A support theorem and a large deviation principle for Kunita flows

by

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Version of April 16, 2010

Summary. In this article, stochastic flows driven by Kunita-type stochastic differential equations are studied, focusing on support theorems (SP) and large deviation principles (LDP). We establish a new ST and LDP for Brownian flows with respect to a fine Hölder topology. Our approach is based on recent advances in rough paths theory, which is the natural framework for proving ST and LDP. Nevertheless, while rigorous, our presentation stays rather clear from the rough paths technicalities and is accessible for readers not familiar with them. We view the localized Brownian stochastic flow as a projection of the solution of a rough path differential equation implying the ST and LDP. In a second step the results are generalized for the global flow.

Keywords. Kunita-type stochastic differential equation, Brownian flow, rough paths, support theorem, large deviation principle.

2000 Mathematics Subject Classification. 60H10, 60F10

1 Introduction

The objective of this article is to derive a support theorem and a large deviation principle for Brownian flows generated by Kunita stochastic differential equations. These type of SDE's are indispensable when studying stochastic flows from the perspective of their generating SDE. The study of stochastic flows has been going along two main directions - in the probabilistic one stochastic flows have been observed mainly through the SDE which generates them. The other approach is rooted back in the ergodic theory and the theory of dynamical systems and deals mainly with iterated random functions having certain properties independently of their generation. If one confines oneself to stochastic differential equations driven by a finite number of Brownian motions (or more generally continuous martingales) there appears a gap between the two lines since there are Brownian stochastic flows (i.e. having infinitesimally independent Gaussian increments) which cannot be generated via a finite dimensional noise SDE. A notorious example are the isotropic Brownian flows studied among others by Baxendale and Harris [1], Le Jan [9] and Yaglom [12]. To retain a proper one-to-one relationship between Brownian stochastic flows in continuous time and SDEs one has to consider Kunita-type SDEs driven by possibly infinitely many Brownian motions. The interested reader can find an in-depth treatment of this relationship in the monograph by Kunita [8].

Our approach is based on recent results from the theory of rough paths [4] (see also [10, 5, 7]). To be more precise we show that the latter results can be applied to the *localized flow* and we then deduce our results by an approximation argument.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ denote a filtered probability space satisfying the usual conditions and denote by $C^\alpha(\mathbb{R}^d, \mathbb{R}^d)$ $(\alpha \geq 0)$ the Fréchet space of $\lfloor \alpha \rfloor$-times continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}^d$ whose differentials are locally $\alpha - \lfloor \alpha \rfloor$-Hölder continuous. As driving process we consider a $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$-valued Wiener process $X = (X_t)_{t\in[0,1]} : \Omega \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ with $\gamma > 2$ being fixed throughout the article. The reproducing kernel Hilbert spaces of X and X_1 will be denoted by $(H, |\cdot|_H)$ and $(H_1, |\cdot|_{H_1})$ respectively. We consider the flow Y generated by the Kunita SDE

$$Y_t(\xi) = \xi + \int_0^t X(Y_t(\xi), \circ\, \mathrm{d}t) + \int_0^t a(Y_t(\xi))\, \mathrm{d}t, \qquad \xi \in \mathbb{R}^d, t \in [0, 1], \tag{1}$$

where $a \in C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$ is an additional drift term.
Our assumptions guarantee the existence of a unique maximal solution for one trajectory (for each fixed ξ) up to a possibly finite explosion time and we need to impose as additional assumptions that none of the trajectories explodes in finite time, Assumption (A1), see [8, Theorem 4.5.1] for sufficient conditions for (A1).

The Kunita flow is closely related to the ordinary differential equation

$$\dot{y}_t(\xi) = \dot{x}_t(y_t(\xi)) + a(y_t(\xi)), \quad y_0(\xi) = \xi, \tag{2}$$

where $\dot{x}_t(\xi)$ denotes the time derivative in $\xi \in \mathbb{R}^d$ of a sufficiently smooth $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$-valued process $x = (x_t)_{t\in[0,1]}$. In the case where x has compact support and is of bounded variation, the Theorem of Picard-Lindelöf [11, Thm. 1.3] implies the existence of a unique solution of the corresponding integral equation quoted explicitly in Section 2. Its solution operator will be denoted by I. Note that for non-compactly supported x this solution operator is well defined only up to some explosion time and we need to impose further assumptions to ensure global existence. We will see that Assumption (A2) below is sufficient to guarantee non-explosion of the solution when applied to elements from the reproducing kernel Hilbert space H.

We consider $W = C^m(\mathbb{R}^d, \mathbb{R}^d)$ as *state space* of the solution Y of (1), where $m \in \mathbb{N}_0$ with $m + 2 < \gamma$. Moreover, let $\varphi : (0, 1] \to (0, \infty)$ be a monotonically increasing function with

$$\lim_{\delta \downarrow 0} \frac{\varphi(\delta)}{\sqrt{-\delta \log \delta}} = \infty. \tag{3}$$

Then we view Y as $C_\varphi([0, 1], W)$-valued random element, where $C_\varphi([0, 1], W)$ denotes the Fréchet space of φ-Hölder continuous W-valued paths, see below for an exact definition.

Theorem 1.1. *Suppose Assumptions (A1) and (A2) are satisfied. Every $h \in H$ admits a unique solution $I(h)$ of the Young integral equation related to (2) and*

$$\operatorname{range}_{C_\varphi([0,1],W)}(Y) = \overline{I(H)}.$$

For the large deviation principle, we consider the family $\{X^\varepsilon : \varepsilon > 0\}$ as driving signals, where $X^\varepsilon = (\varepsilon \cdot X_t)_{t \in [0,1]}$. The corresponding flows are denoted by Y^ε.

Theorem 1.2. *Under Assumptions (A1) and (A2), the family $\{Y^\varepsilon : \varepsilon > 0\}$ satisfies a large deviation principle in $C_\varphi([0, 1], W)$ of speed (ε^2) with good rate function*

$$J(y) = \inf\Big\{\frac{1}{2}|h|_H^2 : h \in H \text{ with } I(h) = y\Big\}. \tag{4}$$

Here and elsewhere the infimum of the empty set is assumed to be infinity.

In general, the rate of a path y is related to an integral equation, and, in general, it is a hard problem to find minimizers on a given set. However, if the set is triggered by the evolution of one particular trajectory $y_\cdot(\xi)$, one can find the minimizer for the single evolution and then conclude back on the rate minimizing evolution for the whole flow.

Theorem 1.3. *Suppose that Assumption (A2) is valid. For a path $\gamma \in C([0, 1], \mathbb{R}^d)$, we consider*

$$A_\gamma = \{y \in C([0, 1], W) : \gamma_t = y_t(\gamma_0) \text{ for all } t \in [0, 1]\}.$$

If A_γ contains an element with finite J-energy, then it possesses a unique J-minimizer y which admits the following representation: For almost all $t \in [0, 1]$, $\dot\gamma_t - a(\gamma_t)$ lies in the image of the covariance kernel $K(\gamma_t, \gamma_t)$ of X_1 (see (8)) and we denote by $K^{-1}(\gamma_t, \gamma_t)\,(\dot\gamma_t - a(\gamma_t))$ the element $v \in \mathbb{R}^d$ with minimal Euclidean norm that satisfies

$$K(\gamma_t, \gamma_t)\, v = \dot\gamma_t - a(\gamma_t).$$

Then $y = I(f)$, where $f \in H$ is given by

$$\dot f_t(\xi) = K(\xi, \gamma_t)\, K^{-1}(\gamma_t, \gamma_t)\, \dot\gamma_t - a(\gamma_t), \qquad \text{for } \xi \in \mathbb{R}^d. \tag{5}$$

In particular, the minimal energy is

$$J(y) = \frac{1}{2}\int_0^1 \langle \dot\gamma_t - a(\gamma_t), K^{-1}(\gamma_t, \gamma_t)\, \dot\gamma_t - a(\gamma_t)\rangle_{\mathbb{R}^d}\, \mathrm{d}t. \tag{6}$$

Remark 1.4. If we only consider the evolution of a single particle, say $\xi \in \mathbb{R}^d$, then the classical large deviation principle for $(Y_\cdot^\varepsilon(\xi) : \varepsilon > 0)$ associates the evolution $(\gamma_t)_{t \in [0,1]}$ with the rate quoted in (6) (provided that (γ_t) is absolutely continuous with derivative in the image of the kernel). Theorem 1.3 additionally gives a representation for the corresponding minimizing flow as solution of an integral equation via (5).

Topologies

Let us now introduce the relevant spaces together with their topologies. Note that we consider the Fréchet space $C^\gamma(\mathbb{R}^d,\mathbb{R}^d)$ of all $\lfloor\gamma\rfloor$-times continuously differentiable functions $f:U\to\mathbb{R}^d$ with all differentials of order $\lfloor\gamma\rfloor$ being locally $\gamma-\lfloor\gamma\rfloor$-Hölder continuous as state space for the driving signal. It is equipped with the semi-norms

$$\|f\|_{C^\gamma_\zeta} := \|\zeta\cdot f\|_{C^\gamma} \qquad (\zeta\in C_c^\infty),$$

where C_c^∞ denotes the class of infinitely often differentiable real functions that are compactly supported and $\|\cdot\|_{C^\gamma}$ is a canonical norm defined as follows: for $n\in\mathbb{N}$ and for a n-times continuously differentiable function $f\in C^n(U,\mathbb{R}^d)$ ($U\subset\mathbb{R}^d$ domain), we set

$$\|f\|_{C^n} = \sum_{|\alpha|\le n}\sup_{x\in U}|\partial^\alpha f(x)|.$$

Moreover, for $\gamma=n+\eta$ with $n\in\mathbb{N}_0$ and $\eta\in(0,1)$, and $f\in C^n(U,\mathbb{R}^d)$, we set

$$\|f\|_{C^\gamma} = \|f\|_{C^n} + \sum_{|\alpha|=n}\sup_{x\neq y}\frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|}{|x-y|^\eta}. \tag{7}$$

Clearly, these seminorms can be used to define a general Fréchet space $C^\gamma(U,\mathbb{R}^d)$ for each domain $U\subset\mathbb{R}^d$. Moreover, if U is bounded, then the space $C^\gamma(U,\mathbb{R}^d)$ is even a Banach space and we can directly work with the norm $\|\cdot\|_{C^\gamma}$.
With $C_0^\gamma(U,\mathbb{R}^d)$ we define the subset of $C^\gamma(\mathbb{R}^d,\mathbb{R}^d)$ comprising all functions vanishing outside U. Again, if U is bounded, then the space $C_0^\gamma(U,\mathbb{R}^d)$ is even a Banach space.

Conversely, the flow Y is viewed as a φ-Hölder continuous W-valued random process with $W=C^m(\mathbb{R}^d,\mathbb{R}^d)$. We now give the precise definition of the corresponding space $C_\varphi([0,1],W_U)$ for $W_U:=C^m(U,\mathbb{R}^d)$ and a general open set $U\subset\mathbb{R}^d$. It is the subspace of continuous W_U-valued paths $y:[0,1]\to W_U$ that is induced by the seminorms $y\mapsto\|\zeta\cdot y\|_\varphi$ ($\zeta\in C_c^\infty$), where

$$\|y\|_\varphi = \sup_{0\le t\le 1}\|y\|_{C^m} + \sup_{0\le s<t\le 1}\frac{\|y_t-y_s\|_{C^m}}{\varphi(t-s)}.$$

If U is bounded, $C_\varphi([0,1],W_U)$ is again a Banach space and we can directly work with the norm $\|\cdot\|_\varphi$. Let us remark that the seminorms

$$y\mapsto \left\|y\big|_U\right\|_\varphi \qquad (U\subset\mathbb{R}^d \text{ bounded and open})$$

are an alternative family of seminorms that induce the space $C_\gamma([0,1],W)$.

Assumptions related to the Kunita SDE

When analyzing the Kunita flow the driving signal $(X_t)_{t\in[0,1]}$ is a continuous Wiener process attaining elements from the Fréchet space $C^\gamma(\mathbb{R}^d,\mathbb{R}^d)$, with $\gamma>2$. Since $C^\gamma(\mathbb{R}^d,\mathbb{R}^d)$ is a subspace of $C(\mathbb{R}^d,\mathbb{R}^d)$ the distribution of the process is characterized by the covariance kernels $K^{i,j}$ ($i,j=1,\dots,d$) given by

$$K^{i,j}(\xi_1,\xi_2) = \mathbb{E}X_1^i(\xi_1)X_1^j(\xi_2) \qquad (\xi_1,\xi_2\in\mathbb{R}^d). \tag{8}$$

Now fix $\gamma' = n + \eta \geq 2$ with $n \in \mathbb{N}$ and $\eta \in [0, 1)$, and note that by Kunita [8, Thm. 3.1.2 and 3.1.3] the assumption that the process X attains values in $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$ for some $\gamma > \gamma'$ can be rephrased in terms of the covariance kernel: Equivalently one may demand that for $i = 1, \dots, d$ the covariance kernel $(\xi_1, \xi_2) \mapsto K^{i,i}(\xi_1, \xi_2)$ is n-times continuously differentiable with respect to each variable ξ_1 and ξ_2 and satisfies for a fixed $\delta > \eta$, all compact sets $K \subset \mathbb{R}^d$ and any multiindex α with $|\alpha| = n$ that

$$\sup_{\substack{\xi_1,\xi_1',\xi_2,\xi_2' \in K \\ \xi_1 \neq \xi_1', \xi_2 \neq \xi_2'}} \frac{\partial^\alpha_{\xi_1}\partial^\alpha_{\xi_2} K^{i,i}(\xi_1,\xi_2) - \partial^\alpha_{\xi_1}\partial^\alpha_{\xi_2} K^{i,i}(\xi_1',\xi_2) - \partial^\alpha_{\xi_1}\partial^\alpha_{\xi_2} K^{i,i}(\xi_1,\xi_2') + \partial^\alpha_{\xi_1}\partial^\alpha_{\xi_2} K^{i,i}(\xi_1',\xi_2')}{|\xi_1 - \xi_1'|^\delta |\xi_2 - \xi_2'|^\delta} < \infty\,.$$

The conditions that we have imposed so far imply unique existence of a solution to (1) up to some possibly finite explosion time ([8], Theorem 3.4.7), and we shall in the following assume Assumption (A1):

(A1) X generates a flow Y that does not explode in finite time. See [8, Theorem 4.5.1] for sufficient conditions for (A1).

For $\xi_0, \xi_1 \in \mathbb{R}^d$ we set

$$J(\xi_0, \xi_1) = \inf_{(\gamma_t)} \int \langle \dot\gamma_t, K^{-1}(\gamma_t, \gamma_t)\,\dot\gamma_t \rangle_{\mathbb{R}^d}\,\mathrm{d}t,$$

where the infimum is taken over all absolutely continuous paths $(\gamma_t) \in C([0,1], \mathbb{R}^d)$ (in the sense that it admits a representation $\gamma_t - \gamma_0 = \int_0^t \dot\gamma_s\,\mathrm{d}s$ for a $\dot\gamma \in L^1([0,1], \mathbb{R}^d)$) with $\gamma_i = \xi_i$ $(i = 0, 1)$ and $\dot\gamma_t \in \mathrm{im}K(\gamma_t, \gamma_t)$ for Lebesgue almost all $t \in [0, 1]$. In the definition of J, $K^{-1}(\gamma_t, \gamma_t)\,\dot\gamma_t$ denotes the element $v \in \mathbb{R}^d$ with minimal Euclidean norm that satisfies $K(\gamma_t, \gamma_t)\,v = \dot\gamma_t$.

When dealing with Kunita flows we furthermore impose the following assumption.

(A2) One has for all bounded sets $U \subset \mathbb{R}^d$

$$\lim_{\kappa \to \infty} \inf_{\xi_0 \in U} \inf_{\xi_1 \in B(0,\kappa)^c} J(\xi_0, \xi_1) = \infty.$$

Agenda

The article is organized as follows. In the next section we show that in the case the driving noise X is compactly supported the flow Y generated by (1) admits a Wong-Zakai approximation. In Section 3 we use a localization argument in order to lift the implications of Section 2 to general X. In this section the main results - a support theorem and a large deviation principle for stochastic flows generated by Kunita SDE are proved.

2 Stochastic flows viewed as Wong-Zakai solutions

Throughout this section we confine ourselves on the analysis of evolutions starting in points $\xi \in U$ with $U \subset \mathbb{R}^d$ being a bounded, open and convex set. Moreover, we restrict attention to $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$-valued Wiener processes X that have compact support. We also confine ourselves with sufficiently smooth, i.e. $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$, and compactly supported drift terms a. Hence both the driving process X and the solution Y can be viewed as Banach space-valued processes and we choose $V_0 = C_0^\gamma(\tilde U, \mathbb{R}^d)$ and $W_U = C^m(U, \mathbb{R}^d)$ as state spaces for X and Y respectively. In

this section, we write V and W for V_0 and W_U, respectively, for ease of notation.
Again we assume that $\gamma > m+2$. Later on we will relate the general case to this restricted setting via a localization argument.
As we see below the stochastic flow Y is closely related to the solution operator of the following Young differential equation

$$\mathrm{d}y_t = f(y_t)\,\mathrm{d}[x_t + t\,a], \quad y_0 = \mathrm{id}_U, \tag{9}$$

where

$$f : W_U \to L(V; W_U), y \mapsto T_y \quad \text{and} \quad T_y(x)(\xi) = x(y(\xi)), \quad \text{for } y \in W_U, x \in V, \xi \in U.$$

By Picard's theorem [11, Thm. 1.28], for f being a Lip(α)-function for a $\alpha > 1$ in the sense of [11, Def. 1.21] the differential equation possesses a unique solution $I(x)$ for any function $x : [0,1] \to V_0$ of bounded variation.
The Lipschitz property of f is also needed for the application of Lyons' universal limit theorem and therefore we verify it in the following lemma. Before proceeding with the statement of the lemma let us briefly introduce some prerequisites on the projective tensor product of Banach spaces:
For two Banach spaces V_1 and V_2 we denote by $V_1 \otimes_a V_2$ the algebraic tensor product of V_1 and V_2 and we let $V_1 \otimes V_2$ denote the *projective tensor product*, that is the completion of $V_1 \otimes_a V_2$ under the *projective tensor norm* $|\cdot|_{V_1 \otimes V_2}$ given by

$$|v|_{V_1 \otimes V_2} = \inf \sum_{i=1}^{n} |f_i|_{V_1} |g_i|_{V_2},$$

where the infimum is taken over all representations

$$v = \sum_{i=1}^{n} f_i \otimes g_i \qquad (n \in \mathbb{N}, f_i \in V_1, g_i \in V_2 \text{ for } i = 1, \ldots, n).$$

For Banach spaces $V_1, \ldots, V_n, W$ we denote by $L(V_1, \ldots, V_n; W)$ the set of multilinear functionals g with

$$|g|_{L(V_1,\ldots,V_n;W)} := \sup_{|v_1|,\ldots,|v_n| \le 1} |g(v_1, \ldots, v_n)|_W < \infty$$

Note that any g admits a canonical extension $\bar{g} : L(V_1 \otimes \cdots \otimes V_n; W)$ which has equal norm. The extension will in the following also be denoted by g.

Lemma 2.1. *The operator f is of class* $\mathrm{Lip}(\gamma - m)$ *in the sense of [11, Def. 1.21] and its Fréchet derivatives are*

$$\mathrm{d}^j f : W \to L(W^j, V; W) = y \mapsto T_y^j, \qquad \text{for } j \in \mathbb{N} \cap [0, \gamma - m],$$

where

$$T_y^j(z_1, \ldots, z_j; x)(\xi) := \nabla^j x(y(\xi))(z_1(\xi), \ldots, z_j(\xi)), \qquad \text{for } x \in V, z_1, \ldots, z_j \in W, \xi \in \mathbb{R}^d$$

and $\nabla^j x$ denotes the j-th total derivative of x.

Proof. We write $\gamma - m = n + \eta$ with $n \in \mathbb{N}_0$ and $\eta \in (0,1]$. It suffices to show that, for $j \leq n$, $w_1, w_2 \in W$, $w \in W^{\otimes j}$ and $v \in V$,

$$\big|T^j_{w_2}(w;v) - \sum_{l=0}^{n-j} \frac{1}{l!} T^{j+l}_{w_1}(w \otimes (y_2 - y_1)^{\otimes l}; v)\big|_W \leq \mathrm{const}\, |w_2 - w_1|_W^{\gamma-m-j}\, |w|_{W^{\otimes j}}\, |v|_V. \tag{10}$$

Now a multiindex $\alpha = i_1 \dots i_m$ of length less than or equal to m is associated to $\bar{w} = e_{i_1} \otimes \cdots \otimes e_{i_m}$, where $(e_1, \dots, e_d)$ denotes the standard basis of $\mathbb{R}^d$. We compute, for $\xi \in U$,

$$\partial^\alpha T^j_{w_2}(w;v)(\xi) = \partial^\alpha \nabla^j v(w_2(\xi))(w(\xi)) = \nabla^{j+m} v(w_2(\xi))(w(\xi) \otimes \bar{w}).$$

Similarly one can rewrite the other terms in (10) and one gets

$$\begin{aligned}
&\partial^\alpha \big[T^j_{w_2}(w;v) - \sum_{l=0}^{n-j} \frac{1}{l!} T^{j+l}_{w_1}(w \otimes (y_2 - y_1)^{\otimes l}; v)\big](\xi) \\
&\quad = \nabla^{j+m} v(w_2(\xi))(w(\xi) \otimes \bar{w}) - \sum_{l=0}^{n-j} \frac{1}{l!} \nabla^{j+m+l} v(w_1(\xi))(w(\xi) \otimes \bar{w} \otimes (w_2(\xi) - w_1(\xi))^{\otimes l}).
\end{aligned}$$

By Taylor, the k-th coordinate ($k \in \{1, \dots, d\}$) of the latter expression coincides with the k-th coordinate of

$$\frac{1}{(n-j)!} \big(\nabla^{m+n} v(w'_k) - \nabla^{m+n} v(w_1(\xi))\big)(w(\xi) \otimes \bar{w} \otimes (w_2(\xi) - w_1(\xi))^{\otimes(n-j)}),$$

where w'_k is an appropriate element on the segment joining $w_1(\xi)$ and $w_2(\xi)$. Moreover,

$$\begin{aligned}
&\Big|\frac{1}{(n-j)!} \big(\nabla^{m+n} v(w'_k) - \nabla^{m+n} v(w_1(\xi))\big)(w(\xi) \otimes \bar{w} \otimes (w_2(\xi) - w_1(\xi))^{\otimes(n-j)})\Big| \\
&\qquad \leq \frac{1}{(n-j)!} |v|_V\, |w_2 - w_1|_W^\eta\, |w|_{W^{\otimes j}}\, |w_2 - w_1|_W^{n-j} \\
&\qquad = \frac{1}{(n-j)!} |w_2 - w_1|_W^{\gamma-m-j}\, |w|_{W^{\otimes j}}\, |v|_V.
\end{aligned}$$

Consequently,

$$\Big|\partial^\alpha \big[T^j_{w_2}(w;v) - \sum_{l=0}^{n-j} \frac{1}{l!} T^{j+l}_{w_1}(w \otimes (y_2 - y_1)^{\otimes l}; v)\big](\xi)\Big| \leq \frac{d}{(n-j)!} |w_2 - w_1|_W^{\gamma-m-j}\, |w|_{W^{\otimes j}}\, |v|_V$$

which proves (10). □

We now show the link between the solution operator $I(x)$ of (9) and the flow Y. Consider the dyadic piecewise linear interpolation $X(n)$ of X with breakpoints $D_n = [0,1] \cap 2^{-n}\mathbb{Z}$ and the corresponding solution $Y(n) = I(X(n))$ of (9) with respect to the interpolated Brownian noise $X(n)$. If $(Y(n))_{n \in \mathbb{N}}$ converges almost surely in $C_\varphi([0,1], W_U)$ to a random process Y^{WZ}, we call Y^{WZ} the Wong-Zakai solution of (9) with control X.

By Lemma 2.1 and [4, Thm. 1.5], X induces a Wong-Zakai solution Y^{WZ} and the solution admits a support theorem and a large deviation principle, see [4, Thm. 1.7]. As we show next the Wong-Zakai solution Y^{WZ} of (9) is precisely the flow generated by (1). In other words the infinite dimensional stochastic flow admits a Wong-Zakai approximation and we immediately have a support theorem and a large deviation principle.

Theorem 2.2. *Let X be a V_0 valued Wiener process. Then the Kunita type Stratonovich SDE (1) with driving noise X possesses a continuous pathwise unique solution Y coinciding with the Wong-Zakai solution Y^{WZ} of (9).*

Proof. The existence and the pathwise uniqueness of the solution to the Kunita SDE (1) follows from Theorem 3.4.1 in [8].
If X was a finite dimensional Brownian noise the statement would boil down to a standard Wong-Zakai theorem. Our approach is to truncate the noise to a finite dimensional one and then let the number of dimensions grow to infinity.
Let $X = \sum_{i=1}^{\infty} \xi^i e_i$, with (ξ^i) being independent real-valued Wiener processes and (e_i) being a complete orthonormal system of the reproducing kernel Hilbert space H_1 of X_1. Consider the truncated sum $X^{(m)} = \sum_{i=1}^{m} \xi^i e_i$ $(m \in \mathbb{N})$ and its dyadic piecewise-linear interpolations $X(n,m)$.
Additionally, we denote by $Y^{\mathrm{Strat}}(m)$ and $Y^{\mathrm{Strat}}(n,m)$ the solutions to the Stratonovich differential equation with driving noise $X^{(m)}$ and $X(n,m)$, respectively.
The Wong-Zakai solutions of (9) with respect to controls $X^{(m)}$ and X are denoted by $Y^{\mathrm{WZ}}(m)$ and Y^{WZ} respectively.
The following diagram represents the identification steps in the proof. On the left hand side we find the solutions of the (possibly truncated) SDE (1), while on the right hand side are the Wong-Zakai solutions of (9) with respect to the various truncations of the Wiener process X:

$$\begin{tikzcd}
Y \arrow[r, phantom, "="] & Y^{\mathrm{WZ}} \\
Y^{\mathrm{Strat}}(m) \arrow[u, "(5)"', "m\to\infty"] \arrow[r, phantom, "\overset{(4)}{=}"] & Y^{\mathrm{WZ}}(m) \arrow[u, "(3)"', "m\to\infty"] \\
Y^{\mathrm{Strat}}(n,m) \arrow[u, "(2)"', "n\to\infty"] \arrow[r, phantom, "\overset{(1)}{=}"] & I(X(n,m)) \arrow[u, "(3)"', "n\to\infty"]
\end{tikzcd}$$

(1) We obviously have $I(X(n,m) = Y^{\mathrm{Strat}}(n,m)$ since $X(n,m)$ is of bounded variation and therefore the Stratonovich and the random differential equations coincide trivially.

(2) The convergence $Y^{\mathrm{Strat}}(n,m) \to Y^{\mathrm{Strat}}(m)$ for $n \to \infty$ is a standard Wong-Zakai type result - see for instance Theorem 7.2 on p. 410 of [6] for a very general version. The convergence mode is the standard convergence in probability uniformly in time on compact sets (sometimes referred to as the UCP convergence).

(3) According to Theorem 1.7 in [4], the smoothness of the Banach space-valued Wiener processes $X^{(m)}$ and X implies, that the equation (9) with control $X^{(m)}$ admits a Wong-Zakai solution, that is $(I(X(n,m)))_{n\in\mathbb{N}}$ converges in $C^{\varphi}([0,1], W_U)$ to a random process $Y^{\mathrm{WZ}}(m)$, which itself converges to Y^{WZ} when letting m tend to infinity.

(4) Clearly, identifying the elements of the sequences implies the identification of the corresponding limits.

(5) The convergence of the solutions $Y^{\mathrm{Strat}}(m)$ of the truncated SDE to the solution Y of the original SDE (1) follows with a standard argument involving the Burkholder-Davis-Gundy

inequality and the Gronwall Lemma. It can also be verified as a consequence of the much more general result in [8] (Theorem 5.3.6).

□

Remark 2.3. The convergences on the right hand side of the diagram is where the Lyons' universal limit theorem enters the picture. The Wong-Zakai solutions of the random differential equation (9) with controls X, $X^*(m)$ and $X^*(n,m)$ are the first coordinates of the solutions of the corresponding rough path equations whose Itô maps are continuous according to this celebrated result.

3 Applications to Kunita's stochastic differential equation

In the following, $X = (X_t)_{t\in[0,1]}$ denotes a general $C^{\bar\gamma}(\mathbb{R}^d,\mathbb{R}^d)$-valued Wiener processes with $\bar\gamma > 2$, that satisfies the non-explosion assumptions (A1) and (A2). Moreover, we choose $W = C^m(\mathbb{R}^d,\mathbb{R}^d)$ with $m \in \mathbb{N}_0$ and $m+2 < \bar\gamma$, and let Y denote a solution to the Kunita SDE (1). We need to apply a localization in order to be able to apply the results from before.

Localization

For each $n \in \mathbb{N}$, we choose a real-valued compactly supported function $\zeta_n : \mathbb{R}^d \to \mathbb{R}$ with $\zeta_n|_{B(0,n)} = 1$, and we call

$$X^{\zeta_n} := D_{\zeta_n}(X) := (\zeta_n \cdot X_t)_{t\in[0,1]}$$

the ζ_n-localized control. Suppose ζ_n is supported on a bounded and open set $\tilde U_n$. Then X^{ζ_n} is a $C_0^{\bar\gamma}(\tilde U_n,\mathbb{R}^d)$-valued Wiener process with reproducing kernel Hilbert space $H^{\zeta_n} = D_{\zeta_n}(H)$. Note that the embedding D_{ζ_n} has norm one and admits for each $h \in H^{\zeta_n}$ an element $\bar h \in H$ with

$$D_{\zeta_n}(\bar h) = h \text{ and } |\bar h|_H = |h|_{H^{\zeta_n}},$$

see [2, Thm. 3.7.3]. The corresponding ζ_n-localized solutions are denoted by Y^{ζ_n} that is

$$Y_t^{\zeta_n}(\xi) = \int_0^t X^{\zeta_n}(Y_s^{\zeta_n}(\xi), \circ\, \mathrm{d}s) + \int_0^t a^{\zeta_n}(Y_s^{\zeta_n}(\xi))\, \mathrm{d}s, \qquad \xi \in \mathbb{R}^d, t \in [0,1], \tag{11}$$

where we briefly write $a^{\zeta_n} = D_{\zeta_n}(a)$.

Furthermore, we choose a bounded, open and convex set U and we consider the U-confined solution Y^U of Y which is $Y^U := Y|_U$. Similarly, we write $Y^{\zeta_n,U}$ for the U-confined solution of Y^{ζ_n}. By the previous section $Y^{\zeta_n,U}$ is just the Wong-Zakai solution in the sense of [4] and, in particular, Theorem 1.7 of [4] is applicable for $Y^{\zeta_n,U}$.

So far we have not yet defined the Itô-map $I(x)$ for general elements $x \in H$. We use the same localization technique: We denote by $I_{\zeta_n,U}(x) = y^{\zeta_n,U}$ the U-confined solution of the Young analog of (11) and we set

$$I_U(x) := \lim_{n\to\infty} I_{\zeta_n,U}(x)$$

provided the limit exists. Finally, $I(x)$ is defined as the W-valued path whose restrictions to all spatial domains U coincide with $I_U(x)$. As the following theorem shows, Assumption (A2) guarantees that the approximating solutions do not explode and that the approximations again become identical for sufficiently large n. Hence, $I(x)$ is well-defined for all $x \in H$.
Note that the following statement also comprises Theorem 1.3.

Theorem 3.1. *Under Assumption (A2) the following statement is true:*

- *For $x \in H$ and a bounded, open and convex set $U \subset \mathbb{R}^d$, the sequence $(I_{\zeta_n,U}(x))_{n\in\mathbb{N}}$ becomes constant for sufficiently large n so that $I(x)$ (see above) is well-defined. Moreover, for $\xi \in \mathbb{R}^d$ and $y = I(x)$, the path $y_\cdot(\xi)$ is absolutely continuous and satisfies*

$$y_t(\xi) = \xi + \int_0^t \dot{x}_s(y_s(\xi))\, \mathrm{d}s + \int_0^t a(y_s(\xi))\, \mathrm{d}s.$$

- *For a path $\gamma \in C([0,1], \mathbb{R}^d)$, we consider*

$$A_\gamma = \{y \in C([0,1], W) : \gamma_t = y_t(\gamma_0) \text{ for all } t \in [0,1]\}, \tag{12}$$

If $A_\gamma \cap I(H)$ is non-empty, then there exists a unique minimal element $f \in H$ with $I(f) \in A_\gamma$. It admits the following representation: For almost all $t \in [0,1]$, $\dot\gamma_t - a(\gamma_t)$ lies in the image of $K(\gamma_t, \gamma_t)$ and we denote by $K^{-1}(\gamma_t,\gamma_t)\,(\dot\gamma_t - a(\gamma_t))$ the element $v \in \mathbb{R}^d$ with minimal Euclidean norm that satisfies

$$K(\gamma_t,\gamma_t)\, v = \dot\gamma_t - a(\gamma_t).$$

Then $f \in H$ satisfies

$$\dot f_t(\xi) = K(\xi,\gamma_t)\, K^{-1}(\gamma_t,\gamma_t)\,(\dot\gamma_t - a(\gamma_t)), \qquad \text{for } \xi \in \mathbb{R}^d \text{ and a.a. } t \in [0,1]$$

and

$$|f|_H^2 = \int_0^1 \langle \dot\gamma_t - a(\gamma_t), K^{-1}(\gamma_t,\gamma_t)\,(\dot\gamma_t - a(\gamma_t))\rangle_{\mathbb{R}^d}\, \mathrm{d}t.$$

Proof. Fix $x \in H$, a bounded, open and convex set U, $\xi \in U$ and $n \in \mathbb{N}$, and consider $y^{\zeta_n,U} = I_{\zeta_n,U}(x)$, $\gamma_t := y_t^{\zeta_n,U}(\xi)$ and $\bar x_t = x_t^{\zeta_n} + t a^{\zeta_n}$ for $t \in [0,1]$. By definition of the Young integral, one has

$$\gamma_t - \gamma_0 = \lim_{m\to\infty} \sum_{i=0}^{m-1} \bar x_{t_i^m, t_{i+1}^m}(\gamma_{t_i}) = \lim_{m\to\infty} \int_0^t \sum_{i=0}^{m-1} \mathbb{1}_{[t_i^m, t_i^m+1)}(s)\, \dot{\bar x}_s(\gamma_{t_i^m})\, \mathrm{d}s, \tag{13}$$

where the limit is taken over a sequence of partitions $0 = t_0^m < \cdots < t_m^m = t$ with mesh tending to zero. Due to the localization, the path (γ_t) does not leave an appropriately chosen open and bounded set $\tilde U$ and there exists a constant $c = c(\tilde U, H_1)$ such that

$$|\dot{\bar x}_s(\xi_1) - \dot{\bar x}_s(\xi_2)| \le (c\, |\dot x_s|_{H_1} + |a|_{C^\gamma(U,\mathbb{R}^d)})\, |\xi_1 - \xi_2|, \qquad \text{for } s \in [0,1] \text{ and } \xi_1, \xi_2 \in \tilde U.$$

Hence, by the continuity of (γ_t) the integrand in (13) converges for Lebesgue almost all times s to $\dot{\bar x}_t^{\zeta_n}(y_t^{\zeta_n}(\xi))$ and it is straight forward to verify via Lebesgue's dominated convergence theorem that $y^{\zeta_n}(\xi)$ is absolutely continuous with $\dot y_t^{\zeta_n,U}(\xi)=\dot\gamma_t=\dot x_t^{\zeta_n}(y_t^{\zeta_n,U}(\xi))+a^{\zeta_n}(y_t^{\zeta_n,U}(\xi))$ for almost all t.

Next, we derive a lower bound for $|x|_H$ solely by looking at (γ_t). Let us denote by $\langle\cdot,\cdot\rangle$ the inner product on $H_1^{\zeta_n}$. We use that for $\phi\in H_1^{\zeta_n}$ and $v,w\in\mathbb{R}^d$

$$\langle\phi, K_n(\cdot,w)\,v\rangle=\langle\phi(w),v\rangle_{\mathbb{R}^d}, \tag{14}$$

where K_n denotes the covariance kernel that belongs to the Gaussian random element $X_1^{\zeta_n}$. Choose $v\in\mathbb{R}^d$ orthogonal to $\operatorname{im} K_n(w,w)$. Since $K_n(w,w)$ is symmetric, we have $v\in\ker K_n(w,w)$ so that by (14)

$$\langle K_n(\cdot,x)\,v,K_n(\cdot,x)\,v\rangle=\langle K_n(x,x)\,v,v\rangle_{\mathbb{R}^d}=0.$$

Hence, $\langle\phi(w),v\rangle_{\mathbb{R}^d}=\langle\phi,K_n(\cdot,w)\,v\rangle=0$, and $\phi(w)$ is in the image of $K_n(w,w)$. Therefore, $K_n^{-1}(\gamma_t,\gamma_t)\,(\dot\gamma_t-a(\gamma_t))$ is well defined in the sense explained above. We set $\dot{\bar\gamma}_t=\dot\gamma_t-a(\gamma_t)$ and choose

$$\dot g_t(\xi')=K_n(\xi',\gamma_t)\,K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t,\qquad\text{for }\xi'\in\mathbb{R}^d.$$

Then again by (14)

$$\begin{aligned}|\dot g_t|^2_{H_1^{\zeta_n}}&=\langle K_n(\cdot,\gamma_t)\,K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t,K_n(\cdot,\gamma_t)\,K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t)\rangle\\&=\langle\dot{\bar\gamma}_t,K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t\rangle_{\mathbb{R}^d}.\end{aligned}\tag{15}$$

On the other hand, we have for almost all $t\in[0,1]$ that

$$|\dot x_t|_{H_1}\ge|\dot x_t^{\zeta_n}|_{H_1^{\zeta_n}}\ge\frac{\langle\dot x_t^{\zeta_n},\dot g_t\rangle}{|\dot g_t|_{H_1^{\zeta_n}}}=\frac{\langle\dot x_t^{\zeta_n}(\gamma_t),K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t\rangle_{\mathbb{R}^d}}{|\dot g_t|_{H_1^{\zeta_n}}}=|\dot g_t|_{H_1^{\zeta_n}},\tag{16}$$

since $\dot{\bar\gamma}_t=\dot x_t^{\zeta_n}(\gamma_t)$ a.e. Note that the second inequality is a strict inequality, whenever $\dot g_t\neq\dot x_t^{\zeta_n}$ (and $\dot{\bar\gamma}_t=\dot x_t^{\zeta_n}(\gamma_t)$). Consequently,

$$|x|_H^2\ge\int_0^1\langle\dot{\bar\gamma}_t,K_n^{-1}(\gamma_t,\gamma_t)\,\dot{\bar\gamma}_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t\tag{17}$$

and the inequality is a strict inequality, whenever $\dot x_t^{\zeta_n}\neq\dot g_t$ on a set of nonvanishing Lebesgue measure or equivalently, if $x^{\zeta_n}\neq g$.

Next, fix $n_0\in\mathbb{N}$ and suppose that $y^{\zeta_n}(\xi)$ leaves $B(0,n_0)$ for some $n\ge n_0$ and $\xi\in U$. We set $\gamma_t=y_t^{\zeta_n}(\xi)$ and denote by T the first exit time. We get

$$\begin{aligned}|x|_H^2&\ge\int_0^T\langle\dot\gamma_t,K_n^{-1}(\gamma_t,\gamma_t)\,\dot\gamma_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t=\int_0^T\langle\dot\gamma_t,K^{-1}(\gamma_t,\gamma_t)\,\dot\gamma_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t\\&\ge\inf_{\xi_1\in U}\inf_{\xi_2\in B(0,n_0)^c}J(\xi_1,\xi_2).\end{aligned}$$

Here we used that K and K_n coincide on $B(0,n)\supset B(0,n_0)$. Thus, Assumption (A2) guarantees that for given $x\in H$, the corresponding solutions $(y^{\zeta_n})_{n\in\mathbb{N}}$ coincide on U for all sufficiently large $n\in\mathbb{N}$. This proves the first statement of the lemma.

Now the second statement follows easily from part one. Indeed, all estimates (15), (16), and (17) remain valid for the nonlocalized controls. $\square$

The support theorem for flows

Theorem 3.2. *Under Assumptions (A1) and (A2), one has*

$$\operatorname{range}_{C_\varphi([0,1],W)}(Y) = \overline{I(H)},$$

where the closure is taken in $C_\varphi([0,1],W)$ and $\varphi:(0,1]\to(0,\infty)$ is increasing and satisfies (3).

Proof. For open and bounded sets, we interpret $\|\cdot\|_{C^m(U,\mathbb{R}^d)}$ as semi-norms on $C^m(\mathbb{R}^d,\mathbb{R}^d)$. Since these semi-norms generate the topology of the Fréchet space, it suffices to prove the support theorem for a fixed set U and the U-confined solution Y^U.
By [4] and Theorem 2.2, the localized solution $Y^{\zeta_n,U}$ admits the following representation of its range (note that $H^{\zeta_n} = D_{\zeta_n}(H)$):

$$\operatorname{range}(Y^{\zeta_n,U}) = \overline{I_{\zeta_n,U}(H)}. \tag{18}$$

We start with proving $I_U(H)\subset \operatorname{range}(Y^U)$. We fix $f = I_U(h)$ with $h\in H$, and note that

$$\mathcal{I}(f) := \{f_t(\xi) : \xi\in U, t\in[0,1]\} \tag{19}$$

is a bounded set by Theorem 3.1 and Assumption (A2). Next, we fix $n\in\mathbb{N}$ large enough such that $\mathcal{I}(f)\subset B(0,n-\varphi(1))$. Then, $f = I_{\zeta_n,U}(h)\in I_{\zeta_n,U}(H)$ and we conlude with (18) that, for each $\varepsilon\in(0,1)$,

$$\mathbb{P}(\|Y^U - f\|_\varphi < \varepsilon) = \mathbb{P}(\|Y^{\zeta_n,U} - f\|_\varphi < \varepsilon) > 0.$$

For the opposite direction, we fix $f\in \operatorname{range}(Y^U)$ and let $n\in\mathbb{N}$ again be large enough such that $\mathcal{I}(f)\subset B(0,n-\varphi(1))$. Then, for all $\varepsilon\in(0,1)$, one has

$$0 < \mathbb{P}(\|Y^U - f\|_\varphi < \varepsilon) = \mathbb{P}(\|Y^{\zeta_n,U} - f\|_\varphi < \varepsilon).$$

By equation (18), one has $f\in\overline{I_{\zeta_n,U}(H)}$ and we fix a sequence $(f_k)_{k\in\mathbb{N}} = (I_{\zeta_n,U}(h_k))$ such that $\|f_k - f\|_\varphi$ converges to 0 and is always less than one. Then $I_{\zeta_n,U}(h_k) = I_U(h_k)$ and $f\in\overline{I_U(H)}$. □

The large deviation principle for flows

Theorem 3.3. *Under Assumptions (A1) and (A2), the family $(Y^\varepsilon : \varepsilon>0)$ of processes satisfies in $C_\varphi([0,1],W)$ a large deviation principle with good rate function*

$$J(y) = \inf\Big\{\frac{1}{2}|h|_H^2 : h\in H \text{ with } I(h) = y\Big\},$$

where $\varphi:(0,1]\to(0,\infty)$ is increasing and satisfies (3).

Before we prove the theorem, we recall the consequences of Theorem 1.7 of [4] and the previous section. Let U be a bounded, open and convex set, let $\varepsilon>0$ and $n\in\mathbb{N}$, and consider the U-confined flows $Y^{\zeta_n,U,\varepsilon}$ which one gets when applying a ζ_n-localization and choosing $X^\varepsilon = \varepsilon\cdot X$ as control. Then the family $(Y^{\zeta_n,U,\varepsilon} : \varepsilon>0)$ satisfies a LDP with good rate function

$$J^{\zeta_n,U}(y) = \inf\{\frac{1}{2}|D_{\zeta_n}(h)|^2_{H^{\zeta_n}} : h\in H \text{ with } I_{\zeta_n,U}(h) = y\}$$

Lemma 3.4. *If $n \in \mathbb{N}$ satisfies $\mathcal{I}(y) \in B(0,n)$, then*

$$J^{\zeta_n,U}(y) = \inf\Big\{\frac{1}{2}|h|_H^2 : h \in H \text{ with } I_U(h) = y\Big\} =: J^U(y). \tag{20}$$

Proof. For given y, we choose $h \in H$ with $I_{\zeta_n,U}(h) = y$. Then there exists $\bar{h} \in H$ with $D_{\zeta_n}(\bar{h}) = D_{\zeta_n}(h)$ and $|\bar{h}|_H = |D_{\zeta_n}(h)|_{H^{\zeta_n}}$. By assumption, $I_U(\bar{h}) = I_{\zeta_n,U}(h)$ so that the right hand side of (20) is not larger than the left hand side. The opposite direction follows similarly when recalling that $D_{\zeta_n} : H \to H^{\zeta_n}$ is a contraction. □

Proof of Theorem 3.3. By the Dawson-Gärtner theorem [3, Thm. 4.6.1] it suffices to consider a cofined solution Y^U for a bounded, open and convex set U, and to prove a large deviation principle for the family $(Y^{U,\varepsilon} : \varepsilon > 0)$ with rate function J^U. We start with proving the upper bound. Let $A_0 \subset C_\varphi([0,1], W_U)$ be a closed subset and set

$$m = \inf_{y\in A_0} J^U(y),$$

if the infimum is finite, and otherwise we pick $m > 0$ arbitrarily large. By Assumption (A2) and Theorem 3.1, there exists a sufficiently large $n \in \mathbb{N}$ such that for any $y \in C_\varphi([0,1], W_U)$, one of the following two properties is valid:

- $\overline{\mathcal{I}(y)} \subset B(0,n)$ or
- $J^U(y) \geq m$,

where $\mathcal{I}$ is as in (19). We consider

$$A = \{y \in C_\varphi([0,1], W_U) : y \in A_0 \text{ or } \overline{\mathcal{I}(y)} \not\subset B(0,n)\}.$$

Certainly, A is a closed set including A_0 and one has

$$\mathbb{P}(Y^{U,\varepsilon} \in A_0) \leq \mathbb{P}(Y^{U,\varepsilon} \in A) = \mathbb{P}(Y^{\zeta_n,U,\varepsilon} \in A).$$

Consequently, the LDP stated above implies that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^2 \log \mathbb{P}(Y^{U,\varepsilon} \in A) \leq -\inf_{y\in A} J^{\zeta_n,U}(y). \tag{21}$$

On the one hand a path $y \in A$ with $\overline{\mathcal{I}(y)} \not\subset B(0,n)$ satisfies $J^{\zeta_n,U}(y) \geq m$ by assumption. On the other hand, one has for a path $y \in A_0$ with $\overline{\mathcal{I}(y)} \subset B(0,n)$

$$J^{\zeta_n,U}(y) = J^U(y) \geq m$$

by the previous lemma. This proves the upper bound.

For the converse direction, we choose $f = I(h) \in I(H)$, fix $\varepsilon > 0$, and set $A = B_{C_\varphi([0,1],W_U)}(f, \varepsilon)$. Again, we and choose $n \in \mathbb{N}$ such that, for all $y \in A$, $\mathcal{I}(y) \subset B(0,n)$. Then

$$\mathbb{P}(Y^{U,\varepsilon} \in A) = \mathbb{P}(Y^{\zeta_n,U,\varepsilon} \in A),$$

and

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^2 \log \mathbb{P}(Y^{U,\varepsilon} \in A) \geq -J^{\zeta_n,U}(y).$$

Since $\mathcal{I}(y) \subset B(0,n)$, we conclude again by Lemma 3.4 that $J^{\zeta_n,U}(y) = J^U(y)$. □

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