The high resolution vector quantization problem with Orlicz norm distortion

by

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Summary. We derive a high-resolution formula for the quantization problem under Orlicz norm distortion. In this setting, the optimal point density solves a variational problem which comprises a function $g : \mathbb{R}_+ \to [0, \infty)$ characterizing the quantization complexity of the underlying Orlicz space. Moreover, asymptotically optimal codebooks induce a tight sequence of empirical measures. The set of possible accumulation points is characterized and in most cases it consists of a single element. In that case, we find convergence as in the classical setting.

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1 Introduction

For $d \in \mathbb{N}$, consider an $\mathbb{R}^d$-valued random vector X (*the original*) defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote by $\mu = \mathcal{L}(X)$ the law of X.

We consider the quantization problem, that is for a given natural number $N \in \mathbb{N}$ and a loss function $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ we ask for a codebook $\mathcal{C} \subset \mathbb{R}^d$ consisting of at most N elements and minimizing the average loss

$$\mathbb{E}\rho(X, \mathcal{C}),$$

where $\rho(x, A) = \inf_{y \in A} \rho(x, y)$, for all $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. The quantization problem arises naturally when discretizing analog signals, and it first gained practical importance

in the context of pulse-code-modulation. Research on it started in the 1940's and one finds numerous articles dedicated to the study of this problem in the engineering literature. For an overview on these developments, one may consult Gray & Neuhoff (1998) (see also Cover & Thomas (1991) and Gersho & Gray (1992)). The quantization problem is also related to numerical integration (Pagès, Pham & Printems 2004), and, more recently, the mathematical community became attracted by the field. In the last years a number of new publications appeared treating finite dimensional as well as infinite dimensional signals (see for instance Graf & Luschgy (2000), Gruber (2004) for vector quantization and Luschgy & Pagès (2004), Dereich, Fehringer, Matoussi & Scheutzow (2003) for functional quantization).

In this article, we consider asymptotic properties of the quantization problem when the size N of the codebook tends to infinity, the *high-resolution quantization problem.* First asymptotic formulae for vector quantization were found by Zador (1966, 1982) and Bucklew & Wise (1982).

In the classical setting (*norm based distortion*), one considers

$$\rho(x,y) = \|x-y\|^p$$

for some norm $\|\cdot\|$ and a moment $p \geqslant 1$. As long as the distribution $\mu := \mathcal{L}(X)$ has thin tails (in an appropriate sense) one can describe asymptotically optimal codebooks via an optimal point density function: the empirical measures associated to optimal codebooks $\mathcal{C}(N)$ of size N

$$\frac{1}{N}\sum_{\hat{x}\in\mathcal{C}} \delta_{\hat{x}}$$

converge to a continuous probability measure that has density proportional to $\left(\frac{d\mu_c}{d\lambda^d}\right)^{d/(d+p)}$. The density will be called *optimal point density.* Here and elsewhere, λ^d denotes d-dimensional Lebesgue measure and μ_c denotes the absolutely continuous part of μ w.r.t. λ^d. When considering point densities we will always assume that μ_c does not vanish. Optimal codebooks for the uniform distribution can then be used to define asymptotically optimal codebooks for general X: roughly speaking, one partitions the space $\mathbb{R}^d$ into appropriate cubes and chooses in each cube an optimal codebook for the uniform distribution of appropriate size (according to the optimal point density). In particular, the non-continuous part of μ has no effect on the asymptotic problem. The concept of a point density will play a crucial role in the following discussion. Its importance in the classical setting was firstly conjectured by Lloyd and Gersho (see Gersho (1979)). First rigorous proofs are due to Bucklew (1984). For a recent account on the theory of high resolution quantization and point density functions one may consult the monograph by Graf & Luschgy (2000).

Nowadays, the asymptotic quantization problem is well understood for loss functions ρ that are shift invariant and look locally like a power of a norm based distance, that is

$$\rho(x,y) = \rho(x-y,0) \quad \text{and} \quad \rho(x,0) = \|x\|^p + o(\|x\|^p) \text{ as } x \to 0$$

for some norm $\|\cdot\|$ on $\mathbb{R}^d$ and a power $p \geqslant 1$ (see Delattre, Graf, Luschgy & Pagès (2004)). Here and thereafter o and $\mathcal{O}$ denote the Landau symbols. For all these distortion measures

one regains the same optimal point density as for the corresponding norm based distortion measures. Since the optimal point density only depends on the behavior of $\rho(\cdot, 0)$ close to 0, quantization schemes based on this density may show bad performance for moderate N. This will occur, when for the original X and the approximation $\hat{X}$ the distances $\rho(X, \hat{X})$ and $\|X - \hat{X}\|^p$ differ significantly.

As a generalization of the above setting, we suggest to use Orlicz norms as a measure for the loss inferred when approximating the original X by the closest point in a codebook $\mathcal{C}$. Orlicz norms offer a wide range of distortion measures. Let us introduce the main notations.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a monotonically increasing, left continuous function with $\lim_{t \downarrow 0} \varphi(t) = 0$. Note that this implies that φ is lower semicontinuous. We assume that $\varphi \neq 0$, let $E = (\mathbb{R}^d, \|\cdot\|)$ denote an arbitrary Banach space, and denote by $d(\cdot, \cdot)$ the associated distance. For any $\mathbb{R}^d$-valued r.v. Z, the Orlicz norm $\|\cdot\|_\varphi$ is defined as

$$\|Z\|_\varphi = \inf\Big\{t \geqslant 0 : \mathbb{E}\,\varphi\Big(\frac{\|Z\|}{t}\Big) \leqslant 1\Big\},$$

with the convention that the infimum of the empty set is equal to infinity. Actually, the left continuity of φ together with monotone convergence imply that the infimum is attained, whenever the set is nonempty. We set

$$L^\varphi(\mathbb{P}) = \{Z : Z\ \mathbb{R}^d\text{-valued r.v. with } \|Z\|_\varphi < \infty\}.$$

Note that $\|\cdot\|_\varphi$ defines a norm on $L^\varphi(\mathbb{P})$ when φ is convex, whereas otherwise, the triangle inequality does not hold. For our analysis we do not require that φ be convex. Nevertheless, with slight misuse of notation, we will allow ourselves to call $\|\cdot\|_\varphi$ an Orlicz norm. Choosing $\varphi(t) = t^p$, $p \geqslant 1$, yields the usual $L^p(\mathbb{P})$-norm, which will be denoted by $\|\cdot\|_p$.

For $N \geqslant 1$, we consider the *quantization error* given by

$$\delta(N|X, \varphi) = \inf_{\hat{X}} \|X - \hat{X}\|_\varphi,$$

where the infimum is taken over all r.v.'s $\hat{X}$ (*reconstructions*) satisfying the range constraint $|\text{range}\,(\hat{X})| \leqslant N$.

Let us compare the Orlicz norm distortion with the classical setting. Suppose that $\hat{X}$ is an optimal N-point quantizer under the distortion $\rho(x, y) = f(\|x - y\|)$, where f is a left continuous and strictly increasing function with $f(0) = \lim_{t \downarrow 0} f(t) = 0$. Then one can easily verify that $\hat{X}$ is also an optimal N-point quantizer in the Orlicz norm setting when choosing $\varphi(t) = f(t)/\Delta$ and $\Delta = \mathbb{E}[\rho(X, \hat{X})]$. Thus optimal quantizers in the classical setting correspond to optimal quantizers in the Orlicz-norm setting. As we will see later, in most cases each choice of Δ (or φ) leads to a unique optimal point density function. Thus the optimal point density function in the classical setting is replaced by a whole family of densities: good descriptions are now obtained by choosing the parameter Δ accordingly. We believe that the optimal Orlicz point density is a favourable description of good codebooks for moderate N. Let us illustrate this in the case where $f(t) = \varphi(t) = \exp(t) - 1$ (meaning that $\Delta = 1$). Whereas the Orlicz norm point density attains rather

large values even at points where the density $\frac{d\mu_c}{d\lambda^d}$ is very small (see Example 1.3), the classical setting neglects the growth of f and one retrieves the optimal density of the norm based distortion to the power 1.

In our notation, the asymptotics under the classical L^p-norm distortion reads as follows (see Graf & Luschgy (2000)): Let $p \geqslant 1$, U denote a uniformly distributed r.v. on $[0,1)^d$, and set

$$q(E,p) = \inf_{N \geqslant 1} N^{1/d}\, \delta(N|U,p).$$

If for some $\tilde{p} > p$, $\mathbb{E}\|X\|^{\tilde{p}} < \infty$ (*concentration assumption*), then

$$\lim_{N\to\infty} N^{1/d}\, \delta(N|X,p) = q(E,p) \left\| \frac{d\mu_c}{d\lambda^d} \right\|^{1/p}_{L^{d/(d+p)}(\mathbb{R}^d)}, \tag{1}$$

where μ_c denotes the absolutely continuous part of μ.

The analysis of the Orlicz norm setting is based on the concept of a point allocation density. We shall see that an optimal point allocation density is given as a minimizer of a variational problem. Unfortunately, in this context the minimization problem cannot be solved in closed form. We will prove the existence and a dual characterization of the solution. The quantity $q(E,p)$ corresponds in the general setting to a convex decreasing function $g : (0,\infty) \to [0,\infty)$ which may be defined via

$$g(\eta) = \lim_{N\to\infty} \inf_{\mathcal{C}(N)} \mathbb{E}\, \varphi\big((N/\eta)^{1/d}\, d(U, \mathcal{C}(N))\big), \tag{2}$$

where the infima are taken over all finite sets $\mathcal{C}(N) \subset \mathbb{R}^d$ with $|\mathcal{C}(N)| \leqslant N$ and U denotes a uniformly distributed r.v. on the unit cube $[0,1)^d$. Moreover, we set $g(0) = \liminf_{\eta\downarrow 0} g(\eta) \in [0,\infty]$. The function g depends on the Banach space and the Orlicz norm. It will be analyzed in Section 2. g can be represented as a particular integral in the situations where $E = l^d_\infty$ or $E = l^2_2$. If φ induces the $L^p(\mathbb{P})$-norm (i.e. $\varphi(t) = t^p$), then due to (1)

$$g(\eta) = q(E,p)^p\, \eta^{-p/d}.$$

If the measure μ is not compactly supported, our analysis relies on a concentration property. As in the $L^p(\mathbb{P})$-setting, this can be done by assuming the finiteness of an integral $\mathbb{E}\Psi(\|X\|)$ for some function Ψ satisfying a growth condition (Condition (G), see Definition 7.2). If φ induces the usual $L^p(\mathbb{P})$-norm, admissible functions Ψ are, for instance, all functions $\Psi(t) = t^{\tilde{p}}$ with $\tilde{p} > p$, and one recovers the classical result.
Let us state the main theorem.

Theorem 1.1. *Assume that Ψ satisfies the growth condition (G) and that $\mathbb{E}\Psi(\|X\|) < \infty$. Then*

$$\lim_{N\to\infty} N^{1/d}\, \delta(N|X,\varphi) = I^{1/d},$$

where I is the finite minimal value in the point allocation problem. It is given by

$$I = \inf_{\xi} \int \xi(x)\, dx, \tag{3}$$

where the infimum is taken over all non-negative Lebesgue integrable functions ξ with

$$\int_{\mathbb{R}^d} g(\xi(x))\, d\mu_c(x) \leqslant 1.$$

Alternatively, one can represent I by the dual formula

$$I = \sup_{\kappa>0} \frac{1}{\kappa}\Big(\int_{\mathbb{R}^d} \bar{g}\big(\frac{\kappa}{h(x)}\big)\, d\mu_c(x) - 1\Big),$$

where $\bar{g}(t) = \inf_{\eta>0}[g(\eta) + \eta t]$ and $h(x) = \frac{d\mu_c}{d\lambda^d}(x)$. Moreover, one has $I > 0$ if and only if

$$\mu_c(\mathbb{R}^d) \sup_{t \geqslant 0} \varphi(t) > 1,$$

where we use the convention that $0 \cdot \infty = 0$.

We keep I as the minimal value in the point allocation problem given by (3). If I is strictly bigger than 0, we denote by $\mathcal{M}$ the set of probability measures on the Borel sets of $\mathbb{R}^d$ associated to the minimizers of the point allocation problem, i.e.,

$$\mathcal{M} = \Big\{\nu : \frac{d\nu}{d\lambda^d} = \bar{\xi},\ \int_{\mathbb{R}^d} g(I\,\bar{\xi}(x))\, d\mu_c(x) = 1,\ \int_{\mathbb{R}^d} \bar{\xi}(x)\, dx = 1\Big\}. \tag{4}$$

Theorem 1.2. *Assume that $I \in (0, \infty)$ and denote by $\mathcal{C}(N)$, $N \in \mathbb{N}$, asymptotically optimal codebooks of size N, that is*

$$\limsup_{N\to\infty} N^{1/d}\, \|d(X, \mathcal{C}(N))\|_\varphi \leqslant I^{1/d}.$$

Then the empirical measures ν_N given by

$$\nu_N = \frac{1}{N} \sum_{\hat{x}\in\mathcal{C}(N)} \delta_{\hat{x}}$$

form a tight sequence of probability measures, and any accumulation point of $(\nu_N)_{N\in\mathbb{N}}$ lies in $\mathcal{M}$. If g is strictly convex, then the set $\mathcal{M}$ contains exactly one measure ν, and

$$\lim_{N\to\infty} \nu_N = \nu \qquad \textit{weakly.}$$

As an example we present implications of our results for the standard normal distribution under a particular φ growing exponentially fast:

Example 1.3. Let μ denote the standard normal distribution, $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ and $\varphi : [0, \infty) \to [0, \infty)$, $x \mapsto \exp(x) - 1$. Then $\Psi(x) = \exp(x^{3/2})$ satisfies the growth condition (G) (see Example 7.3), and Theorem 1.1 is thus applicable. Moreover, (see Remark 2.2 and Lemma 2.3)

$$g(\eta) = 2\int_0^{1/2} \varphi(t/\eta)\, dt = \eta e^{1/(2\eta)} - 2\eta - 1.$$

Note that g ist strictly convex, so that there has to exist a unique normalized optimal point density $\bar{\xi}$. In order to approximate the optimal value for κ, we used numerical methods to obtain $\kappa \approx 0.699$. Due to (23) and (24), the optimal point density is given by

$$\xi(x) = (-g')^{-1}\big(\frac{\kappa}{h(x)}\big), \qquad x \in \mathbb{R},$$

and $I = \int_{\mathbb{R}} \xi(x)\, dx \approx 2.88$. Next, elementary calculus gives

$$(-g')^{-1}(t) = \frac{1}{2}\frac{1}{\log(t) - \log(2\log t) + o(1)} \quad \text{as } t \to \infty$$

so that the normalized point density $\bar{\xi} = \xi/I$ satisfies

$$\bar{\xi}(x) \sim \frac{1}{I}\frac{1}{x^2} \quad \text{as } |x| \to \infty.$$

Hence, $\bar{\xi}$ decays to zero much more slowly than in the classical setting.

The article is outlined as follows. In Section 2, we begin with an analysis of the function g. In Section 3 we construct asymptotically good codebooks based on a given point allocation measure. Up to this stage, we are restricting ourselves to absolutely continuous measures with compact support. In Section 4, we turn things around and prove a lower bound based on a given point density measure. This bound implies the lower bound in Theorem 1.1 and proves a part of Theorem 1.2. The estimates of Sections 3 and 4 lead to the variational problem characterising the point density, and this is treated in Section 5. In the last two sections, we treat the upper bounds in the quantization problem for singular and non-compactly supported measures. In particular, we derive a concentration analog which guarantees that the quantization error is of order $\mathcal{O}(N^{-1/d})$. Finally, we combine the estimates and prove the general upper bound in Theorem 1.1.

It is convenient to use the symbols $\sim$, $\lesssim$ and $\approx$. We write $f \sim g$ iff $\lim \frac{f}{g} = 1$, while $f \lesssim g$ stands for $\limsup \frac{f}{g} \leqslant 1$. Finally, $f \approx g$ means $0 < \liminf \frac{f}{g} \leqslant \limsup \frac{f}{g} < \infty$.

2 First estimates for the uniform distribution

In this section, X denotes a uniformly distributed r.v. on $[0,1)^d$. For $\eta > 0$ and $N \geqslant 1$, we consider

$$f_N(\eta) := \inf_{\mathcal{C}(N)} \mathbb{E}\,\varphi\Big((N/\eta)^{1/d} \min_{\hat{x}\in\mathcal{C}(N)} \|X - \hat{x}\|\Big), \tag{5}$$

where the infimum is taken over all codebooks $\mathcal{C}(N) \subset \mathbb{R}^d$ of size $\lfloor N \rfloor$. Here and elsewhere, $\lfloor N \rfloor$ denotes the largest integer smaller or equal to N. By a straightforward argument, the lower semicontinuity of φ implies that the function

$$\big(\mathbb{R}^d\big)^{\lfloor N \rfloor} \ni (\hat{x}_1, \ldots, \hat{x}_{\lfloor N \rfloor}) \mapsto \mathbb{E}\,\varphi\Big((N/\eta)^{1/d} \min_{i=1,\ldots,\lfloor N \rfloor} \|X - \hat{x}_i\|\Big) \in [0,\infty)$$

is also lower semicontinuous. In the minimization problem (5), it suffices to allow for codebook entries that are elements of a sufficiently large compact set. So the lower semicontinuity implies the existence of an optimal codebook. We usually denote by $\hat{X}^{(N)}$ or $\hat{X}^{(N,\eta)}$ an optimal reconstruction attaining at most N different values, that is $\hat{X} = \hat{X}^{(N)}$ is a minimizer of

$$\mathbb{E}\,\varphi\Big((N/\eta)^{1/d}\,\|X - \hat{X}\|\Big),$$

among all r.v.'s satisfying the range constraint $|\mathrm{range}\,(\hat{X})| \leqslant N$. Now define the function g by

$$g(\eta) = \inf_{N \geqslant 1} f_N(\eta), \qquad \eta > 0.$$

We start with a derivation of the structural properties of g. In particular, we show the validity of (2).

Theorem 2.1. *The function $g : \mathbb{R}_+ \to [0, \infty)$ is monotonically decreasing and convex, and satisfies $\lim_{\eta\to\infty} g(\eta) = 0$ and $\lim_{\eta\downarrow 0} g(\eta) = \sup_{t \geqslant 0} \varphi(t)$. Moreover, for $\eta > 0$,*

$$\lim_{N\to\infty} \mathbb{E}\,\varphi\Big((N/\eta)^{1/d}\,\|X - \hat{X}^{(N)}\|\Big) = g(\eta). \tag{6}$$

We will sometimes use the convention $g(0) = \lim_{\eta\downarrow 0} g(\eta)$. Note that $g(0)$ is finite iff φ is bounded. Moreover, we set $f_N(\eta) = \infty$ for $N \in [0, 1)$.

Remark 2.2. In general, computing the function g explicitly constitutes a hard problem. However, as for the classical L^q-norm distortion, one can calculate g when $E = \mathbb{R}^d$ is endowed with supremum-norm, and in the case where E is the two dimensional Euclidean space. In such cases, the same lattice quantizers can be used to construct asymptotically optimal codebooks and to compute the function g. The case of the suremum-norm is trivial since the unit ball is space filling.

Lemma 2.3. *Let U be uniformly distributed on a centered regular hexagon V in $\mathbb{R}^2$ having unit area, and assume that E is the 2-dimensional Euclidean space. One has*

$$g(\eta) = \mathbb{E}\,\varphi\big(\eta^{-1/2}\,\|U\|\big).$$

The proof is similar as in the classical setting and therefore ommitted, see Graf & Luschgy (2000, Theorem 8.15) and Fejes Tóth (1972).

In order to prove Theorem 2.1, we use the inequality below. It is essentially a consequence of the self similarity of X.

Proposition 2.4. *Let $M \in \{k^d : k \in \mathbb{N}\}$, $\eta_2 > \eta_1 > 0$ and let $\eta = \alpha\eta_1 + \beta\eta_2$ be a convex combination of η_1 and η_2. Then for any $N \geqslant 1$ one has*

$$f_N(\eta) \leqslant a(M)\, f_{N_1/M}(\eta_1) + b(M)\, f_{N_2/M}(\eta_2), \tag{7}$$

where

- $N_1 = \frac{\eta_1}{\eta} N$ *and* $N_2 = \frac{\eta_2}{\eta} N$,

- $b(M) = \lfloor \beta M \rfloor / M$ *and* $a(M) = 1 - b(M)$.

Additionally, one has for $N \geqslant 1$ *and* $\eta > 0$,

$$f_N(\eta) \leqslant f_{N/M}(\eta). \tag{8}$$

Proof. Fix $N \in \mathbb{N}$ and let $\eta_1, \eta_2, \alpha, \beta, N_1, N_2, M = k^d$ be as in the proposition. Let $C_1 = [0, 1/k)^d$. We decompose the cube $[0,1)^d$ into an appropriate union $\bigcup_{i=1}^M C_i$ of disjoint sets $C_1, \ldots, C_M$, where each set $C_2, \ldots, C_M$ is a translate of C_1. Moreover, let X_i denote a uniformly distributed r.v. on C_i. Since $\mathcal{U}([0,1)^d) = \frac{1}{M} \sum_{i=1}^M \mathcal{U}(C_i)$, one has, in analogy to Lemma 4.14 in Graf & Luschgy (2000),

$$f_N(\eta) = \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \|X - \hat{X}^{(N)}\| \Big) \leqslant \frac{1}{M} \sum_{i=1}^M \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \|X_i - \hat{X}_i^{(\tilde{N}_i)}\| \Big)$$

for any $[1, \infty)$-valued sequence $(\tilde{N}_i)_{i=1,\ldots,M}$ with $\sum_i \tilde{N}_i \leqslant N$. Here, $\hat{X}_i^{(\tilde{N}_i)}$ denotes an optimal quantizer for X_i among all quantizers attaining at most $\tilde{N}_i$ different values. Since the distributions $\mathcal{U}(C_i)$ can be transformed into $\mathcal{U}(C_1)$ through a translation, one obtains

$$f_N(\eta) \leqslant \frac{1}{M} \sum_{i=1}^M \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \|X_1 - \hat{X}_1^{(\tilde{N}_i)}\| \Big).$$

Self similarity ($\mathcal{L}(X_1) = \mathcal{L}(\frac{1}{k} X)$) then implies that

$$f_N(\eta) \leqslant \frac{1}{M} \sum_{i=1}^M \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \frac{1}{k} \, \|X - \hat{X}^{(\tilde{N}_i)}\| \Big).$$

Note that the assertion of the proposition is trivial if $N_1/M < 1$, so that we may assume $N_2/M \geqslant N_1/M \geqslant 1$. We now choose for $\lfloor \beta M \rfloor$ indices $\tilde{N}_i = N_2/M$ and for $M - \lfloor \beta M \rfloor$ indices $\tilde{N}_i = N_1/M$. Then $\sum_i \tilde{N}_i \leqslant N$, so that

$$\begin{aligned} f_N(\eta) &\leqslant a(M)\, \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \frac{1}{k} \, \|X - \hat{X}^{(N_1/M)}\| \Big) \\ &\quad + b(M)\, \mathbb{E}\, \varphi\Big((N/\eta)^{1/d} \, \frac{1}{k} \, \|X - \hat{X}^{(N_2/M)}\| \Big) \\ &= a(M)\, f_{N_1/M}(\eta_1) + b(M)\, f_{N_2/M}(\eta_2), \end{aligned}$$

where $b(M) = \lfloor \beta M \rfloor / M$ and $a(M) = 1 - b(M)$. Analogously, setting $\tilde{N}_i = N/M$ for $i = 1, \ldots, M$, we obtain that $f_N(\eta) \leqslant f_{N/M}(\eta)$. □

Proof of Theorem 2.1. First we prove that for arbitrary $\eta > 0$,

$$g(\eta) \leqslant \limsup_{N \to \infty} f_N(\eta) \leqslant g_-(\eta). \tag{9}$$

Fix $\varepsilon > 0$ and choose $\eta_0 \in (0, \eta)$ so that $g(\eta_0) \leqslant g_-(\eta) + \varepsilon/2$. Moreover, fix $N_0 \geqslant 1$ with $f_{N_0}(\eta) \leqslant g(\eta_0) + \varepsilon/2$. For $N \geqslant N_0$, we decompose N into $N = N_0\, k^d + \tilde{N}$, where $k = k(N) \in \mathbb{N}$ and $\tilde{N} = \tilde{N}(N) \in \mathbb{N}_0$ are chosen so that $N < (k+1)^d N_0$. Then

$$\begin{aligned} f_N(\eta) &= \mathbb{E}\,\varphi\Big((N/\eta)^{1/d}\,\|X - \hat{X}^{(N)}\|\Big) \\ &\leqslant \mathbb{E}\,\varphi\Big(\big(N_0\, k^d/(N_0\, k^d \eta/N)\big)^{1/d}\,\|X - \hat{X}^{(N_0\, k^d)}\|\Big) = f_{N_0\, k^d}(\eta N_0\, k^d/N), \end{aligned}$$

and inequality (8) implies that for $M = M(N) = k^d$:

$$f_N(\eta) \leqslant f_{N_0}(\eta N_0 M/N).$$

Note that $N_0 k^d \leqslant N < N_0(k+1)^d$, hence: $\lim_{N\to\infty} \eta N_0 k^d/N = \eta$. Consequently, there exists $N_1 \geqslant N_0$ such that for all $N \geqslant N_1$ one has: $\eta N_0 M/N \geqslant \eta_0$, and

$$f_N(\eta) \leqslant f_{N_0}(\eta N_0 M/N) \leqslant f_{N_0}(\eta_0) \leqslant g_-(\eta) + \varepsilon$$

for all $N \geqslant N_1$. Since $\varepsilon > 0$ was arbitrary statement (9) follows.

We now prove that g_- is convex. Let $\eta_2 > \eta_1 > 0$ and let $\eta = \alpha\eta_1 + \beta\eta_2$ be a convex combination of η_1 and η_2 and suppose that $g_-(\eta_1)$ is finite. Fix $k \in \mathbb{N}$ and let $M = k^d$, $a(M)$ and $b(M)$ be as in Proposition 2.4. Moreover, for given $N \in \mathbb{N}$ we let $N_1 = N_1(N)$ and $N_2 = N_2(N)$ be as in the previous proposition. Then inequality (7) implies that

$$f_N(\eta) \leqslant a(M)\, f_{N_1/M}(\eta_1) + b(M)\, f_{N_2/M}(\eta_2).$$

Therefore, formula (9) and the left continuity of g_- give

$$g(\eta) \leqslant \limsup_{N\to\infty} f_N(\eta) \leqslant a(M)\, g_-(\eta_1) + b(M)\, g_-(\eta_2).$$

Recall that $M \in \{k^d : k \in \mathbb{N}\}$ was arbitrary. Since $\lim_{M\to\infty} a(M) = \alpha$ and $\lim_{M\to\infty} b(M) = \beta$ we conclude that

$$g(\eta) \leqslant \alpha\, g_-(\eta_1) + \beta\, g_-(\eta_2).$$

For the general statement, observe that

$$g_-(\eta) = \lim_{\delta\downarrow 0} g(\eta - \delta) \leqslant \limsup_{\delta\downarrow 0}\big[\alpha\, g_-(\eta_1 - \delta) + \beta\, g_-(\eta_2 - \delta)\big] = \alpha\, g_-(\eta_1) + \beta\, g_-(\eta_2).$$

Consequently, g_- is convex, and a fortiori it is continuous. Therefore, the functions g and g_- coincide, which proves (6).

It remains to prove the asymptotic statements for g. First note that

$$g(\eta) \leqslant f_1(\eta) \leqslant \mathbb{E}\,\varphi(\eta^{-1/d}\,\|X\|) \leqslant \varphi(\eta^{-1/d} \sup_{x\in[0,1)^d} \|x\|) \;\longrightarrow\; 0$$

as $\eta \to \infty$. On the other hand, one has for $\eta > 0$, $\varepsilon > 0$ and $N \in \mathbb{N}$,

$$\begin{aligned} f_N(\eta) &= \mathbb{E}\,\varphi\Big((N/\eta)^{1/d}\,\|X - \hat{X}^{(N)}\|\Big) \\ &\geqslant \mathbb{E}\Big[1_{\{\|X - \hat{X}^{(N)}\| \,\geqslant\, \varepsilon/N^{1/d}\}}\,\varphi\Big((N/\eta)^{1/d}\,\|X - \hat{X}^{(N)}\|\Big)\Big] \\ &\geqslant (1 - N\,\lambda^d(B(0, \varepsilon/N^{1/d})))\,\varphi(\varepsilon/\eta^{1/d}) \\ &= (1 - \lambda^d(B(0,\varepsilon)))\,\varphi(\varepsilon/\eta^{1/d}), \end{aligned}$$

so that

$$g(\eta) \geqslant (1-\lambda^d(B(0,\varepsilon)))\,\varphi(\varepsilon/\eta^{1/d}) \underset{\eta\downarrow 0}{\longrightarrow} (1-\lambda^d(B(0,\varepsilon)))\sup_{t\geqslant 0}\varphi(t).$$

Since $g(\eta) \leqslant \sup_{t\geqslant 0}\varphi(t)$ and $\varepsilon > 0$ was arbitrary, the assertion follows. □

3 The upper bound (1st step)

In this section, we consider an original X with law $\mu \ll \lambda^d$. Moreover, we assume that μ is compactly supported and fix $l > 0$ large enough so that $\mu(C) = 1$ for $C = [-l, l)^d$.

Based on a given integrable function $\xi : \mathbb{R}^d \to [0,\infty)$ (*point density*), we define codebooks and control their efficiency.

Proposition 3.1. *There exist codebooks $\mathcal{C}(N)$, $N \geqslant 1$, such that $\lim_{N\to\infty}\frac{1}{N}|\mathcal{C}(N)| = \|\xi\|_{L^1(\mathbb{R}^d)}$ and*

$$\limsup_{N\to\infty}\mathbb{E}\,\varphi(N^{1/d}\,d(X,\mathcal{C}(N))) \leqslant \int g(\xi(x))\,d\mu(x).$$

Proof. It suffices to prove the assertion for functions ξ that are uniformly bounded away from 0 on C. If this is not the case, one can consider $\bar{\xi} = \xi + \varepsilon\,1_C$ for some $\varepsilon > 0$. Then the statement says that there exist codebooks $\mathcal{C}^\varepsilon(N)$, $N \geqslant 1$, with $|\mathcal{C}^\varepsilon(N)| \sim N(\|\xi\|_{L^1(\mathbb{R}^d)} + \varepsilon\,\lambda^d(C))$ satisfying

$$\limsup_{N\to\infty}\mathbb{E}\,\varphi(N^{1/d}\,d(X,\mathcal{C}^\varepsilon(N))) \leqslant \int g(\bar{\xi}(x))\,d\mu(x) \leqslant \int g(\xi(x))\,d\mu(x),$$

and a diagonalization argument for $\varepsilon \downarrow 0$ proves the general assertion.

Fix $m \in \mathbb{N}$, let $C_1 = [0, l/2^m)^d$ and decompose C into a finite disjoint union

$$C = \bigcup_{i=1}^M C_i,$$

where $M = 2^{(m+1)d}$ and $C_2, \ldots, C_M$ are translates of C_1. For $i = 1, \ldots, M$, we denote by X_i a uniformly distributed r.v. on C_i, and let $\mu^m = \sum_{i=1}^M \mu(C_i)\,\mathcal{U}(C_i)$. Moreover, we let

$$h_m = \frac{d\mu^m}{d\lambda^d} = \sum_{i=1}^M \frac{\mu(C_i)}{\lambda^d(C_i)}\cdot 1_{C_i},$$

and denote by ν the measure given by $\nu(A) = \int_A \xi\,d\lambda^d$, $A \in \mathcal{B}(\mathbb{R}^d)$.

We introduce the codebooks of interest. For some fixed $\kappa > 0$, let

$$\tilde{\mathcal{C}}(N) = (\kappa\,N^{-1/d}\mathbb{Z}^d)\cap C, \qquad N \geqslant 1,$$

and let $\mathcal{C}_i(N)$ denote codebooks of size $N_i = N_i(N) = N\,\nu(C_i)$ minimizing $\mathbb{E}\,\varphi(N^{1/d}\,d(X_i,\mathcal{C}_i(N)))$. We consider the efficiency of the codebooks

$$\mathcal{C}(N) = \tilde{\mathcal{C}}(N) \cup \bigcup_{i=1}^M \mathcal{C}_i(N), \qquad N \geqslant 1.$$

First, note that

$$|\tilde{\mathcal{C}}(N)| \leqslant \big(1 + 2\frac{l}{\kappa} N^{1/d}\big)^d \sim \Big(\frac{2l}{\kappa}\Big)^d N, \qquad N \to \infty,$$

hence:

$$|\mathcal{C}(N)| \lesssim \Big(\|\xi\|_{L^1(\mathbb{R}^d)} + \Big(\frac{2l}{\kappa}\Big)^d\Big) N, \qquad N \to \infty. \tag{10}$$

It remains to estimate the expectation $\mathbb{E}\,\varphi(N^{1/d} d(X, \mathcal{C}(N)))$ for large $N \geqslant 1$. Observe that $d(x, \tilde{\mathcal{C}}(N)) \leqslant \kappa\, N^{-1/d} \sup_{x\in[0,1)^d} \|x\|$ for all $x \in C$ so that

$$\begin{aligned} &\Big|\mathbb{E}\,\varphi\big(N^{1/d}\, d(X, \mathcal{C}(N))\big) - \int \varphi\big(N^{1/d}\, d(x, \mathcal{C}(N))\big)\, d\mu^m(x)\Big| \\ &\quad = \Big|\int \varphi\big(N^{1/d}\, d(x, \mathcal{C}(N))\big)(h(x) - h_m(x))\, dx\Big| \leqslant \varphi(c\kappa)\, \|h - h_m\|_{L^1(\mathbb{R}^d)}, \end{aligned} \tag{11}$$

where $c = \sup_{x\in[0,1)^d} \|x\|$ is a universal constant. Moreover,

$$\begin{aligned} \int \varphi(N^{1/d}\, d(x, \mathcal{C}(N))\, d\mu^m(x) &= \sum_{i=1}^M \mu(C_i)\, \mathbb{E}\,\varphi\big(N^{1/d}\, d(X_i, \mathcal{C}(N))\big) \\ &\leqslant \sum_{i=1}^M \mu(C_i)\, \mathbb{E}\,\varphi\big(N^{1/d}\, d(X_i, \mathcal{C}_i(N))\big). \end{aligned}$$

Now let U denote a $\mathcal{U}([0,1)^d)$-distributed r.v. Due to the optimality assumption on the choice of $\mathcal{C}_i(N)$, a translation and scaling then yields

$$\mathbb{E}\,\varphi\big(N^{1/d}\, d(X_i, \mathcal{C}_i(N))\big) = \mathbb{E}\,\varphi\Big(N^{1/d} \frac{l}{2^{m+1}}\, \|U - \hat{U}^{(N_i)}\|\Big),$$

where $\hat{U}^{(N_i)}$ denotes a reconstruction minimizing the latter expectation among all r.v. with a range of size N_i. Next, rewriting the previous expectation as

$$\mathbb{E}\,\varphi\Big(N^{1/d} \frac{l}{2^{m+1}}\, \|U - \hat{U}^{(N_i)}\|\Big) = f_{N_i}(\nu(C_i)/\lambda^d(C_i)),$$

it follows that

$$\int \varphi\big(N^{1/d}\, d(x, \mathcal{C}(N))\big)\, d\mu^m(x) \leqslant \sum_{i=1}^M \mu(C_i)\, f_{N_i}(\nu(C_i)/\lambda^d(C_i)).$$

As $N \to \infty$, every N_i, $i = 1, \ldots, M$, converges to ∞, and one has

$$\sum_{i=1}^M \mu(C_i)\, f_{N_i}(\nu(C_i)/\lambda^d(C_i)) \;\longrightarrow\; \sum_{i=1}^M \mu(C_i)\, g\Big(\frac{\nu(C_i)}{\lambda^d(C_i)}\Big) = \int g(\xi_m(x))\, d\mu(x),$$

where $\xi_m = \sum_{i=1}^M \frac{\nu(C_i)}{\lambda^d(C_i)} \cdot 1_{C_i}$. Putting everything together yields

$$\limsup_{N\to\infty} \mathbb{E}\,\varphi\big(N^{1/d}\, d(X,\mathcal{C}(N))\big) \leqslant \int g(\xi_m(x))\, d\mu(x) + \kappa\, \|h - h_m\|_{L^1(\mathbb{R}^d)} \sup_{x\in[0,1)^d} \|x\|.$$

The function ξ_m converges to ξ as $m \to \infty$ in λ^d-a.a. points x (see Cohn (1980), Theorem 6.2.3). Recall that by construction ξ_m is bounded from below on C, and hence dominated convergence gives

$$\lim_{m\to\infty} \int g(\xi_m(x))\, d\mu(x) = \int g(\xi(x))\, d\mu(x).$$

Analogously, h_m converges to h in λ^d-a.a. points x and due to Scheffé's theorem (see Billingsley (1979), Theorem 16.11) h_m converges to h in $L^1(\mathbb{R}^d)$ as $m \to \infty$.

For arbitrary $\varepsilon > 0$, we can choose $\kappa > 0$ sufficiently large to ensure that the size of $\mathcal{C}(N)$ (see (10)) satisfies

$$|\mathcal{C}(N)| \lesssim (1+\varepsilon)\, \|\xi\|_{L^1(\mathbb{R}^d)}\, N.$$

Finally, it remains to pick $m \in \mathbb{N}$ sufficiently large so that

$$\limsup_{N\to\infty} \mathbb{E}\,\varphi\big(N^{1/d}\, d(X,\mathcal{C}(N))\big) \leqslant \int g(\xi(x))\, d\mu(x) + \varepsilon,$$

and the general statement then follows from a diagonalization argument. □

4 The lower bound

From now on, let X be an arbitrary random vector on $\mathbb{R}^d$ with law μ. In this section, we change our viewpoint: for an index set $\mathbb{I} \subset [1,\infty)$ with $\sup \mathbb{I} = \infty$, we consider arbitrary finite codebooks $\mathcal{C}(N)$, $N \in \mathbb{I}$, and ask for asymptotic lower bounds of

$$\mathbb{E}\,\varphi(N^{1/d}\, d(X,\mathcal{C}(N)))$$

as $N \to \infty$. Our computations are based on the assumption that the empirical measures

$$\nu^N = \frac{1}{N} \sum_{\hat{x}\in\mathcal{C}(N)} \delta_{\hat{x}}, \qquad N \in \mathbb{I},$$

associated to $\mathcal{C}(N)$ converge vaguely to some locally finite measure ν on $\mathbb{R}^d$.

Proposition 4.1. *Letting ν_c denote the absolutely continuous part of ν, one has*

$$\liminf_{N\to\infty} \mathbb{E}\,\varphi\big(N^{1/d}\, d(X,\mathcal{C}(N))\big) \geqslant \int g\big(\frac{d\nu_c}{d\lambda^d}\big)\, d\mu_c(x).$$

Proof. It suffices to prove that for an arbitrary $l > 0$:

$$\liminf_{N\to\infty} \mathbb{E}\,\varphi\big(N^{1/d}\, d(X,\mathcal{C}(N))\big) \geqslant \int_{[-l,l)^d} g\big(\frac{d\nu_c}{d\lambda^d}\big)\, d\mu_c(x).$$

Indeed, the assertion then follows immediately by monotone convergence. For a given $m \in \mathbb{N}$, just as in the proof of the upper bound, we decompose the set $C = [-l, l)^d$ into a disjoint union $C = \bigcup_{i=1}^M C_i$, where $M = 2^{m+1}$, $C_1 = [0, l/2^m)^d$, and $C_2, \dots, C_M$ are translates of C_1. Again we consider the measure $\mu^m = \sum_{i=1}^M \mu_c(C_i)\,\mathcal{U}(C_i)$ and the density

$$h_m = \sum_{i=1}^M \frac{\mu_c(C_i)}{\lambda^d(C_i)} \cdot 1_{C_i}.$$

Analogously, we let $\xi_m = \sum_{i=1}^M \frac{\nu(\bar{C}_i)}{\lambda^d(C_i)} \cdot 1_{C_i}$. For some fixed $\kappa > 0$, we extend the codebooks $\mathcal{C}(N)$ to

$$\mathcal{C}^{(1)}(N) = \mathcal{C}(N) \cup \big((\kappa N^{-1/d}\mathbb{Z}^d) \cap C\big).$$

Then, just as in (11), one has

$$\Big|\int_C \varphi(N^{1/d}\, d(x, \mathcal{C}^{(1)}(N)))\, d\mu_c(x) - \int_C \varphi(N^{1/d}\, d(x, \mathcal{C}^{(1)}(N)))\, d\mu^m(x)\Big| \leqslant \varphi(c\,\kappa)\, \|h - h_m\|_{L^1(C)}, \tag{12}$$

where $c = \sup_{x \in [0,1)^d} \|x\|$.

Next, we control the approximation efficiency of $\mathcal{C}^{(1)}(N)$ for the measure μ^m. We fix $\varepsilon \in (0, l/2^{m+1})$ and consider for $i = 1, \dots, M$, the closed cubes

$$C_i^\varepsilon = \{x \in \mathbb{R}^d : d_2(x, (C_i)^c) \geqslant \varepsilon\} \subset C_i.$$

Here d_2 denotes the standard Euclidean metric on $\mathbb{R}^d$. Now observe that there exists a finite set $\mathcal{K} = \mathcal{K}(\varepsilon) \subset \mathbb{R}^d$ such that for $x \in C_i^\varepsilon$, $i = 1, \dots, M$,

$$d(x, \mathcal{K} \cap C_i) \leqslant d(x, (C_i)^c). \tag{13}$$

We extend the codebooks $\mathcal{C}^{(1)}(N)$ to $\mathcal{C}^{(2)}(N) = \mathcal{C}^{(1)}(N) \cup \mathcal{K}$ and let $\mathcal{C}_i(N) = \mathcal{C}^{(2)}(N) \cap C_i^\varepsilon$ for $N \in \mathbb{I}$ and $i = 1, \dots, M$. Note that property (13) guarantees that any point x in an arbitrary cube C_i^ε has as best $\mathcal{C}^{(2)}(N)$-approximant an element in $\mathcal{C}_i(N)$. Moreover, none of the codebooks $\mathcal{C}_i(N)$ is empty, i.e. the number $N_i = N_i(N)$ defined as $N_i = |\mathcal{C}_i(N)|$, is greater or equal to 1. Consequently, letting X_i denote $\mathcal{U}(C_i^\varepsilon)$-distributed r.v.'s, one obtains

$$\begin{aligned} \int \varphi(N^{1/d} d(x, \mathcal{C}^{(2)}(N)))\, d\mu^m(x) &\geqslant \int_{\bigcup_{i=1}^M C_i^\varepsilon} \varphi(N^{1/d} d(x, \mathcal{C}^{(2)}(N)))\, d\mu^m(x) \\ &= \sum_{i=1}^M \mu^m(C_i^\varepsilon)\, \mathbb{E}\, \varphi(N^{1/d} d(X_i, \mathcal{C}_i(N))). \end{aligned} \tag{14}$$

Let U be a $\mathcal{U}([0,1)^d$-distributed r.v., and fix an arbitrary $i \in \{1, \dots, M\}$. Note that the cube C_i^ε has side length $2^{-m}l - 2\varepsilon$, so that a shifting and rescaling yields

$$\begin{aligned} \mathbb{E}\, \varphi(N^{1/d} d(X_i, \mathcal{C}_i(N))) &\geqslant \mathbb{E}\, \varphi(N^{1/d}\, (2^{-m}l - 2\varepsilon)\, \|U - \hat{U}^{(N_i)}\|) \\ &= \mathbb{E}\, \varphi((N\lambda^d(C_i^\varepsilon))^{1/d}\, \|U - \hat{U}^{(N_i)}\|), \end{aligned}$$

where $\hat{U}^{(N_i)}$ denotes an optimal approximation satisfying the range constraint $|\text{range}\,(\hat{U}^{(N_i)})| \leqslant N_i$. With $f_N(\eta)$ as in (5), we arrive at

$$\mathbb{E}\,\varphi(N^{1/d} d(X_i, \mathcal{C}_i(N))) \geqslant f_{N_i}(N_i/(N\,\lambda^d(C_i^\varepsilon))).$$

We need to control the quantity N_i/N for N large. Recall that $\mathcal{C}^{(2)}(N)$ is the union of the sets $\mathcal{C}(N)$, $\mathcal{K}$ and $(\kappa N^{-1/d}\mathbb{Z}^d) \cap C$, and the vague convergence of ν^N to ν implies that

$$\limsup_{N\to\infty} \frac{|\mathcal{C}(N) \cap C_i|}{N} \leqslant \nu(\bar{C}_i).$$

Moreover, the set $(\kappa N^{-1/d}\mathbb{Z}^d) \cap C_i$ contains at most $\big(\frac{l\,2^{-m}}{\kappa N^{-1/d}} + 1\big)^d$ elements, so that

$$\limsup_{N\to\infty} \frac{N_i}{N} \leqslant \nu(\bar{C}_i) + \frac{\lambda^d(C_i)}{\kappa^d}.$$

Consequently, Theorem 2.1 implies that

$$\mathbb{E}\,\varphi(N^{1/d} d(X_i, \mathcal{C}_i(N))) \gtrsim g\Big(\Big(\nu(\bar{C}_i) + \frac{\lambda^d(C_i)}{\kappa^d}\Big)\Big/\lambda^d(C_i^\varepsilon)\Big).$$

Combining this estimate with (12) and (14) yields

$$\begin{aligned}\int_C \varphi(N^{1/d}\, d(x, \mathcal{C}))\, d\mu_c(x) &\geqslant \int_C \varphi(N^{1/d}\, d(x, \mathcal{C}^{(1)}))\, d\mu^m(x) - \varphi(c\,\kappa)\, \|h - h_m\|_{L^1(C)} \\ &\gtrsim \sum_{i=1}^M \mu^m(C_i^\varepsilon)\, g\Big(\Big(\nu(\bar{C}_i) + \frac{\lambda^d(C_i)}{\kappa^d}\Big)\Big/\lambda^d(C_i^\varepsilon)\Big) \\ &\qquad - \varphi(c\,\kappa)\, \|h - h_m\|_{L^1(C)}\end{aligned}$$

as $N \to \infty$. Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows that

$$\begin{aligned}\int_C \varphi(N^{1/d} d(x, \mathcal{C}^{(2)}))\, d\mu_c(x) &\gtrsim \sum_{i=1}^M \mu_c(C_i)\, g\Big(\frac{\nu(C_i)}{\lambda^d(C_i)} + \frac{1}{\kappa^d}\Big) - \varphi(c\,\kappa)\, \|h - h_m\|_{L^1(C)} \\ &= \int_C g\Big(\xi_m(x) + \frac{1}{\kappa^d}\Big)\, d\mu_c(x) - \varphi(c\,\kappa)\, \|h - h_m\|_{L^1(C)}\end{aligned}$$

As $m \to \infty$, the densities ξ_m converge pointwise to $\xi = \frac{d\nu_c}{d\lambda^d}$ for λ^d-a.a. x, and h_m converges to h in $L^1(C)$. Consequently, Fatou's Lemma implies that

$$\liminf_{N\to\infty} \int_C \varphi(N^{1/d} d(x, \mathcal{C}^{(2)}))\, d\mu_c \geqslant \int_C g\Big(\xi(x) + \frac{1}{\kappa^d}\Big)\, d\mu_c(x).$$

Finally, observing that $\kappa > 0$ was arbitrary and applying monotone convergence yields the general result. □

The above proposition enables us to give a partial proof of Theorem 1.2. For the remainder of this section, let I be given by (3), assume that $I \in (0, \infty)$, and denote by $\mathcal{M}$ the set of measures associated to the minimizers of the point allocation problem as defined in (4). So far we have not proven that $\mathcal{M}$ is non-empty.

Proposition 4.2. *Suppose that the codebooks $\mathcal{C}(N)$, $N \in \mathbb{N}$, are of size N and satisfy*

$$\limsup_{N\to\infty} N^{1/d} \, \|d(X, \mathcal{C}(N))\|_\varphi \leqslant I^{1/d}, \tag{15}$$

and consider the associated empirical measures

$$\nu^N = \frac{1}{N} \sum_{\hat{x}\in\mathcal{C}(N)} \delta_{\hat{x}}, \qquad N \in \mathbb{N}.$$

Then $(\nu^N)_{N\in\mathbb{N}}$ is a tight sequence of probability measures and any accumulation point of (ν^N) lies in $\mathcal{M}$ (in the weak topology).

Proof. Fix an arbitrary vaguely convergent subsequence $(\nu^N)_{N\in\mathbb{I}}$ of $(\nu^N)_{N\in\mathbb{N}}$ and denote by ν its limiting measure. Let $\varepsilon > 0$. As long as the Orlicz norm $\|d(X, \mathcal{C}(N))\|_\varphi$ is finite, one has in general

$$\mathbb{E}\varphi\Big(\frac{d(X, \mathcal{C}(N))}{\|d(X, \mathcal{C}(N))\|_\varphi}\Big) \leqslant 1.$$

Note that (15) implies that for all sufficiently large $N \in \mathbb{N}$

$$\|d(X, \mathcal{C}(N))\|_\varphi \leqslant ((1+\varepsilon)I/N)^{1/d}$$

so that

$$\limsup_{N\to\infty} \mathbb{E}\, \varphi((N/(1+\varepsilon)I)^{1/d} \, d(X, \mathcal{C}(N))) \leqslant 1.$$

We consider the codebooks $\tilde{\mathcal{C}}(\tilde{N}) = \mathcal{C}((1+\varepsilon)\tilde{N}I)$ for

$$\tilde{N} \in \tilde{\mathbb{I}} := \{N/((1+\varepsilon)I) : N \in \mathbb{I}\}.$$

Then

$$\limsup_{\tilde{N}\to\infty} \mathbb{E}\, \varphi(\tilde{N}^{1/d} \, d(X, \tilde{\mathcal{C}}(\tilde{N}))) \leqslant 1. \tag{16}$$

On the other hand, the empirical measures

$$\tilde{\nu}^{\tilde{N}} := \frac{1}{\tilde{N}} \sum_{\tilde{x}\in\tilde{\mathcal{C}}(\tilde{N})} \delta_{\hat{x}} = (1+\varepsilon)I\nu^{(1+\varepsilon)\tilde{N}I}$$

converge vaguely to $(1+\varepsilon)I\nu$ so that by Theorem 4.1,

$$\liminf_{\tilde{N}\to\infty} \mathbb{E}\, \varphi(\tilde{N}^{1/d} \, d(X, \tilde{\mathcal{C}}(\tilde{N}))) \geqslant \int g((1+\varepsilon)I\, \xi(x)) \, d\mu_c(x),$$

where $\xi = \frac{d\nu_c}{d\lambda^d}$. Combining this with (16), and noticing that $\varepsilon > 0$ is arbitrary, one obtains

$$\int g(I\, \xi(x)) \, d\mu_c(x) \leqslant 1.$$

Consequently, the point allocation $\tilde{\xi}(x) = I\,\xi(x)$ solves the allocation problem:

$$\int g(\tilde{\xi}(x))\,d\mu_c(x) \leqslant 1 \text{ and } \int \tilde{\xi}\,d\lambda^d \leqslant I. \tag{17}$$

Due to the definition of I, the right inequality is actually an equality.

Assume now that $\int g(\tilde{\xi}(x))\,d\mu_c(x) < 1$, and fix $\delta > 0$ (small) so that the set $A := \{x \in \mathbb{R}^d : \tilde{\xi}(x) \geqslant \delta\}$ has positive Lebesgue measure. Since g restricted to $[\delta/2, \infty)$ is Lipschitz continuous, we can lower the density $\tilde{\xi}$ on A in such a way that the point allocation constraint remains valid, thus contradicting the optimality of I. Consequently, the inequalities in (17) are even equalities, and we immediately obtain that $\nu_c \in \mathcal{M}$. Since ν_c has mass 1, we also have that $\nu = \nu_c \in \mathcal{M}$. Moreover, $(\nu^N)_{N\in\mathbb{I}}$ converges to ν in the weak topology. We finish the proof by noticing that the sequence (ν^N) is tight, since it has no vaguely convergent subsequence loosing some of its mass. □

5 The point allocation problem

We decompose the original measure μ into its absolutely continuous part $\mu_c = h\,d\lambda^d$ and singular component μ_s. The singular component will have no influence on the asymptotics of the quantization error.

In this section we use standard methods for convex optimization problems to treat the point allocation problem, i.e. the minimization of

$$\int_{\mathbb{R}^d} \xi(x)\,dx \tag{18}$$

over all positive integrable functions $\xi : \mathbb{R}^d \to [0, \infty)$ satisfying

$$\int_{\mathbb{R}^d} g(\xi(x))\,d\mu_c(x) \leqslant 1. \tag{19}$$

We shall use the convex conjugate of g, i.e.

$$g^*(a) = \sup_{\eta \geqslant 0} [a\eta - g(\eta)], \qquad a \leqslant 0,$$

and the concave function $\bar{g} : [0, \infty) \to [0, \infty), a \mapsto -g^*(-a)$. Alternatively, one can define $\bar{g}$ as $\bar{g}(a) = \inf_{\eta \geqslant 0}[a\eta + g(\eta)]$.

The function $\bar{g}$ is continuous and satisfies $\bar{g}(0) = \inf_{\eta \geqslant 0} g(\eta) = 0$. The right continuity in 0 is a consequence of the lower semicontinuity of g^*. Moreover, since g is lower semicontinuous, one has

$$g(\eta) = \sup_{a \leqslant 0} [a\eta - g^*(a)] = \sup_{a \geqslant 0} [\bar{g}(a) - a\eta], \qquad \eta \geqslant 0. \tag{20}$$

Theorem 5.1. *1. The minimal value I satisfies the dual formula*

$$I = \sup_{\kappa > 0} \frac{1}{\kappa} \Big(\int \bar{g}\big(\frac{\kappa}{h(x)}\big)\,d\mu_c(x) - 1 \Big). \tag{21}$$

2. *The optimization problem has an integrable solution iff the integral*

$$\int \bar{g}\big(\frac{\kappa}{h(x)}\big)\, d\mu_c(x) \tag{22}$$

is finite for some $\kappa > 0$. In such a case there exists an optimal point density ξ.

3. *Suppose that (22) is finite and that*

$$\mu_c(\mathbb{R}^d) \sup_{t \geqslant 0} \varphi(t) > 1.$$

Then $I > 0$ and there exists an optimal point density. Moreover, all optimal point densities ξ satisfy

$$\int g(\xi(x))\, d\mu_c(x) = 1 \quad \textit{and} \quad \bar{g}'_+\big(\frac{\kappa}{h(x)}\big) \leqslant \xi(x) \leqslant \bar{g}'_-\big(\frac{\kappa}{h(x)}\big) \textit{ for a.a. } x \in \mathbb{R}^d, \tag{23}$$

where κ is a maximizer of the right hand side of (21). (Here we make use of the convention that $\bar{g}'_+(\infty) = \bar{g}'_-(\infty) = 0$). In particular, the supremum in the dual formula is attained.

4. *If $\mu_c(\mathbb{R}^d) \sup_{t \geqslant 0} \varphi(t) \leqslant 1$, then $I = 0$ and $\xi = 0$ is an optimal point density.*

Remark 5.2.

- It follows from the concavity of $\bar{g}$ that the integral (22) is either finite or infinite for all $\kappa > 0$.
- Assume that $I > 0$ and that (22) is finite. In that case, (23) gives an efficient description of optimal point densities. In particular, if $\bar{g}$ is differentiable, one obtains a one parameter family of candidates. Note that by standard results from convex analysis (see Rockafellar (1970), Theorem 26.3), $\bar{g}$ is differentiable if and only if g is strictly convex. Moreover, in that case $\bar{g}'$ is given by

$$\bar{g}'(a) = \inf\{b > 0 : -g'(b) \leqslant a\}. \tag{24}$$

Proof of Theorem 5.1. Let us first show the " $\geqslant$ " inequality in the duality formula. Note that by definition of $\bar{g}$, $a\,b \geqslant \bar{g}(a) - g(b)$ for $a, b \geqslant 0$. Therefore, for $\kappa > 0$ and ξ satisfying (19), it is true that

$$\begin{aligned}\int \xi(x)\, dx &\geqslant \int_{\{h>0\}} \xi(x)\, dx = \frac{1}{\kappa} \int \frac{\kappa}{h(x)}\, \xi(x)\, d\mu_c(x) \\ &\geqslant \frac{1}{\kappa}\Big(\int \bar{g}\Big(\frac{\kappa}{h(x)}\Big)\, d\mu_c(x) - \int g(\xi(x))\, d\mu_c(x)\Big) \geqslant \frac{1}{\kappa}\Big(\int \bar{g}\Big(\frac{\kappa}{h(x)}\Big)\, d\mu_c(x) - 1\Big).\end{aligned} \tag{25}$$

In order to have equalities in the above estimates, we need to find a density ξ and $\kappa > 0$ such that

$$\xi(x) = 0 \text{ for } \lambda^d \text{ a.a. } x \in \{h = 0\}, \tag{26}$$

$$\bar{g}\Big(\tfrac{\kappa}{h(x)}\Big) - g(\xi(x)) = \tfrac{\kappa}{h(x)}\, \xi(x), \qquad \text{for } \mu_c \text{ a.a. } x \tag{27}$$

and

$$\int g(\xi(x))\, d\mu_c(x) = 1. \tag{28}$$

In the case $\mu_c(\mathbb{R}^d) \sup_{t \geqslant 0} \varphi(t) \leqslant 1$, it is easily seen that $\xi = 0$ is an optimal point density so that $I = 0$ which proves assertion 4. Moreover, the term on the right hand side of (21) tends to 0 when letting $\kappa \to \infty$ so that the dual formula is valid in that case.

From now on, we assume that $\mu_c(\mathbb{R}^d) \sup_{t \geqslant 0} \varphi(t) > 1$. Next, we derive a density ξ satisfying the three abovementioned conditions. Then estimate (25) implies optimality for this choice of ξ which proves assertion 1.

Since the remaining arguments are standard in the treatment of convex optimization problems, we will only give a sketch of the remaining proof. First one needs to verify that for each $\kappa > 0$, the point densities

$$\xi_+^\kappa(x) = \bar{g}_+'\Big(\frac{\kappa}{h(x)}\Big), \qquad x \in \mathbb{R}^d,$$

and

$$\xi_-^\kappa(x) = \bar{g}_-'\Big(\frac{\kappa}{h(x)}\Big), \qquad x \in \mathbb{R}^d$$

meet requirements (26) and (27). Next, one notices that the integral on the left hand side of (28) is increasing in κ when choosing $\xi = \xi_-^\kappa$ or $\xi = \xi_+^\kappa$. Moreover, for $\xi = \xi_-^\kappa$, letting $\kappa \downarrow 0$ and $\kappa \to \infty$, the integral converges to 0 and some value bigger than 1, respectively. Finally, one proves that condition (28) is satified for an appropriate interpolation of ξ_-^κ and ξ_+^κ, where $\kappa > 0$ is the largest value for which the integral in (28) is less than or equal to 1 for $\xi = \xi_-^\kappa$. □

6 The singular case

In this section, we consider an original X with law $\mu \perp \lambda^d$ or, equivalently, $\mu_c = 0$. Moreover, we again assume that μ is compactly supported.

Proposition 6.1. *There exist codebooks $\mathcal{C}(N)$, $N \geqslant 1$, with $\lim_{N\to\infty} \frac{1}{N}|\mathcal{C}(N)| = 0$ such that*

$$\lim_{N\to\infty} \mathbb{E}\, \varphi(N^{1/d}\, d(X, \mathcal{C}(N))) = 0.$$

Proof. We fix $\varepsilon > 0$. As the measure μ is singular with respect to the Lebesgue measure λ^d, there exists an open set $A \subset \mathbb{R}^d$ with $\mu(A) = 1$ and $\lambda^d(A) \leqslant \varepsilon$. Due to Lemma 1.4.2 in Cohn (1980), one can represent the open set A as a countable disjoint union of half-open cubes $(C_i)_{i\in\mathbb{N}}$ in $\bigcup_{m=1}^\infty \mathcal{B}_m$, where

$$\mathcal{B}_m = \big\{[i_1 2^{-m}, (i_1+1)2^{-m}) \times \cdots \times [i_d 2^{-m}, (i_d+1)2^{-m}) : i_1, \ldots, i_d \in \mathbb{Z}\big\} \subset \mathbb{R}^d$$

for $m \in \mathbb{N}$. Due to monotone convergence, we obtain that there exists $M \in \mathbb{N}$ with

$$\mu\Big(\bigcup_{i=1}^M C_i\Big) \geqslant \mu(A) - \varepsilon = 1 - \varepsilon. \tag{29}$$

Set $C = \bigcup_{i=1}^{M} C_i$.

Let us introduce the codebooks; fixing $l > 0$ such that $\operatorname{supp}(\mu) \subset [-l, l)^d$, the construction depends upon two parameters $\kappa_1, \kappa_2 > 0$:

$$\mathcal{C}(N) = \big((\kappa_1 N^{-1/d} \mathbb{Z}^d) \cap C\big) \cup \big((\kappa_2 N^{-1/d} \mathbb{Z}^d) \cap [-l, l)^d\big), \qquad N \geqslant 1.$$

We need to control the size of $\mathcal{C}(N)$. For $i \in \{1, \ldots, M\}$, let $m_i \in \mathbb{N}$ denote the unique number with $C_i \in \mathcal{B}_{m_i}$, and observe that

$$|(\kappa_1 N^{-1/d} \mathbb{Z}^d) \cap C_i| \leqslant \Big(\frac{2^{-m_i}}{\kappa_1 N^{-1/d}} + 1\Big)^d \sim \lambda^d(C_i)\, \kappa_1^{-d} N$$

as $N \to \infty$. Analogously,

$$|(\kappa_2 N^{-1/d} \mathbb{Z}^d) \cap [-l, l)^d| \leqslant \Big(\frac{2l}{\kappa_2 N^{-1/d}} + 1\Big)^d \sim (2l)^d \kappa_2^{-d} N.$$

Consequently,

$$\begin{aligned} |\mathcal{C}(N)| &\leqslant \sum_{i=1}^{M} |(\kappa_1 N^{-1/d} \mathbb{Z}^d) \cap C_i| + |(\kappa_2 N^{-1/d} \mathbb{Z}^d) \cap [-l, l)^d| \\ &\lesssim \big(\lambda^d(C)\, \kappa_1^{-d} + (2l)^d \kappa_2^{-d}\big) N \leqslant \big(\varepsilon\, \kappa_1^{-d} + (2l)^d \kappa_2^{-d}\big) N. \end{aligned}$$

Next, we estimate the approximation error. Suppose that $N \geqslant 1$ is sufficiently large so that $C_i \cap \mathcal{C}(N) \neq \emptyset$ for all $i = 1, \ldots, M$. Let $c = \sup_{x \in [0,1)^d} \|x\|$, and observe that for all $x \in C$, $d(x, \mathcal{C}(N)) \leqslant c\, \kappa_1 N^{-1/d}$. Moreover, for any $x \in [-l, l)^d$, $d(x, \mathcal{C}(N)) \leqslant c\, \kappa_2 N^{-1/d}$. Consequently,

$$\begin{aligned} \mathbb{E}\, \varphi(N^{1/d} d(X, \mathcal{C}(N))) &\leqslant \mu(C)\, \varphi(c\, \kappa_1) + (1 - \mu(C))\, \varphi(c\, \kappa_2) \\ &\leqslant \varphi(c\, \kappa_1) + \varepsilon\, \varphi(c\, \kappa_2). \end{aligned} \tag{30}$$

Now, for $\delta > 0$ arbitrary, pick $\kappa_1, \kappa_2 > 0$ satisfying $\varphi(c\, \kappa_1) \leqslant \delta/2$ and $(2l)^d/\kappa_2^d \leqslant \delta/2$, and choose $\varepsilon > 0$ so that $\varepsilon \kappa_1^{-d} \leqslant \delta/2$ and $\varepsilon\, \varphi(c\, \kappa_2) \leqslant \delta/2$. Then the corresponding codebooks $\mathcal{C}(N)$ satisfy

$$|\mathcal{C}(N)| \lesssim \delta\, N \quad \text{and} \quad \mathbb{E}\, \varphi(N^{1/d} d(X, \mathcal{C}(N))) \lesssim \delta,$$

and the assertion of the lemma follows by a diagonalization argument.

□

7 Extension to the non-compact setting

In order to treat the non-compact quantization problem, we need to control the impact of realizations lying outside large cubes. For $L^p(\mathbb{P})$-norm distortions, Pierce (1970) (see also Graf & Luschgy (2000, Lemma 6.6)) discovered that the quantization error can be

estimated against a higher moment $\tilde{p} > p$ of $\|X\|$. His result can be easily extended to the inequality

$$\delta(N|X,p) \leqslant C\,\mathbb{E}[\|X\|^{\tilde{p}}]^{1/\tilde{p}} N^{-1/d},$$

where X is an arbitrary original in $\mathbb{R}^d$, $N \in \mathbb{N}$ and C is a universal constant depending only on E, p and $\tilde{p}$. Pierce's proof is based on a random coding argument. In contrast to his approach, we will use ε-nets to establish a similar result.

The construction is based on several parameters. Let $\Psi : [0,\infty) \to [0,\infty)$ denote a monotonically increasing function, $(r_n)_{n\in\mathbb{N}_0}$ an increasing sequence, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a positive summable sequence.

Lemma 7.1. *Let $J \in \mathbb{N}_0$ and denote by X a $(B(0,r_J)^c \cup \{0\})$-valued r.v. For $N \geqslant 0$ there exists a codebook $\mathcal{C}(N)$ of size $1 + N\sum_{n=J}^{\infty}\alpha_n$ satisfying*

$$\mathbb{E}\,\varphi\big(N^{1/d}\,d(X,\mathcal{C}(N))\big) \leqslant \mathbb{E}[\Psi(\|X\|)] \sum_{n=J}^{\infty} \frac{1}{\Psi(r_n)}\,\varphi(c_E\,\alpha_n^{-1/d}\,r_{n+1}), \tag{31}$$

where c_E is a finite constant depending on the norm $\|\cdot\|$ only.

Proof. First, observe that the sum in estimate (31) diverges whenever $\liminf_{n\to\infty} r_n < \infty$. Thus we can assume without loss of generality that $\lim_{n\to\infty} r_n = \infty$. Fix $J \in \mathbb{N}_0$ and $N \in \mathbb{N}$. For $n \in \mathbb{N}_0$, let $V_n = B(0,r_n)$ and $N_n = \alpha_n N$. Moreover, we denote by $I \in \mathbb{N}_0$ the smallest index n with $N_n < 1$, and let for $n \in \mathbb{N}_0$ with $J \leqslant n < I$, $\mathcal{C}_n$ denote an optimal ε-net for $B(0,r_{n+1})$ consisting of N_n elements. As is well known there exists a constant c_E only depending on the norm $\|\cdot\|$ such that $d(x,\mathcal{C}_n) \leqslant c_E r_{n+1} N_n^{-1/d}$ for all $x \in B(0,r_{n+1})$. Thus the codebook $\mathcal{C} = \{0\} \cup \bigcup_{n=J}^{I-1}\mathcal{C}_n$ contains at most $1 + N\sum_{n=J}^{\infty}\alpha_n$ elements, we get

$$\begin{aligned}
\mathbb{E}\,\varphi\big(N^{1/d}\,d(X,\mathcal{C})\big) &= \sum_{n=J}^{\infty} \mathbb{E}\Big[1_{V_{n+1}\setminus V_n}(X)\,\varphi\big(N^{1/d}\,d(X,\mathcal{C})\big)\Big] \\
&\leqslant \sum_{n=J}^{\infty} \mathbb{P}(X \notin V_n)\,\varphi\big(c_E\,r_{n+1}\,N^{1/d}/(1\vee N_n)^{1/d}\big) \\
&\leqslant \sum_{n=J}^{\infty} \mathbb{P}(X \notin V_n)\,\varphi(c_E\,r_{n+1}\,\alpha_n^{-1/d}) \\
&\leqslant \mathbb{E}[\Psi(\|X\|)] \sum_{n=J}^{\infty} \frac{1}{\Psi(r_n)}\,\varphi(c_E\,r_{n+1}\,\alpha_n^{-1/d}).
\end{aligned}$$

□

Definition 7.2. We say that an increasing function $\Psi : [0,\infty) \to [0,\infty)$ satisfies the growth condition (G) for φ iff there exists a summable sequence (α_n) and an increasing sequence (r_n) with

$$\sum_{n=0}^{\infty} \frac{1}{\Psi(r_n)}\,\varphi(\alpha_n^{-1/d}\,r_{n+1}) < \infty.$$

Suppose that Ψ satisfies condition (G) for φ. We shall see that the condition that $\mathbb{E}\Psi(\|X\|) < \infty$ is sufficient to conclude that the quantization problem for X under the Orlicz norm induced by φ is of order $N^{-1/d}$. Moreover, non-compact quantization can then be approximated by the compact quantization.

Example 7.3. Suppose that φ satisfies $\varphi(t) \leqslant c \exp\{t^\kappa\}$ for all $t \geqslant 0$, where $c, \kappa \in \mathbb{R}_+$ are appropriate constants. Then, for any $\tilde{\kappa} > \kappa$, the function

$$\Psi(t) = \exp\{t^{\tilde{\kappa}}\}$$

satisfies (G) for φ, as can be verified easily for $(\alpha_n)_{n\in\mathbb{N}_0} = ((n+2)^{-2})_{n\in\mathbb{N}_0}$ and $(r_n)_{n\in\mathbb{N}_0} = ((n+1)^s)_{n\in\mathbb{N}_0}$ for $s > 0$ with $s\tilde{\kappa} > (\frac{2}{d} + s)\kappa$.

Remark 7.4. The proof of the upper bound in Theorem 1.1 relies on the assumption that $\mathbb{E}\Psi(\|X\|) < \infty$ for some Ψ satisfying the growth condition (G). As we shall see below, this assumption can be replaced by the equivalent condition that $X \in L^\Psi(\mathbb{P})$ for some Ψ satisfying (G). First, assume that $\mathbb{E}\Psi(\|X\|) < \infty$ for some Ψ satisfying (G). Then $\tilde{\Psi} = 1_{[1,\infty)}\,\Psi$ satisfies (G), and since by monotone convergence

$$\lim_{\kappa\to\infty} \mathbb{E}\Psi(\|X\|/\kappa) = 0,$$

it follows that $X \in L^{\tilde{\Psi}}(\mathbb{P})$. On the other hand, assuming that $X \in L^\Psi(\mathbb{P})$ for some Ψ satisfying (G) implies the existence of a $\kappa > 0$ for which

$$\mathbb{E}\Psi(\|X\|/\kappa) < \infty.$$

Now, let $\tilde{\Psi}(t) = \Psi(t/\kappa)$, and denote by (α_n) and (r_n) sequences as in Definition 7.2. Then $\mathbb{E}\tilde{\Psi}(\|X\|) < \infty$, and

$$\sum_{n=0}^{\infty} \frac{1}{\tilde{\Psi}(\tilde{r}_n)}\,\varphi(\tilde{\alpha}_n^{-1/d}\,\tilde{r}_{n+1}) = \sum_{n=0}^{\infty} \frac{1}{\Psi(r_n)}\,\varphi(\alpha_n^{-1/d}\,r_{n+1}) < \infty$$

for $\tilde{\alpha}_n = \kappa^d\alpha_n$ and $\tilde{r}_n = \kappa r_n$, $n \in \mathbb{N}_0$.

We now combine the quantization results for continuous, singular, and unbounded measures to finish the proof of the upper bound in Theorem 1.1.

Proof of Theorem 1.1. Let Ψ be a function satisfying (G). It is easy to see that there exist also a summable sequence (α_n) and an increasing sequence (r_n) such that

$$\sum_{n=0}^{\infty} \frac{1}{\Psi(r_n)}\,\varphi(c_E\,\alpha_n^{-1/d}\,r_{n+1}) < \infty,$$

where c_E is as in Lemma 7.1. We denote by ξ an optimal point density, so that ξ satisfies

$$\int_{\mathbb{R}^d} g(\xi(x))\,d\mu_c(x) = 1 \quad\text{and}\quad \int_{\mathbb{R}^d} \xi(x)\,dx = I$$

or $\xi = 0$ (in the case $I = 0$). We fix $\varepsilon > 0$ and let $\tilde{\xi}(x) = \xi(x) + \varepsilon h(x)$, $x \in \mathbb{R}^d$. This point density satisfies

$$\int_{\mathbb{R}^d} g(\tilde{\xi}(x))\, d\mu_c(x) < 1$$

since g is strictly decreasing on $\{\eta > 0 : g(\eta) > 0\}$. Now fix $J \in \mathbb{N}_0$ such that

$$\int_{\mathbb{R}^d} g(\tilde{\xi}(x))\, d\mu_c(x) + \mathbb{E}[\Psi(\|X\|)] \sum_{n=J}^{\infty} \frac{1}{\Psi(r_n)} \varphi(c_E\, \alpha_n^{-1/d} r_{n+1}) < 1 \tag{32}$$

and $\sum_{n=J}^{\infty} \alpha_n < \varepsilon$. Next, decompose the measure μ into the sum $\mu = \tilde{\mu}_c + \tilde{\mu}_s + \mu_u$, where $\tilde{\mu}_c$ and $\tilde{\mu}_s$ are the absolutely continuous and singular part of μ restricted to $B(0, r_J)$, respectively, and μ_u contains the rest of the mass of μ.

It remains to combine the former results. Due to Proposition 3.1 there exist codebooks $\mathcal{C}_1(N)$, $N \geqslant 1$, with $\lim_{N\to\infty} \frac{1}{N}|\mathcal{C}_1(N)|$ and

$$\limsup_{N\to\infty} \int \varphi(N^{1/d} d(x, \mathcal{C}_1(N)))\, d\tilde{\mu}_c \leqslant \int_{\mathbb{R}^d} g(\tilde{\xi}(x))\, d\mu_c(x).$$

Moreover, Proposition 6.1 implies the existence of codebooks $\mathcal{C}_2(N)$, $N \geqslant 1$, with $\lim_{N\to\infty} \frac{1}{N}|\mathcal{C}_2(N)| = 0$ and

$$\lim_{N\to\infty} \int \varphi(N^{1/d} d(x, \mathcal{C}_2(N)))\, d\tilde{\mu}_s = 0.$$

Finally, Lemma 7.1 (applied to $\tilde{X} = 1_{B(0,r_J)^c}(X) \cdot X$) yields the existence of codebooks $\mathcal{C}_3(N)$, $N \geqslant 1$, for which

$$\limsup_{N\to\infty} \frac{1}{N}|\mathcal{C}_3(N)| \leqslant \sum_{n=J}^{\infty} \alpha_n < \varepsilon$$

and

$$\limsup_{N\to\infty} \int \varphi(N^{1/d} d(x, \mathcal{C}_3(N)))\, d\mu_u \leqslant \mathbb{E}[\Psi(\|X\|)] \sum_{n=J}^{\infty} \frac{1}{\Psi(r_n)} \varphi(C\, \alpha_n^{-1/d} r_{n+1}).$$

Now consider the codebooks $\mathcal{C}(N) = \mathcal{C}_1(N) \cup \mathcal{C}_2(N) \cup \mathcal{C}_3(N)$. Due to the above estimates and (32), one has

$$\begin{aligned} &\limsup_{N\to\infty} \int \varphi(N^{1/d} d(x, \mathcal{C}(N)))\, d\mu \\ &\qquad \leqslant \int_{\mathbb{R}^d} g(\tilde{\xi}(x))\, d\mu_c(x) + \mathbb{E}[\Psi(\|X\|)] \sum_{n=J}^{\infty} \frac{1}{\Psi(r_n)} \varphi(C\, \alpha_n^{-1/d} r_{n+1}) < 1, \end{aligned}$$

so that for sufficiently large N it is true that $\|d(X, \mathcal{C}(N))\|_\varphi \leqslant N^{-1/d}$. On the other hand,

$$\limsup_{N\to\infty} \frac{1}{N}|\mathcal{C}(N)| < (1+\varepsilon)I + \varepsilon$$

and, for sufficiently large N, it holds that $|\mathcal{C}(N)| \leqslant (I+\varepsilon I+\varepsilon)N$. Consequently, it follows that for large N

$$\delta((I+\varepsilon I+\varepsilon)N|X,\varphi) \leqslant N^{-1/d}.$$

Switching from N to $M = (I+\varepsilon I+\varepsilon)N$ one obtains

$$\delta(M|X,\varphi) \leqslant (I+\varepsilon I+\varepsilon)^{1/d}\, M^{-1/d},$$

for M large. Since $\varepsilon > 0$ was arbitrary, it follows that

$$\limsup_{M\to\infty} M^{1/d}\,\delta(M|X,\varphi) \leqslant I^{1/d}$$

and we are done. □

References

Billingsley, P. 1979. *Probability and measure.* Wiley Series in Probability and mathematical Statistics.

Bucklew, J. A. 1984. "Two results on the asymptotic performance of quantizers." *IEEE Trans. Inform. Theory* 30(2, part 2):341–348.

Bucklew, J. A. & G. L. Wise. 1982. "Multidimensional asymptotic quantization theory with rth power distortion measures." *IEEE Trans. Inf. Theory* 28:239–247.

Cohn, D. L. 1980. *Measure theory.* Mass.: Birkhäuser Boston.

Cover, T. M. & J. A. Thomas. 1991. *Elements of information theory.* Wiley Series in Telecommunications. New York: John Wiley & Sons, Inc.

Delattre, S., S. Graf, H. Luschgy & G. Pagès. 2004. "Quantization of probability distributions under norm-based distortion measures." *Statist. Decision* 22:261–282.

Dereich, S., F. Fehringer, A. Matoussi & M. Scheutzow. 2003. "On the link between small ball probabilities and the quantization problem for Gaussian measures on Banach spaces." *J. Theoret. Probab.* 16(1):249–265.

Fejes Tóth, L. 1972. *Lagerungen in der Ebene, auf der Kugel und im Raum.* 2nd ed. Springer-Verlag.

Gersho, A. 1979. "Asymptotically optimal block quantization." *IEEE Trans. Inform. Theory* 25(4):373–380.

Gersho, A. & R. M. Gray. 1992. *Vector quantization and signal compression.* Boston, MA: Kluwer Academic Publishers.

Graf, S. & H. Luschgy. 2000. *Foundations of quantization for probability distributions.* Lecture Notes in Mathematics 1730, Berlin: Springer.

Gray, R. M. & D. L. Neuhoff. 1998. "Quantization." *IEEE Trans. Inf. Theory* 44(6):2325–2383.

Gruber, P. M. 2004. "Optimum quantization and its applications." *Adv. Math.* 186(2):456–497.

Luschgy, H. & G. Pagès. 2004. "Sharp asymptotics of the functional quantization problem for Gaussian processes." *Ann. Probab.* 32(2):1574–1599.

Pagès, Gilles, Huyên Pham & Jacques Printems. 2004. Optimal quantization methods and applications to numerical problems in finance. In *Handbook of computational and numerical methods in finance.* Boston, MA: Birkhäuser Boston pp. 253–297.

Pierce, J. N. 1970. "Asymptotic quantizing error for unbounded random variables." *IEEE Trans. Inf. Theory* 16:81–83.

Rockafellar, R. T. 1970. *Convex analysis.* 2nd ed. Princeton, N. J.

Zador, P. L. 1966. "Topics in the asymptotic quantization of continuous random variables." Bell Laboratories Technical Memorandum.

Zador, P. L. 1982. "Asymptotic quantization error of continuous signals and the quantization dimension." *IEEE Trans. Inf. Theory* 28:139–149.