^>=i\}<\infty$ and $i\in\mathscr{S}^>$. Furthermore, $(M_n^>,\sigma_n^>)_{n\geq 0}$ forms a MRW with $\mathbb{E}_{\pi^>}\sigma^><\infty$, since $|\mathscr{S}^>|<\infty$. Finally, we use the occupation measure formula (2.1) to conclude

$$\mathbb{E}_i\tau^>(i)\ \leq\ \mathbb{E}_i\sigma^{>,i}\ =\ \mathbb{E}_{\pi^>}\sigma^>\cdot\mathbb{E}_i\kappa\ <\ \infty,$$

which is equivalent to (ii).

"(ii)⇔(iii)" Suppose $\mathbb{E}_\pi|X_1|<\infty$. By the second assertion of Lemma 9.1, (ii) and (iii) reduce to the respective first condition. An application of (2.2) yields

$$\pi_i\lim_{y\to\infty}A_i(y)\ =\ \pi_i\,\mathbb{E}_i S_{\tau(i)}\ =\ \mathbb{E}_\pi X_1\ =\lim_{y\to\infty}A_\pi(y)$$

and thus the equivalence of (i) and (ii).

Suppose $\mathbb{E}_i|S_{\tau(i)}|=\infty$. Then, as remarked after Theorem 1.2, (ii) and (iii) reduce to the respective second condition and these are equivalent by Lemma 9.2 for $\gamma=1$.

"(ii)⇒(i)" (ii) states that $(S_{\tau_n(i)})_{n\geq 0}$ is positive divergent for all $i\in\mathscr{S}$. Since $|\mathscr{S}|<\infty$, this does clearly imply positive divergence of $(S_n)_{n\geq 0}$. □

**Theorem 9.4** *Let* $(M_n,S_n)_{n\geq 0}$ *be a positive divergent MRW with* $|\mathscr{S}|<\infty$ *and* $\alpha>0$*. (a) and (b) contain sets of equivalent conditions, which themselves are equivalent to* $\mathbb{E}_\pi(X_1^-)^{1+\alpha}<\infty$ *if* $\mathbb{E}_\pi|X_1|<\infty$*.*

*(a)* *(i)* $\mathbb{E}_i|\min_{n\geq 0}S_n|^\alpha<\infty$ *for some (hence all)* $i\in\mathscr{S}$*.*

*(ii)* $\mathbb{E}_i[(S^-_{\tau(i)})^\alpha\,J_i(S^-_{\tau(i)})]<\infty$ *for some (hence all)* $i\in\mathscr{S}$*.*

*(iii)* $\mathbb{E}_\pi[(X_1^-)^\alpha\,J_\pi(X_1^-)]<\infty$*.*

*(iv)* $\mathbb{E}_i|S_{\sigma^\leqslant(-x)}|^\alpha\,\mathbf{1}_{\{\sigma^\leqslant(-x)<\infty\}}<\infty$ *for all* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$*.*

*(v)* $\mathbb{E}_i(\max_{0\leq n\leq\rho(x)}|S_n|)^\alpha<\infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$*.*

*(b)* *(i)* $\mathbb{E}_i\rho(x)^\alpha<\infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_\geq\times\mathscr{S}$*.*

*(ii)* $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha}<\infty$ *for some (hence all)* $i\in\mathscr{S}$*.*

*(iii)* $\mathbb{E}_\pi J_\pi(X_1^-)^{1+\alpha}<\infty$*.*

*(iv)* $\mathbb{E}_i\sigma_{\min}^{\alpha} < \infty$.

*(v)* $\mathbb{E}_i\sigma^{\leqslant}(-x)^{\alpha}\,\mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}} < \infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$ *satisfying* $\mathbb{P}_i(\sigma^{\leqslant}(-x)=\infty)>0$.

*(vi)* $\mathbb{E}_i\Lambda(x)^{\alpha}<\infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$.

*(vii)* $\sum_{n\geq 1}n^{\alpha-1}\,\mathbb{P}_i(S_n\leq x)<\infty$ *for some (hence all)* $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$.

*(viii)* $\mathbb{E}_i\sigma^{>}(x)^{1+\alpha}<\infty$ *for all* $x\in\mathbb{R}_{\geq}$ *for some (hence all)* $i\in\mathscr{S}$.

**Proof.** Concerning the additional comment on the case $\mathbb{E}_\pi|X_1|<\infty$, notice that the respective third condition of (a) and (b) reduces to $\mathbb{E}_\pi(X_1^-)^{1+\alpha}<\infty$.

(a) The equivalence of (i), (ii) and (iii) follows immediately from Theorem 6.1 and Lemma 9.2 for $\gamma=1/(1+\alpha)$.

"(i)⇒(v)" As in [29, (vii)⇒(viii)], we derive

$$\begin{aligned}\mathbb{P}_i\Big(\max_{0\leq n\leq\rho(x)}S_n>y\Big) &\leq \mathbb{P}_i\Big(\min_{n\geq 0}(S_{\sigma^>(y)+n}-S_{\sigma^>(y)})<x-y\Big)\\ &\leq \mathbb{P}_i\Big(\Big|\min_{n\geq 0}(S_{\sigma^>(y)+n}-S_{\sigma^>(y)})\Big|>y-x\Big)\end{aligned}$$

for all $y>x$. Together with the obvious estimation

$$\begin{aligned}\mathbb{P}_i(\max_{0\leq n\leq\rho(x)}-S_n>y) \;=\; \mathbb{P}_i\Big(\min_{0\leq n\leq\rho(x)}S_n<-y\Big) &\leq \mathbb{P}_i\Big(\min_{n\geq 0}S_n<-y\Big)\\ &\leq \mathbb{P}_i\Big(\Big|\min_{n\geq 0}S_n\Big|>y-x\Big)\end{aligned}$$

(see also [33, p. 30]), $|\mathscr{S}|<\infty$ implies

$$\mathbb{P}_i\Big(\max_{0\leq n\leq\rho(x)}|S_n|>y\Big)\leq 2\max_{j\in\mathscr{S}}\mathbb{P}_j\Big(\Big|\min_{n\geq 0}S_n\Big|>y-x\Big),$$

which easily yields (v) for all $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$.

"(v)⇒(ii)" Suppose (v) is true for some $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$. Use

$$\rho[i] \;:=\; \sup\{n\geq 0: S_{\tau_n(i)}\leq 0\} \;\leq\; \rho(x)$$

to infer

$$\mathbb{E}_i\Big(\max_{0\leq n\leq\rho[i]}|S_{\tau_n(i)}|\Big)^{\alpha} \;\leq\; \mathbb{E}_i\Big(\max_{0\leq n\leq\rho(x)}|S_n|\Big)^{\alpha} \;<\; \infty.$$

An appeal to Theorem 1.4 yields (ii).

"(i)⇒(iv)" is obviously true.

"(iv)⇒(i)" Let $x\in\mathbb{R}_{\geq}$ be large enough for $\min_{j\in\mathscr{S}}\mathbb{P}_j(\sigma^{\leqslant}(-x)<\infty)=:p>0$. Set $\kappa:=\inf\{n\geq 1:\sigma_n^{\leqslant}(-x)=\infty\}$, where $\mathbb{P}_i(\kappa=\ell+1)\leq(1-p)^{\ell}$. We derive

$$\begin{aligned}\mathbb{E}_i\Big|\min_{n\geq 0}S_n\Big|^{\alpha} &\leq \mathbb{E}_i\Big[x+\Big|\sum_{k=1}^{\kappa-1}\Big(S_{\sigma_k^{\leqslant}(-x)}-S_{\sigma_{k-1}^{\leqslant}(-x)}\Big)\Big|\Big]^{\alpha}\\ &\asymp \sum_{\ell\geq 0}\mathbb{P}_i(\kappa=\ell+1)\,\mathbb{E}_i\Big|\sum_{k=1}^{\ell}\Big(S_{\sigma_k^{\leqslant}(-x)}-S_{\sigma_{k-1}^{\leqslant}(-x)}\Big)\Big|^{\alpha}\end{aligned}$$

$$\leq \sum_{\ell\geq 0}(1-p)^{\ell}\,\ell^{\alpha\wedge 1}\left[\max_{j\in\mathscr{S}}\mathbb{E}_j(|S_{\sigma^{\leqslant}(-x)}|^{\alpha}|\mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}}\right]$$
$$< \infty$$

(cf. [29, (v)$_0$ ⇒(vii)]).

(b) By Theorem 6.6 and Lemma 9.2, (i)–(iii) are equivalent. Hence, the considerations at the beginning of Section 6 entail that (i) is the strongest assertion in (b) and any assertion in (b) implies (viii). Consequently, it remains to prove "(viii)⇒(ii)".

Suppose (viii) is true. Positive divergence ensures $\mathscr{S}^{>}\neq\emptyset$ and Lemma 6.17 yields

$$\mathbb{E}_i\sigma^{>}(x)^{1+\alpha} < \infty \qquad \text{for all } (x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}.$$

Pick some $i\in\mathscr{S}^{>}$ and set $\kappa := \inf\{n\geq 0 : S_n = H_1^i\}$. Using $|\mathscr{S}^{>}|<\infty$ results in

$$\begin{aligned}
&\mathbb{E}_i\overline{\sigma}^{>}(0)^{1+\alpha}\\
&\leq \mathbb{E}_i\left(\sigma^{>}\,\mathbf{1}_{\{H_1^i=0\}}+\kappa\,\mathbf{1}_{\{H^i>0\}}+\sum_{j\in\mathscr{S}^{>}}\mathbf{1}_{\{M_\kappa=j,H_1^i>0\}}\inf\{n\geq 1: S_{\kappa+n}-S_\kappa>0\}\right)^{1+\alpha}\\
&\leq (|\mathscr{S}^{>}|+2)^{\alpha}\left(\mathbb{E}_i(\sigma^{>})^{1+\alpha}+\mathbb{E}_i\tau(i)^{1+\alpha}+\sum_{j\in\mathscr{S}^{>}}\mathbb{E}_j(\sigma^{>})^{1+\alpha}\right)\\
&<\infty,
\end{aligned}$$

which is equivalent to (ii) by Proposition 6.19. □

Notice that $\mathbb{E}_i|S_{\sigma^{\leqslant}(-x)}|^{\alpha}\,\mathbf{1}_{\{\sigma^{\leqslant}(-x)<\infty\}}<\infty$ for some $(x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$ can trivially hold, when $X_1=-x$ $\mathbb{P}_i$-a.s.

The proof revealed that a positive divergent MRW only needs $|\mathscr{S}^{>}|<\infty$ for

$$\mathbb{E}_i\sigma^{>}(x)^{1+\alpha} < \infty \qquad \text{for all } (x,i)\in\mathbb{R}_{\geq}\times\mathscr{S}$$

and $\mathbb{E}_iJ_i(S^-_{\tau(i)})^{1+\alpha}<\infty$ to be equivalent for any $\alpha>0$.

## 9.2. Certain Markov Random Walks with Stochastically Bounded Increments

Not all results from fluctuation theory of ordinary random walks can be generalised to non-trivial MRWs, which mostly results from extremal excursion within a cycle being of greater magnitude than the increment cumulated over a cycle. In our examples, this is caused by an embedded null-homologous MRW with dominating extreme values. This motivates to assume the increments of the MRW to be stochastically bounded. Unfortunately, we can not dispense with a strong stochastic boundedness condition. However, we find out that in regards to positive divergent MRWs it suffices to assume only some stochastic boundedness of the positive increments. Referring to the integral criteria on positive divergent MRWs, we strive for the validity of

$$\mathbb{P}_i(D^i>y) \lesssim \mathbb{P}_i(S^-_{\tau(i)}>cy) \qquad \text{as } y\to\infty \tag{9.1}$$

for some constant $c \in \mathbb{R}_>$ and thus $\mathbb{E}_i J_{i,\gamma}(D^i)^{1+\alpha} < \infty$ if and only if $\mathbb{E}_i J_{i,\gamma}(S^-_{\tau(i)})^{1+\alpha} < \infty$ for any $(\alpha,\gamma) \in \mathbb{R}_\geq \times [0,1]$. Recall that given a positive divergent MRW with $\mathbb{E}_i \tau(i)^{1+\alpha} < \infty$, $\alpha > 0$, the following implications are true

$$\mathbb{E}_i J_i(D^i)^{1+\alpha} \ < \ \infty \qquad \Rightarrow \qquad \mathbb{E}_i \Lambda(0)^\alpha \ < \ \infty \qquad \Rightarrow \qquad \mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} \ < \ \infty.$$

Hence, (9.1) does generally also require

$$\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} \ < \ \infty \qquad \Rightarrow \qquad \int J_i(y) \ \mathbb{V}_i^\alpha(dy) \ < \ \infty,$$

(cf. Theorem 6.9 and the remark after its proof). Example 6.12 indicates that this implication can be false if some power moment of $\tau(i)$ is infinite. This motivates assumption (A1) below. Our assumptions are:

(A1) $\mathbb{E}_i \tau(i)^{1+\beta} < \infty$ for all $\beta \geq 0$,

(A2) there is a non-negative random variable $W$ such that

$$\sup_{i,j \in \mathscr{S}} \mathbb{P}_i(X > y | M_1 = j, X > 0) \ \leq \ \mathbb{P}(W > y) \quad \text{ for all } y \in \mathbb{R}_>,$$

(A3) $\mathbb{E}(W)^\varepsilon < \infty$ for some $\varepsilon > 0$.

Define $S^\oplus_{\tau(i)} := \sum_{k=1}^{\tau(i)} X_k^+$ and $S^\ominus_{\tau(i)} := \sum_{k=1}^{\tau(i)} X_k^-$. We will prove that even the tails of $S^-_{\tau(i)}$ and $S^\ominus_{\tau(i)}$ are of the same magnitude under (A1)–(A3), but we are still not able to relate the results to conditions in terms of the stationary increments. Moreover, the assumptions do not enable us to find an equivalent condition for finiteness of power moments of $\sigma^>$.

Our main theorems on MRWs fulfilling the aforementioned assumptions are:

**Theorem 9.5** *Let $(M_n,S_n)_{n\geq 0}$ be a non-trivial MRW fulfilling (A1)–(A3). Consider the following assertions:*

*(i) $(S_n)_{n\geq 0}$ is positive divergent.*

*(ii) $A_i(y) > 0$ for all sufficiently large $y$ and $\mathbb{E}_i J_i(S^-_{\tau(i)}) < \infty$ for some (hence all) $i \in \mathscr{S}$.*

*(iii) $\sum_{n\geq 1} n^{-1} \mathbb{P}_i(S_n \leq x) < \infty$ for some (hence all) $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.*

*(iv) $\mathbb{E}_i \sigma^>(x) < \infty$ for all $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.*

*Then, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).*

**Theorem 9.6** *Let $(M_n,S_n)_{n\geq 0}$ be a positive divergent MRW fulfilling (A1)–(A3). For $\alpha > 0$, (a) and (b) contain sets of equivalent conditions, which themselves are equivalent to $\mathbb{E}_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ if $\mathbb{E}_\pi |X_1| < \infty$.*

*(a) (i) $\mathbb{E}_i |\min_{n\geq 0} S_n|^\alpha < \infty$ for some (hence all) $i \in \mathscr{S}$.*

*(ii)* $\mathbb{E}_i[(S^-_{\tau(i)})^\alpha J_i(S^-_{\tau(i)})] < \infty$ *for some (hence all)* $i \in \mathscr{S}$.

*(b)* *(i)* $\mathbb{E}_i \rho(x)^\alpha < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.

*(ii)* $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$ *for some (hence all)* $i \in \mathscr{S}$.

*(iii)* $\mathbb{E}_i \sigma^\alpha_{\min} < \infty$.

*(iv)* $\mathbb{E}_i \sigma^\leqslant(-x)^\alpha \mathbf{1}_{\{\sigma^\leqslant(-x)<\infty\}} < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$ *satisfying* $\mathbb{P}_i(\sigma^\leqslant(-x) = \infty) > 0$.

*(v)* $\mathbb{E}_i \Lambda(x)^\alpha < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.

*(vi)* $\sum_{n\geq 1} n^{\alpha-1} \mathbb{P}_i(S_n \leq x) < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.

*(vii)* $\mathbb{E}_i \overline{\sigma}^>(x)^{1+\alpha} < \infty$ *for some (hence all)* $(x,i) \in \mathbb{R}_\geq \times \mathscr{S}$.

In regard to the latter two theorems, we can point out that the fluctuation behaviour of the MRW, its embedded random walk and its dual MRW coincide. For the latter recall that ${}^\# S_{{}^\#\tau(i)} \stackrel{d}{=} S_{\tau(i)}$ under $\mathbb{P}_i$.

The moment assumption on $W$ is needed for the application of the following lemma.

**Lemma 9.7** *Let* $(T_n)_{n\geq 0}$ *be an i.i.d. sequence of non-negative random variables with* $\mathbb{E}T^\varepsilon < \infty$ *for some* $\varepsilon > 0$*. Then,*

$$\frac{1}{n^\gamma} \sum_{k=1}^n T_k \stackrel{\mathbb{P}}{\to} 0 \tag{9.2}$$

*for some* $\gamma \in \mathbb{R}_>$.

**Proof.** Following [32, Theorem 2.4],

$$\frac{1}{n^\gamma} \sum_{k=1}^n T_k \stackrel{\mathbb{P}}{\to} 0$$

is satisfied if and only if

$$\frac{n\mathbb{E}(T \wedge n^\gamma)}{n^\gamma} = \frac{n \int_0^{n^\gamma} \mathbb{P}(T > y)\, dy}{n^\gamma} \xrightarrow{n\to\infty} 0.$$

W.l.o.g., suppose $\varepsilon < 1$. $\mathbb{E}T^\varepsilon < \infty$ entails $\mathbb{P}(T > y) \lesssim y^{-\varepsilon}$ as $y \to \infty$. Choose $\gamma := 2/\varepsilon > 1$ and thus $\lim_{n\to\infty} n^{-1}(n^\gamma)^\varepsilon = \infty$. Then, (9.2) follows from

$$\limsup_{n\to\infty} \frac{n \int_0^{n^\gamma} \mathbb{P}(T > y)\, dy}{n^\gamma} \lesssim \limsup_{n\to\infty} \frac{n \int_1^{n^\gamma} y^{-\varepsilon}\, dy}{n^\gamma} \asymp \limsup_{n\to\infty} \frac{n}{(n^\gamma)^\varepsilon} = 0.$$

□

The crucial step for the proof of the above theorems is taken by the verification of the subsequent lemma.

**Lemma 9.8** *Let $(M_n, S_n)_{n\ge 0}$ be a non-trivial MRW fulfilling (A1)–(A3) and $\mathbb{E}_i(S^{\ominus}_{\tau(i)})^{1+\beta} = \infty$ for some $\beta \ge 0$ and $i \in \mathscr{S}$. Then,*

$$\mathbb{P}_i(D^i > 2y) \;\lesssim\; \mathbb{P}_i(S^{\ominus}_{\tau(i)} > 2y) \;\lesssim\; \mathbb{P}_i(S^{-}_{\tau(i)} > y) \qquad \textit{as } y \to \infty.$$

**Proof.** Obviously, it suffices to prove $\mathbb{P}_i(S^{\ominus}_{\tau(i)} > 2y) \lesssim \mathbb{P}_i(S^{-}_{\tau(i)} > y)$ as $y \to \infty$. Let $(W_n)_{n\ge 1}$ denote an i.i.d. sequence of copies of $W$, which is independent of all other occurring random variables. At first, notice the existence of $n(y) \in \mathbb{N}$, $n(y) \uparrow \infty$, such that

$$\mathbb{P}_i(S^{\ominus}_{\tau(i)} > y) \;\asymp\; \mathbb{P}_i(S^{\ominus}_{\tau(i)} > y, \tau(i) \le n(y)) \qquad \text{as } y \to \infty.$$

For $\ell \in \mathbb{N}$, set $I_\ell := \{1 \le k \le \ell : X_k \le 0\}$ and $I^c_\ell := \{1, \ldots, \ell\} \setminus I_\ell$. We infer from the independence of $(X_1, \ldots, X_n)$ given $(M_0, \ldots, M_n)$ and (A2)

$$\begin{aligned}
&\mathbb{P}_i(S^{-}_{\tau(i)} > y)\\
&\ge \mathbb{P}_i\Big(S^{\ominus}_{\tau(i)} - S^{\oplus}_{\tau(i)} > y, \tau(i) \le n(2y)\Big)\\
&= \sum_{\ell=1}^{n(2y)} \sum_{m=1}^{\ell} \mathbb{P}_i\Big(\sum_{k\in I_\ell} X^-_k - \sum_{k \in I^c_\ell} X^+_k > y, \tau(i) = \ell, |I_\ell| = m\Big)\\
&= \mathbb{E}_i\Bigg[\sum_{\ell=1}^{n(2y)} \sum_{m=1}^{\ell} \mathbb{P}_i\Big(\sum_{k\in I_\ell} X^-_k - \sum_{k \in I^c_\ell} X^+_k > y, \tau(i) = \ell, |I_\ell| = m \Big| M_0, \ldots, M_{n(2y)}\Big)\Bigg]\\
&\ge \sum_{\ell=1}^{n(2y)} \sum_{m=1}^{\ell} \mathbb{P}_i\Big(\sum_{k\in I_\ell} X^-_k - \sum_{k=1}^{n(2y)} W_k > y, \tau(i) = \ell, |I_\ell| = m\Big)\\
&\ge \int \mathbb{P}_i\Big(\sum_{k=1}^{n(2y)} W_k < x - y\Big)\, \mathbb{P}_i\Big(S^{\ominus}_{\tau(i)} \in dx, \tau(i) \le n(2y)\Big)\\
&\ge \mathbb{P}_i\Big(\frac{1}{y}\sum_{k=1}^{n(2y)} W_k < 1\Big) \cdot \mathbb{P}_i\Big(S^{\ominus}_{\tau(i)} > 2y, \tau(i) \le n(2y)\Big)\\
&\asymp \mathbb{P}\Big(\frac{1}{y}\sum_{k=1}^{n(2y)} W_k < 1\Big) \cdot \mathbb{P}_i(S^{\ominus}_{\tau(i)} > 2y).
\end{aligned}$$

Consequently, it remains to show positiveness of the first factor for all large $y$. By (A3), an appeal to Lemma 9.7 yields

$$\frac{1}{n^\gamma}\sum_{k=1}^{n} W_k \;\xrightarrow{\mathbb{P}}\; 0 \tag{9.3}$$

for some $\gamma \in \mathbb{R}_>$. W.l.o.g., let $\gamma > 1$. (A1) entails

$$\int y^\beta\, \mathbb{P}_i(S^{\ominus}_{\tau(i)} > 2y, \tau(i) > y^{1/\gamma})\; dy \;\le\; \int y^\beta\, \mathbb{P}_i(\tau(i)^\gamma > y)\; dy \;\le\; \mathbb{E}_i \tau(i)^{\gamma(1+\beta)} \;<\; \infty.$$

Then, since $\int y^\beta\, \mathbb{P}_i(S^{\ominus}_{\tau(i)} > 2y)\; dy = \infty$, we can suppose $n(2y)^\gamma = y$. A combination with (9.3) results in

$$\mathbb{P}\Big(\frac{1}{y}\sum_{k=1}^{n(2y)} W_k < 1\Big) \;=\; 1 - \mathbb{P}\Big(\frac{1}{n(2y)^\gamma}\sum_{k=1}^{n(2y)} W_k \ge 1\Big) \;\xrightarrow{y\to\infty}\; 1.$$

□

**Proof of Theorem 9.5.** Referring to Theorem 4.4, we only have to deal with the case $\mathbb{E}_\pi X_1^+ = \mathbb{E}_\pi X_1^- = \infty$. Notice that $\mathbb{E}_i S^{\ominus}_{\tau(i)} = \pi_i^{-1}\mathbb{E}_\pi X_1^-$. Therefore, Lemma 9.8 yields $\mathbb{E}_i J_i(D^i) < \infty$ if and only if $\mathbb{E}_i J_i(S^-_{\tau(i)}) < \infty$. Moreover, since the latter is an implication of condition (iii), the assertion follows from Theorem 4.1. □

**Proof of Theorem 9.6.** Preliminarily, notice that $\mathbb{E}_\pi|X_1| < \infty$ if and only if $\mathbb{E}_i|S_{\tau(i)}| < \infty$. As mentioned before, it remains to show that $\mathbb{E}_\pi X_1^+ \vee \mathbb{E}_\pi X_1^- = \infty$ implies $\mathbb{E}_i|S_{\tau(i)}| = \infty$. The occupation measure formula yields that $\mathbb{E}_\pi|X_1| = \infty$ if and only if $\mathbb{E}_i S^{\oplus}_{\tau(i)} + \mathbb{E}_i S^{\ominus}_{\tau(i)} = \infty$, which already implies $\mathbb{E}_i|S_{\tau(i)}| = \infty$ by an application of Lemma 9.8.

Suppose $\mathbb{E}_i (S^{\ominus}_{\tau(i)})^{1+\beta} < \infty$ for all $\beta \geq 0$. Then, trivially

$$\mathbb{E}_i J_i(D^i)^{1+\alpha} \ \lesssim\ \mathbb{E}_i[(D^i)^\alpha J_i(D^i)] \ \lesssim\ \mathbb{E}_i(D^i)^{1+\alpha} \ \lesssim\ \mathbb{E}_i(S^{\ominus}_{\tau(i)})^{1+\alpha} \ <\ \infty.$$

Given $\mathbb{E}_i (S^{\ominus}_{\tau(i)})^{1+\beta} = \infty$ for some $\beta \geq 0$, an appeal to Lemma 9.8 yields

$$\mathbb{E}_i[(D^i)^\alpha J_i(D^i)] \ <\ \infty \qquad \Leftrightarrow \qquad \mathbb{E}_i[(S^-_{\tau(i)})^\alpha J_i(S^-_{\tau(i)})] \ <\ \infty$$

and

$$\mathbb{E}_i J_i(D^i)^{1+\alpha} \ <\ \infty \qquad \Leftrightarrow \qquad \mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} \ <\ \infty.$$

Moreover, remember that condition (v) and (vi) in (b) each imply $\mathbb{E}_i J_i(S^-_{\tau(i)})^{1+\alpha} < \infty$. The remaining parts follow from Theorem 6.1, Theorem 6.6 and Proposition 6.19. □

## 9.3. Tail-Homogeneous Markov Random Walks

Another class of interest are *tail-homogeneous* MRWs, whose fluctuation behaviour is entirely determined by the stationary increments as we will see. A MRW $(M_n, S_n)_{n\geq 0}$ is called *tail-homogeneous* if there exists distribution functions $F$ and $G$ such that for all $y \in \mathbb{R}$

$$\sup_{i,j\in\mathscr{S}} \mathbb{P}_i(X_1 \leq y | M_1 = j) \ \leq\ F(y) \qquad \text{and} \qquad \sup_{i,j\in\mathscr{S}} \mathbb{P}_i(X_1 > y | M_1 = j) \ \leq\ 1 - G(y)$$

as well as $1 - F(y) \asymp 1 - G(y)$ and $F(-y) \asymp G(-y)$ as $y \to \infty$. In other words, the positive and negative tails of the conditional increments of the MRWs behave homogeneously. This homogeneity entails directly some crucial properties. Either all increments are non-negative or none, which makes the study of the first passage time simpler. Moreover, we can directly conclude $1 - F(y) \asymp \mathbb{P}_\pi(X_1^+ > y)$ and $F(-y) \asymp \mathbb{P}_\pi(X_1^- \geq y)$ as $y \to \infty$.

The proof does mainly rely on the subsequent Lemma (see also [1, Lemma 3.1]).

**Lemma 9.9** *Given a tail-homogeneous MRW, there exists a distributional copy* $(\widehat{X}_n)_{n\geq 1}$ *of* $(X_n)_{n\geq 1}$ *and i.i.d. sequences* $(W_n)_{n\geq 1}$ *and* $(W_n')_{n\geq 0}$ *with*

$$W_n \;\leq\; \widehat{X}_n \;\leq\; W_n' \qquad a.s.,$$

*where* $\mathbb{P}(W>y)\asymp\mathbb{P}(W'>y)\asymp\mathbb{P}_\pi(X_1^+>y)$ *and* $\mathbb{P}(W<-y)\asymp\mathbb{P}(W'<-y)\asymp\mathbb{P}_\pi(X_1^->y)$ *for* $y\to\infty$ *is satisfied.*

**Proof.** By assumption, we have for any $i,j\in\mathscr{S}$ and $n\geq 1$

$$G(y) \;\leq\; Q_{i,j}(y) \;:=\; \mathbb{P}_i(X_1\leq y|M_1=j) \;\leq\; F(y) \qquad \text{for all } y\in\mathbb{R}.$$

Let $(U_n)_{n\geq 1}$ be an i.i.d. sequence of random variables uniformly distributed on $(0,1)$, which is independent of all other occurring random variables. Now, it is standard knowledge that $\widehat{X}_n := Q^{-1}_{M_{n-1},M_n}(U_n) \stackrel{d}{=} X_n$ under any initial distribution and

$$W_n \;:=\; F^{-1}(U_n) \;\leq\; \widehat{X}_n \;\leq\; G^{-1}(U_n) \;=:\; W_n' \qquad \mathbb{P}_i\text{-a.s.}$$

for all $i\in\mathscr{S}$. The remaining assertions are immediate. □

The following theorem gathers the main results on tail-homogeneous MRWs.

**Theorem 9.10** *Let* $(M_n,S_n)_{n\geq 0}$ *be a non-trivial, tail-homogeneous MRW. The following assertions are true:*

*(i) Positive divergence of* $(S_n)_{n\geq 0}$ *is equivalent to* $A_\pi(y)>0$ *for sufficiently large* $y$ *and* $\mathbb{E}_\pi J_\pi(X_1^-)<\infty$.

*If* $(S_n)_{n\geq 0}$ *is positive divergent and* $\alpha>0$, *then*

*(ii)* $\mathbb{E}_i|\min_{n\geq 0}S_n|^\alpha<\infty$ *for some (hence all)* $i\in\mathscr{S}$ *is equivalent to* $\mathbb{E}_\pi[(X_1^-)^\alpha\,J_\pi(X_1^-)]<\infty$.

*(iii)* $\mathbb{E}_i\rho(0)^\alpha<\infty$ *for some (hence all)* $i\in\mathscr{S}$ *is equivalent to* $\mathbb{E}_\pi J_\pi(X_1^-)^{1+\alpha}<\infty$.

**Proof.** Lemma 9.9 entails that we can deal with $\widehat{S}_n := \sum_{k=1}^n \widehat{X}_k$ instead of $(S_n)_{n\geq 0}$ and further have

$$\sum_{k=1}^n W_k \;\leq\; \widehat{S}_n \;\leq\; \sum_{k=1}^n W_k' \qquad \text{a.s.}$$

with $(W_n)_{n\geq 1}$ and $(W_n')_{n\geq 1}$ as in Lemma 9.9. Since $(\sum_{k=1}^n W_k)_{n\geq 0}$ and $(\sum_{k=1}^n W_k')_{n\geq 0}$ form ordinary random walks, the results follow easily from fluctuation theory of random walks. □

Similarly, one can derive a full generalisation of Theorems 1.2–1.4 for tail-homogeneous MRWs. For the sake of brevity, we just point out that even

$$\mathbb{E}_i\sigma^>(x)^{1+\alpha} \;<\; \infty \qquad \text{for some (hence all) } (x,i)\in\mathbb{R}_\geq\times\mathscr{S}$$

is equivalent to positive divergence and $\mathbb{E}_\pi J_\pi(X_1^-)^{1+\alpha}<\infty$ for $\alpha\geq 0$.

# 10. Arcsine Law for Markov Random Walks

In the ordinary setup, the arcsine law is proved with the help of Spitzer's formulas. As mentioned before, we have to use a different approach for generalising it to the MRW setup. Define

$$\Lambda_n^{>} := \sum_{k=1}^{n} \mathbf{1}_{\{S_k>0\}} \qquad \text{and} \qquad \Lambda_n^{\leqslant} := \sum_{k=1}^{n} \mathbf{1}_{\{S_k\leq 0\}} = n-\Lambda_n^{>}, \qquad n\geq 1.$$

After we have learned that the extremal excursions within a cycle have a great influence on the almost sure asymptotic behaviour of a MRW, it may appear a suprising that these are irrelevant for the arcsine law. In fact, the proof bases on the validity of the arcsine law for the embedded random walks.

**Theorem 10.1 (Arcsine law for MRWs)** *Let $(M_n,S_n)_{n\geq 0}$ be a non-trivial MRW.*

*(i) Suppose the MRW fulfils*

$$\exists \theta\in[0,1]: \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}_i(S_{\tau_k(i)}>0) = \theta, \tag{10.1}$$

*for some $i\in\mathscr{S}$. Then,*

$$\frac{\Lambda_n^{>}}{n} \stackrel{d}{\to} AR(\theta) \qquad \text{and} \qquad \frac{\Lambda_n^{\leqslant}}{n} \stackrel{d}{\to} AR(1-\theta)$$

*under any $\mathbb{P}_j$, $j\in\mathscr{S}$.*

*(ii)* (10.1) *is equivalent to*

$$\exists \theta\in[0,1]: \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}_i(S_k>0) = \theta \tag{10.2}$$

*for some $i\in\mathscr{S}$. Moreover, both conditions hold either for all $i\in\mathscr{S}$ or none.*

First of all, we need a generalisation of the arcsine law for ordinary random walks.

**Lemma 10.2** *Let $(X_n,Z_n)_{n\geq 1}$ be an i.i.d. sequence with $\mathbb{P}(X=0)<1$ and $\mathbb{E}Z=\mu\in\mathbb{R}$. Set $S_n:=\sum_{k=1}^{n}X_k$, $n\geq 1$, and $S_0:=0$. If $(S_n)_{n\geq 0}$ fulfils*

$$\exists \theta\in[0,1]: \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}(S_k>0) = \theta, \tag{10.3}$$

*then*

$$\frac{1}{n}\sum_{k=1}^{n} Z_k\mathbf{1}_{\{S_{k-1}>0\}} \stackrel{d}{\to} \mu W,$$

*where $W \stackrel{d}{=} AR(\theta)$.*

**Proof.** W.l.o.g., we assume $\theta \in (0,1]$, since otherwise one can proceed with $\frac{1}{n}\sum_{k=1}^{n} Z_k \mathbf{1}_{\{S_k \le 0\}}$ and use $\mu - \mu W \stackrel{d}{=} \mu W'$, where $W' \stackrel{d}{=} AR(1-\theta)$. Hence, $(S_n)_{n\ge 0}$ is not negative divergent and $\Lambda_n' := \sum_{k=1}^{n} \mathbf{1}_{\{S_{k-1}>0\}}$ tends to $\infty$ a.s. Moreover, define

$$\kappa(n) \ := \ \inf\{k \ge 1 : \Lambda_k' = n\}.$$

Then, $(Z_{\kappa(n)})_{n\ge 1}$ is still i.i.d. with $Z_{\kappa(1)} \stackrel{d}{=} Z_1$ and

$$\frac{1}{\Lambda_n'}\sum_{k=1}^{\Lambda_n'} Z_{\kappa(k)} \xrightarrow{n\to\infty} \mu \qquad \text{a.s.}$$

by the strong law of large numbers. Consequently,

$$\frac{1}{n}\sum_{k=1}^{n} Z_k \mathbf{1}_{\{S_{k-1}>0\}} \ = \ \frac{1}{n}\sum_{k=1}^{\Lambda_n'} Z_{\kappa(k)} \ = \ \frac{\Lambda_n'}{n}\frac{1}{\Lambda_n'}\sum_{k=1}^{\Lambda_n'} Z_{\kappa(k)} \tag{10.4}$$

and the arcsine law for ordinary random walks entail the distributional convergence to $\mu W$. ☐

**Proof of Theorem 10.1.** (i) Again, let $W$ be a $AR(\theta)$-distributed random variable. Because $1-W \stackrel{d}{=} AR(1-\theta)$, we only deal with $(n^{-1}\Lambda_n^{>})_{n\ge 1}$.

STEP 1. In the first step, we want to prove

$$\frac{1}{n}\sum_{k=1}^{n} \chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}-D_k^i>0\}} \ \stackrel{d}{\to} \ \mathbb{E}_i\tau(i)\cdot W \tag{10.5}$$

under any $\mathbb{P}_j$. By Lemma 10.2 and

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} \chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\\ &= \frac{1}{n}\sum_{k=1}^{n} \chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}-D_k^i>0\}} + \frac{1}{n}\sum_{k=1}^{n} \chi_k(i)\,\mathbf{1}_{\{0<S_{\tau_{k-1}(i)}\le D_k^i\}},\end{aligned}$$

it remains to show

$$L_n \ := \ \frac{1}{n}\sum_{k=1}^{n} \chi_k(i)\,\mathbf{1}_{\{0<S_{\tau_{k-1}(i)}\le D_k^i\}} \ \stackrel{d}{\to} \ 0 \tag{10.6}$$

for verifying (10.5) under $\mathbb{P}_i$. (10.6) follows under $\mathbb{P}_i$, if we prove $\mathbb{E}_i L_n \xrightarrow{n\to\infty} 0$, since $L_n \ge 0$ and

$$\mathbb{E}_i L_n \ \ge \ \varepsilon\,\mathbb{P}_i(L_n > \varepsilon)$$

for all $\varepsilon > 0$. It is well-known that the absolute value of a non-trivial random walk tends stochastically to $\infty$ (e.g., see the second part of Theorem B.1.1 for $\mathbb{P}(A=1)=1$). Hence,

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_i(0 < S_{\tau_{k-1}(i)} \le y) \ \le \ \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_i(|S_{\tau_{k-1}(i)}| \le y) \ \xrightarrow{n\to\infty} \ 0$$

for all $y \in \mathbb{R}_{\geq}$. Then, dominated convergence entails

$$\begin{aligned}\mathbb{E}_i L_n &= \frac{1}{n}\sum_{k=1}^{n}\int m\,\mathbb{P}_i(0 < S_{\tau_{k-1}(i)} \leq y)\ \mathbb{P}_i((\tau(i), D^i) \in d(m,y)) \\ &= \int \frac{1}{n}\sum_{k=1}^{n} m\,\mathbb{P}_i(0 < S_{\tau_{k-1}(i)} \leq y)\ \mathbb{P}_i((\tau(i), D^i) \in d(m,y)) \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

For the truth of (10.5) under any $\mathbb{P}_j$, notice that one analogously obtains

$$\begin{aligned}\frac{1}{n}\sum_{k=2}^{n}\chi_k(i)\,\mathbf{1}_{\{(S_{\tau_{k-1}(i)}-S_{\tau_1(i)})-D_k^i+S_{\tau_1(i)}>0\}} &\overset{d}{\simeq} \frac{1}{n}\sum_{k=2}^{n}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}-S_{\tau_1(i)}>0\}} \\ &\xrightarrow{d} \mathbb{E}_i\tau(i)\cdot W\end{aligned}$$

under any $\mathbb{P}_j$ and thus (10.5) under any $\mathbb{P}_j$.

In addition, one also obtains

$$\frac{1}{n}\sum_{k=1}^{n}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}+H_k^i>0\}} \xrightarrow{d} \mathbb{E}_i\tau(i)\cdot W$$

under any $\mathbb{P}_j$ with the same approach.

Step 2. We proceed with a proof of

$$\frac{1}{n}\sum_{k=1}^{\tau_n(i)}\mathbf{1}_{\{S_k>0\}} \xrightarrow{d} \mathbb{E}_i\tau(i)\,W. \tag{10.7}$$

Observe that

$$\frac{1}{n}\sum_{k=1}^{\tau_n(i)}\mathbf{1}_{\{S_k>0\}} = \frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{\chi_k(i)}\mathbf{1}_{\{S_{\tau_{k-1}(i)+\ell}>0\}} \begin{cases} \leq \frac{1}{n}\sum_{k=1}^{n}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}+H_k^i>0\}} \\ \geq \frac{1}{n}\sum_{k=1}^{n}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}-D_k^i>0\}}.\end{cases}$$

By Step 1 the upper and lower bound converge in distribution to $\mathbb{E}_i\tau(i)\cdot W$, which yields (10.7).

Step 3. We start with

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{S_k>0\}} \begin{cases} \leq \frac{1}{n}\sum_{k=1}^{\tau_{N(n)+1}(i)}\mathbf{1}_{\{S_k>0\}} \\ \geq \frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)}\mathbf{1}_{\{S_k>0\}}.\end{cases}$$

If the lower bound converges in distribution, so does the upper bound and the limit distributions coincide, because $\mathbb{E}_i\tau(i) < \infty$. An application of $n^{-1}N(n) \to \pi_i = 1/\mathbb{E}_i\tau(i)$ a.s., yields

$$\frac{\tau_{N(n)}(i)}{n}\mathbf{1}_{\{N(n)\in[n(\pi_i-\varepsilon),n(\pi_i+\varepsilon)]^c\}} \leq \frac{\tau_n(i)}{n}\mathbf{1}_{\{N(n)\in[n(\pi_i-\varepsilon),n(\pi_i+\varepsilon)]^c\}} \to 0 \quad \text{a.s.} \tag{10.8}$$

for all $\varepsilon > 0$. We rewrite

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}} \\
&= \begin{cases} \left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)<n(\pi_i-\varepsilon)\}} + \left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)\geq n(\pi_i-\varepsilon)\}} \\ \left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)>n(\pi_i+\varepsilon)\}} + \left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)\leq n(\pi_i+\varepsilon)\}}, \end{cases}
\end{aligned}$$

and the first summand of the both alternatives converges to 0 a.s. by (10.8). Moreover, it follows by (10.7) and standard arguments that

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)\geq n(\pi_i-\varepsilon)\}} \\
&\geq \frac{\lceil n(\pi_i-\varepsilon)\rceil}{n}\,\frac{1}{\lceil n(\pi_i-\varepsilon)\rceil}\left(\sum_{k=1}^{\tau_{\lceil n(\pi_i-\varepsilon)\rceil}(i)} \mathbf{1}_{\{S_k>0\}}\right)\mathbf{1}_{\{N(n)\geq n(\pi_i-\varepsilon)\}} \\
&\xrightarrow{d} (\pi_i-\varepsilon)\,\mathbb{E}_i\tau(i)\cdot W.
\end{aligned}$$

Analogous steps for the other identity result in

$$\begin{aligned}
\mathbb{P}_i\Big((\pi_i-\varepsilon)\,\mathbb{E}_i\tau(i)\cdot W\leq x\Big) \;&\leq\; \liminf_{n\to\infty}\mathbb{P}_i\left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\leq x\right) \\
&\leq\; \limsup_{n\to\infty}\mathbb{P}_i\left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}}\leq x\right) \\
&\leq \mathbb{P}_i\Big((\pi_i+\varepsilon)\,\mathbb{E}_i\tau(i)\cdot W\leq x\Big)
\end{aligned}$$

for all $\varepsilon > 0$. Hence, $\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)} \mathbf{1}_{\{S_k>0\}} \xrightarrow{d} W$.

(ii) By assertion (i) and $(\Lambda_n^{>}/n)_{n\geq 1}$ being obviously uniformly integrable, we derive that (10.1) implies

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}_j(S_k>0) \;=\; \lim_{n\to\infty}\mathbb{E}_j\left(\frac{\Lambda_n^{>}}{n}\right) \;=\; \theta$$

for all $j \in \mathscr{S}$.

It remains to prove that (10.2) implies (10.1). Suppose (10.2) is true. By Lemma C.7, $(\chi_{N(n)+1}(i))_{n\geq 0}$ converges in distribution, hence $n^{-1}\chi_{N(n)+1}(i) \xrightarrow{d} 0$. Moreover, the sequence is uniformly integrable due to being bounded by $(n^{-1}\tau_{n+1}(i))_{n\geq 1}$, which implies $n^{-1}\,\mathbb{E}_i\chi_{N(n)+1}(i) \xrightarrow{n\to\infty} 0$ and thus

$$\lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{S_k>0\}}\right) \;=\; \lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)}\mathbf{1}_{\{S_k>0\}}\right).$$

Analogously, since

$$\left(\frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\left(\mathbf{1}_{\{0<S_{\tau_{k-1}(i)}\le D_k^i\}}+\mathbf{1}_{\{S_{\tau_{k-1}(i)}\le 0,\,S_{\tau_{n-1}(i)}+H_k^i>0\}}\right)\right)_{n\ge 1}$$

converges in distribution to 0 and is uniformly integrable due to being bounded by $(n^{-1}\,2\,\tau_n(i))_{n\ge 1}$, we obtain

$$\lim_{n\to\infty}\mathbb{E}_i\Big[\frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\left(\mathbf{1}_{\{0<S_{\tau_{k-1}(i)}\le D_k^i\}}+\mathbf{1}_{\{S_{\tau_{k-1}(i)}\le 0,\,S_{\tau_{n-1}(i)}+H_k^i>0\}}\right)\Big] \;=\; 0.$$

Combined with

$$\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)}\mathbf{1}_{\{S_k>0\}}\begin{cases}\le & \frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}+H_k^i>0\}}\\ \ge & \frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}-D_k^i>0\}},\end{cases}$$

we derive

$$\lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{\tau_{N(n)}(i)}\mathbf{1}_{\{S_k>0\}}\right) \;=\; \lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\right).$$

From here, we just remark that $n^{-1}N(n)\to\pi_i$ a.s. and Lemma 10.2 further entail

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\right) \;&=\; \lim_{n\to\infty}\mathbb{E}_i\left(\frac{N(n)}{n}\,\frac{1}{N(n)}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\right)\\ &= \pi_i\lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{N(n)}\sum_{k=1}^{N(n)}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\right)\\ &= \pi_i\lim_{n\to\infty}\mathbb{E}_i\left(\frac{1}{n}\sum_{k=1}^{n}\chi_k(i)\,\mathbf{1}_{\{S_{\tau_{k-1}(i)}>0\}}\right)\\ &= \pi_i\,\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_i\tau(i)\,\mathbb{P}_i(S_{\tau_{k-1}(i)}>0)\\ &= \frac{1}{n}\sum_{k=1}^{n}\mathbb{P}_i(S_{\tau_{k-1}(i)}>0).\end{aligned}$$

□

# B. On Markov-Modulated Random Difference Equations

## 1. An Overview of I.I.D. Random Difference Equations

In applications a stochastic process $(Z_n)_{n\geq 0}$ is often recursively defined by

$$Z_n \;=\; A_n Z_{n-1} + B_n, \qquad n \geq 1,$$

where $(A_n, B_n)_{n\geq 1}$ is an i.i.d. sequence and $Z_0$ a random variable which is independent of all other occurring random variables. A random variable with the latter properties is called *admissable*. Alternatively, we can write

$$Z_n \;=\; \Psi_n(Z_{n-1}), \qquad n \geq 1,$$

where $\Psi_n(x) := A_n x + B_n$, $x \in \mathbb{R}$. This so-called *random difference equation* appears in various settings. Mostly known are applications in insurance ruin theory and in financial time series, e.g. in form of ARCH(1) and GARCH(1,1)-processes (e.g. see [38] and [18]).

In order to find assumptions that guarantee distributional convergence of $(Z_n)_{n\geq 0}$, the continuous mapping theorem entails that possible limit distributions correspond to random variables $R$ solving the SFPE

$$R \;\stackrel{d}{=}\; \Psi(R) \;= A R + B, \tag{1.1}$$

where $(A, B)$ and $R$ are independent. In other words, the problem is associated with finding assumptions for the existence of a distribution $Q$ which stays fixed under $\Psi$, when $\Psi$ is interpreted as a map on $\mathscr{P}(\mathbb{R})$, the set of probabilty measures on $(\mathbb{R}, \mathfrak{B})$.

For convenience, define

$$\Pi_0 \;:= 1, \qquad \Pi_n \;:=\; \prod_{k=1}^{n} A_k \qquad \text{and} \qquad \Psi_{k:n} \;:=\; \Psi_k \circ \ldots \circ \Psi_n \tag{1.2}$$

for all $k, n \in \mathbb{N}$. In particular, determine $\prod_{\ell=n+1}^{n} A_\ell := 1$ for all $n \geq 0$, and $\Psi_{1:0}$ and $\Psi_{0:1}$ as the identity map on $\mathbb{R}$.

The independence assumptions yield

$$\Psi_{n:1}(Z_0) \;=\; \sum_{k=1}^{n} \Big( \prod_{\ell=k+1}^{n} A_\ell \Big) B_k + \Pi_n Z_0 \;\stackrel{d}{=}\; \sum_{k=1}^{n} \Pi_{k-1} B_n + \Pi_n Z_0 \;=\; \Psi_{1:n}(Z_0), \tag{1.3}$$

where $(A_k, B_k)_{1\le k\le n}$ is replaced with the distributional copy $(A_{n+1-k}, B_{n+1-k})_{1\le k\le n}$. Hence, in search for a distributional limit of the forward iterations $(\Psi_{n:1}(Z_0))_{n\ge0} = (Z_n)_{n\ge0}$ one may as well study distributional convergence of the backward iterations $(\Psi_{1:n}(Z_0))_{n\ge0}$. Given $\lim_{n\to\infty}\Pi_n = 0$ a.s., (1.3) entails the natural limit candidate as the distribution of

$$\widehat{Z}_\infty := \sum_{n\ge1}\Pi_{n-1}B_n$$

if it exists.

$\widehat{Z}_\infty$ is called *perpetuity*, which is the actuarial notion of the present value of infinite future payments $(B_n)_{n\ge1}$, where $(\Pi_n)_{n\ge1}$ denote the associated discount rates. Therefore, $\widehat{Z}_\infty$ being interpreted as the future financial obligations of a company motivates studying its tail decay. This has been done by many authors, most notably by Kesten [31] and Goldie [22], by using that $\widehat{Z}_\infty$ solves (1.1). Further properties of the distribution of $\widehat{Z}_\infty$ extracted from (1.1) can be found for example in [6] and [24].

Naturally, the asymptotic behaviour of $(\Pi_n)_{n\ge0}$ has a major influence on the asymptotics of the backward iterations. Given $\mathbb{P}(B=0)=1$, $\Psi_{1:n}(Z_0)=\Pi_n Z_0$ a.s. and thus convergence of $(\Psi_{1:n}(Z_0))_{n\ge0}$ reduces to convergence of $(\Pi_n)_{n\ge0}$. Given $\mathbb{P}(A=0)>0$, it is easy to verify that $\sum_{n=1}^{\kappa}\Pi_{n-1}B_n$ is the unique fixed point of (1.1), where $\kappa=\inf\{n\ge 1: A_n=0\}$. Under the assumption of $\mathbb{P}(A=0)=0$, $S_n := -\log|\Pi_n|$, $n\ge0$, defines an ordinary random walk with increments $X_n := -\log|A_n|$, $n\ge1$, which indicates the influence of fluctuation theory of random walks. In this context, we define as before

$$J(y) := \begin{cases} \frac{y}{\mathbb{E}(X^+\wedge y)}, & \text{if } \mathbb{P}(X^+=0)<1,\\ y, & \text{if } \mathbb{P}(X^+=0)=1,\end{cases}$$

for $y\in\mathbb{R}_\ge$, where $0/\mathbb{E}(X^+\wedge 0) := 1$.

Necessary and sufficient conditions for the existence of fixed points of (1.1) have finally been achieved by Vervaat [45], and Goldie and Maller [23], whose results are summarised in Theorem 1.2 below. The theorem is due to Vervaat [45, Theorem 1.5] except for the case $\lim_{n\to\infty}\Pi_n=0$ a.s., where he only proved that the law of $\widehat{Z}_\infty$ is the only possible solution to (1.1). In this case, Goldie and Maller [23, Theorem 2.1] succeeded establishing necessary and sufficient conditions for $\widehat{Z}_\infty$ to exist in the distributional sense, which we state here in an adjusted version. Notice that we put $\log^+ 0 := 0$.

**Theorem 1.1** *Suppose* $\mathbb{P}(A=0)=0$ *and* $\mathbb{P}(B=0)<1$. *The following conditions are equivalent:*

*(i)* $\lim_{n\to\infty}\Psi_{1:n}(0)=\widehat{Z}_\infty=\sum_{n\ge1}\Pi_{n-1}B_n$ *a.s. and* $\widehat{Z}_\infty$ *is a proper random variable.*

*(ii)* $((\Psi_{1:n}(Z_0))_{n\ge0}$ *converges almost surely to a proper random variable for any admissable* $Z_0$.

*(iii)* $\lim_{n\to\infty}\Pi_n=0$ *a.s. and* $\mathbb{E}\,J(\log^+|B|)<\infty$.

*(iv)* $\mathbb{P}(|A|=1)<1$ *and* $\limsup_{n\to\infty}|\Pi_{n-1}B_n|<\infty$ *a.s.*

*(v)* $\lim_{n\to\infty}\Pi_{n-1}B_n=0$ *a.s.*

*(vi)* $\sum_{n\geq 1}|\Pi_{n-1}B_n|<\infty$ *a.s.*

*Moreover, if*

$$\mathbb{P}(B=c(1-A)) \;=\; 1 \qquad \textit{for some } c\in\mathbb{R} \tag{1.4}$$

*and (i) both fail, then*

$$|\Psi_{1:n}(Z_0)| \;\stackrel{\mathbb{P}}{\to}\; \infty$$

*for any admissable* $Z_0$*.*

**Proof.** We have adjusted the statement of Theorem 2.1 from [23] by adding the equivalence to (i) and (ii). "(iii)⇒(ii)" is already part of the original version and "(ii)⇒(i)" is trivially valid. Therefore, it suffices to verify the truth of "(i)⇒(v)". Due to the second part of [22, Theorem 2.1], we only have to handle the degenerate case, i.e. when (1.4) is true. In addition, we have $c\neq 0$ by assumption. Hence,

$$\Psi_{1:n}(0) \;=\; \sum_{k=1}^{n}\Pi_{k-1}\,c(1-A_k) \;=\; c(1-\Pi_n) \qquad \text{a.s.}$$

converges almost surely to a proper random variable if either $\lim_{n\to\infty}\Pi_n=0$ a.s. or $\mathbb{P}(A=1)=1$. But $\mathbb{P}(A=1)=1$ together with (1.4) yields $\mathbb{P}(B=0)=1$, which is ruled out in the assumptions of the theorem. Moreover, $\lim_{n\to\infty}\Pi_n=0$ a.s. combined with (1.4) shows

$$\lim_{n\to\infty}\Pi_{n-1}B_n \;=\; c\lim_{n\to\infty}(\Pi_{n-1}-\Pi_n) \;=\; 0 \qquad \text{a.s.},$$

i.e. (v). □

Here is the result on fixed points of (1.1). Given $\mathbb{P}(A=0)=0$, corresponding to the fluctuation behaviour of the ordinary random walk $(-\log|\Pi_n|)_{n\geq 0}$, either $\lim_{n\to\infty}\Pi_n=0$ a.s., $\mathbb{P}(|A|=1)=1$ or $\limsup_{n\to\infty}|\Pi_n|=\infty$ a.s. prevails.

**Theorem 1.2 ([23], Theorem 3.1)** *Suppose* $\mathbb{P}(A=0)=0$*. There exists a fixed point* $Q\in\mathscr{P}(\mathbb{R})$ *of* (1.1) *if and only if one of the following conditions is satisfied:*

*(i)* $\lim_{n\to\infty}\Pi_n=0$ *a.s. and* $\mathbb{E}\,J(\log^+|B|)<\infty$*. Moreover, Q is given by the distribution of* $\widehat{Z}_\infty$*.*

*(ii)* $\mathbb{P}(|A|=1)=\mathbb{P}(B=c(1-A))=1$ *for some* $c\in\mathbb{R}$ *and either*

*(ii.1)* $\mathbb{P}(A=1)<1$ *and Q is some arbitrary distribution, which is symmetric about c, or*

*(ii.2)* $\mathbb{P}(A=1)=1$ *and Q is arbitrary.*

*(iii)* $\limsup_{n\to\infty}|\Pi_n|=\infty$ *a.s.,* $\mathbb{P}(B=c(1-A))=1$ *and* $Q=\delta_c$*.*

In addition, to draw the connection to distributional convergence of $(\Psi_{n:1}(Z_0))_{n\geq 0}$, it is easily seen that

$$\mathbb{P}(\Psi_{n:1}(Z_0)\in\cdot) \ \stackrel{w}{\to}\ Q, \tag{1.5}$$

$Q$ as in Theorem 1.2, is true for any admissable $Z_0$ if $\lim_{n\to\infty}\Pi_n = 0$ a.s., $1=\mathbb{P}(|A|=1) > \mathbb{P}(A=-1)$, or $\limsup_{n\to\infty}|\Pi_n| = \infty$ a.s. and $\Pi_n \stackrel{\mathbb{P}}{\to} 0$. Conversely, for (1.5) to be true, $Z_0 - c$ must be symmetric if $\mathbb{P}(A=-1)=1$, and $Z_0 = c$ a.s. is necessary in case (iii) if $\Pi_n \stackrel{\mathbb{P}}{\not\to} 0$.

Another survey on random difference equations and perpetuites with a different range of information can be found in [17], but dates from before the publication of [23].

# 2. The Markov-modulation Setup and Further Organisation

As in Chapter A, let $(M_n)_{n\geq 0}$ be a positive recurrent, aperiodic Markov chain on some countable set $\mathscr{S}$ with transition matrix $\mathbf{P}=(p_{ij})_{i,j\in\mathscr{S}}$ and unique stationary distribution $\pi := (\pi_i)_{i\in\mathscr{S}}$. Let $(\tau_n(i))_{n\geq 0}$, $(\chi_n(i))_{n\geq 1}$ and $(N(n))_{n\geq 1}$ be defined as before. Again, we use "a.s." synonymous for $\mathbb{P}_i$-a.s. for all $i\in\mathscr{S}$. Furthermore, let $(A_n,B_n)_{n\geq 1}$ be a Markov-modulated sequence defined by a stochastic kernel $K:(\mathscr{S}^2\times\mathfrak{B}^2)\to[0,1]$ in the way that

$$K_{M_{n-1}M_n} \ :=\ K(M_{n-1},M_n,\cdot)\ :=\ \mathbb{P}((A_n,B_n)\in\cdot|M_{n-1},M_n).$$

Then, set

$$\Psi_n(x)\ :=\ A_n x + B_n, \qquad x\in\mathbb{R}, n\geq 1.$$

We will study convergence of the backward iterations $(\Psi_{1:n}(Z_0))_{n\geq 0}$, where $Z_0$ is an *admissable* random variable. In this setup, a random variable is called *admissable* if it is independent of all other occurring random variables given $M_0$.

Suppose $\lim_{n\to\infty}\Psi_{1:n}(0) = \widehat{Z}_\infty = \sum_{n\geq 1}\Pi_{n-1}B_n$ exists in the distributional sense under any $\mathbb{P}_i$, $i\in\mathscr{S}$. Then, we use the continuity of $\Psi_1$ to infer from

$$\lim_{n\to\infty}\Psi_{1:n}(0)\ =\ \Psi_1(\lim_{n\to\infty}\Psi_{2:n}(0))$$

that $\widehat{Z}_\infty$ must satisfy

$$\widehat{Z}_\infty\ \stackrel{d}{=}\ \Psi_1(\widehat{Z}'_\infty)\ =\ A_1\widehat{Z}'_\infty + B_1,$$

where $\widehat{Z}'_\infty$ has the same distribution as $\widehat{Z}_\infty$ and is independent of $(A_1,B_1)$ given $(M_0,M_1)$. More precisely, the law of the perpetuity can be characterised as a fixed point of an action on $\mathscr{P}(\mathscr{S},\mathbb{R})$, which denotes the set of stochastic kernels $Q:\mathscr{S}\times\mathfrak{B}\to[0,1]$. This action is defined by

$$(\Psi_1\bullet Q)(i,\cdot)\ :=\ \mathbb{P}_i(A_1R_1+B_1\in\cdot), \qquad i\in\mathscr{S}, \tag{2.1}$$

where $(A_1,B_1)$ and $R_1$ are independent given $(M_0,M_1)$ and $\mathbb{P}(R_1\in\cdot|M_0=i, M_1=j) = Q(j,\cdot)$. We will give a full characterisation of the associated fixed points.

Moreover, we will examine distributional convergence of the forward iterations $(\Psi_{n:1}(Z_0))_{n\geq 0}$ for some admissable $Z_0$, i.e. a process $(Z_n)_{n\geq 0}$ recursively defined by

$$Z_n \;=\; A_n Z_{n-1} + B_n, \qquad n \geq 1. \tag{2.2}$$

Hence, the recursive structure of $(Z_n)_{n\geq 0}$ is formed by *Markov-modulated random difference equations*. (2.2) appears in the definition of Markov-switching autoregressive models, which was initially proposed in [28] and is now a popular topic in econometrics.

In our Markovian setup, the distribution of the backward and the forward iterations do not coincide. Besides studying distributional convergence of $(\Psi_{n:1}(Z_0))_{n\geq 0}$ under some $\mathbb{P}_i$, $i \in \mathscr{S}$, we will prove that possible limit distributions coincide with $\mathbb{P}_\pi(R_0 \in \cdot)$, where $(R_n)_{n\geq 0}$ forms a proper stationary solution to (2.2). In general, a sequence $(R_n)_{n\geq 0}$ is called a stationary solution to (2.2) if it is stationary under $\mathbb{P}_\pi$ and satisfies

$$R_n \;\overset{d}{=}\; A_n R_{n-1} + B_n \tag{2.3}$$

under $\mathbb{P}_\pi$ for all $n \geq 1$. In our setup, we additionally demand independence of $R_n$ and $(A_k, B_k)_{k\geq n+1}$ given $(M_0, M_n)$ for all $n \geq 0$, thus $R_0$ being admissable. The proof of this correspondence does mainly reduce to showing that if $\mathbb{P}_i(\Psi_{n:1}(Z_0) \in \cdot) \xrightarrow{w} \widehat{Q}$, then $\widehat{Q} = \sum_{i\in\mathscr{S}} \pi_i Q(i,\cdot)$, where $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ solves $\Psi_1 \star Q = Q$. $\Psi_1 \star Q$ is defined by

$$(\Psi_1 \star Q)(i,\cdot) \;:=\; \mathbb{P}_\pi(A_1 R_0 + B_1 \in \cdot\,|M_1 = i), \qquad i \in \mathscr{S}, \tag{2.4}$$

where $(M_1, A_1, B_1)$ and $R_0$ are independent given $M_0$ and $\mathbb{P}(R_0 \in \cdot\,|M_0 = i) = Q(i,\cdot)$ for all $i \in \mathscr{S}$.

The remainder of this chapter is structured as follows. Section 3 forms the basis for the main results, where a degeneracy condition induced by (1.4) is studied. Ruling out the degenerate case, the backward iterations either converge stochastically or its modulus tends stochastically to $\infty$. In contrast to the ordinary setup, stochastic convergence does not imply almost sure convergence. This can be concluded from Sections 4 and 5, where almost sure and distributional convergence of the backward iterations is examined. Section 6 characterises all fixed points of (2.1). The last section is devoted to the study of the forward iterations, which will yield an equivalent criterion for distributional convergence of $(\Psi_{n:1}(Z_0))_{n\geq 0}$.

Literature on Markov-modulated random difference equations with the above focus reduces to articles of Brandt [10] and Elton [16]. Given a stationary, ergodic sequence $(A_n, B_n)_{n\geq 1}$ with $\mathbb{E}\log|A_1| < 0$ and $\mathbb{E}\log^+|B_1| < \infty$, Brandt proved the existence of a unique stationary process $(R_n)_{n\geq 0}$ fulfilling (2.3). The same can be concluded from Theorem 3 from [16], since the assumption provides that $(\Psi_{1:n}(Z_0))_{n\geq 0}$ has a negative (top) Liapunov exponent. The associated assumption $\mathbb{E}_\pi\log|A_1| < 0$ and $\mathbb{E}_\pi\log^+|B_1| < \infty$ is used in several works on Markov-modulated random difference equations. This has been done explicitly in [11], [15], [42] and indirectly in [14]. In particular, [14], [15] and [42] study the tail decay of $Q(i,\cdot)$ and of $\sum_{i\in\mathscr{S}} \pi_i Q(i,\cdot)$ respectively, where $Q$ solves $\Psi_1 \star Q = Q$.

Again, the analysis differs with the asymptotic behaviour of $(\Pi_n)_{n\geq 0}$. Given $\mathbb{P}_\pi(A_1 = 0) = 0$, $S_n := -\log|\Pi_n|$, $n \geq 0$, defines a MRW, which emphasises the connection with the first chapter. Nevertheless, this chapter rather relies just on the basic results from fluctuation theory of MRWs.

The forward and backward iterations contain subsequences that are of an easier recursive structure. Here, we focus on an illustration for the backward iterations. For $n \geq 1$, define

$$\begin{aligned} A_n^i &:= \prod_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} A_k, \\ B_n^i &:= \Psi_{\tau_{n-1}(i)+1:\tau_n(i)}(0) \;=\; \sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} \Big( \prod_{\ell=\tau_{n-1}(i)+1}^{k-1} A_\ell \Big) B_k \end{aligned}$$

and

$$\Psi_n^i(x) \;:=\; \Psi_{\tau_{n-1}(i)+1:\tau_n(i)}(x) \;=\; A_n^i x + B_n^i, \qquad x \in \mathbb{R}.$$

Now, $(A_n^i, B_n^i)_{n\geq 1}$ are independent and further identically distributed for $n \geq 2$ under any initial distribution. Consequently, when studying almost sure convergence of the backward iterations, we can conclude from

$$\Psi_{1:\tau_n(i)}(Z_0) \;=\; \Pi_{\tau_1(i)} \Psi_{2:n}^i(Z_0) + B_1^i$$

that $(\Psi_{2:n}^i(Z_0))_{n\geq 0}$ converges almost surely, which leads to Theorem 1.1. In particular, this leads to the degeneracy condition

$$\mathbb{P}_i(B^i = c_i(1 - A_i)) \;=\; 1 \qquad \text{for some } c_i \in \mathbb{R},$$

which is examined in the next section.

# 3. The Degeneracy Condition

Given degenerate $(A_n^i, B_n^i)_{n\geq 1}$ under $\mathbb{P}_i$, namely

$$\mathbb{P}_i(B^i = c_i(1 - A_i)) \;=\; 1 \qquad \text{for some } c_i \in \mathbb{R}, \tag{3.1}$$

one supposes that $(A_n^j, B_n^j)_{n\geq 1}$ is degenerate under $\mathbb{P}_j$ for any $j \in \mathscr{S}$. Besides confirming the latter conjecture, we will derive an equivalent degeneracy condition in terms of $(A_1, B_1)$.

**Lemma 3.1** (3.1) *holds either for all $i \in \mathscr{S}$ or none.*

The proof is based on a few auxiliary lemmata.

**Lemma 3.2** *Suppose $\mathbb{P}_\pi(A_1 = 0) = 0$ and $|\mathscr{S}| > 1$. If there is a sequence $(b_i)_{i\in\mathscr{S}}$ such that*

$$\mathbb{P}_i((A^i, B^i) = (1, b_i)) \;=\; 1 \qquad \textit{for all } i \in \mathscr{S},$$

*then $b_i = 0$ and* (3.1) *are satisfied for all $i \in \mathscr{S}$.*

**Proof.** For all $i \in \mathscr{S}$, there exist $j \in \mathscr{S}$ and $n_0, n_1, n_2 \in \mathbb{N}$ such that

$$\mathbb{P}_i(\tau(i) = n_1, \tau(j) > n_1) \ > \ 0 \qquad \text{and} \qquad \mathbb{P}_j(\tau(j) = n_0 + n_2, M_{n_0} = i) \ > \ 0.$$

Notice that the first condition is guaranteed by aperiodicity of $(M_n)_{n \geq 0}$. On $E_1 := \{M_0 = j, \tau(j) = n_0 + n_2\}$, the assumption provides

$$b_j \ = \ B_1^j \ = \ \sum_{k=1}^{n_0+n_2} \Pi_{k-1} B_k \qquad \text{a.s.}$$

Moreover, on $E_2 := \{M_0 = j, M_{n_0} = M_{n_0+n_1} = i, \tau(j) = n_0 + n_1 + n_2\}$, the assumption entails

$$\begin{aligned} b_j \ &= \ B_1^j \\ &= \ \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} B_2^i + \Pi_{n_0} A_2^i \sum_{k=1}^{n_2} \Big( \prod_{\ell=1}^{k-1} A_{n_0+n_1+\ell} \Big) B_{n_0+n_1+k} \\ &= \ \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} b_i + \Pi_{n_0} \sum_{k=1}^{n_2} \Big( \prod_{\ell=1}^{k-1} A_{n_0+n_1+\ell} \Big) B_{n_0+n_1+k} \qquad \text{a.s.} \end{aligned}$$

Markov-modulation results in

$$\begin{aligned} &\mathbb{P}\Big( b_j = \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} \sum_{k=1}^{n_2} \Big( \prod_{\ell=1}^{k-1} A_{n_0+n_1+\ell} \Big) B_{n_0+n_1+k} \Big| E_2 \Big) \\ &= \mathbb{P}\Big( b_j = \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \sum_{k=n_0+1}^{n_0+n_2} \Pi_{k-1} B_k \Big| E_1 \Big) \ = \ 1 \end{aligned}$$

Consequently, $\Pi_{n_0} b_i = 0$ $\mathbb{P}_j$-a.s., which further yields $b_i = 0$ by assumption. Now, (3.1) for all $i \in \mathscr{S}$ is trivial. □

The assertion of Lemma 3.1 and the subsequent results are trivial if $|\mathscr{S}| = 1$. Therefore, $|\mathscr{S}| > 1$ is assumed in the proofs of this section without further mentioning of it.

**Lemma 3.3** $\mathbb{P}_i(A^i = 1) = 1$ *holds either for all* $i \in \mathscr{S}$ *or none.*

**Proof.** Suppose $\mathbb{P}_i(A^i = 1) = 1$ for some $i \in \mathscr{S}$ and thus $\Pi_{\tau_n(i)} = 1$ $\mathbb{P}_i$-a.s. for all $n \geq 1$. Set

$$\tau_\theta^*(i) \ := \ \inf\{n \geq 1 : M_n = i, n > \tau_\theta(j)\}$$

for $\theta \in \{1, 2\}$ and some $j \in \mathscr{S}$. Using

$$\mathbb{P}_i\Big( \Pi_{\tau_1(j)} \cdot \prod_{\ell=\tau_2(j)+1}^{\tau_2^*(i)} A_\ell \in \cdot \Big) \ = \ \mathbb{P}_i(\Pi_{\tau_1^*(i)} \in \cdot) \ = \ \delta_1,$$

we derive

$$A_2^j \ = \ \Pi_{\tau_1(j)} \cdot A_2^j \cdot \prod_{\ell=\tau_2(j)+1}^{\tau_2^*(i)} A_\ell \ = \ \Pi_{\tau_2^*(i)} \ = \ 1 \qquad \mathbb{P}_i\text{-a.s.},$$

which yields $\mathbb{P}_j(A^j = 1) = 1$. □

A fundamental result for the discussion of these degeneracy conditions is the following result due to Grincevičius.

**Proposition 3.4 ([25], Prop. 1)** *Let $(A_n, B_n)_{n\geq 1}$ be a sequence of i.i.d. random variables. The following conditions are equivalent:*

*a) $B_1 + A_1 B_2 = f(A_1 A_2)$ a.s. for some measurable function $f$.*

*b) Either $B = c(1-A)$ or $(A,B) = (1,c)$ a.s. for some $c \in \mathbb{R}$.*

**Proof of Lemma 3.1.** Suppose (3.1) holds for some $i \in \mathscr{S}$. Then,

$$\sum_{k=1}^{\tau_n(i)} \Pi_{k-1} B_k \;=\; \sum_{k=1}^{n} \Pi_{\tau_{k-1}(i)} c_i (1 - A_n^i) \;=\; c_i (1 - \Pi_{\tau_n(i)}) \qquad \mathbb{P}_i\text{-a.s.}$$

for all $n \geq 1$. In other words, $\sum_{k=1}^{\tau_n(i)} \Pi_{k-1} B_k = f(\Pi_{\tau_n(i)})$ $\mathbb{P}_i$-a.s. for some measurable function $f$. Pick some arbitrary $j \in \mathscr{S} \setminus \{i\}$ and set

$$\tau^*(i) \;:=\; \inf\{n \geq 1 : M_n = i, n > \tau_3(j)\}.$$

We have

$$f(\Pi_{\tau^*(i)}) \;=\; B_1^j + \Pi_{\tau_1(j)} \left[ B_2^j + A_2^j B_3^j \right] + B^* \qquad \mathbb{P}_i\text{-a.s.}, \tag{3.2}$$

where $B^* := \sum_{k=\tau_3(j)+1}^{\tau^*(i)} \Pi_{k-1} B_k$. The left-hand term is deterministic given $\Pi_{\tau_1(j)}$, $\prod_{k=\tau_1(j)+1}^{\tau_3(j)} A_k$ and $\prod_{k=\tau_3(j)+1}^{\tau^*(i)} A_k$. Trivially, this remains true, when one additionally conditions on $B_1^j$ and $B^*$. In this setting, we see that the term in the square brackets of (3.2) must be determistic. As this term is independent of the given random variables except for $\prod_{k=\tau_1(j)+1}^{\tau_3(j)} A_k = A_2^j A_3^j$, we derive

$$B_2^j + A_2^j B_3^j \;=\; h(A_2^j A_3^j) \qquad \mathbb{P}_i\text{-a.s.}$$

for some measurable function $h$. An appeal to Proposition 3.4 yields that either $B^j = c_j(1 - A^j)$ or $(A^j, B^j) = (1, c_j)$ $\mathbb{P}_j$-a.s. for some $c_j \in \mathbb{R}$. Since $j$ was arbitrary, one of both alternatives must hold for each $j \in \mathscr{S}$.

If the first alternative is satisfied for all $j \in \mathscr{S}$, there is nothing left to show. Now, suppose the second alternative is true for some $j \in \mathscr{S}$ and thus $\mathbb{P}_j(A^j = 1) = 1$ for all $j \in \mathscr{S}$ by Lemma 3.3. Hence, regardless of the corresponding alternative, there exists a sequence $(b_j)_{j\in\mathscr{S}}$ such that

$$\mathbb{P}_j((A^j, B^j) = (1, b_j)) \;=\; 1 \qquad \text{for all } j \in \mathscr{S}.$$

Consequently, the proof finishes by an appeal to Lemma 3.2. □

Now, we turn to results on structural implications of (3.1) on $(A_n, B_n)_{n\geq 1}$. The cases $\mathbb{P}_i(\Pi_{\tau(i)} = 1) < 1$ and $\mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1$ are handled separately.

**Proposition 3.5** *Suppose* $\mathbb{P}_i(\Pi_{\tau(i)}=1)<1$ *and* (3.1) *are satisfied for some (hence all)* $i\in\mathscr{S}$. *Then,*

$$\mathbb{P}_\pi(B_1=c_{M_0}-A_1\,c_{M_1}) \;=\; 1 \tag{3.3}$$

*and, more generally,*

$$\mathbb{P}_\pi\Big(\Psi_{1:n}(c_{M_n})=c_{M_0}\Big) \;=\; 1 \tag{3.4}$$

*for all* $n\geq 1$.

**Proof.** Pick any $i,j\in\mathscr{S}$ with $p_{ij}>0$. The assumptions entail that either $\delta_{-1}=\mathbb{P}_i(\Pi_{\tau(i)}\in\cdot)$, hence

$$\mathbb{P}_i(\Pi_{\tau(i)}\in\cdot\,|M_1=j) \;\neq\; \delta_1,$$

or $\delta_{-1}\neq\mathbb{P}_i(\Pi_{\tau(i)}\in\cdot)$ and

$$\mathbb{P}_i(\Pi_{\tau_2(i)}\in\cdot\,|M_1=j) \;=\; \int\mathbb{P}_i(x\cdot\Pi_{\tau(i)}\in\cdot\,|M_1=j)\;\mathbb{P}_i(\Pi_{\tau(i)}\in dx) \;\neq\; \delta_1.$$

For the subsequent proof, the two cases provide just notational differences. Hence, we assume

$$\mathbb{P}_i(\Pi_{\tau(i)}\in\cdot\,|M_1=j) \;\neq\; \delta_1. \tag{3.5}$$

On $E:=\{M_0=i,\,M_1=M_{\tau(i)+1}=j\}$, the degeneracy condition implies $\Psi_{1:\tau(i)}(c_i)=c_i$ and $\Psi_{2:\tau(i)+1}(c_j)=c_j$ a.s. and thus

$$\begin{aligned}\Psi_1(c_j)-c_i &\;=\; \Psi_1\left(\Psi_{2:\tau(i)+1}(c_j)\right)-c_i \;=\; \Psi_{1:\tau(i)}\left(\Psi_{\tau(i)+1}(c_j)\right)-\Psi_{1:\tau(i)}(c_i)\\ &\;=\; \Pi_{\tau(i)}\Big(\Psi_{\tau(i)+1}(c_j)-c_i\Big) \qquad \text{a.s.}\end{aligned}$$

Given $E$, $\Psi_1(c_j)\overset{d}{=}\Psi_{\tau(i)+1}(c_j)$, $\Pi_{\tau(i)}$ is unequal 1 with positive probability by (3.5) and $\Psi_{\tau(i)+1}(c_j)$ is independent of the other two random variables. Therefore, the above almost sure equality can only be satisfied if

$$1 \;=\; \mathbb{P}_i(\Psi_1(c_j)=c_i|E) \;=\; \mathbb{P}_i(\Psi_1(c_j)=c_i|M_1=j) \;=\; \mathbb{P}_i(A_1\,c_j+B_1=c_i|M_1=j),$$

which yields the first assertion. The second one follows by iteration. □

While (3.1) does determine $(c_i)_{i\in\mathscr{S}}$ uniquely if $\mathbb{P}_i(\Pi_{\tau(i)}=1)<1$, this fails when $\mathbb{P}_i(\Pi_{\tau(i)}=1)=1$. That the assertion of the latter proposition is still true for some sequence $(c_i)_{i\in\mathscr{S}}$ is our next result.

**Proposition 3.6** *Suppose* $\mathbb{P}_i(\Pi_{\tau(i)}=1)=1$ *and* (3.1) *are satisfied for some (hence all)* $i\in\mathscr{S}$. *The following assertions are true:*

*(i) There exist functions* $f_A,\,f_B:\mathscr{S}^2\to\mathbb{R}^2$ *such that*

$$(A_1,B_1) \;=\; \Big(f_A(M_0,M_1),\,f_B(M_0,M_1)\Big) \qquad a.s.$$

*(ii) There exists a group of affine functions* $\Phi_{ij}$, $i,j\in\mathscr{S}$, *such that* $\Phi_{ij}=\Phi_{ji}^{-1}$, $\Phi_{ii}(x)=x$ *for all* $x\in\mathbb{R}$ *and, if* $\mathbb{P}_i(M_n=j)>0$,

$$\mathbb{P}_i(\Psi_{1:n}=\Phi_{ij}|M_n=j) \;=\; 1.$$

*(iii) For each $(c,j)\in\mathbb{R}\times\mathscr{S}$, there exists a sequence $(c_i)_{i\in\mathscr{S}}$ such that $c_j=c$,* (3.3) *and* (3.4) *are satisfied.*

**Proof.** (i) $\mathbb{P}_i(A^i=1, B^i=0)=1$ implies $\Psi_{1:\tau_n(i)}(x)=x$ $\mathbb{P}_i$-a.s. for all $(x,i)\in\mathbb{R}\times\mathscr{S}$ and $n\geq 1$. Again, pick any $i,j\in\mathscr{S}$ with $p_{ij}>0$ and define $E:=\{M_0=i,\, M_1=M_{\tau(i)+1}=j\}$. Hence,

$$A_1x+B_1 \;=\; \Psi_1(x) \;=\; \Psi_{1:\tau(i)+1}(x) \;=\; \Psi_{\tau(i)+1}(x) \;=\; A_{\tau(i)+1}x+B_{\tau(i)+1} \qquad \text{a.s.}$$

for all $x\in\mathbb{R}$ on $E$. $(A_1,B_1)$ and $(A_{\tau(i)+1},B_{\tau(i)+1})$ being i.i.d. on $E$, yields the assertion.

(ii) For any $i,j\in\mathscr{S}$, pick a path $\{M_0=i,\, M_1=i_1,\ldots,\, M_n=j\}$ of positive probability from $i$ to $j$ of minimal length $n$. Conditioned on this path, put $\Phi_{ij}:=\Psi_{1:n}$. Given any path $\{M_0=j,\, M_1=j_1,\ldots,\, M_m=i\}$ of positive probability from $j$ to $i$, we additionally assume $\{M_{m+1}=i_1,\,\ldots,\, M_{m+n}=j\}$ to derive

$$x \;=\; \Psi_{1:m}(\Psi_{m+1:m+n}(x)) \;=\Psi_{1:m}(\Phi_{ij}(x)) \qquad \text{a.s.}$$

for all $x\in\mathbb{R}$, hence $\Psi_{1:m}=\Phi_{ij}^{-1}$ a.s. The assertion follows now from the arbitrariness of $m$.

(iii) For arbitrary $(c,j)\in\mathbb{R}\times\mathscr{S}$, set $c_i:=\Phi_{ij}(c)$ for all $i\in\mathscr{S}$. Therefore, $c_{M_0}=\Phi_{M_0j}(c)$ and

$$A_1c_{M_1}+B_1 \;=\; \Phi_{M_0M_1}(c_{M_1}) \;=\; \Phi_{M_0M_1}(\Phi_{M_1j}(c)) \;=\; \Phi_{M_0j}(c) \qquad \text{a.s.}$$

shows (3.3) and then (3.4) by iteration. □

Consequently, the degeneracy condition yields

$$\mathbb{P}_\pi(B_1=c_{M_0}-A_1c_{M_1}) \;=\; 1 \tag{3.6}$$

and

$$\Psi_{1:n}(Z_0) \;=\; c_{M_0}+\Pi_n(Z_0-c_{M_n}) \qquad \text{a.s.} \qquad \text{for all } n\geq 1$$

for some not necessarily uniquely determined $(c_i)_{i\in\mathscr{S}}$. (3.6) also appears in [42, (1.6)].

# 4. Almost Sure Convergence of the Backward Iterations

It is the aim of this section to shed some light on the almost sure asymptotic behaviour of the backward iterations $(\Psi_{1:n}(Z_0))_{n\geq 0}$ in the Markov-modulated setup, where $Z_0$ is an admissable random variable. In particular, we examine in how far Theorem 1.1 can be generalised.

We begin with a short note on the cases $\mathbb{P}_\pi(A_1=0)>0$ and $\mathbb{P}_\pi(B_1=0)=1$. Given $\mathbb{P}_\pi(A_1=0)>0$, it is easily seen that $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges almost surely to the proper random variable $\sum_{n=1}^{\kappa}\Pi_{n-1}B_n$, where $\kappa:=\inf\{n\geq 1: A_n=0\}$.

Given $\mathbb{P}_\pi(B_1=0)=1$, $\Psi_{1:n}(Z_0)=\Pi_n Z_0$ a.s. and its almost sure convergence relies on the one of $(\Pi_n)_{n\geq 0}$. If neither $\mathbb{P}_\pi(A_1=0)>0$ nor $\mathbb{P}_\pi(A_1=1)=1$, this reduces to positive divergence of $(S_n)_{n\geq 0}=(-\log|\Pi_n|)_{n\geq 0}$, which is characterised in Theorem A.4.1.

Since

$$\Psi_{1:n}(0) \;=\; \mathbf{1}_{\{\tau(i)<n\}}\left(\Pi_{\tau(i)}\,\Psi_{\tau(i)+1:n}(0)+B^i\right)+\mathbf{1}_{\{\tau(i)\geq n\}}\Psi_{1:n}(0),$$

$\mathbf{1}_{\{\tau(i)<n\}}\to 1$ a.s. and $\Psi_{\tau(i)+1:n}(0)$ is independent of $(\Pi_{\tau(i)},B^i)$ given $\tau(i)$, almost sure convergence of $(\Psi_{1:n}(0))_{n\geq 0}$ holds either under any $\mathbb{P}_i$, $i\in\mathscr{S}$, or none. The next theorem shows that the same holds for $(\Psi_{1:n}(Z_0))_{n\geq 0}$ given $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ a.s.

Suppose

$$\mathbb{P}_\pi(A_1=0) \;=\; 0 \qquad \text{and} \qquad \mathbb{P}_\pi(B_1=0)\;<\;1. \tag{4.1}$$

Then, observe that corresponding to the fluctuation behaviour of $(S_{\tau_n(i)})_{n\geq 0}$, either $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ a.s, $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$ or $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}|=\infty$ a.s. and these conditions hold either for all $i\in\mathscr{S}$ or none.

We need further definitions. For $i\in\mathscr{S}$ and $y\in\mathbb{R}_{\geq}$, introduce

$$J_i(y) \;:=\; \begin{cases} \frac{y}{\mathbb{E}_i(S^+_{\tau(i)}\wedge y)}, & \text{if } \mathbb{P}_i(S^+_{\tau(i)}=0)<1,\\ y, & \text{if } \mathbb{P}_i(S^+_{\tau(i)}=0)=1,\end{cases}$$

where $0/[\mathbb{E}_i(S^+_{\tau(i)}\wedge 0)]:=1$, and

$$W_n^i \;:=\; \max_{1\leq k\leq \chi_n(i)}|(\Pi_{\tau_n(i)+k-1}/\Pi_{\tau_{n-1}(i)})\,B_k|, \qquad n\geq 1.$$

In contrast to the ordinary setup, almost sure convergence of $(\Psi_{1:n}(0))_{n\geq 0}$ does not imply almost sure convergence of $(\Psi_{1:n}(Z_0))_{n\geq 0}$ for any admissable $Z_0$.

**Theorem 4.1** *Suppose* (4.1)*. The following conditions are equivalent:*

*(i)* $\lim_{n\to\infty}\Psi_{1:n}(0)=\widehat{Z}_\infty=\sum_{n\geq 1}\Pi_{n-1}B_n$ *a.s. and* $\widehat{Z}_\infty$ *is a proper random variable.*

*(ii)* $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ *a.s. and* $\mathbb{E}_iJ_i(\log^+W^i)<\infty$ *for some (hence all)* $i\in\mathscr{S}$.

*(iii)* $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)<1$ *for some (hence all)* $i\in\mathscr{S}$ *and* $\limsup_{n\to\infty}|\Pi_{n-1}B_n|<\infty$ *a.s.*

*(iv)* $\lim_{n\to\infty}\Pi_{n-1}B_n=0$ *a.s.*

*(v)* $\sum_{n\geq 1}|\Pi_{n-1}B_n|<\infty$ *a.s.*

*Moreover,* $(\Psi_{1:n}(Z_0))_{n\geq 0}$ *converges almost surely for any admissable* $Z_0$ *to a proper random variable if and only if* $\lim_{n\to\infty}\Pi_n=0$ *a.s. and* $\mathbb{E}_iJ_i(\log^+W^i)<\infty$*. In this case,* $\lim_{n\to\infty}\Psi_{1:n}(Z_0)=\widehat{Z}_\infty$ *a.s.*

**Proof.** Observe that the implications "(v)⇒(i)⇒(iv)⇒(iii)" are trivial. For the first part of the theorem, it remains to show "(iii)⇒(ii)" and "(ii)⇒(v)".

"(iii)⇒(ii)" Suppose $(\Pi_{\tau_n(i)})_{n\geq 0}$ does not converge to 0 a.s. Consequently, $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}|=\infty$ a.s. is the only choice left, since $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)<1$ is assumed. W.l.o.g., let $i$ be such that $\mathbb{P}_i(B_1=0)<1$. Then,

$$\limsup_{n\to\infty}|\Pi_{n-1}B_n| \;\geq\; \limsup_{n\to\infty}|\Pi_{\tau_n(i)}\,B_{\tau_n(i)+1}| \;=\; \infty \qquad \text{a.s.},$$

which contradicts (iii).

Conversely, if $\mathbb{E}_i J_i(\log^+ W^i) = \infty$ and thus $\mathbb{E}_i \log^+(W^i) = \infty$, Lemma A.4.2 entails

$$\limsup_{n\to\infty} \frac{\log^+(W^i_{n+1})}{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+} \;=\; \infty \qquad \text{a.s.}$$

and thus

$$\begin{aligned}\infty \;&=\; \limsup_{n\to\infty} \left[\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+ \left(-1 + \frac{\log^+(W^i_{n+1})}{\sum_{k=1}^n (S_{\tau_k(i)} - S_{\tau_{k-1}(i)})^+}\right)\right] \\ &\le\; \limsup_{n\to\infty} \left[-S_{\tau_n(i)} + \log^+(W^i_{n+1})\right] \\ &=\; \limsup_{n\to\infty} \left[\log^+ |\Pi_{\tau_n(i)} W^i_{n+1}|\right] \qquad \text{a.s.}\end{aligned}$$

We infer $\limsup_{n\to\infty} |\Pi_{\tau_n(i)} W^i_{n+1}| = \infty$ a.s. and therefore $\limsup_{n\to\infty} |\Pi_{n-1} B_n| = \infty$ a.s., which contradicts (iii).

"(ii)⇒(v)" The implication of (v) is self-evident if one shows

$$\lim_{n\to\infty} e^{cn} \Pi_n B_{n+1} \;=\; 0 \qquad \text{a.s.} \tag{4.2}$$

for some $c \in \mathbb{R}_>$. $n/\tau_{N(n)}(i) \to 1$ a.s. implies

$$e^{cn} \Pi_n B_{n+1} \;\le\; e^{cn} \Pi_{\tau_{N(n)}(i)} W^i_{N(n)+1} \;\asymp\; e^{c\tau_{N(n)}(i)} \Pi_{\tau_{N(n)}(i)} W^i_{N(n)+1} \qquad \text{a.s.}$$

as $n \to \infty$. Furthermore, since $(e^{c\tau_n(i)} \Pi_{\tau_n(i)} W^i_{n+1})_{n\ge 0}$ is a subsequence of $(e^{cn} \Pi_n B_{n+1})_{n\ge 0}$, (4.2) is equivalent to

$$\lim_{n\to\infty} e^{c\tau_n(i)} \Pi_{\tau_n(i)} W^i_{n+1} \;=\; 0 \qquad \text{a.s.}$$

for some $c \in \mathbb{R}_>$. A logarithmic transformation shows that it suffices to prove

$$S_{\tau_n(i)} - c\tau_n(i) - \log^+(W^i_{n+1}) \;\xrightarrow{n\to\infty}\; \infty \qquad \text{a.s.}$$

for some $c \in \mathbb{R}_>$.

By assumption, $\lim_{n\to\infty} S_{\tau_n(i)} = \infty$ a.s. Hence, either $0 < \mathbb{E}_i S_{\tau(i)} < \infty$ or $\mathbb{E}_i |S_{\tau(i)}| = \infty$. In the first case, we choose $c = \mathbb{E}_i S_{\tau(i)}/2$. In the second case, we pick $c = 1$. Therefore, Theorem A.4.5 yields

$$\lim_{n\to\infty} \frac{\tau_n(i)}{S_{\tau_n(i)}} \;\le\; \frac{1}{2c} \qquad \text{a.s.}$$

As seen in the proof of Theorem A.4.1 "(ii)⇒(i)", (ii) yields

$$\limsup_{n\to\infty} \frac{\log^+(W^i_{n+1})}{S_{\tau_n(i)}} \;=\; 0 \qquad \text{a.s.}$$

Consequently,

$$\liminf_{n\to\infty} \left[S_{\tau_n(i)} - c\tau_n(i) - \log^+(W^i_{n+1})\right] \;=\; \liminf_{n\to\infty} S_{\tau_n(i)} \left(1 - c\frac{\tau_n(i)}{S_{\tau_n(i)}} - \frac{\log^+ |W^i_n|}{S_{\tau_n(i)}}\right)$$

$$\geq \liminf_{n\to\infty} S_{\tau_n(i)}\left(1/2 - \frac{\log^+|W_n^i|}{S_{\tau_n(i)}}\right)$$
$$= \infty \qquad \text{a.s.}$$

Concerning the additional assertion, notice that almost sure convergence of $(\Psi_{1:n}(0))_{n\geq 0}$ and $(\Psi_{1:n}(Z_0))_{n\geq 0}$ for some admissable, non-zero $Z_0$ leads to (ii) being satisfied and almost sure convergence of

$$\Psi_{1:n}(Z_0) - \Psi_{1:n}(0) \;=\; \Pi_n Z_0$$

to a proper random variable. Hence, either $\lim_{n\to\infty}\Pi_n = 0$ a.s. or $\mathbb{P}_\pi(A_1 = 1) = 1$. The proof is complete if we exclude the second possibility. Given $\mathbb{P}_\pi(A_1 = 1) = 1$, $\Psi_{1:n}(0) = \sum_{k=1}^n B_k$ forms a MRW, which converges almost surely to a proper random variable if and only if $\mathbb{P}_\pi(B_1 = 0) = 1$, which is ruled out in (4.1). □

Indeed, (iv) does not generally imply $\lim_{n\to\infty}\Pi_n = 0$ a.s. Let $(M_n)_{n\geq 0}$ be the Markov chain defined in Example 2.7, where we dispense with modelling aperiodicity. In addition, for all $i \in \mathbb{N}$ and given $(M_0, M_1, M_2) = (0, i, 0)$, we define

$$(A_1, B_1, A_2, B_2) \;:=\; (\exp(p_{0i}^{-1}), 1, \exp(-p_{0i}^{-1})/2, \exp(-p_{0i}^{-1})).$$

Obviously, $\lim_{n\to\infty}\Pi_{\tau_n(0)} = 0$ a.s. and $\lim_{n\to\infty}\Pi_{n-1}B_n = 0$ a.s., but $\limsup_{n\to\infty}\Pi_n = \infty$ a.s. (cf. Example 2.7).

# 5. Distributional Convergence of the Backward Iterations

In terms of distributional convergence, we will show that $(\Psi_{1:n}(0))_{n\geq 0}$ and $(\Psi_{1:n}(Z_0))_{n\geq 0}$ behave equally for any admissable $Z_0$ given $\lim_{n\to\infty}\Pi_{\tau_n(i)} = 0$ a.s. Moreover, when ruling out (3.6), $(\Psi_{1:n}(Z_0))_{n\geq 0}$ either converges stochastically to a proper random variable or its modulus tends stochastically to $\infty$ under any $\mathbb{P}_i$. Concerning the cases $\mathbb{P}_\pi(A_1 = 0) > 0$ and $\mathbb{P}_\pi(B_1 = 0) = 1$, the considerations in the previous section have shown that distributional convergence coincides with almost sure convergence.

Before we state the theorem, we introduce

$$\widehat{\tau}(i) \;:=\; \inf\{\tau_n(i) : \Pi_{\tau_n(i)} = 1\}.$$

**Theorem 5.1** *Suppose* (4.1) *and let* $Z_0$ *be an admissable random variable.* $\mathbb{P}_i(\Psi_{1:n}(Z_0) \in \cdot) \xrightarrow{w} Q(i,\cdot)$, $i \in \mathscr{S}$, *for some* $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ *is satisfied if and only if one of the following conditions is fulfilled:*

*(i)* $\lim_{n\to\infty}\Pi_{\tau_n(i)} = 0$ *a.s. and* $\mathbb{E}_i J_i(\log^+|B^i|) < \infty$*. Then,* $Q(i,\cdot) = \mathbb{P}_i(\widehat{Z}_\infty \in \cdot)$ *and* $\Psi_{1:n}(Z_0) \xrightarrow{\mathbb{P}_\pi} \widehat{Z}_\infty$.

*(ii)* $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$, (3.6) *is true and one of the following cases prevails:*

*(ii.1)* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *is aperiodic.*

*(ii.2)* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *is 2-periodic and*

$$\lim_{n\to\infty} \mathbb{P}_i(\Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot) = \lim_{n\to\infty} \mathbb{P}_i(-\Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot).$$

*(iii)* $\limsup_{n\to\infty} |\Pi_n| = \infty$ *a.s. and one of the following cases prevails:*

*(iii.1)* $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$ *and* (3.6) *are true. Then,* $Q(i,\cdot) = \delta_{c_i}$ *and* $\Psi_{1:n}(Z_0) \xrightarrow{\mathbb{P}_\pi} c_{M_0}$.

*(iii.2)* $\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > 0) > 0$, $\mathbb{P}_\pi(B_1 = c(1 - A_1)) = 1$ *and* $Z_0 = c$ $\mathbb{P}_i$*-a.s. for some* $c \in \mathbb{R}$*. Then,* $Q(i,\cdot) = \delta_c$ *and* $\Psi_{1:n}(Z_0) = c$ $\mathbb{P}_i$*-a.s. for all* $n \geq 0$.

*Moreover, if* (3.6) *and (i) both fail, then*

$$|\Psi_{1:n}(Z_0)| \xrightarrow{\mathbb{P}_\pi} \infty$$

*for any admissable* $Z_0$.

Since $\Psi_{1:n}(Z_0) \xrightarrow{\mathbb{P}_\pi} \widehat{Z}_\infty$ in case (i), we infer that $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ a.s. and $\mathbb{E}_i J_i(\log^+ |B^i|) < \infty$ are either true for all $i \in \mathscr{S}$ or none.

The description of the limit distributions in (ii) is given in Section 5.3.

We emphasise that $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution either for all admissable $Z_0$ or none in the cases (i), (ii.1) and (iii.1). In contrast, we directly see that $(\Psi_{1:n}(Z_0))_{n\geq 0}$ does not necessarily converge in distribution under $\mathbb{P}_j$ if it does under $\mathbb{P}_i$ in case (iii.2). The same can be verified for the case (ii.2). Suppose $\mathbb{P}_\pi((A_1, B_1) = (-1, c)) = 1$ for some $c \in \mathbb{R}$. As noted after Theorem 1.2, $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$ if and only $Z_0 - c$ is symmetric under $\mathbb{P}_i$. The latter does obviously not imply $Z_0 - c$ to be symmetric under $\mathbb{P}_j$.

In the case of a finite state space $\mathscr{S}$, when (3.6) is ruled out, $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution if and only if it converges almost surely. This follows since Theorem 5.1 (i) entails almost sure convergence of $(\Psi^i_{1:n}(Z_0))_{n\geq 0}$ to $\widehat{Z}_\infty$ for all $i \in \mathscr{S}$.

The second part of the theorem particularly yields that the modulus of a non-trivial MRW tends stochastically to $\infty$. For explanation, given $\mathbb{P}_\pi(A_1 = 1) = 1$, $\Psi_{1:n}(0) = \sum_{k=1}^n B_k$ forms a MRW, where non-triviality is equivalent to the failure of (3.6).

Necessary conditions for distributional convergence will be obtained from an application of Theorem 1.1 on $(\Psi^i_{1:n}(Z_0))_{n\geq 0}$ for some $i \in \mathscr{S}$, but we have to ensure that $\mathbb{P}_i(B^i = 0) < 1$.

**Lemma 5.2** *Suppose* $\mathbb{P}_\pi(A_1 = 0) = 0$*. The following assertions are true:*

*(i)* $\mathbb{P}_i(B^i = 0) = 1$ *for some* $i \in \mathscr{S}$ *implies* (3.6).

*(ii) If* $\mathbb{P}_i(B^i = 0) = 1$ *for all* $i \in \mathscr{S}$*, then either* $\mathbb{P}_\pi(B_1 = 0) = 1$ *or* $\mathbb{P}_i(A^i = 1) = 1$ *for all* $i \in \mathscr{S}$.

**Proof.** (i) $\mathbb{P}_i(B^i = 0) = 1$ for some $i \in \mathscr{S}$ is equivalent to $\mathbb{P}_i(B^i = c_i(1 - A^i)) = 1$ with $c_i = 0$, which yields (3.6).

(ii) By assumption, we have

$$\sum_{k=1}^{\tau_n(i)} \Pi_{k-1} B_k \;=\; \sum_{k=1}^{n} \Pi_{\tau_{k-1}(i)} B_n^i \;=\; 0 \qquad \mathbb{P}_i\text{-a.s.}$$

for all $n \geq 0$ and $i \in \mathscr{S}$. Assume $\mathbb{P}_\pi(B_1 = 0) < 1$ and $\mathbb{P}_i(A^i = 1) < 1$ for all $i \in \mathscr{S}$. Then, we can find $j, s \in \mathscr{S}$ such that $p_{sj} > 0$ and

$$\mathbb{P}_s(B_1 = 0 | M_1 = j) \;<\; 1.$$

Moreover, there exist $n_0, n_1 \in \mathbb{N}$ such that

$$\mathbb{P}_j(\tau(s) = n_0) \;>\; 0 \qquad \text{and} \qquad \mathbb{P}_s(\Pi_{n_1} \neq 1, \tau(s) = n_1) \;>\; 0.$$

Now, on $E_1 := \{M_0 = j, \tau(s) = n_0, M_{n_0+1} = j\}$, it holds that

$$0 \;=\; \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} B_{n_0+1} \qquad \text{a.s.,}$$

while on $E_2 := \{M_0 = j,\, \tau_1(s) = n_0,\, \tau_2(s) = n_0 + n_1,\, M_{n_0+n_1+1} = j\}$,

$$\begin{aligned} 0 \;&=\; \sum_{k=1}^{n_0+n_1+1} \Pi_{k-1} B_k \;=\; \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} B_2^s + \Pi_{n_0} A_2^s B_{n_0+n_1+1} \\ &=\; \sum_{k=1}^{n_0} \Pi_{k-1} B_k + A_2^s \Pi_{n_0} B_{n_0+n_1+1} \qquad \text{a.s.} \end{aligned}$$

is satisfied. Markov-modulation yields

$$\begin{aligned} &\mathbb{P}\Big( \sum_{k=1}^{n_0} \Pi_{k-1} B_k + A_2^s \Pi_{n_0} B_{n_0+n_1+1} = 0 \Big| E_2 \Big) \\ &= 1 \\ &= \mathbb{P}\Big( \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} B_{n_0+1} \Big| E_1 \Big) \\ &= \mathbb{P}\Big( \sum_{k=1}^{n_0} \Pi_{k-1} B_k + \Pi_{n_0} B_{n_0+n_1+1} = 0 \Big| E_2 \Big) \end{aligned}$$

Thus, $\mathbb{P}(\Pi_{n_0} B_{n_0+n_1+1} (1 - A_2^s) = 0 | E_2) = 1$, but this is not possible by construction. □

We give a short example to show that $\mathbb{P}(B^i = 0) = 1$ for some $i \in \mathscr{S}$ does not imply $\mathbb{P}_j(B^j = 0) = 1$ for all $j \in \mathscr{S}$. Set $\mathscr{S} = \{0, 1, 2, 3\}$, $0 < p_{01} = 1 - p_{02} < 1$, $p_{23} = p_{30} = p_{10} = 1$, $\mathbb{P}_\pi(B_1 = 1)$,

$$\mathbb{P}(A_1 = -1,\, A_2 = 1 | M_0 = 0, M_1 = 1) \;=\; 1$$

and

$$\mathbb{P}(A_1 = -3/2,\, A_2 = -1/3,\, A_3 = 1 | M_0 = 0,\, M_1 = 2) \;=\; 1.$$

One easily computes $B^i = \sum_{k=1}^{\tau(0)} \Pi_{k-1} = 0$ a.s. just for $i = 0$.

The theorem will be proved separately for the cases $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ a.s., $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$ and $\limsup_{n\to\infty} |\Pi_{\tau_n(i)}| = \infty$ a.s.

## 5.1. The Case $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ a.s.

We begin with some simple implications of $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ a.s. By an appeal to Lemma C.7,

$$\left(\max_{1\le k\le \chi_{N(n)+1}(i)} |\Pi_{\tau_{N(n)}(i)+k}/\Pi_{\tau_{N(n)}(i)}|\right)_{n\ge 1}$$

and

$$\left(\max_{1\le \ell\le \chi_{N(n)+1}(i)} \left|\sum_{k=1}^{\ell} (\Pi_{\tau_{N(n)}(i)+k-1}/\Pi_{\tau_{N(n)}(i)})\, B_{\tau_{N(n)}(i)+k}\right|\right)_{n\ge 1}$$

converge in distribution. As $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ a.s. implies $\lim_{n\to\infty} \Pi_{\tau_{N(n)}(i)} = 0$ a.s., Slutsky's theorem entails

$$|\Pi_n| \;\le\; |\Pi_{\tau_{N(n)}(i)}| \cdot \max_{1\le k\le \chi_{N(n)+1}(i)} |\Pi_{\tau_{N(n)}(i)+k}/\Pi_{\tau_{N(n)}(i)}| \;\xrightarrow{\mathbb{P}_\pi}\; 0,$$

thus $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$, and

$$\max_{1\le \ell\le \chi_{N(n)+1}(i)} \left|\sum_{k=1}^{\ell} \Pi_{\tau_{N(n)}(i)+k-1}\, B_{\tau_{N(n)}(i)+k}\right| \;\xrightarrow{\mathbb{P}_\pi}\; 0. \tag{5.1}$$

The following lemma uses the approach of [23, Lemma 5.5]. Furthermore, we make use of the notation $T_n \overset{d}{\simeq} Y_n$ as shorthand for sequences $(T_n)_{n\ge 1}$ and $(Y_n)_{n\ge 1}$ to have the same distributional limit.

**Lemma 5.3** *Suppose* (4.1) *and* $\lim_{n\to\infty} \Pi_{\tau_n(i)} = 0$ *a.s. for some* $i \in \mathscr{S}$. *If* $\mathbb{E}_i J_i(\log^+ |B^i|) = \infty$, *then* $|\Psi_{1:n}(Z_0)| \xrightarrow{\mathbb{P}_\pi} \infty$ *for any admissable* $Z_0$.

**Proof.** Since $\Pi_n Z_0 \xrightarrow{\mathbb{P}_\pi} 0$, it suffices to consider $(|\Psi_{1:n}(0)|)_{n\ge 0}$. Suppose $|\Psi_{1:n}(0)|$ does not tend stochastically to $\infty$ under $\mathbb{P}_i$. Then, there exists $x_1 \in \mathbb{R}_>$ such that

$$0 \;<\; \limsup_{n\to\infty} \mathbb{P}_i(|\Psi_{1:n}(0)| \le x_1) \;\le\; \limsup_{n\to\infty} \mathbb{P}_i(\Psi_{1:n}(0) \le x_1)$$

and thus there is a sequence $(n'_k)_{k\ge 1}$ of natural numbers such that $(\mathbb{P}_i(|\Psi_{1:n'_k}(0)| \le x_1))_{k\ge 1}$ and $(\mathbb{P}_i(\Psi_{1:n'_k}(0) \le x_1))_{k\ge 1}$ have a positive limit. By the usual diagonal sequence argument, we obtain a subsequence $(n_k)_{k\ge 1}$ such that $(\mathbb{P}_i(\Psi_{1:n_k}(0) \le x_\ell))_{k\ge 1}$ converges for all $\ell \ge 1$, where $I := \{x_1, x_2, \dots\}$ denotes a countable dense set in $\mathbb{R}$. For $x \in \mathbb{R}$, define the right-continuous function

$$F(x) \;:=\; \lim_{z\in I, z\downarrow x} \; \lim_{k\to\infty} \mathbb{P}_i(\Psi_{1:n_k}(0) \le z),$$

which exists since $z \mapsto \lim_{k\to\infty} \mathbb{P}_i(\Psi_{1:n_k}(0) \le z)$, $z \in I$, is a non-decreasing, bounded function. In addition, put

$$F(\infty) \;:=\; \lim_{x\uparrow\infty} F(x) \qquad \text{and} \qquad F(-\infty) \;:=\; \lim_{x\downarrow-\infty} F(x).$$

Consequently, $(\Psi_{1:n_k}(0))_{k\geq 1}$ converges in distribution to some $Z_\infty$ under $\mathbb{P}_i$, which has the possibly unproper distribution function $F$ with

$$F(\infty)-F(-\infty) \;\geq\; \lim_{k\to\infty} \mathbb{P}_i(|\Psi_{1:n'_k}(0)| \leq x_1) \;>\; 0.$$

Using that $(\Psi_{n_k-m+1:n_k}(0))_{k\geq 1}$ is a sequence of identically distributed random variables under $\mathbb{P}_\pi$ and $\Pi_{n_k-m} \xrightarrow{\mathbb{P}_\pi} 0$, we obtain

$$|\Pi_{n_k-m}\,\Psi_{n_k-m+1:n_k}(0)| \;\xrightarrow{\mathbb{P}_i}\; 0.$$

Consequently,

$$\Psi_{1:n_k-m}(0) \;\overset{d}{\simeq}\; \Psi_{1:n_k-m}(0)+\Pi_{n_k-m}\,\Psi_{n_k-m+1:n_k}(0) \;=\; \Psi_{1:n_k}(0) \;\xrightarrow{d}\; Z_\infty$$

for all $m \geq 1$ under $\mathbb{P}_i$. An application yields

$$\begin{aligned}
\mathbb{P}_i(Z_\infty \in \cdot\,) \;&=\; \mathbb{P}_i(\lim_{k\to\infty} \Psi_{1:n_k}(0) \in \cdot\,) \\
&=\; \mathbb{P}_i\Big(A^i \lim_{k\to\infty} \Psi_{\tau(i)+1:n_k}(0) + B^i \in \cdot\,\Big) \\
&= \int \mathbb{P}_i\Big(a \lim_{k\to\infty} \Psi_{m+1:n_k}(0) + b \in \cdot\,|\tau(i)=m\Big)\; \mathbb{P}_i((A^i,B^i,\tau(i)) \in d(a,b,m)) \\
&= \int \mathbb{P}_i\Big(a \lim_{k\to\infty} \Psi_{1:n_k-m}(0) + b \in \cdot\,\Big)\; \mathbb{P}_i((A^i,B^i,\tau(i)) \in d(a,b,m)) \\
&= \int \mathbb{P}_i(a\,Z_\infty + b \in \cdot\,)\; \mathbb{P}_i((A^i,B^i) \in d(a,b)) \\
&= \mathbb{P}_i(A^i\,Z_\infty + B^i \in \cdot\,),
\end{aligned}$$

where $Z_\infty$ is independent of $(A^i,B^i)$. Moreover, since $\{|Z_\infty|<\infty\} = \{|A^i\,Z_\infty+B^i|<\infty\}$, we obtain that the proper random variable $Z$ with distribution function $G(x) := F(x) - F(-\infty)/[F(\infty)-F(-\infty)]$ satisfies

$$Z \;\overset{d}{=}\; A^i Z + B^i,$$

where $Z$ and $(A^i,B^i)$ are independent. Due to Theorem 1.2, $\mathbb{E}_i J_i(\log^+|B^i|)<\infty$ must be satisfied. □

The following lemma is essentially a reformulation of the assertion in Theorem 5.1 for the present case.

**Lemma 5.4** *Suppose* (4.1), $\lim_{n\to\infty}\Pi_{\tau_n(i)} = 0$ *a.s. and let* $Z_0$ *be an admissable random variable.* $\mathbb{P}_i(\Psi_{1:n}(Z_0) \in \cdot\,) \xrightarrow{w} Q(i,\cdot\,)$, $i \in \mathscr{S}$, *for some* $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ *is satisfied if and only if* $\mathbb{E}_i J_i(\log^+|B^i|) < \infty$. *Then,* $Q(i,\cdot\,) = \mathbb{P}_i(\widehat{Z}_\infty \in \cdot\,)$ *and* $\Psi_{1:n}(Z_0) \xrightarrow{\mathbb{P}_\pi} \widehat{Z}_\infty$ *for any admissable* $Z_0$.

**Proof.** Lemma 5.3 entails the necessity of $\mathbb{E}_i J_i(\log^+|B^i|)<\infty$. Moreover, as argued before, it suffices to consider $(\Psi_{1:n}(0))_{n\ge 0}$.

Suppose $\mathbb{P}_i(B^i=0)=1$. By Lemma 5.2, (3.6) is satisfied and thus

$$\Psi_{1:n}(0) \;=\; c_{M_0}-\Pi_n\, c_{M_n} \qquad \text{a.s.}$$

for all $n\ge 1$. Since $(c_{M_n})_{n\ge 1}$ is stationary under $\mathbb{P}_\pi$, we infer $\Psi_{1:n}(0) \xrightarrow{\mathbb{P}_\pi} c_{M_0}$. Additionally, notice that $\mathbb{E}_i J_i(\log^+|B^i|)<\infty$ holds trivially if $\mathbb{P}_i(B^i=0)=1$.

Suppose $\mathbb{E}_i J_i(\log^+|B^i|)<\infty$ and $\mathbb{P}_i(B^i=0)<1$. Then, Theorem 1.1 implies almost sure convergence of $(\Psi^i_{1:n}(0))_{n\ge 0}$ and particularly almost sure convergence of $(\Psi^i_{1:N(n)}(0))_{n\ge 0}$ under $\mathbb{P}_i$. Using (5.1) and

$$\Psi_{1:n}(0) \;=\; \Psi^i_{1:N(n)}(0)+\sum_{k=\tau_{N(n)}(i)+1}^{n}\Pi_{k-1}\,B_k,$$

we obtain stochastic convergence of $(\Psi_{1:n}(0))_{n\ge 0}$ under $\mathbb{P}_i$.

Since

$$\Psi_{1:n}(0) \;=\; \mathbf{1}_{\{\tau(i)<n\}}\,(\Pi_{\tau(i)}\,\Psi_{\tau(i)+1:n}(0)+B^i)+\mathbf{1}_{\{\tau(i)\ge n\}}\,\Psi_{1:n}(0),$$

$\Psi_{\tau(i)+1:n}(0)$ is independent of $(\Pi_{\tau(i)},B^i)$ given $\tau(i)$ under $\mathbb{P}_\pi$ and

$$\mathbb{P}_\pi(\Psi_{\tau(i)+1:\tau(i)+n}(0)\in\cdot\,) \;=\; \mathbb{P}_i(\Psi_{1:n}(0)\in\cdot\,),$$

we derive $\Psi_{1:n}(0)\xrightarrow{\mathbb{P}_\pi}\widehat{Z}_\infty$. □

## 5.2. The Case $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$

First of all, given $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$ for some (hence all) $i\in\mathscr{S}$, $(\log|\Pi_n|)_{n\ge 0}$ forms a null-homologous MRW and thus

$$|\Pi_n| \;=\; \exp(g(M_n)-g(M_0)) \qquad \text{a.s.}$$

for some function $g:\mathscr{S}\to\mathbb{R}$. Remember that such function is not uniquely determined. We fix such $g$ and set

$$a_i \;:=\; \exp(-g(i)), \qquad i\in\mathscr{S} \tag{5.2}$$

and therefore $|\Pi_n|=a_{M_0}/a_{M_n}$ a.s.

Further results on the structure of $(\Pi_n)_{n\ge 0}$ will follow, but the current knowledge already suffices to verify that $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$ implies $|\Psi_{1:n}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$ if (3.6) is not satisfied. This is the consequence of the next two lemmata.

**Lemma 5.5** *Suppose $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$ for some $i\in\mathscr{S}$ is true and* (3.6) *is not satisfied. If $\Psi_{1:n}(0)=f(n,M_0,M_n,\Pi_n)$ a.s. for some measurable function $f$ for all $n\ge 1$, then $|\Psi_{1:n}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$.*

**Proof.** The assertion is clear if we show $|\Psi_{1:n}(Z_0)| \xrightarrow{\mathbb{P}_i} \infty$ for arbitrary $i \in \mathscr{S}$. Fix some arbitrary $i \in \mathscr{S}$. By the previous considerations, $\Pi_n \in \{\pm a_i/a_j\}$ a.s. on $\{M_0 = i, M_n = j\}$ for all $n \in \mathbb{N}$ and $j \in \mathscr{S}$. $(M_n')_{n\geq 0} := (M_n, \Pi_n)_{n\geq 0}$ forms a Markov chain with countable state space $\mathscr{S}' := \{(j,a) \in \mathscr{S} \times \mathbb{R} : a \in \{\pm a_i/a_j\}\}$ and initial distribution $\delta_{(i,1)}$. Moreover,

$$\mathbb{P}\Big( \prod_{k=1}^{\tau(j,a)} A_k = 1 \Big| M_0' = (j,a) \Big) \;=\; 1 \qquad \text{for all } (j,a) \in \mathscr{S}', \tag{5.3}$$

where $(\tau_n(j,a))_{n\geq 1}$ denote the successive return times to $(j,a)$. Given $M_0 = i$, $(A_n, B_n)_{n\geq 1}$ is clearly a Markov-modulated sequence with driving chain $(M_n')_{n\geq 0}$. Moreover, since $f(n, M_0, M_n, \Pi_n) = f'(n, M_0', M_n')$ for some measurable function $f'$ and (5.3) are satisfied, we can assume w.l.o.g. that $\mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1$ for some (hence all) $i \in \mathscr{S}$ and $\Psi_{1:n}(0) = f(n, M_0, M_n)$ a.s. for some measurable function $f$ for all $n \geq 1$.

The assumption entails $\Psi_{1:m}(0) = f(m,i,i)$ a.s. on $\{M_0 = M_m = i\}$ for all $m \in I := \{n \in \mathbb{N} : \mathbb{P}_i(M_n = i) > 0\}$. Fix some $m_1 \in I$. Now, for any $m \in I$, we derive that $m_1 m \in I$ and on $\{M_0 = M_{m_1 m} = i\}$

$$\Psi_{1:m_1 m}(0) \;=\; \begin{cases} m\, f(m_1, i, i), & \text{if } M_{m_1} = M_{2m_1} = \ldots = M_{m_1 m} = i, \\ m_1\, f(m, i, i), & \text{if } M_m = M_{2m} = \ldots = M_{m_1 m} = i, \end{cases}$$

almost surely. Since $\Psi_{1:m_1 m}(0) = f(m_1 m, M_0, M_{m_1 m})$ a.s., we infer

$$f(m,i,i) \;=\; \mu_i\, m \qquad \text{for all } i \in \mathscr{S}, \tag{5.4}$$

where $\mu_i = f(m_1, i, i)/m_1$. The failure of (3.6) implies $\mu_i \neq 0$ and thus

$$\mathbb{E}_i B^i \;=\; \mathbb{E}_i \Psi^i(0) \;=\; \mu_i\, \mathbb{E}_i \tau(i) \;\neq\; 0.$$

Hence,

$$|\Psi^i_{1:n}(0)| \;=\; \Big| \sum_{k=1}^n B^i_k \Big| \;=\; |\mu_i|\, \tau_n(i) \;\to\; \infty \qquad \mathbb{P}_i\text{-a.s.}$$

and particularly

$$|\Psi^i_{1:N(n)}(0)| \;\to\; \infty \qquad \mathbb{P}_i\text{-a.s.}$$

By Lemma C.7 and admissibility of $Z_0$,

$$(B_n^*)_{n\geq 1} \;:=\; \Big( \max_{1\leq k\leq \chi_{N(n)+1}(i)} |\Psi_{\tau_{N(n)}(i)+1:\tau_{N(n)}(i)+k}(Z_0)| + |Z_0| \Big)_{n\geq 1}$$

converges in distribution. Then, $\mathbb{P}_i(|\Pi_{\tau_{N(n)}(i)}| = 1) = 1$,

$$|\Psi^i_{1:N(n)}(0)| - B_n^* \;\leq\; |\Psi_{1:n}(Z_0)| \;\leq\; |\Psi^i_{1:N(n)}(0)| + B_n^* \qquad \mathbb{P}_i\text{-a.s},$$

and Slutsky's theorem yield $|\Psi_{1:n}(Z_0)| \xrightarrow{\mathbb{P}_i} \infty$. □

The proof of the next lemma is inspired by the proof of the corresponding assertion in [23] (see their Lemma 5.8).

**Lemma 5.6** *Suppose* (4.1) *is satisfied, but* $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ *a.s. and* (3.6) *both fail. If there is some* $m\in\mathbb{N}$ *such that* $\Psi_{1:m}(0)$ *is not almost surely a measurable function of* $(M_0,M_m,\Pi_m)$*, then* $|\Psi_{1:n}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$.

**Proof.** Notice that $|\Psi_{1:mn}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$ implies $|\Psi_{\ell+1:\ell+mn}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$ and thus

$$|\Psi_{1:\ell+mn}(Z_0)| \;=\; |\Psi_{1:\ell}(0)+\Pi_\ell\,\Psi_{\ell+1:\ell+mn}(Z_0)| \;\xrightarrow{\mathbb{P}_\pi}\; \infty$$

for all $1\le\ell<m$, which further yields $|\Psi_{1:n}(Z_0)|\xrightarrow{\mathbb{P}_\pi}\infty$. Hence, it suffices to prove that the modulus of $(\Psi^{(m)}_{1:n}(Z_0))_{n\ge0}:=(\Psi_{1:mn}(Z_0))_{n\ge0}$ tends stochastically to $\infty$ under $\mathbb{P}_\pi$. Observe that $(A^{(m)}_n,B^{(m)}_n)_{n\ge1}$,

$$A^{(m)}_n \;:=\; \prod_{k=(n-1)m+1}^{nm} A_k \qquad\text{and}\qquad B^{(m)}_n \;:=\; \Psi_{(n-1)m+1:nm}(0),$$

is a Markov-modulated sequence with driving chain $(M_{nm})_{n\ge1}$ and

$$\Psi^{(m)}_n(x) \;:=\; A^{(m)}_n\,x+B^{(m)}_n, \qquad x\in\mathbb{R},\, n\ge1.$$

Consequently, we can assume that $B_1$ is not almost surely a measurable function of $(M_0,M_1,A_1)$ and prove

$$|\Psi_{1:n}(Z_0)| \;\xrightarrow{\mathbb{P}_\pi}\; \infty.$$

For $a\in\mathbb{R}$ and $i,j\in\mathscr{S}$, define

$$F_{i,j,a}(x) \;:=\; \mathbb{P}(B_1\le x|M_0=i,\,M_1=j,\,A_1=a)$$

and $F^{-1}_{i,j,a}$ as its pseudo-inverse. Let $(U_n)_{n\ge1}$ be an i.i.d. sequence of random variables, which are uniformly distributed on $(0,1)$ and independent of all other occurring random variables. Setting $B'_n:=F^{-1}_{M_{n-1},M_n,A_n}(U_n)$, $(A_n,B_n)_{n\ge1}$ and $(A_n,B'_n)_{n\ge1}$ are identically distributed. Moreover, $Y_1,\ldots,Y_n$,

$$Y_k \;:=\; B_k-B'_k, \qquad k\ge1,$$

are independent and symmetric given $M_0,(M_k,A_k)_{1\le k\le n}$. Suppose

$$Y_1 \;=\; c'_{M_0}-A_1\,c'_{M_1} \qquad \text{a.s.} \tag{5.5}$$

for some sequence $(c'_j)_{j\in\mathscr{S}}$ of real numbers. $Y_1$ being symmetric given $M_0,M_1,A_1$, implies $c'_j=0$ for all $j\in\mathscr{S}$, hence $\mathbb{P}_\pi(Y_1=0)=1$. But as $\mathbb{P}_\pi(Y_1=0)=1$ is equivalent to $B_1=f(M_0,M_1,A_1)$ a.s. for some measurable function $f$, we infer that (5.5) fails. Then, Lemma 5.2 (i) yields $\mathbb{P}_i(\sum_{k=1}^{\tau(i)}\Pi_{k-1}Y_k=0)<1$ for all $i\in\mathscr{S}$, whereupon an appeal to Theorem 1.1 entails $|\sum_{k=1}^{\tau_n(i)}\Pi_{k-1}Y_k|\xrightarrow{\mathbb{P}_i}\infty$ for all $i\in\mathscr{S}$. Particularly, we obtain

$$\Bigg|\sum_{k=1}^{\tau_n(i)}\Pi_{k-1}\,Y_k\Bigg| \;\xrightarrow{\mathbb{P}_\pi}\; \infty. \tag{5.6}$$

Set $b := \pi_i/2$. By the elementary renewal theorem, we have

$$\mathbb{P}_\pi(N(n) \geq \lceil n\, b \rceil) \;\to\; 1.$$

For $n \geq 1$, set

$$W_n = \sum_{k=1}^{\tau_{\lceil nb \rceil}(i)} \Pi_{k-1}\, B_k' + (\Psi_{1:n}(0) - \Psi^i_{1:\lceil nb \rceil}(0)) + \Pi_n\, Z_0$$

and

$$\mathcal{G}_n \;:=\; \sigma\Big(Z_0, M_0, (M_k, A_k)_{1\leq k\leq n}, \Psi_{1:n}(0) - \Psi^i_{1:\lceil nb \rceil}(0)\Big).$$

Using Jensen's inequality and that $\Psi_{1:n}(Z_0)$ and $W_n$ are i.i.d. given $\mathcal{G}_n$ and $N(n) \geq \lceil n\, b \rceil$, we infer for all $x \in \mathbb{R}_\geq$

$$\begin{aligned}
\mathbb{P}_\pi\Big(\Big|\sum_{k=1}^{\tau_{\lceil nb \rceil}(i)} \Pi_{k-1}\, Y_k\Big| \leq x\Big) \;&\geq \mathbb{P}_\pi\Big(\Big|\sum_{k=1}^{\tau_{\lceil nb \rceil}(i)} \Pi_{k-1}\, B_k - \sum_{k=1}^{\tau_{\lceil nb \rceil}(i)} \Pi_{k-1}\, B_k'\Big| \leq x,\, N(n) \geq \lceil n\, b \rceil\Big)\\
&= \mathbb{P}_\pi(|\Psi_{1:n}(Z_0) - W_n| \leq x,\, N(n) \geq \lceil n\, b \rceil)\\
&\geq \mathbb{P}_\pi(|\Psi_{1:n}(Z_0) - W_n| \leq x, |W_n| \leq x/2,\, N(n) \geq \lceil n\, b \rceil)\\
&\geq \mathbb{E}_\pi\Big(\mathbf{1}_{\{N(n) > \lceil n\, b \rceil\}}\, \mathbb{P}_\pi(|\Psi_{1:n}(Z_0)| \leq x/2,\, |W_n| \leq x/2 | \mathcal{G}_n)\Big)\\
&= \mathbb{E}_\pi\Big(\mathbf{1}_{\{N(n) > \lceil n\, b \rceil\}}\, \mathbb{P}_\pi(|\Psi_{1:n}(Z_0)| \leq x/2 | \mathcal{G}_n)^2\Big)\\
&\geq \Big[\mathbb{P}_\pi(|\Psi_{1:n}(Z_0)| \leq x/2,\, N(n) \geq \lceil n\, b \rceil)\Big]^2\\
&\asymp \Big[\mathbb{P}_\pi(|\Psi_{1:n}(Z_0)| \leq x/2)\Big]^2,
\end{aligned}$$

which yields the assertion by (5.6). □

A combination of the preceding two lemmata shows the necessity of (3.6) for distributional convergence of $(\Psi_{1:n}(Z_0))_{n\geq 0}$ in the prevailing case. Now, we turn to sufficient conditions and proceed with further results on $(\Pi_n)_{n\geq 0}$.

**Lemma 5.7** *Suppose $\mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1$ for some $i \in \mathscr{S}$. There is a sequence of $\{\pm 1\}$-valued integers $(\theta_i)_{i\in\mathscr{S}}$ such that*

$$\Pi_n \;=\; \theta_{M_0}\, a_{M_0}\, \theta_{M_n}/a_{M_n} \qquad a.s.$$

*for all $n \geq 0$.*

**Proof.** The assertion is immediate if we prove

$$\operatorname{sign}(\Pi_n) \;=\; \theta_{M_0}\, \theta_{M_n}$$

for some sequence $(\theta)_{i\in\mathscr{S}}$ as claimed.

The argument is similar to the one used in the proof of Proposition 3.6 (ii). For any $i,j \in \mathscr{S}$, pick a path $\{M_0 = i, M_1 = i_1, \ldots, M_n = j\}$ of positive probability from $i$ to $j$ of minimal length $n$. Conditioned on this path, $\operatorname{sign}(\Pi_n)$ is obviously deterministic and denoted as $\theta^*(i,j)$. Given any path $\{M_0 = j, M_1 = j_1, \ldots, M_m = i\}$ of positive probability from $j$ to $i$, we additionally assume $\{M_{m+1} = i_1, \ldots, M_{m+n} = j\}$ to derive

$$1 \;=\; \operatorname{sign}(\Pi_{m+n}) \;=\; \operatorname{sign}(\Pi_m) \cdot \theta^*(i,j),$$

i.e. $\operatorname{sign}(\Pi_m) = \theta^*(i,j)^{-1} = \theta^*(i,j)$. More generally, we infer $\operatorname{sign}(\Pi_n) = \theta^*(M_0, M_n)$ a.s. for all $n \geq 0$. Furthermore, one easily verifies $\theta^*(i,j) = \theta^*(j,i)$ and $\theta^*(i,j) = \theta^*(i,s) \cdot \theta^*(s,j)$ for all $i,j,s \in \mathscr{S}$. Now, define for all $i \in \mathscr{S}$

$$\theta_i \;:=\; \theta^*(s,i)$$

for some fixed $s \in \mathscr{S}$. As a consequence,

$$\operatorname{sign}(\Pi_n) \;=\; \theta^*(M_0,s) \cdot \theta^*(s,M_n) \;=\; \theta_{M_0} \cdot \theta_{M_n} \qquad \text{a.s.}$$

for all $n \geq 0$. □

**Lemma 5.8** *Suppose* $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$ *for some* $i \in \mathscr{S}$. *Then,* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *is either aperiodic or 2-periodic.*

**Proof.** Suppose $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ is not aperiodic. Aperiodicity of $\mathbb{P}_i(\tau(i) \in \cdot)$ implies the existence of $m_1, m_2 \in \mathbb{N}$, $m_1$ even, $m_2$ odd, such that $\gcd(m_1, m_2) = 1$ and

$$\mathbb{P}_i(\Pi_{m_1} = 1,\, \tau_1(i) = m_1,\, \Pi_{m_1+m_2} = -1,\, \tau_2(i) = m_1 + m_2) \;>\; 0.$$

Then, $m_1/2$ and $m_2$ are coprime and Dirichlet's prime number theorem provides an $k \in \mathbb{N}$ such that $m_1/2 + k m_2$ is prime. Obviously, $\gcd(m_1, m_1 + 2k m_2) = 2$ and thus

$$\mathbb{P}_i(\Pi_{m_1+2km_2} = 1,\, M_{m_1+2km_2} = i) \;>\; 0$$

entails the 2-periodicity of $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$. □

Of course, one can show that aperiodicity of $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ is a solidarity property, but this is of no further relevance.

We introduce the following notation for the case $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$. If $\mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1$, let $(\theta_j)_{j \in \mathscr{S}}$ denote the sequence provided by Lemma 5.7. If $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ is 2-periodic, then $\mathbb{P}_i(\Pi_{2\ell} = 1, \inf\{n \geq 1 : M_{2n} = i\} = \ell) = 1$ and we define $(\theta_j)_{j \in \mathscr{S}}$ as the corresponding sequence provided by Lemma 5.7 applied on $(M_{2n}, \Pi_{2n})_{n \geq 0}$.

**Lemma 5.9** *Suppose* $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$, $\mathbb{P}_i(\Pi_{\tau(i)} = 1) < 1$ *and aperiodicity of* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *for some* $i \in \mathscr{S}$. *Then,*

$$\lim_{n \to \infty} \mathbb{P}_i(\operatorname{sign}(\Pi_n) = 1,\, M_n = j) \;=\; \lim_{n \to \infty} \mathbb{P}_i(\operatorname{sign}(\Pi_n) = -1,\, M_n = j) \;=\; \frac{1}{2}\pi_j$$

*for all* $j \in \mathscr{S}$.

**Proof.** $(M_n, \text{sign}(\Pi_n))_{n\geq 0}$ is a regenerative process with regeneration times $(\widehat{\tau}_n(i))_{n\geq 0}$ under $\mathbb{P}_i$. The aperiodicity of $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ implies distributional convergence of $(M_n, \text{sign}(\Pi_n))_{n\geq 0}$ (e.g., see [7, Cor. VI.1.5 (i)]). Set $\tau^*(i) := \inf\{n \geq 1 : M_n = i, \Pi_n = -1\}$. Then,

$$\mathbf{1}_{\{\text{sign}(\Pi_n)=1, M_n=j\}} \overset{d}{\simeq} \mathbf{1}_{\{\tau^*(i)<n\}} \mathbf{1}_{\{-\text{sign}(\Pi_n/\Pi_{\tau^*(i)})=1, M_n=j\}} \overset{d}{\simeq} \mathbf{1}_{\{\text{sign}(\Pi_n)=-1, M_n=j\}}$$

under $\mathbb{P}_i$, which yields the assertion. □

**Lemma 5.10** *Suppose* (4.1), $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$ *for some* $i \in \mathscr{S}$ *and let* $Z_0$ *be an admissable random variable. Then,* $\mathbb{P}_i(\Psi_{1:n}(Z_0) \in \cdot) \xrightarrow{w} Q(i, \cdot)$ *for some* $Q \in \mathscr{P}(\mathscr{S}, \mathbb{R})$ *is satisfied if and only if* (3.6) *and one of the following cases prevails:*

*(i)* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *is aperiodic. Then,*

$$Q(i, \cdot) = \begin{cases} \sum_{j\in\mathscr{S}} \pi_j \, \mathbb{P}_i(c_i + (\theta_i \, a_i \, \theta_j / a_j)(Z_0 - c_j) \in \cdot), & \text{if } \mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1, \\ \sum_{j\in\mathscr{S}} \pi_j \, \mathbb{P}_i(c_i + (a_i/a_j) \, Y_j \in \cdot), & \text{if } \mathbb{P}_i(\Pi_{\tau(i)} = 1) < 1, \end{cases}$$

*where* $\mathbb{P}_i(Y_j \in \cdot) = \frac{1}{2}[\mathbb{P}_i(Z_0 - c_j \in \cdot) + \mathbb{P}_i(-(Z_0 - c_j) \in \cdot)]$.

*(ii)* $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ *is 2-periodic and*

$$\lim_{n\to\infty} \mathbb{P}_i(\Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot) = \lim_{n\to\infty} \mathbb{P}_i(-\Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot). \tag{5.7}$$

*Then,* $Q(i, \cdot) = \sum_{j\in\mathscr{S}} \pi_j \, \mathbb{P}_i(c_i + (\theta_i \, a_i \, \theta_j / a_j)(Z_0 - c_j) \in \cdot)$.

**Proof.** By Lemma 5.5 and Lemma 5.6, $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$ for some admissable $Z_0$ only if (3.6) is satisfied. Now, suppose (3.6) is true. We distinguish between $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ being aperiodic or 2-periodic.

Suppose $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ is aperiodic. Then, for all $n \geq 0$

$$\Psi_{1:n}(Z_0) = c_i + (\text{sign}(\Pi_n) \, a_i / a_{M_n})(Z_0 - c_{M_n}) \qquad \mathbb{P}_i\text{-a.s.}$$

As a consequence, ergodicity of $(M_n)_{n\geq 0}$ combined with Lemma 5.7 and Lemma 5.9 entails the identity of $Q(i, \cdot)$ as claimed.

Now, suppose $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot)$ is 2-periodic. Recalling

$$\Psi_{1:n}(Z_0) = c_{M_0} + \Pi_n \, (Z_0 - c_{M_n}) \qquad \text{a.s.} \qquad \text{for all } n \geq 0,$$

we conclude that $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$ if and only if $(\Pi_n \, (Z_0 - c_{M_n}))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$. The assumed 2-periodicity entails that $(\Pi_{2n} \, (Z_0 - c_{M_{2n}}))_{n\geq 0}$ and $(\Pi_{2n+1} \, (Z_0 - c_{M_{2n+1}}))_{n\geq 0}$ are regenerative processes with aperiodic regeneration times and thus converge in distribution. Consequently, it remains to show that their limit distributions coincide if and only if (5.7) is satisfied. Set $\tau^*(i) := \inf\{n \geq 1 : M_n = i, \Pi_n = -1\}$ and notice that $\mathbb{P}_i(\tau^*(i) \in 2\mathbb{N}_0 + 1) = 1$. We infer that under $\mathbb{P}_i$

$$\Pi_{2n+1} \, (Z_0 - c_{M_{2n+1}}) \overset{d}{\simeq} \mathbf{1}_{\{\tau^*(i)<2n+1\}} \Big( - \prod_{k=\tau^*(i)+1}^{2n+1} A_k \Big) (Z_0 - c_{M_{2n+1}})$$

$$\stackrel{d}{\simeq} \Big(-\prod_{k=\tau^*(i)+1}^{2n+1} A_k\Big)(Z_0 - c_{M_{2n+1}})$$
$$\stackrel{d}{\simeq} -\Pi_{2n}(Z_0 - c_{M_{2n}}),$$

which yields the necessity and sufficiency of (5.7). Then, since

$$c_i + \Pi_{2n}(Z_0 - c_{M_{2n}}) \;=\; c_i + (\theta_i\, a_i\, \theta_{M_{2n}}/a_{M_{2n}})(Z_0 - c_{M_{2n}}) \qquad \text{a.s.}$$

we obtain $Q(i,\cdot)$ as claimed. □

At first glance, condition (5.7) appears unsatisfying. One may suspect necessity of the convergence of $(M_n, \Psi_{1:n}(Z_0))_{n\geq 0}$ under $\mathbb{P}_i$ as in the other cases and thus symmetry of $Z_0 - c_j$ for all $j \in \mathscr{S}$ under $\mathbb{P}_i$, which further results in $c_j = c_i$ for all $j \in \mathscr{S}$. The following example reveals its falsity.

**Example 5.11** Let $a \in \mathbb{R}\setminus\{0\}$ and $c_0, c_1 \in \mathbb{R}$. We define a Markov chain $(M_n)_{n\geq 0}$ with state space $\{0,1,2\}$ by

$$p_{01} \;=\; p_{02} \;=\; \frac{1}{4}, \qquad p_{00} \;=\; \frac{1}{2} \qquad \text{and} \qquad p_{10} \;=\; p_{20} \;=\; 1.$$

The Markov-modulated sequence $(A_n, B_n)_{n\geq 1}$ is given by

$$(A_1, B_1) \;=\; \begin{cases} (-1,\, 2c_0), & \text{if } i = j = 0,\\ (a,\, c_0 - a\,c_1), & \text{if } i = 0, j = 1,\\ (a^{-1},\, c_1 - a^{-1}\,c_0), & \text{if } i = 1, j = 0,\\ (-a,\, c_0 + a\,c_1), & \text{if } i = 0, j = 2,\\ (-a^{-1},\, c_1 + a^{-1}\,c_0), & \text{if } i = 2, j = 0. \end{cases}$$

Consequently, we have a degenerate $(A_n, B_n)_{n\geq 1}$ with $\mathbb{P}_0(|\Pi_{\tau(0)}| = 1) = 1$, $\mathbb{P}_0(\widehat{\tau}(0) \in \cdot)$ 2-periodic, $c_1 = c_2$ and $\pi_1 = \pi_2$. Moreover, since $\mathbb{P}_0(\Pi_{2n} = 1 | M_{2n} = 0) = 1$, Lemma 5.7 shows that $\operatorname{sign}(\Pi_{2n})$ and $|\Pi_{2n}|$ are functions of $M_0$ and $M_{2n}$ thus

$$\mathbb{P}_0(\Pi_{2n} = -a | M_{2n} = 1) \;=\; \mathbb{P}_0(\Pi_{2n} = a | M_{2n} = 2) \;=\; 1.$$

Therefore, if $Z_0$ is symmetric under $\mathbb{P}_0$ one easily derives

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}_0(c_0 + \Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot) \\ &= \pi_1\,\mathbb{P}_0(c_0 - a(Z_0 - c_1) \in \cdot) + \pi_2\,\mathbb{P}_0(c_0 + a(Z_0 - c_1) \in \cdot) + \pi_0\,\mathbb{P}_0(Z_0 \in \cdot) \\ &= \pi_1\,\mathbb{P}_0(c_0 + a(Z_0 - c_1) \in \cdot) + \pi_2\,\mathbb{P}_0(c_0 - a(Z_0 - c_1) \in \cdot) + \pi_0\,\mathbb{P}_0(-Z_0 \in \cdot) \\ &= \lim_{n\to\infty} \mathbb{P}_0(c_0 - \Pi_{2n}(Z_0 - c_{M_{2n}}) \in \cdot) \\ &= \lim_{n\to\infty} \mathbb{P}_0(c_0 + \Pi_{2n+1}(Z_0 - c_{M_{2n+1}}) \in \cdot), \end{aligned}$$

which suffices for distributional convergence of $(\Psi_{1:n}(Z_0))_{n\geq 0}$ under $\mathbb{P}_0$.

## 5.3. The Case $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}| = \infty$ a.s.

For showing that $|\Psi_{1:n}(Z_0)| \xrightarrow{\mathbb{P}_\pi} \infty$ if $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}| = \infty$ a.s., but (3.6) fails, we embark on the use of Lemma 5.6. The subsequent result states that the associated assumptions are satisfied.

**Lemma 5.12** *Suppose* $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}| = \infty$ *a.s. for some (hence all)* $i \in \mathscr{S}$ *is true and* (3.6) *is not satisfied. Then, there exists* $m \in \mathbb{N}$ *such that* $\Psi_{1:m}(0)$ *is not almost surely a measurable function of* $(M_0, M_m, \Pi_m)$.

**Proof.** Suppose there is a measurable function $f$ such that

$$\Psi_{1:2n}(0) \;=\; f(M_0, M_{2n}, \Pi_{2n}) \qquad \text{a.s.} \tag{5.8}$$

for some $n \in \mathbb{N}$ with $\mathbb{P}_i(M_n = i) > 0$. Hence,

$$\Psi_{1:n}(0) + \Pi_n\,\Psi_{n+1:2n}(0) \;=\; f(M_0, M_{2n}, \Pi_{2n}) \qquad \text{a.s.}$$

Proposition 3.4 entails that on $\{M_0 = M_n = M_{2n} = i\}$ either $\Psi_{1:n}(0) = c(1-\Pi_n)$ a.s. or $(\Pi_n, \Psi_{1:n}(0)) = (1, c)$ a.s. for some $c \in \mathbb{R}$. Consequently, it suffices to show the existence of $m \in \mathbb{N}$ such that $\mathbb{P}_i(M_m = i, \Pi_m \neq 1) > 0$ and

$$\mathbb{P}_i(\Psi_{1:m}(0) \;=\; c(1-\Pi_m)|M_m = i) \;<\; 1 \qquad \text{for all } c \in \mathbb{R}.$$

Suppose there is $c_m \in \mathbb{R}$ such that $\mathbb{P}_i(\Psi_{1:m}(0) = c_m(1-\Pi_m)|M_m = i) = 1$ for all $m \in I := \{n \in \mathbb{N} : \mathbb{P}_i(M_n = i, \Pi_n \neq 1) > 0\}$.

By assumption, there is $m_1 \in I$ with $\mathbb{P}_i(M_{m_1} = i, |\Pi_{m_1}| \neq 1) > 0$. Hence,

$$p \;:=\; \max\Big\{\mathbb{P}_i(M_{m_1} = i, |\Pi_{m_1}| > 1), \mathbb{P}_i(M_{m_1} = i, |\Pi_{m_1}| < 1)\Big\} \;>\; 0.$$

For any other $m \in I$, we derive $m_1 m \in I$, since

$$\mathbb{P}_i(M_{m_1 m} = i, |\Pi_{m_1 m}| \neq 1) \;\geq\; p^m \;>\; 0.$$

On $\{M_0 = M_{m_1 m} = i\}$, we have $\Psi_{1:m_1 m}(0) = c_{m_1 m}(1-\Pi_{m_1 m})$ a.s., but also

$$\Psi_{1:m_1 m}(0) \;=\; \begin{cases} c_{m_1}(1-\Pi_{m_1 m}), & \text{if } M_{m_1} = M_{2m_1} = \ldots = M_{m_1 m} = i, \\ c_m(1-\Pi_{m_1 m}), & \text{if } M_m = M_{2m} = \ldots = M_{m_1 m} = i. \end{cases}$$

Hence $c_m = c$ for all $m \in I$ for some $c \in \mathbb{R}$.

By the above assumption and $\mathbb{P}_i(B^i = c(1-A^i)) < 1$, there is $\ell \in \mathbb{N}$ with $\mathbb{P}_i(M_\ell = i, \Pi_\ell = 1) > 0$ and

$$\mathbb{P}_i(\Psi_{1:\ell}(0) = c(1-\Pi_\ell)\,|\,M_\ell = i, \Pi_\ell = 1) \;<\; 1,$$

i.e.

$$\mathbb{P}_i(\Psi_{1:\ell}(0) \neq 0\,|\,M_\ell = i, \Pi_\ell = 1) \;>\; 0. \tag{5.9}$$

However, $\mathbb{P}_i(M_{\ell+m} = i, \Pi_{\ell+m} \neq 1) > 0$ for $m \in I$ yields

$$\Psi_{1:\ell+m}(0) \;=\; c(1-\Pi_{\ell+m}) \qquad \mathbb{P}_i\text{-a.s.}$$

given $M_{\ell+m} = i$. Therefore, we infer that on $\{M_\ell = i, \Pi_\ell = 1, M_{\ell+m} = i, \Pi_{\ell+m} \neq 1\}$

$$\begin{aligned}\Psi_{1:\ell}(0) &= \Psi_{1:\ell+m}(0) - \Pi_\ell \Psi_{\ell+1:\ell+m}(0) = \Psi_{1:\ell+m}(0) - \Psi_{\ell+1:\ell+m}(0) \\ &= c(1-\Pi_{\ell+m}) - c\Big(1 - \prod_{k=\ell+1}^{\ell+m} A_k\Big) = 0 \qquad \mathbb{P}_i\text{-a.s.,}\end{aligned}$$

which contradicts (5.9). □

Given $\limsup_{n\to\infty} |\Pi_{\tau_n(i)}| = \infty$ a.s., either $\Pi_n \xrightarrow{\mathbb{P}_i} 0$ or $\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > 0) > 0$. Moreover, since $\Pi_{\tau(i)}$ is a proper random variable under $\mathbb{P}_\pi$, it is easily seen that $\Pi_n \xrightarrow{\mathbb{P}_i} 0$ and $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$ are equivalent.

**Lemma 5.13** *Suppose* $\limsup_{n\to\infty} |\Pi_{\tau_n(i)}| = \infty$ *a.s. for some* $i \in \mathscr{S}$ *and let* $Z_0$ *be an admissable random variable.* $\mathbb{P}_i(\Psi_{1:n}(Z_0) \in \cdot\,) \xrightarrow{w} Q(i,\cdot\,)$ *for some* $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ *is satisfied if and only if one of the following cases prevails:*

*(i)* $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$ *and* (3.6) *are true. Then,* $Q(i,\cdot\,) = \delta_{c_i}$ *and* $\Psi_{1:n}(Z_0) \xrightarrow{\mathbb{P}_\pi} c_{M_0}$.

*(ii)* $\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > 0) > 0$, $\mathbb{P}_\pi(B_1 = c(1-A_1)) = 1$ *for some* $c \in \mathbb{R}$ *and* $Z_0 = c$ $\mathbb{P}_i$*-a.s. Then,* $Q(i,\cdot\,) = \delta_c$ *and* $\Psi_{1:n}(Z_0) = c$ $\mathbb{P}_i$*-a.s. for all* $n \geq 0$.

**Proof.** By the previous considerations, (3.6) is necessary for distributional convergence.

We infer from

$$\Psi_{1:n}(Z_0) \overset{d}{\simeq} \mathbf{1}_{\{\tau(i)<n\}} \Psi^i(\Psi_{\tau(i)+1:n}(Z_0))$$

that a possible limit distribution $Q(i,\cdot\,)$ solves the SFPE $R \overset{d}{=} \Psi^i(R)$, where $R$ and $(A^i, B^i)$ are independent. Hence, an appeal to Theorem 1.2 (iii) shows $Q(i,\cdot\,) = \delta_{c_i}$.

Suppose $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$. Then, $\Psi_{1:n}(Z_0) = c_{M_0} + \Pi_n(Z_0 - c_{M_n}) = c_{M_0} + \Pi_n Z_0 - \Pi_n c_{M_n}$ converges stochastically to $c_{M_0}$ under $\mathbb{P}_\pi$ for all $i \in \mathscr{S}$ by Slutsky's theorem, since $(c_{M_n})_{n\geq 0}$ is stationary under $\mathbb{P}_\pi$.

Suppose $\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > 0) > 0$. Then, there exists $j \in \mathscr{S}$ and $\varepsilon > 0$ with

$$\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > \varepsilon, M_n = j) > 0.$$

$\limsup_{n\to\infty} |\Pi_{\tau_n(j)}| = \infty$ a.s. ensures that for every $x \in \mathbb{R}_>$ there exists some $m(x) \in \mathbb{N}$ such that $\mathbb{P}_j(|\Pi_{m(x)}| > x/\varepsilon, M_{m(x)} = j) > 0$. Consequently, $\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > x, M_n = j) > 0$ for all $x \in \mathbb{R}_\geq$, which is easily seen to imply

$$\limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > x, M_n = j) > 0 \qquad \text{for all } (x,j) \in \mathbb{R}_\geq \times \mathscr{S}.$$

We infer from $Q(i,\cdot\,) = \delta_{c_i}$ that

$$\Psi_{1:n}(Z_0) - c_i = \Pi_n(Z_0 - c_{M_n}) \xrightarrow{\mathbb{P}_i} 0$$

must be satisfied. Assume $\mathbb{P}_i(Z_0 = c_j) < 1$ for some $j \in \mathscr{S}$. Then, we obtain a contradiction from

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathbb{P}_i(|\Pi_n (Z_0 - c_{M_n})| > \varepsilon) \\ &\geq \limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n (Z_0 - c_j)| > \varepsilon, M_n = j) \\ &= \limsup_{n\to\infty} \mathbb{P}_i(|\Pi_n| > \varepsilon / |Z_0 - c_j|, Z_0 \neq c_j, M_n = j) \ > \ 0. \end{aligned}$$

Consequently, $\mathbb{P}_i(Z_0 = c_j) = 1$ for all $j \in \mathscr{S}$, which implies all $c_j$, $j \in \mathscr{S}$, being equal to some $c \in \mathbb{R}$ and $Z_0 = c$ $\mathbb{P}_i$-a.s. □

# 6. Fixed Points of $\Psi_1 \bullet Q = Q$

This section is devoted to giving necessary and sufficient conditions for $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ to solve

$$\Psi_1 \bullet Q \ = \ Q \tag{6.1}$$

defined in (2.1).

The following observation is essential. Let $Q$ be a fixed point of (6.1). Then,

$$Q(i,\cdot) \ = \ \mathbb{P}_i(\Psi_1(R_1) \in \cdot) \ = \ \sum_{j\in\mathscr{S}} p_{ij} \int \mathbb{P}_i(\Psi_1(r) \in \cdot | M_1 = j) \ Q(j, dr) \tag{6.2}$$

entails

$$R^i \ \overset{d}{=} \ \Psi_1(R^{M_1}) \ = \ \sum_{j\in\mathscr{S}} \mathbf{1}_{\{M_1 = j\}} \Psi_1(R^j) \qquad \text{under } \mathbb{P}_i$$

for all $i \in \mathscr{S}$, where $R^j$ has distribution $Q(j,\cdot)$, $j \in \mathscr{S}$, and is independent of all other occurring random variables under $\mathbb{P}_i$. Furthermore, an iteration of the above argument shows

$$R^i \ \overset{d}{=} \ \Psi^i(R^i) \ \overset{d}{=} \ A^i R^i + B^i, \tag{6.3}$$

where $(A^i, B^i)$ and $R^i$ are independent, and these fixed points are characterised in Theorem 1.2.

Let $(Z_n)_{n\geq 0}$ be a sequence of random variables such that $Z_n$ is independent of all other occurring random variables given $M_n$ and $\mathbb{P}(Z_n \in \cdot | M_n = i) = \mathbb{P}(Z_0 \in \cdot | M_0 = i)$. We call such sequence *admissable*. Then, if $\mathbb{P}_i(\Psi_{1:n}(Z_n) \in \cdot) \xrightarrow{w} Q(i,\cdot)$, $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$, for some (hence all) $i \in \mathscr{S}$, $Q$ is a fixed point of (6.1). That this convergence is a solidarity property follows with the usual arguments from

$$\Psi_{1:n}(Z_n) \ = \ \mathbf{1}_{\{\tau(j)<n\}} (\Pi_{\tau(j)} \Psi_{\tau(j)+1:n}(Z_n) + B_1^j) + \mathbf{1}_{\{\tau(j)\geq n\}} \Psi_{1:n}(Z_n)$$

for all $n \geq 1$. Then, the fixed point characterisation results from

$$\lim_{n\to\infty} \Psi_{1:n}(Z_n) \ = \ \Psi_1(\lim_{n\to\infty} \Psi_{2:n}(Z_n)).$$

Conversely, let $Q$ be a solution to (6.1). Then, define

$$R_n \ :=\ \sum_{j\in\mathscr{S}} \mathbf{1}_{\{M_n=j\}} R_n^j, \qquad n\ge 0,$$

where $R_n^j$ is independent of all other occurring random variables and has distribution $Q(j,\cdot)$. Clearly, $(R_n)_{n\ge0}$ is admissable and (6.2) shows that

$$R_n \ \stackrel{d}{=}\ \Psi_{n+1}(R_{n+1})$$

under any $\mathbb{P}_i$ for all $n\ge 0$. Consequently,

$$\mathbb{P}_i(\Psi_{1:n}(R_n)\in\cdot\,)\ =\ \mathbb{P}_i(R_0\in\cdot\,)\ =\ Q(i,\cdot\,)$$

converges weakly.

Suppose $\mathbb{P}_\pi(A_1=0)>0$. By the above characterisation, it follows immediately that a unique fixed point of (6.1) is given by $Q$, where $Q(i,\cdot\,) := \mathbb{P}_i(\sum_{k=1}^{\kappa}\Pi_{k-1}B_k\in\cdot\,)$ and $\kappa := \inf\{n\ge 1 : A_n=0\}$.

Concerning the remaining cases, we formulate the next theorem.

**Theorem 6.1** *Suppose $\mathbb{P}_\pi(A_1=0)=0$. There exists a fixed point $Q\in\mathscr{P}(\mathscr{S},\mathbb{R})$ of (6.1) if and only if one of the following conditions is satisfied for some (hence all) $i\in\mathscr{S}$:*

*(i)* $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ *a.s. and* $\mathbb{E}_iJ_i(\log^+|B^i|)<\infty$. *Then,* $Q(j,\cdot)=\mathbb{P}_j(\widehat{Z}_\infty\in\cdot\,)$ *for all* $j\in\mathscr{S}$.

*(ii)* $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$, (3.6) *is true and one of the following cases prevails:*

*(ii.1)* $\mathbb{P}_i(\Pi_{\tau(i)}=1)<1$. *Then,* $Q(j,\cdot\,)=\mathbb{P}(c_j+a_jY\in\cdot\,)$ *for all* $j\in\mathscr{S}$ *and some symmetric random variable* $Y$.

*(ii.2)* $\mathbb{P}_i(\Pi_{\tau(i)}=1)=1$. *Then,* $Q(j,\cdot\,)=\mathbb{P}(c_j+a_j\theta_jY\in\cdot\,)$ *for all* $j\in\mathscr{S}$ *and some random variable* $Y$, *where* $(c_j)_{j\in\mathscr{S}}$ *is not uniquely determined by* (3.6).

*(iii)* $\limsup_{n\to\infty}|\Pi_n|=\infty$ *a.s. and* (3.6) *are satisfied. Then,* $Q(j,\cdot\,)=\delta_{c_j}$ *for all* $j\in\mathscr{S}$.

**Proof.** Case $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ a.s. Suppose there is a fixed point $Q$ and hence that $Q(i,\cdot)$ solves (6.3). By Theorem 1.2, $\mathbb{E}_iJ_i(\log^+|B^i|)<\infty$ and $Q(i,\cdot)=\mathbb{P}_i(\lim_{n\to\infty}\Psi^i_{1:n}(0)\in\cdot\,)$, which equals $\mathbb{P}_i(\widehat{Z}_\infty\in\cdot\,)$ by Theorem 5.1 (i) for all $i\in\mathscr{S}$.

Conversely, given $\mathbb{E}_iJ_i(\log^+|B^i|)<\infty$, $Q\in\mathscr{P}(\mathscr{S},\mathbb{R})$ defined by $Q(i,\cdot\,):=\mathbb{P}_i(\widehat{Z}_\infty\in\cdot)$ is a fixed point of (6.1), because $\Psi_{1:n}(Z_n)\stackrel{d}{\to}\widehat{Z}_\infty$ for $Z_n:=0$ for all $n\ge0$.

Suppose $(\Pi_{\tau_n(i)})_{n\ge0}$ does not converge to 0 almost surely. Since a fixed point $Q$ of (6.1) exists only if $Q(i,\cdot\,)$ solves (6.3), we infer from Theorem 1.2 that (3.6) must be satisfied. From now on, we assume (3.6) to be valid and, given a fixed point $Q$, $R^i$, $i\in\mathscr{S}$, denotes a random variable with law $Q(i,\cdot\,)$, which is independent of all other occurring random variables.

Case $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$. Suppose $\mathbb{P}_i(\Pi_{\tau(i)}=1)<1$. Let $Q$ be a fixed point of (6.1). Theorem 1.2 entails that $Q(i,\cdot\,)$ is symmetric about $c_i$. Hence,

$$Y^i\ :=\ R^i-c_i$$

is a symmetric random variable. An appeal to Proposition 3.5 shows

$$Y^i \overset{d}{=} \Psi_{1:n}(R^{M_n}) - c_i = \Psi_{1:n}(R^{M_n}) - \Psi_{1:n}(c_{M_n}) = \Pi_n Y^{M_n} \tag{6.4}$$

for all $n \geq 0$ under $\mathbb{P}_i$. In particular,

$$|Y^i| \overset{d}{=} |\Pi_n||Y^{M_n}| = a_i \frac{|Y^{M_n}|}{a_{M_n}} \qquad \text{under } \mathbb{P}_i$$

with $(a_j)_{j\in\mathscr{S}}$ as in (5.2). Ergodicity of $(M_n)_{n\geq 0}$ implies distributional convergence of $(|Y^{M_n}|/a_{M_n})_{n\geq 0}$ to some $Y'$, which is independent of $M_0$. Consequently,

$$R^i \overset{d}{=} c_i + a_i Y,$$

where $Y$ is symmetric.

Conversely, we show that $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ with

$$Q(j,\cdot) := \mathbb{P}(c_j + a_j Y \in \cdot), \qquad j \in \mathscr{S},$$

solves the SFPE (6.1), where $Y$ is symmetric, $(c_j)_{j\in\mathscr{S}}$ given by (3.6) and $(a_j)_{j\in\mathscr{S}}$ as above. Let $R_1$ be a random variable with law $\mathbb{P}(R_1 \in \cdot|M_0 = i, M_1 = j) = Q(j,\cdot)$, which is independent of $(A_1, B_1)$ given $(M_0, M_1)$. It holds that

$$\begin{aligned}\Psi_1(R_1) &= A_1 R_1 + c_i - A_1 c_{M_1} = c_i + A_1 (R_1 - c_{M_1}) \overset{d}{=} c_i + |A_1| (R_1 - c_{M_1}) \\ &= c_i + \frac{a_i}{a_{M_1}} (R_1 - c_{M_1}) \overset{d}{=} c_i + a_i Y\end{aligned}$$

under $\mathbb{P}_i$ for all $i \in \mathscr{S}$.

Suppose $\mathbb{P}_i(\Pi_{\tau(i)} = 1) = 1$. Let $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ be a fixed point $Q$ of (6.1). Theorem 1.2 entails that $Q(i,\cdot)$ is arbitrary. By Proposition 3.6, there exists an infinite class of sequences $(c_i)_{i\in\mathscr{S}}$ such that $\Psi_1(c_{M_1}) = c_{M_0}$ a.s. For any such sequence

$$Y^i := R^i - c_i \overset{d}{=} \Pi_n Y^{M_n}$$

remains true under $\mathbb{P}_i$ (cf. (6.4)). Using Lemma 5.7, we derive

$$Y^i \overset{d}{=} a_i \theta_i \frac{\theta_{M_n} Y^{M_n}}{a_{M_n}} \qquad \text{for all } n \geq 0.$$

Again, ergodicity of $(M_n)_{n\geq 0}$ implies distributional convergence of $(\theta_{M_n} Y^{M_n}/a_{M_n})_{n\geq 0}$ to some random variable $Y$. Thus,

$$R^i \overset{d}{=} c_i + a_i \theta_i Y$$

for all $i \in \mathscr{S}$.

Conversely, with $R_1$ having the same dependencies as above, but $\mathbb{P}(R_1 \in \cdot|M_0 = i, M_1 = j) = \mathbb{P}(c_j + a_j \theta_j Y \in \cdot)$, we obtain

$$\Psi_1(R_1) = c_i + A_1 (R_1 - c_{M_1}) = c_i + a_i \theta_i \frac{\theta_{M_1}}{a_{M_1}} (R_1 - c_{M_1}) \overset{d}{=} c_i + a_i \theta_i Y$$

under $\mathbb{P}_i$ for all $i \in \mathscr{S}$.

Case $\limsup_{n\to\infty} |\Pi_{\tau_n(i)}| = \infty$ a.s. Due to Theorem 1.2 a fixed point $Q$ must satisfy $Q(i,\cdot) = \delta_{c_i}$ for all $i \in \mathscr{S}$. The so-defined stochastic kernel does in fact solve the SFPE (6.1) by Proposition 3.5. □

In a closing remark, we briefly discuss conditions on $(Z_n)_{n\geq 0}$ that are necessary and sufficient for distributional convergence of $(\Psi_{1:n}(Z_n))_{n\geq 0}$, where we omit the proofs, which are analogous to those of Theorem 5.1. We emphasise that $(\Psi_{1:n}(Z_n))_{n\geq 0}$ converges in distribution for arbitrary admissable $(Z_n)_{n\geq 0}$ in the cases (i), $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$ and $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot\,)$ aperiodic, and $\limsup_{n\to\infty} |\Pi_{\tau_n(i)}| = \infty$ a.s. and $\Pi_n \xrightarrow{\mathbb{P}_\pi} 0$. In contrast, $\mathbb{P}(Z_n \in \cdot\,|M_n = i) = \delta_{c_i}$ must be satisfied in (iii) if $(\Pi_n)_{n\geq 0}$ does not tend stochastically to 0. Moreover, analogous to the proof of Lemma 5.10 (ii), one verifies that

$$\lim_{n\to\infty} \mathbb{P}_i(\Pi_{2n}(Z_{2n} - c_{M_{2n}}) \in \cdot\,) \;=\; \lim_{n\to\infty} \mathbb{P}_i(-\Pi_{2n}(Z_{2n} - c_{M_{2n}}) \in \cdot\,) \tag{6.5}$$

is necessary and sufficient if $\mathbb{P}_i(|\Pi_{\tau(i)}| = 1) = 1$ and $\mathbb{P}_i(\widehat{\tau}(i) \in \cdot\,)$ is 2-periodic. For example, (6.5) is satisfied if $\mathbb{P}_j(Z_0 \in \cdot\,) = \mathbb{P}(c_j + Y \in \cdot\,)$ for all $j \in \mathscr{S}$, where $Y$ is symmetric.

# 7. On the Forward Iterations

The main result of this section is a theorem giving equivalent conditions for $(\Psi_{n:1}(Z_0))_{n\geq 0}$ to converge in distribution under $\mathbb{P}_i$, $i \in \mathscr{S}$, for some admissable $Z_0$. First of all, we mention that $(\Psi_{n:1}(Z_0))_{n\geq 0}$ can converge in distribution under $\mathbb{P}_i$, but not under $\mathbb{P}_j$, $i,j \in \mathscr{S}$, as we will see. Nevertheless, given $(\Psi_{n:1}(Z_0))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$,

$$\Psi_{n:1}(Z_0) \;\overset{d}{\simeq}\; \mathbf{1}_{\{\tau(j)<n\}}\,\Psi_{n:\tau(j)+1}(\Psi_{\tau(j):1}(Z_0))$$

shows that $(\Psi_{n:1}(\widetilde{Z}_0))_{n\geq 0}$ converges in distribution to the same limit under any $\mathbb{P}_j$, where $\widetilde{Z}_0$ is admissable and has conditional distribution

$$\mathbb{P}_j(\widetilde{Z}_0 \in \cdot\,) \;=\; \begin{cases} \mathbb{P}_i(\Psi_{\tau(j):1}(Z_0) \in \cdot\,), & \text{if } j \neq i, \\ \mathbb{P}_i(Z_0 \in \cdot\,), & \text{if } j = i. \end{cases}$$

Now, let $({}^{\#}A_n, {}^{\#}B_n)_{n\geq 1}$ be the dual process of $(A_n, B_n)_{n\geq 1}$, i.e.

$${}^{\#}K_{ij} \;:=\; \mathbb{P}(({}^{\#}A_1, {}^{\#}B_1) \in \cdot\,|{}^{\#}M_0 = i, {}^{\#}M_1 = j) \;:=\; K_{ji},$$

where $({}^{\#}M_n)_{n\geq 0}$ is defined as in Chapter A. Using

$$\begin{aligned} &\mathbb{P}((A_k, B_k)_{1\leq k\leq n} \in \cdot\,|M_0 = i_0, \ldots, M_n = i_n) \\ &= \bigotimes_{k=1}^{n} K_{i_{k-1}i_k} \;=\; \bigotimes_{k=1}^{n} {}^{\#}K_{i_k i_{k-1}} \\ &= \mathbb{P}(({}^{\#}A_{n-k+1}, {}^{\#}B_{n-k+1})_{1\leq k\leq n} \in \cdot\,|{}^{\#}M_0 = i_n, \ldots, {}^{\#}M_n = i_0) \end{aligned}$$

and

$$\mathbb{P}_{i_0}(M_1=i_1,\ldots,M_n=i_n) \;=\; \frac{\pi_{i_n}}{\pi_{i_0}}\,\mathbb{P}_{i_n}({}^{\#}M_1=i_{n-1},\ldots,{}^{\#}M_n=i_0)$$

for all $n\ge 1$ and $i_0,\ldots,i_n\in\mathscr{S}$, we derive

$$\mathbb{P}_\pi((\widetilde{Z}_0,(A_k,B_k)_{1\le k\le n})\in\cdot\,) \;=\; \mathbb{P}_\pi(({}^{\#}Z_n,({}^{\#}A_{n-k+1},{}^{\#}B_{n-k+1})_{1\le k\le n})\in\cdot\,),$$

where $({}^{\#}Z_n)_{n\ge0}$ is admissable in terms of the dual process, i.e. ${}^{\#}Z_n$ is independent of all other occurring random variables given ${}^{\#}M_n$ and $\mathbb{P}({}^{\#}Z_n\in\cdot\,|{}^{\#}M_n=j)=\mathbb{P}_j(\widetilde{Z}_0\in\cdot\,)$. As a consequence,

$$\begin{aligned}\Psi_{n:1}(\widetilde{Z}_0) \;&=\; \sum_{k=1}^{n}\Big(\prod_{\ell=k+1}^{n}A_\ell\Big)B_k+\Pi_n\,\widetilde{Z}_0\\ &\overset{d}{=}\; \sum_{k=1}^{n}{}^{\#}\Pi_{k-1}\,{}^{\#}B_k+{}^{\#}\Pi_n\,{}^{\#}Z_n \;=\; {}^{\#}\Psi_{1:n}({}^{\#}Z_n)\end{aligned}$$

is true under $\mathbb{P}_\pi$, where

$${}^{\#}\Psi_n(x) \;:=\; {}^{\#}A_n\,x+{}^{\#}B_n,\qquad x\in\mathbb{R},\,n\ge1.$$

Consequently, if $(\Psi_{n:1}(Z_0))_{n\ge0}$ converges in distribution under $\mathbb{P}_i$, then

$$\lim_{n\to\infty}\mathbb{P}_i(\Psi_{n:1}(Z_0)\in\cdot\,) \;=\; \lim_{n\to\infty}\mathbb{P}_\pi(\Psi_{n:1}(\widetilde{Z}_0)\in\cdot\,) \;=\; \lim_{n\to\infty}\mathbb{P}_\pi({}^{\#}\Psi_{1:n}({}^{\#}Z_n)\in\cdot\,). \tag{7.1}$$

Hence, if $\mathbb{P}_\pi(A_1=0)>0$, then

$$\mathbb{P}_i(\Psi_{n:1}(Z_0)\in\cdot\,) \;\xrightarrow{w}\; \mathbb{P}_\pi\Big(\sum_{k=1}^{{}^{\#}\kappa}{}^{\#}\Pi_{k-1}\,{}^{\#}B_k\in\cdot\,\Big)$$

for any $i\in\mathscr{S}$, where ${}^{\#}\kappa:=\inf\{n\ge1:{}^{\#}A_n=0\}$.

Moreover, since limit distributions of $({}^{\#}\Psi_{1:n}(\widetilde{Z}_n))_{n\ge0}$ correspond to fixed points of

$${}^{\#}\Psi_1\bullet Q \;=\; Q,$$

the possible limit distributions can easily be inferred from Theorem 6.1.

Before stating the main theorem, we discuss some aspects of the dual process $({}^{\#}A_n,{}^{\#}B_n)_{n\ge1}$. Let $({}^{\#}A_n^i,{}^{\#}B_n^i)_{n\ge1}$ and ${}^{\#}\widehat{Z}_\infty$ be the analogues of $(A_n^i,B_n^i)_{n\ge1}$ and $\widehat{Z}_\infty$. $({}^{\#}A_n,{}^{\#}B_n)_{n\ge1}$ being degenerate, namely

$$\mathbb{P}_\pi({}^{\#}B_1=c_{{}^{\#}M_0}-{}^{\#}A_1\,c_{{}^{\#}M_1}) \;=\; 1$$

for some sequence $(c_i)_{i\in\mathscr{S}}$, is equivalent to

$$\mathbb{P}_\pi(B_1=c_{M_1}-A_1\,c_{M_0}) \;=\; 1. \tag{7.2}$$

In addition, (7.2) yields

$$\Psi_{n:1}(Z_0) \;=\; c_{M_n}+\Pi_n\,(Z_0-c_{M_0})\qquad\text{a.s.}$$

**Theorem 7.1** *Suppose $\mathbb{P}_\pi(A_1=0)=0$ and let $Z_0$ be an admissable random random variable. Then, $\mathbb{P}_i(\Psi_{n:1}(Z_0)\in\cdot)\xrightarrow{w}Q$, $i\in\mathscr{S}$, for some $Q\in\mathscr{P}(\mathbb{R})$ is satisfied if and only if one of the following conditions is fulfilled:*

*(i)* $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ *a.s. and* $\mathbb{E}_i J_i(\log^+|^\# B^i|)<\infty$. *Then,* $Q=\mathbb{P}_\pi(^\#\widehat{Z}_\infty\in\cdot)$.

*(ii)* $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$, (7.2) *is true and one of the following cases prevails:*

*(ii.1)* $\mathbb{P}_i(\widehat{\tau}(i)\in\cdot)$ *is aperiodic. Then,*

$$Q \;=\; \begin{cases}\sum_{j\in\mathscr{S}}\pi_j\,\mathbb{P}_i(c_j+(\theta_i\,a_i\,\theta_j/a_j)\,(Z_0-c_i)\in\cdot), & \text{if } \mathbb{P}_i(\Pi_{\tau(i)}=1)=1,\\ \sum_{j\in\mathscr{S}}\pi_j\,\mathbb{P}_i(c_j+(a_i/a_j)\,Y\in\cdot), & \text{if } \mathbb{P}_i(\Pi_{\tau(i)}=1)<1,\end{cases}$$

*where* $\mathbb{P}_i(Y\in\cdot)=\frac{1}{2}\left[\mathbb{P}_i(Z_0-c_i\in\cdot)+\mathbb{P}_i(-(Z_0-c_i)\in\cdot)\right]$.

*(ii.2)* $\mathbb{P}_i(\widehat{\tau}(i)\in\cdot)$ *is 2-periodic and*

$$\lim_{n\to\infty}\mathbb{P}_i(c_{M_{2n}}+\Pi_{2n}\,(Z_0-c_i)\in\cdot) \;=\; \lim_{n\to\infty}\mathbb{P}_i(c_{M_{2n}}-\Pi_{2n}\,(Z_0-c_i)\in\cdot). \tag{7.3}$$

*Then,*

$$Q \;=\; \sum_{j\in\mathscr{S}}\pi_j\,\mathbb{P}_i(c_j+(\theta_i\,a_i\,\theta_j/a_j)\,(Z_0-c_i)\in\cdot).$$

*(iii)* $\limsup_{n\to\infty}|\Pi_{\tau_n(i)}|=\infty$ *a.s. and* (7.2) *are satisfied. Moreover,* $Z_0=c_i$ $\mathbb{P}_i$*-a.s. if* $\limsup_{n\to\infty}\mathbb{P}_i(|\Pi_n|>0)>0$. *In each case,* $Q=\mathbb{P}_\pi(c_{M_0}\in\cdot)$.

**Proof.** Case $\lim_{n\to\infty}\Pi_{\tau_n(i)}=0$ a.s. Suppose $(\Psi_{1:n}(Z_0))_{n\geq 0}$ converges in distribution under $\mathbb{P}_i$. As $\Pi_n\xrightarrow{\mathbb{P}_\pi}0$, we obtain

$$\begin{aligned}\Psi_{n:1}(Z_0) &\overset{d}{\simeq} \Psi_{n:1}(0)\\ &\overset{d}{\simeq} \mathbf{1}_{\{\tau(j)<n\}}\left[(\Pi_n/\Pi_{\tau(j)})\left[\sum_{k=1}^{\tau(j)}\Big(\prod_{\ell=k+1}^{\tau(j)}A_\ell\Big)B_k\right]+\Psi_{n:\tau(j)+1}(0)\right]\\ &\overset{d}{\simeq} \Psi_{n:\tau(j)+1}(0).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\mathbb{P}_i(\Psi_{n:1}(Z_0)\in\cdot) \;=\; \lim_{n\to\infty}\mathbb{P}_\pi(\Psi_{n:1}(0)\in\cdot) \;=\; \lim_{n\to\infty}\mathbb{P}_\pi(^\#\Psi_{1:n}(0)\in\cdot)$$

so that the assertion follows from Theorem 5.1 (i). The converse follows analogously.

Theorem 6.1 yields that (7.2) must be satisfied if $(\Pi_{\tau_n(i)})_{n\geq 0}$ does not converge to 0 almost surely. Therefore, we assume (7.2) to be true in the remainder of the proof.

Case $\mathbb{P}_i(|\Pi_{\tau(i)}|=1)=1$. Suppose $\mathbb{P}_i(\widehat{\tau}(i)\in\cdot)$ is aperiodic. Using

$$\Psi_{n:1}(Z_0) \;=\; c_{M_n}+(\mathrm{sign}(\Pi_n)\,a_i/a_{M_n})\,(Z_0-c_i) \qquad \mathbb{P}_i\text{-a.s.},$$

Lemma 5.7 and Lemma 5.9 entail the identity of $Q$.

Suppose $\mathbb{P}_i(\widehat{\tau}(i)\in\cdot)$ is 2-periodic. 2-periodicity of $\mathbb{P}_i(\widehat{\tau}(i)\in\cdot)$ implies that $(\Psi_{1:2n}(Z_0))_{n\geq 0}$ and $(\Psi_{1:2n+1}(Z_0))_{n\geq 0}$ are regenerative processes with aperiodic regeneration times and thus converge in distribution under $\mathbb{P}_i$. Therefore, the equality of their limit distribution is equivalent to distributional convergence of $(\Psi_{1:n}(Z_0))_{n\geq 0}$. Analogous to the proof of Lemma 5.10 (ii), one concludes that the latter holds if and only if (7.3) is satisfied and obtains the identity of $Q$.

CASE $\limsup_{n\to\infty} |\Pi_n| = \infty$ a.s. We infer from the correspondence with fixed points of ${}^{\#}\Psi_1 \bullet Q = Q$ that the limit distribution equals $\mathbb{P}_\pi(c_{M_0} \in \cdot)$. The remaining assertion follows easily from

$$\Psi_{n:1}(Z_0) \;=\; c_{M_n} + \Pi_n\,(Z_0 - c_i) \qquad \mathbb{P}_i\text{-a.s.}$$

□

In comparison with Theorem 6.1 (ii), notice that $\mathbb{P}_i(c_j + (\theta_i\, a_i\, \theta_j/a_j)\,(Z_0 - c_i) \in \cdot\,)$ corresponds with $\mathbb{P}_i(c_j + {}^{\#}a_j {}^{\#}\theta_j\, Y \in \cdot\,)$, $Y := \theta_i\, a_i\,(Z_0 - c_i)$, where $({}^{\#}a_j)_{j\in\mathscr{S}}$ and $({}^{\#}\theta_j)_{j\in\mathscr{S}}$ are defined in terms of $({}^{\#}\Pi_n)_{n\geq 0}$.

Moreover, notice that (7.3) is always satisfied for some admissable $Z_0$. By Theorem 6.1, there exists a sequence $({}^{\#}Z_n)_{n\geq 0}$ admissable for the dual process such that $({}^{\#}\Psi_{1:n}({}^{\#}Z_n))_{n\geq 0}$ converges in distribution under $\mathbb{P}_\pi$. Hence, define

$$\mathbb{P}_i(Z_0 \in \cdot\,) \;:=\; \mathbb{P}_\pi(\Psi_{\tau(i):1}(\widetilde{Z}_0) \in \cdot\,),$$

where $\mathbb{P}_j(\widetilde{Z}_0 \in \cdot\,) = \mathbb{P}_j({}^{\#}Z_0 \in \cdot\,)$. and an appeal to (7.1) yields (7.3).

In conclusion, we verify a correspondence between proper stationary solutions to (2.2) and limit distributions of $(\Psi_{n:1}(Z_0))_{n\geq 0}$ under some $\mathbb{P}_i$ for some admissable $Z_0$.

Consider a proper stationary solution $(R_n)_{n\geq 0}$ to (2.2). Then, $(\Psi_{n:1}(R_0))_{n\geq 0}$ converges naturally in distribution under $\mathbb{P}_\pi$, because

$$R_n \;\overset{d}{=}\; A_n\, R_{n-1} + B_n \;=\; \Psi_n(R_{n-1}) \qquad \text{under } \mathbb{P}_\pi$$

for all $n \geq 1$. Furthermore, $(R_n)_{n\geq 0}$ being a stationary solution yields that $R_{\tau(i)}$ is independent of $(A_k, B_k)_{k>\tau(i)+1}$. Together with

$$\Psi_{n:1}(R_0) \;\overset{d}{\simeq}\; \mathbf{1}_{\{\tau(i)<n\}}\, \Psi_{n:\tau(i)+1}(R_{\tau(i)}),$$

we obtain $\Psi_{n:1}(Z_0) \overset{d}{\to} R_0$ under $\mathbb{P}_i$, where $Z_0$ is admissable and $\mathbb{P}_i(Z_0 \in \cdot\,) = \mathbb{P}_\pi(R_{\tau(i)} \in \cdot\,)$.

Conversely, suppose $\mathbb{P}_i(\Psi_{n:1}(Z_0) \in \cdot\,) \overset{w}{\to} \widehat{Q} \in \mathscr{P}(\mathbb{R})$. We will show that there is a stationary solution $(R_n)_{n\geq 0}$ to (2.2) with $\mathbb{P}_\pi(R_0 \in \cdot\,) = \widehat{Q}$. Due to (7.1), $\widehat{Q} = \lim_{n\to\infty} \mathbb{P}_\pi({}^{\#}\Psi_{1:n}({}^{\#}Z_n) \in \cdot\,)$. Referring to Section 6, we can define $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$ by $Q(i,\cdot) := \lim_{n\to\infty} \mathbb{P}_i({}^{\#}\Psi_{1:n}({}^{\#}Z_n) \in \cdot\,)$, $i \in \mathscr{S}$. Then, $Q$ satisfies ${}^{\#}\Psi_1 \bullet Q = Q$ and the subsequent lemma shows that this is equivalent to $\Psi_1 \star Q = Q$.

**Lemma 7.2** *Suppose $Q \in \mathscr{P}(\mathscr{S},\mathbb{R})$. $\Psi_1 \star Q = Q$ if and only if ${}^{\#}\Psi_1 \bullet Q = Q$.*

**Proof.** Let $R_0$ be a random variable independent of $(M_1, A_1, B_1)$ given $M_0$ and ${}^{\#}R_1$ a random variable independent of $({}^{\#}A_1, {}^{\#}B_1)$ given $({}^{\#}M_0, {}^{\#}M_1)$. Furthermore, we suppose

$$\mathbb{P}(R_0 \in \cdot\,|M_0 = j) \;=\; Q(j,\cdot) \;=\; \mathbb{P}({}^{\#}R_1 \in \cdot\,|{}^{\#}M_0 = i, {}^{\#}M_1 = j)$$

for all $i, j \in \mathscr{S}$. Then, the assertion follows from

$$\begin{aligned}
&\mathbb{P}_\pi(A_1 R_0 + B_1 \in \cdot\,|M_1 = i) \\
&= \sum_{j\in\mathscr{S}} \frac{\pi_j}{\pi_i}\mathbb{P}_j(A_1 R_0 + B_1 \in \cdot\,, M_1 = i) \\
&= \sum_{j\in\mathscr{S}} \frac{\pi_j p_{ji}}{\pi_i}\mathbb{P}(A_1 R_0 + B_1 \in \cdot\,|M_0 = j, M_1 = i) \\
&= \sum_{j\in\mathscr{S}} {}^{\#}p_{ij} \int\!\!\int \mathbf{1}_{\{xr+y\in\cdot\}}\ \mathbb{P}((A_1, B_1) \in d(x,y)|M_0 = j, M_1 = i)\ Q(j, dr) \\
&= \sum_{j\in\mathscr{S}} {}^{\#}p_{ij} \int\!\!\int \mathbf{1}_{\{xr+y\in\cdot\}}\ \mathbb{P}(({}^{\#}A_1, {}^{\#}B_1) \in d(x,y)|{}^{\#}M_0 = i, {}^{\#}M_1 = j)\ Q(j, dr) \\
&= \sum_{j\in\mathscr{S}} {}^{\#}p_{ij}\,\mathbb{P}({}^{\#}A_1\,{}^{\#}R_1 + {}^{\#}B_1 \in \cdot\,|{}^{\#}M_0 = i, {}^{\#}M_1 = j) \\
&= \sum_{j\in\mathscr{S}} \mathbb{P}_i({}^{\#}A_1\,{}^{\#}R_1 + {}^{\#}B_1 \in \cdot\,, {}^{\#}M_1 = j) \\
&= \mathbb{P}_i({}^{\#}A_1\,{}^{\#}R_1 + {}^{\#}B_1 \in \cdot\,)
\end{aligned}$$

for all $i \in \mathscr{S}$. □

Now, let $(R_n)_{n\geq 0}$ be a sequence of random variables such that $R_n$ is independent of $(A_k, B_k)_{k>n}$ given $M_n$ and $\mathbb{P}(R_n \in \cdot\,|M_n = i) = Q(i, \cdot\,)$ for all $n \geq 0$. Using $\Psi \star Q = Q$, yields

$$\begin{aligned}
\mathbb{P}_\pi(A_n R_{n-1} + B_n \in \cdot\,) &= \mathbb{P}_\pi(A_1 R_0 + B_1 \in \cdot\,) \\
&= \sum_{i\in\mathscr{S}} \pi_i\,\mathbb{P}_\pi(A_1 R_0 + B_1 \in \cdot|M_1 = i) \\
&= \sum_{i\in\mathscr{S}} \pi_i\,Q(i, \cdot\,) \\
&= \mathbb{P}_\pi(R_n \in \cdot\,)
\end{aligned}$$

for all $n \geq 1$

As mentioned in Section 2, the corresponding assumption of Brandt [10] for the study of a stationary solution to (2.2) is $\mathbb{E}_\pi \log|A_1| < 0$ and $\mathbb{E}_\pi \log^+|B_1| < \infty$. The assumption is equivalent to $\mathbb{E}_\pi \log|{}^{\#}A_1| < 0$ and $\mathbb{E}_\pi \log^+|{}^{\#}B_1| < \infty$. Consequently,

$$\log|{}^{\#}\Pi_n| \;=\; \sum_{k=1}^{n} \log|{}^{\#}A_k| \;\to\; -\infty \qquad \text{a.s.}$$

by Birkhoff's ergodic theorem and

$$\lim_{n\to\infty} \frac{\log^+|{}^{\#}B_n|}{n} \;=\; 0 \qquad \text{a.s.}$$

by the Borel-Cantelli lemma, which entail

$$\lim_{n\to\infty} {}^{\#}\Pi_n\,{}^{\#}B_{n+1} \;=\; 0 \qquad \text{a.s.}$$

By Theorem 4.1, this yields that $({}^{\#}\Psi_{1:n}(0))_{n\geq 0}$ converges almost surely and in particular in distribution. Hence, we are in case (i) of Theorem 7.1 and the unique solution is given by $\mathbb{P}_\pi({}^{\#}\widehat{Z}_\infty \in \cdot)$. [10, Theorem 1] contains a different expression of the limit distribution in terms of the doubly infinite extension $(A_n, B_n)_{n\in\mathbb{Z}}$.

# C. Appendix

## C.1. Consequences of $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$

This chapter gathers some technical results that are used frequently in Chapter A. $(M_n, S_n)_{n\geq 0}$ denotes a MRW if not stated otherwise.

The lemma can easily be proved by applying Theorems 1.2–1.3 on the random walks $(S_n - n(\mu+\varepsilon))_{n\geq 0}$ and $(S_n - n(\mu-\varepsilon))_{n\geq 0}$. However, the result is already known (see [12, Theorem 8.4.5] for $\alpha = 0$ and [9, Theorem 1] for $\alpha > 0$).

**Lemma C.1** *Let $(S_n)_{n\geq 0}$ be a random walk.*

*(i) Suppose $\alpha = 0$. $\mu := \mathbb{E}X \in \mathbb{R}$ if and only if*

$$\sum_{n\geq 1} n^{-1}\,\mathbb{P}(|n^{-1}S_n - \mu| > \varepsilon) \;<\; \infty \qquad \text{for all } \varepsilon > 0.$$

*(ii) Suppose $\mu := \mathbb{E}X \in \mathbb{R}$ and $\alpha > 0$. Then, $\mathbb{E}|X|^{1+\alpha} < \infty$ if and only if*

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}(|n^{-1}S_n - \mu| > \varepsilon) \;<\; \infty \qquad \text{for all } \varepsilon > 0.$$

**Lemma C.2** *Let $\alpha > 1$ and suppose $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some (hence all) $i \in \mathscr{S}$. Then,*

$$\mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{\tau_n(i) > (\mathbb{E}_i\tau(i)+\varepsilon)n\}}\Big) \;<\; \infty \qquad \text{for all } \varepsilon > 0$$

*for all $i \in \mathscr{S}$.*

**Proof.** Set $b := \mathbb{E}_i\tau(i) + \varepsilon$. We begin with

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\geq 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{\tau_n(i)>bn\}}\Big)\\
&\asymp \sum_{n\geq 1}\int_0^\infty x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > bn\vee x)\;dx\\
&= \sum_{n\geq 1}\int_0^{bn} x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > bn)\;dx + \sum_{n\geq 1}\int_{bn}^\infty x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > x)\;dx\\
&\asymp \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_n(i) > bn) + \sum_{n\geq 1}\sum_{k\geq n}\int_{bk}^{b(k+1)} x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > x)\;dx,
\end{aligned}$$

where the first summand of the last line is finite for all $\varepsilon > 0$ by Lemma C.1. Concerning the second summand, we infer

$$\begin{aligned}
&\sum_{n\ge 1}\sum_{k\ge n}\int_{bk}^{b(k+1)} x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > x)\; dx \\
\le\; &\sum_{n\ge 1}\sum_{k\ge n}\int_{bk}^{b(k+1)} x^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > bk)\; dx \\
\asymp\; &\sum_{n\ge 1}\sum_{k\ge n} k^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > bk) \\
=\; &\sum_{k\ge 1}\sum_{n=1}^{k} k^{\alpha-2}\,\mathbb{P}_i(\tau_n(i) > bk) \\
\le\; &\sum_{k\ge 1} k^{\alpha-1}\,\mathbb{P}_i(\tau_k(i) > bk),
\end{aligned}$$

where we used that $(\tau_n(i))_{n\ge 0}$ is a renewal process. The upper bound is again finite by Lemma C.1. □

**Lemma C.3** *Let $\alpha \ge 0$ and suppose $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some (hence all) $i \in \mathscr{S}$. Then,*

$$\sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le y) \;\asymp\; \mathbb{E}_i\Big(\sum_{n\ge 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\le y\}}\Big) \qquad \text{as } y\to\infty$$

*for all $i \in \mathscr{S}$.*

**Proof.** On the one hand, Lemma C.1 yields

$$\mathbb{E}_i\Big(\sum_{n\ge 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{\tau_n(i)>2n\mathbb{E}_i\tau(i)\}}\Big) \;\le\; \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_n(i) > 2n\mathbb{E}_i\tau(i)) \;<\; \infty$$

for $\alpha \in [0,1]$ and, on the other hand, Lemma C.2 entails

$$\infty \;>\; \mathbb{E}_i\Big(\sum_{n\ge 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{\tau_n(i)>2n\mathbb{E}_i\tau(i)\}}\Big) \;\ge\; \sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_n(i) > 2n\mathbb{E}_i\tau(i))$$

for $\alpha > 1$. Consequently,

$$\begin{aligned}
&\mathbb{E}_i\Big(\sum_{n\ge 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\le y\}}\Big) \\
\asymp\; &\mathbb{E}_i\Big(\sum_{n\ge 1}\tau_n(i)^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\le y,\, n\le\tau_n(i)\le 2n\mathbb{E}_i\tau(i)\}}\Big) \\
\asymp\; &\mathbb{E}_i\Big(\sum_{n\ge 1} n^{\alpha-1}\,\mathbf{1}_{\{S_{\tau_n(i)}\le y,\, n\le\tau_n(i)\le 2n\mathbb{E}_i\tau(i)\}}\Big) \\
\asymp\; &\sum_{n\ge 1} n^{\alpha-1}\,\mathbb{P}_i(S_{\tau_n(i)} \le y) \qquad \text{as } y\to\infty
\end{aligned}$$

as claimed. □

**Lemma C.4** *Let $\alpha \geq 0$ and suppose $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some (hence all) $i \in \mathscr{S}$. Then,*

$$\mathbb{E}_i\Big(\sum_{n\geq 1}\chi_n(i)^\alpha\,\mathbf{1}_{\{\chi_n(i)>n\}}\Big) \;<\; \infty$$

*for all $i \in \mathscr{S}$.*

**Proof.** Analogous to the beginning of the proof of Lemma C.2, one derives

$$\mathbb{E}_i\Big(\sum_{n\geq 1}\chi_n(i)^\alpha\,\mathbf{1}_{\{\chi_n(i)>n\}}\Big) \;\asymp\; \sum_{n\geq 1} n^\alpha\,\mathbb{P}_i(\tau(i)>n) + \sum_{n\geq 1}\sum_{k\geq n}\int_k^{k+1} x^{\alpha-1}\,\mathbb{P}_i(\tau(i)>x)\;dx$$

and that

$$\sum_{n\geq 1}\sum_{k\geq n}\int_k^{k+1} x^{\alpha-1}\,\mathbb{P}_i(\tau(i)>x)\;dx \;\lesssim\; \sum_{n\geq 1} n^\alpha\,\mathbb{P}_i(\tau(i)>n).$$

Therefore, the assertion follows from

$$\sum_{n\geq 1} n^\alpha\,\mathbb{P}_i(\tau(i)>n) \;\lesssim\; \mathbb{E}_i\tau(i)^{1+\alpha} \;<\; \infty.$$

□

**Lemma C.5** *Let $\alpha > 0$ and suppose $\mathbb{E}_i\tau(i)^{1+\alpha} < \infty$ for some (hence all) $i \in \mathscr{S}$. Then, for any non-negative random variable $T$*

$$\mathbb{E}_i T^\alpha \;<\; \infty \qquad \textit{if and only if} \qquad \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(T>\tau_n(i)) \;<\; \infty.$$

**Proof.** As seen in the proof of Lemma C.3, we have

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_n(i)>2\,n\,\mathbb{E}_i\tau(i)) \;<\; \infty$$

and thus

$$\begin{aligned}
\mathbb{E}_i T^\alpha \;&\asymp\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(T>n) \\
&\asymp\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(T>2\,n\,\mathbb{E}_i\tau(i)) \\
&\asymp\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(T>2\,n\,\mathbb{E}_i\tau(i),\tau_n(i)\leq 2\,n\,\mathbb{E}_i\tau(i)) \\
&\leq\; \sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(T>\tau_n(i)).
\end{aligned}$$

Analogously, the reverse inequality can be obtained by using

$$\sum_{n\geq 1} n^{\alpha-1}\,\mathbb{P}_i(\tau_n(i)\leq 2^{-1}\,n\,\mathbb{E}_i\tau(i)) \;<\; \infty.$$

□

## C.2. Renewal-Theoretic Auxiliary Results

**Lemma C.6** *Let $(S_n)_{n\ge 0}$ be a positive divergent random walk. Then,*

$$\sum_{n\ge 1}\mathbb{P}(0\le S_n\le y)\ \asymp\ J(y)\qquad \text{as } y\to\infty.$$

**Proof.** Set $\mathbb{U}^{>}((\infty,y]) := \sum_{n\ge 0}\mathbb{P}(S_{\sigma_n^{>}}\le y)$ for $y\in\mathbb{R}_{\ge}$. By Theorem 1.6,

$$\mathbb{U}^{>}((-\infty,y])\ =\ \mathbb{U}^{>}([0,y])\ \asymp\ J(y)\qquad \text{as } y\to\infty.$$

Consequently, one part follows directly from

$$\sum_{n\ge 1}\mathbb{P}(0\le S_n\le y)\ \ge\ \mathbb{U}^{>}([0,y])-1$$

for $y\in\mathbb{R}_{\ge}$. It is well-known from renewal theory that $\mathbb{U}^{>}([-x,y-x])\le \mathbb{U}^{>}(y)$ for all $x\in\mathbb{R}$ and $y\in\mathbb{R}_{\ge}$. Hence, the second part follows from

$$\begin{aligned}
\sum_{n\ge 1}\mathbb{P}(0\le S_n\le y)\ &=\ \sum_{n\ge 1}\mathbb{E}\left(\sum_{k=\sigma_{n-1}^{>}+1}^{\sigma_n^{>}}\mathbf{1}_{\{0\le S_k\le y\}}\right)\\
&= \int\mathbb{E}\left(\sum_{k=1}^{\sigma^{>}}\mathbf{1}_{\{0\le S_k+x\le y\}}\right)\mathbb{U}^{>}(dx)\\
&= \mathbb{E}\left[\sum_{k=1}^{\sigma^{>}}\mathbb{U}^{>}([-S_k,y-S_k])\right]\\
&\le \mathbb{E}\left[\sum_{k=1}^{\sigma^{>}}\mathbb{U}^{>}([0,y])\right]\\
&= \mathbb{E}\sigma^{>}\cdot\mathbb{U}^{>}([0,y]).
\end{aligned}$$

□

**Lemma C.7** *Let $(M_n,X_n)_{n\ge 0}$ be a Markov-modulated sequence, where $X_n$ is assumed to be $\mathbb{R}^d$ valued. Pick some $i\in\mathscr{S}$ and define*

$$T_n\ =\ f(X_{\tau_{n-1}(i)+1},\dots,X_{\tau_n(i)}),\qquad i\in\mathscr{S},\,n\ge 1,$$

*for some measurable, real-valued function $f$. Then, $(T_{N(n)+1})_{n\ge 1}$ converges in distribution.*

**Proof.** Let $(\mu_n)_{n\ge 0}$ denote the discrete renewal density associated with $(\tau_n(i))_{n\ge 0}$, i.e.

$$\mu_n\ :=\ \sum_{k\ge 0}\mathbb{P}_i(\tau_k(i)=n)$$

for $n\ge 0$. Since $(M_n)_{n\ge 0}$ is aperiodic, thus $\mathbb{P}_i(\tau(i)\in\cdot)$ 1-arithmetic, we have $\lim_{n\to\infty}\mu_n = 1/\mathbb{E}_i\tau(i)$. By the key renewal theorem,

$$\mathbb{P}_i(T_{N(n)+1}\le x)\ =\ \sum_{k\ge 1}\mathbb{P}_i(N(n)+1=k,\,T_k\le x)$$

$$\begin{aligned} &= \sum_{k\geq 1}\sum_{\ell=0}^{n} \mathbb{P}_i(\tau_{k-1}(i)=\ell)\,\mathbb{P}_i(\tau(i)>n-\ell, T\leq x) \\ &= \sum_{\ell=0}^{n} \mu_\ell\,\mathbb{P}_i(\tau(i)>n-\ell, T\leq x) \end{aligned}$$

converges to

$$\frac{1}{\mathbb{E}_i\tau(i)}\sum_{\ell\geq 0}\mathbb{P}_i(\tau(i)>\ell, T\leq x) \;=\; \frac{\mathbb{E}_i\tau(i)\,\mathbf{1}_{\{T\leq x\}}}{\mathbb{E}_i\tau(i)}$$

for all $x \in \mathbb{R}$ as claimed.

$\square$

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# Acronyms

**a.s.** almost surely, also used synonymous for $\mathbb{P}_i$-a.s. for all $i \in \mathscr{S}$.
**i.i.d.** independent and identically distributed.
**MRW** Markov random walk.
**PRW** perturbed random walk.
**SFPE** stochastic fixed point equation.
**w.l.o.g.** without loss of generality.

# List of Symbols

$\lesssim$ $f(y) \lesssim g(y)$: $\Leftrightarrow$ $f(y) \le c\,g(y)$ for all sufficiently large $y > 0$ for some $c \in \mathbb{R}_>$. $\asymp$ $f(y) \asymp g(y)$ :$\Leftrightarrow$ $f($
$A(y)$ $A(y) := \mathbb{E}(X^+ \wedge y) - \mathbb{E}(X^- \wedge y)$. $A_i(y)$ $A_i(y) := \mathbb{E}_i(S^+_{\tau(i)} \wedge y) - \mathbb{E}_i(S^-_{\tau(i)} \wedge y)$. $A_\pi(y)$ $A_\pi(y) := \mathbb{E}_\pi$
$\mathfrak{B}$ Borel $\sigma$-algebra on $\mathbb{R}$. $B_n^i$ $B_n^i := \sum_{k=\tau_{n-1}(i)+1}^{\tau_n(i)} (\prod_{\ell=\tau_{n-1}(i)+1}^{k-1} A_\ell) B_k$.
$\chi_n(i)$ $\chi_n(i) := \tau_n(i) - \tau_{n-1}(i)$.
$D_n^i$ $D_n^i := \max_{\tau_{n-1}(i) < k \le \tau_n(i)} (S_k - S_{\tau_{n-1}(i)})^-$. $D_n^{i,>}$ $D_n^{i,>} := \max_{\tau^>_{n-1}(i) < k \le \tau^>_n(i)} (S_k - S_{\tau^>_{n-1}(i)})^-$. $\delta_a$ Di
$\mathbb{E}_i$ expectation symbol of $\mathbb{P}_i$. $\mathbb{E}_\pi$ expectation symbol of $\mathbb{P}_\pi$.
$H_n^i$ $H_n^i := \max_{\tau_{n-1}(i) < k \le \tau_n(i)} (S_k - S_{\tau_{n-1}(i)})^+$.
$J$ $J(y) := \frac{y}{\mathbb{E}(X^+ \wedge y)}$. $J_{i,\gamma}$ $J_{i,\gamma}(y) := \frac{y}{[\mathbb{E}_i(S^+_{\tau(i)} \wedge y)]^\gamma}$. $J_i$ $J_i := J_{i,1}$. $J_{\pi,\gamma}$ $J_{\pi,\gamma}(y) := \frac{y}{[\mathbb{E}_\pi(X_1^+ \wedge y)]^\gamma}$. $J_\pi$ $J_\pi :=$
$K_{ij}$ conditional distribution of $X_1$ (Chapter A) or $(A_1, B_1)$ (Chapter B) given $M_0 = i, M_1 = j$.
$\Lambda(x)$ $\Lambda(x) := \sum_{n \ge 1} \mathbf{1}_{\{S_n \le x\}}$. $\Lambda_n^>$ $\Lambda_n^> := \sum_{k=1}^n \mathbf{1}_{\{S_n > 0\}}$. $\Lambda_n^\leqslant$ $\Lambda_n^\leqslant := \sum_{k=1}^n \mathbf{1}_{\{S_n \le 0\}}$.
$\mathbb{N}$ positive integers. $\mathbb{N}_0$ $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $N(n)$ $N(n) := N(i,n) := \sup\{k \ge 0 : \tau_k(i) \le n\}$. $\nu(x)$ $\nu(x) :=$
$\mathbf{P}$ $\mathbf{P} := (p_{ij})_{i,j \in \mathscr{S}}$. $p_{ij}$ $p_{ij} := \mathbb{P}_i(M_1 = j)$. $\pi$ stationary distribution of $(M_n)_{n \ge 0}$. $\pi^>$ stationary distri
$\mathbb{R}_\ge$ non-negative real numbers. $\mathbb{R}_>$ positive real numbers. $\rho(x)$ $\rho(x) := \sup\{n \ge 0 : S_n \le 0\}$.
$\mathscr{S}$ state space of $(M_n)_{n \ge 0}$. $\mathscr{S}^>$ state space of $(M_n^>)_{n \ge 0}$. $\sigma_1^>(x)$ $\sigma_1^>(x) := \inf\{n \ge 1 : S_n > x\}$. $\sigma_1^>$ $\sigma$
$(\tau_n(i))_{n \ge 1}$ sequence of return times to $i$. $\widehat{\tau}(i)$ $\widehat{\tau}(i) := \inf\{\tau_n(i) : \Pi_{\tau_n(i)} = 1\}$. $\tau^>(i)$ $\tau^>(i) :=$ in
$\mathbb{U}_i$ $\mathbb{U}_i := \sum_{n \ge 0} \mathbb{P}_i(S_{\tau_n(i)} \in \cdot)$. $\upsilon$ $\upsilon := \upsilon(i,j) := \inf\{n \ge 1 : \tau_n(i) > \tau(j)\}$.
$\mathbb{V}_i^\alpha$ the measure induced by $\mathbb{V}_i^\alpha((x,\infty)) := \mathbb{E}_i(\sum_{k=1}^{\tau(i)} \mathbf{1}_{\{S_k^- > x\}})^\alpha$. $\mathbb{V}_i$ $\mathbb{V}_i := \mathbb{V}_i^1$.
$W_n^i$ $W_n^i := \max_{\tau_{n-1}(i) < k \le \tau_n(i)} |(\Pi_{k-1}/\Pi_{\tau_{n-1}(i)}) B_k|$.
$\widehat{Z}_\infty$ $\widehat{Z}_\infty := \sum_{n \ge 1} \Pi_{n-1} B_n$.

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