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Renewal, Recurrence and Regeneration

February 16, 2010

Springer

For Sylviane, Melanie and Daniel

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cdf	(cumulative) distribution function
chf	characteristic function
CLT	central limit theorem
dRi	directly Riemann integrable
FT	Fourier transform
iff	if, and only if
iid	independent and identically distributed
mgf	moment generating function
MRW	Markov random walk
RP	renewal process
RW	random walk
SLLN	strong law of large numbers
SRP	standard renewal process = zero-delayed renewal process
SRW	standard random walk = zero-delayed random walk
ui	uniformly integrable
WLLN	weak law of large numbers

Chapter 1
Introduction

What goes around, comes around.

In Applied Probability, one is frequently facing the task to determine the asymptotic behavior of a real-valued stochastic processes $(R_t)_{t\in\mathbb{T}}$ in discrete $(\mathbb{T}=\mathbb{N}_0)$ or continuous $(\mathbb{T}=\mathbb{R}_\geq)$ time which bears a *regeneration scheme* in the following sense: for an increasing sequence $0<\nu_1\le\nu_2,...$ either

- $\{R_{\nu_k+t}:0\le t<\nu_{k+1}-\nu_k\},\ k\ge 1$ [type A]

or

- $\{R_{\nu_k+t}-R_{\nu_k}:0\le t<\nu_{k+1}-\nu_k\},\ k\ge 1$ [type B]

are independent and identically distributed (iid) random elements, often called *cycles* hereafter. The ν_k, which may be random or deterministic, are called *regeneration epochs* and constitute a so-called *renewal process*, that is a nondecreasing sequence of nonnegative random variables with iid increments. The last assertion is a necessary consequence of each of the two above regeneration properties. Intuitively speaking, regeneration means that $(R_t)_{t\in\mathbb{T}}$, possibly after being reset to 0 (type B), restarts or *regenerates* at $\nu_1,\nu_2,...$ in a distributional sense. For example, if R_t denotes the number of waiting customers at time t in a queuing system, then (under suitable model assumptions) a regeneration scheme of type A is obtained with the help of the epochs ν_k where an arriving customer finds the system idle. For another example, let R_t be the stock level at time t in an (s,S)-inventory model with maximal stock level S and critical level $0<s<S$. Whenever incoming demands cause the stock level to fall below s, denote these epochs as $\nu_1,\nu_2,...$, the stock is immediately refilled to go back to S. If the times between demands and the demand sizes are iid, then it is not difficult to verify that the ν_k are regeneration epochs for (R_t). Simple symmetric random walk $(R_n)_{n\ge0}$ on the integer lattice provides a more theoretical example. Here R_n denotes the position at time n of a particle which in each step moves to one of the two neighboring sites with probability $\frac{1}{2}$ each. It is known and will in fact be shown in ??????? that with probability one this particle visits any site $k\in\mathbb{Z}$ infinitely often. Hence, the epochs $\nu_1,\nu_2,...$ where it returns to 0 provide a regeneration scheme of type A for $(R_n)_{n\ge0}$. A regeneration scheme of type B can also be given for this sequence by letting ν_k be the first time where the particle hits k for any $k\in\mathbb{N}$. In a similar vein, Brownian motion $(R_t)_{t\ge0}$ with positive drift μ, i.e.

a Gaussian process with stationary independent increments, continuous trajectories, $R_0 = 0$, $\mathbb{E}R_t = \mu t$ and $\mathbb{V}\mathrm{ar} R_t = \sigma^2 t$ for some $\sigma^2 > 0$, regenerates in the sense of type B at the consecutive hitting epochs of the line $x \mapsto \mu x$.

Drawing conclusions from the existence of a regeneration scheme for a given stochastic process may be viewed as the ultimate goal of renewal theory, but in a narrower and more classical sense it deals with the analysis of those sequences that are at the bottom of such schemes, namely sums of iid real-valued random variables, called *random walks*, including the afore-mentioned renewal sequences as a special case by having nonnegative increments. However, unlike classical limit theorems which provide information on the asymptotic behavior of a random walk after a suitable normalization, renewal theory strives for the fine structure of random walks by exploring its ubiquitous regenerative pattern. The present text puts a strong emphasis on this latter....

1.1 The renewal problem: a simple example to begin with

Despite its simplistic nature, the following example provides a good framework to motivate some of the most basic questions in connection with a renewal process. Suppose we are given an infinite supply of light bulbs which are used one at a time until they fail. Their lifetimes are denoted as $X_1, X_2, \ldots$ and assumed to be iid random variables with positive mean μ. If the first light bulb is installed at time $S_0 := 0$, then

$$S_n := \sum_{k=1}^{n} X_k \quad \text{for } n \geq 1$$

denotes the time at which the n^{th} bulb fails and is replaced with a new one. In other words, each S_n marks a renewal epoch. Some of the natural problems that come to mind for this model are the following:

(Q1) Is the number of renewals up to time t, denoted as $N(t)$, almost surely finite for all $t > 0$? And what about its expectation $\mathbb{E}N(t)$?

(Q2) What is the asymptotic behavior of $t^{-1}N(t)$ and its expectation as $t \to \infty$, that is the long run average (expected) number of renewals per unit of time?

(Q3) What can be said about the long run behavior of $\mathbb{E}(N(t+h) - N(t))$ for any fixed $h > 0$?

The stochastic process $(N(t))_{t\geq 0}$ is called the *renewal counting process* associated with $(S_n)_{n\geq 1}$ and may be formally defined as

$$N(t) := \sup\{n \geq 0 : S_n \leq t\} \quad \text{for } t \geq 0. \tag{1.1}$$

An equivalent definition is

$$N(t) := \sum_{n\geq 1} \mathbf{1}_{[0,t]}(S_n)$$

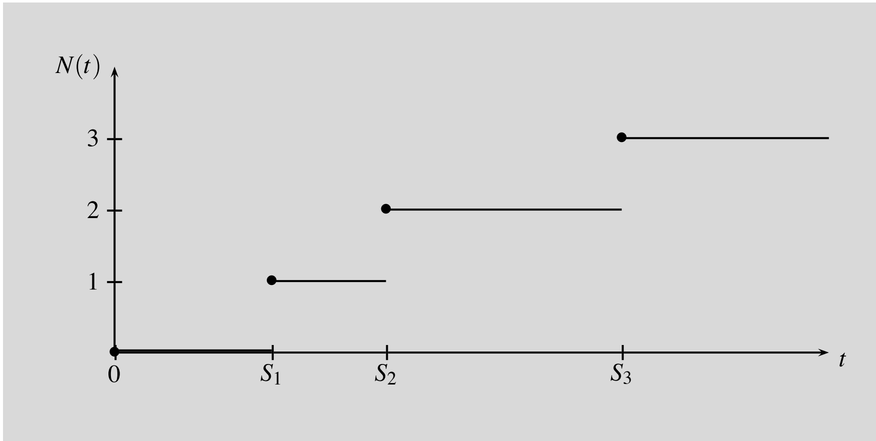

Fig. 1.1 The renewal counting process $(N(t))_{t\geq 0}$ with renewal epochs $S_1, S_2, \ldots$

and has the advantage that it immediately extends to general measurable subsets A of $\mathbb{R}_\geq$ by putting

$$N(A) \;:=\; \sum_{n\geq 1} \mathbf{1}_A(S_n) \;=\; \sum_{n\geq 1} \delta_{S_n}(A). \tag{1.2}$$

Here $\mathbf{1}_A$ denotes the indicator function of A and δ_{S_n} the Dirac measure at S_n. Ignoring measurability aspects here [☞ Subsection 2.1.4], it is clear that $N = \sum_{n\geq 1} \delta_{S_n}$ does in fact constitute a *random counting measure*, also called *point process*, on $(\mathbb{R}_\geq, \mathscr{B}(\mathbb{R}_\geq))$. By further defining its intensity measure

$$\mathbb{U}(A) \;:=\; \mathbb{E}N(A) \;=\; \sum_{n\geq 1} \mathbb{P}(S_n \in A) \quad (A \in \mathscr{B}(\mathbb{R}_\geq)) \tag{1.3}$$

we arrive at the so-called *renewal measure* of $(S_n)_{n\geq 1}$ which measures the expected number of renewals in a set and is one of the central objects in renewal theory. Its "distribution function"

$$[0,\infty) \;\ni\; t \;\mapsto\; \mathbb{U}(t) \;:=\; \mathbb{U}([0,t]) \;=\; \sum_{n\geq 1} \mathbb{P}(S_n \leq t) \tag{1.4}$$

is called *renewal function* of $(S_n)_{n\geq 1}$ and naturally of particular interest.

Turning to question (Q1), we directly infer

$$N(t) < \infty \quad \text{a.s. for all } t \geq 0 \tag{1.5}$$

because $S_n \to \infty$ a.s. as a consequence of the strong law of large numbers (SLLN). The question whether $N(t)$ has finite expectation as well requires only little more work and follows with the help of a stochastic comparison argument.

Theorem 1.1.1. *If $(S_n)_{n\ge 1}$ is a renewal process, then its renewal function is everywhere finite, i.e.* $\mathbb{U}(t) < \infty$ *for all* $t \ge 0$.

Proof. The essence of the subsequent argument leads back to old work by STEIN [42]. Since $\mathbb{P}(X_1 = 0) < 1$ there exists a constant $c > 0$ such that $p := \mathbb{P}(X_1 \le c) < 1$. Consider the renewal process (S_n') with increments defined as $X_n' := c\,\mathbf{1}_{\{X_n > c\}}$ for $n \ge 1$, thus $S_n' = \sum_{k=1}^n X_k'$ for $n \ge 1$. This process moves as follows: Each jump of size c is preceded by a random number of zeroes having a geometric distribution with parameter $1 - p = \mathbb{P}(X_1' = c)$. Consequently, if N', $\mathbb{U}'$ have the obvious meaning, then $N'(nc)$ equals n plus the sum of $n+1$ (actually independent) geometric random variables with parameter p giving

$$\mathbb{U}'(nc) \;=\; \mathbb{E}N'(nc) \;=\; n + (n+1)\frac{p}{1-p} \;=\; \frac{n+p}{1-p} \quad \text{for all } n \in \mathbb{N}.$$

The assertion now follows because $X_n' \le X_n$ for each $n \in \mathbb{N}$ clearly implies $N(t) \le N'(t)$ and hence $\mathbb{U}(t) \le \mathbb{U}'(t)$ for all $t \ge 0$. □

With the previous result at hand we can turn to question (Q2) on the long run average number of renewals per time unit, also called *renewal rate*. Ignoring the random oscillations in the replacement scheme it is natural to expect that this rate should be μ^{-1}, as every installed light bulb is expected to burn for a time interval of length μ. The following result provides a positive answer for $t^{-1}N(t)$ and is based on a neat probabilistic argument that goes back to DOOB [18]. The corresponding assertion for $t^{-1}\mathbb{U}(t)$ is also valid, but its proof requires more work and will be given later [☞ Thm. 2.4.3].

Theorem 1.1.2. *If $(S_n)_{n\ge 1}$ is a renewal process with mean interrenewal time* $0 < \mu \le \infty$, *then*

$$\lim_{t\to\infty} \frac{N(t)}{t} \;=\; \frac{1}{\mu} \quad a.s.$$

with the usual convention that $\infty^{-1} := 0$.

Proof. Using $N(t) \to \infty$ a.s. in combination with the SLLN for $(S_n)_{n\ge 1}$ we infer that both, $N(t)^{-1}S_{N(t)}$ and $N(t)^{-1}S_{N(t)+1}$ converge a.s. to μ as $t \to \infty$. Moreover, (1.1) implies $S_{N(t)} \le t < S_{N(t)+1}$ for all $t \ge 0$. Consequently,

$$\frac{S_{N(t)}}{N(t)} \;\le\; \frac{t}{N(t)} \;\le\; \frac{S_{N(t)+1}}{N(t)} \quad \text{for all } t \ge 0$$

provides the assertion upon letting t tend to infinity. □

Asking for the expected number of renewals in a bounded interval of length h (question (Q3)) the heuristic argument given before the previous result suggests that it should be approximately equal to $\mu^{-1}h$, that is

$$\mathbb{U}((t,t+h]) \;=\; \mathbb{U}(t+h)-\mathbb{U}(t) \;\approx\; \frac{h}{\mu} \tag{1.6}$$

at least for large values of t. On the other hand, it should not take by surprise that for this to show the random fluctuations of $N(t+h)-N(t)$ must be considered more carefully. In fact, an answer to (Q3) cannot be provided at this point and is one of the highly nontrivial blockbuster results to be derived in Chapter 3.

1.2 The Poisson process: a nice example to learn from

So far we have not addressed the question whether the distribution of $N(t)$ or its expectation $\mathbb{U}(t)$ may be computed explicitly in closed form. Let F and F_n denote the cdf of X_1 and S_n, respectively, hence $F_1 = F$ and $F_n = F^{*n}$ for each $n \in \mathbb{N}$, where F^{*n} denotes the n-fold covolution of F defined recursively as

$$F^{*n}(t) \;=\; \int_{[0,t]} F^{*(n-1)}(t-x)\,F(dx) \quad \text{for all } t \geq 0.$$

Now observe that, by (1.1),

$$\{N(t)=n\} \;=\; \{S_n \leq t < S_{n+1}\} \;=\; \{S_n \leq t\}\backslash\{S_{n+1} \leq t\}$$

and thus

$$\mathbb{P}(N(t)=n) \;=\; \mathbb{P}(S_n \leq t)-\mathbb{P}(S_{n+1} \leq t) \;=\; F_n(t)-F_{n+1}(t). \tag{1.7}$$

for all $n \in \mathbb{N}_0$ and $t \geq 0$. Furthermore, by (1.4)

$$\mathbb{U}(t) \;=\; \sum_{n\geq 1}\mathbb{P}(S_n \leq t) \;=\; \sum_{n\geq 1} F_n(t) \quad \text{for all } t \geq 0. \tag{1.8}$$

This shows that closed form expressions require an explicit knowledge of *all* $F_n(t)$ as well as of their infinite sum which is true only in very few cases. The most important one of these will be discussed next.

Suppose that F is an exponential distribution with parameter $\theta > 0$, that is

$$F(t) \;=\; 1-e^{-\theta t} \quad (t \geq 0).$$

It is well-known that S_n then has a Gamma distribution with parameters n and θ, the density of which (with respect to Lebesgue measure $\lambda\!\!\lambda_0$) is

$$f_n(x) \;=\; \frac{\theta^n x^{n-1}}{(n-1)!}\, e^{-\theta x} \quad (x \ge 0)$$

for each $n \in \mathbb{N}$. In order to find $\mathbb{P}(N(t) = n)$, we consider $g_n(t) := e^{\theta t}\, \mathbb{P}(N(t) = n)$ for $t \ge 0$ and any fixed $n \in \mathbb{N}$. A conditioning argument yields

$$\begin{aligned} g_n(t) \;&=\; e^{\theta t}\, \mathbb{P}(S_n \le t < S_n + X_{n+1}) \\ &=\; e^{\theta t} \int_0^t \mathbb{P}(X_1 > t - x)\, f_n(x)\, dx \\ &=\; \int_0^t e^{\theta x} f_n(x)\, dx \\ &=\; \int_0^t \frac{\theta^n x^{n-1}}{(n-1)!}\, dx \quad \text{for all } t \ge 0, \end{aligned}$$

whence

$$\mathbb{P}(N(t) = n) \;=\; \frac{(\theta t)^n}{n!} e^{-\theta t} \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}.$$

For $n = 0$ we obtain more easily

$$\mathbb{P}(N(t) = 0) \;=\; \mathbb{P}(S_1 > t) \;=\; 1 - F(t) \;=\; e^{-\theta} \quad \text{for all } t \ge 0.$$

We have thus shown

Theorem 1.2.1. *If $(S_n)_{n\ge 1}$ is a renewal process having exponential increments with parameter θ, then $N(t)$ has a Poisson distribution with parameter θt for each $t > 0$, in particular*

$$\mathbb{U}(t) \;=\; \mathbb{E}N(t) \;=\; \theta t \quad \textit{for all } t \ge 0, \tag{1.9}$$

that is, $\mathbb{U}$ equals Lebesgue measure on $\mathbb{R}_>$.

We thus see that for the exponential case question (Q3) has an explicit answer in that (1.6) becomes an exact identity:

$$\mathbb{U}(t+h) - \mathbb{U}(t) \;=\; \frac{h}{\mu} \quad \text{for all } t \ge 0,\ h > 0.$$

The previous result allows us also to find the cdf of S_n $(n \in \mathbb{N})$, namely

$$F_n(t) \;=\; \mathbb{P}(N(t) \ge n) \;=\; e^{-\theta t} \sum_{k \ge n} \frac{(\theta t)^k}{k!} \quad \text{for all } t \ge 0. \tag{1.10}$$

As for the renewal counting process $\{N(t) : t \ge 0\}$, the fact that $N(t) \stackrel{d}{=} \mathit{Poisson}(\theta t)$ is actually a piece only of the following more complete result.

Theorem 1.2.2. *If $(S_n)_{n\geq 1}$ is a renewal process having exponential increments with parameter θ, then the associated renewal counting process $(N(t))_{t\geq 0}$ forms a* ***homogeneous Poisson process with intensity (rate)*** *θ, that is:*

(PP1) $N(0) = 0$.

(PP2) $(N(t))_{t\geq 0}$ *has independent increments, i.e.,*

$$N(t_1), N(t_2) - N(t_1), ..., N(t_n) - N(t_{n-1})$$

are independent random variables for each choice of $n \in \mathbb{N}$ and $0 < t_1 < t_2 < ... < t_n < \infty$.

(PP3) $(N(t))_{t\geq 0}$ *has stationary increments, i.e., $N(s+t) - N(s) \stackrel{d}{=} N(t)$ for all $s,t \geq 0$.*

(PP4) $N(t) \stackrel{d}{=} Poisson(\theta t)$ *for each $t \geq 0$.*

We refrain from providing a proof of the result at this point [☞ ??????] and just mention that the crucial fact behind it is the lack of memory property of the exponential distribution.

1.3 Markov chains: a good example to motivate

A good motivation for the theoretical relevance of renewal theory in connection with stochastic processeses with inherent regeneration scheme is provided by a look at an important subclass, namely finite irreducible Markov chains.

A stochastic sequence $(M_n)_{n\geq 0}$ such that all M_n take values in a *finite* set $\mathscr{S}$ is called *finite Markov chain* if it satisfies the *Markov property*, viz.

$$\mathbb{P}(M_{n+1} = j | M_n = i, M_{n-1} = i_{n-1}, ..., M_0 = i_0) = \mathbb{P}(M_{n+1} = j | M_n = i)$$

for all $n \in \mathbb{N}_0$ and $i_0, ..., i_{n-1}, i, j \in \mathscr{S}$, and is *temporally homogeneous*, viz.

$$\mathbb{P}(M_{n+1} = j | M_n = i) \ = \ \mathbb{P}(M_1 = j | M_0 = i) \ =: \ p_{ij}$$

for all $n \in \mathbb{N}_0$ and $i, j \in \mathscr{S}$. The set $\mathscr{S}$ is called the *state space* of $(M_n)_{n\geq 0}$ and $\mathbf{P} = (p_{ij})_{i,j\in\mathscr{S}}$ its *(one-step) transition matrix*. We continue with a summary of some basic properties of such chains. A more detailed exposition will be provided in ????????

First of all, if τ is a stopping time for $(M_n)_{n\geq 0}$, i.e. τ takes values in $\mathbb{N}_0 \cup \{\infty\}$ and $\{\tau = n\} \in \sigma(M_0, ..., M_n)$ for all $n \in \mathbb{N}_0$, then the Markov property persists in the sense that

$$\mathbb{P}(M_{\tau+1}=j|M_\tau=i,M_{\tau-1},...,M_0,\tau<\infty)\ =\ \mathbb{P}(M_{\tau+1}=j|M_\tau=i)\ =\ p_{ij}$$

for all $i,j\in\mathscr{S}$. This is called the *strong Markov property*. For any path $i_0\to i_1\to ...\to i_n$ in $\mathscr{S}$ its probability is easily obtained by multiplying one-step transition probabilities, viz.

$$\mathbb{P}(M_1=i_1,...,M_n=i_n|M_0=i_0)\ =\ \lambda_{i_0}p_{i_0i_1}\cdot...\cdot p_{i_{n-1}i_n},$$

where $\lambda=\{\lambda_i : i\in\mathscr{S}\}$ denotes the distribution of M_0, called *initial distribution* of the chain. In the following, we make use of the common notation $\mathbb{P}_i:=\mathbb{P}(\cdot|M_0=i)$ and $\mathbb{P}_\lambda:=\sum_{i\in\mathscr{S}}\mathbb{P}_i$ for any distribution $\lambda=\{\lambda_i\}$ on $\mathscr{S}$. Hence $(M_n)_{n\geq 0}$ starts at i under $\mathbb{P}_i$ and has initial distribution λ under $\mathbb{P}_\lambda$.

Not surprisingly, temporal homogeneity extends to all n-step transition probabilities $(n\in\mathbb{N})$, i.e. $p_{ij}^{(n)}:=\mathbb{P}(M_n=j|M_0=i)=\mathbb{P}(M_{k+n}=j|M_k=i)$ for all $k\in\mathbb{N}_0$. Moreover, they satisfy the *Chapman-Kolmogorov equations*

$$p_{ij}^{(n)}\ =\ \sum_{k\in\mathscr{S}}p_{ik}^{(m)}p_{kj}^{(n-m)}\quad\text{for all } i,j\in\mathscr{S}\text{ and } m,n\in\mathbb{N}_0,\tag{1.11}$$

where $p_{ij}^{(0)}:=\delta_{ij}$. If $\mathbf{P}^{(n)}:=(p_{ij}^{(n)})_{i,j\in\mathscr{S}}$ denotes the n-step transition matrix, then these equations may be restated in matrix form as

$$\mathbf{P}^{(n)}\ =\ \mathbf{P}^{(m)}\mathbf{P}^{(n-m)}\quad\text{for all } m,n\in\mathbb{N}_0.\tag{1.12}$$

Consequently $\mathbf{P}^{(n)}=\mathbf{P}^n$ for each $n\in\mathbb{N}_0$, i.e. the transition matrices form a semigroup generated by $\mathbf{P}$. Let us note that, under $\mathbb{P}_\lambda$, the distribution of M_n is given by $\lambda\mathbf{P}^n$ for every $n\in\mathbb{N}_0$.

The chain $(M_n)_{n\geq 0}$ and its transition matrix $\mathbf{P}$ are called *irreducible* if all states *communicate* with respect to $\mathbf{P}$, where $i,j\in\mathscr{S}$ are said to be communicating if there exist $m,n\in\mathbb{N}_0$ such that $p_{ij}^{(m)}>0$ and $p_{ji}^{(n)}>0$. In other words, the chain can reach any state from any other state in a finite number of steps with positive probability. As a further prerequisite, some state properties must be defined. Let $T(i):=\inf\{n\geq 1 : M_n=i\}$ denote the first return time to $i\in\mathscr{S}$, where $\inf\emptyset:=\infty$. Then i is called

- recurrent if $\mathbb{P}_i(T(i)<\infty)=1$.
- transient if $\mathbb{P}_i(T(i)=\infty)>0$.
- positive recurrent if i is recurrent and $\mathbb{E}_iT(i)<\infty$.
- null recurrent if i is recurrent and $\mathbb{E}_iT(i)=\infty$.
- aperiodic if $\mathbb{P}_i(T(i)\in d\mathbb{N})<1$ for any integer $d\geq 2$.

It will be shown in ?????? that each of these properties is a *solidarity property* which means that it is shared by communicating states and thus by all states if the chain is irreducible. In the latter case we can therefore attribute any property to the chain as well. Putting $f_{ij}^{(n)}:=\mathbb{P}_i(T(j)=n)$, it is an easy exercise to verify that

$$p_{ij}^{(n)} \;=\; \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad \text{for all } i,j \in \mathscr{S} \text{ and } n \in \mathbb{N}. \tag{1.13}$$

Now consider, for any state $i \in \mathscr{S}$, the sequence $\{T_n(i)\}$ of successive return times, i.e. $T_1(i) := T(i)$ and

$$T_n(i) \;:=\; \begin{cases} \inf\{k > T_{n-1}(i) : M_k = i\}, & \text{if } T_{n-1}(i) < \infty, \\ \infty, & \text{otherwise} \end{cases} \quad \text{for } n \geq 2.$$

The next lemma provides a first indication of how renewal theory enters in Markov chain analysis. A stochastic sequence with iid increments taking values in $\mathbb{R}_{\geq} \cup \{\infty\}$ and having positive mean is called *proper renewal process* if the increments are a.s. finite, and *terminating renewal process* otherwise.

Lemma 1.3.1. *For any $i \in \mathscr{S}$, the sequence $(T_n(i))_{n\geq 1}$ forms a renewal process under $\mathbb{P}_i$. It is proper if i is recurrent and terminating otherwise.*

Proof. Fix any $i \in \mathscr{S}$, write T_n for $T_n(i)$ and put $\beta := \mathbb{P}_i(T_1 < \infty)$. Use the Markov property and temporal homogeneity to find that

$$\begin{aligned}
&\mathbb{P}_i(T_1 = m, T_2 - T_1 = n) \\
&\qquad = \mathbb{P}_i(T_1 = m, M_m = i, T_2 - T_1 = n) \\
&\qquad = \mathbb{P}(M_{m+n} = i, M_{m+k} \neq i \text{ for } 1 \leq k < n | M_m = i)\,\mathbb{P}_i(T_1 = m) \\
&\qquad = \mathbb{P}_i(M_n = i, M_k \neq i \text{ for } 1 \leq k < n)\,\mathbb{P}_i(T_1 = m) \\
&\qquad = \mathbb{P}_i(T_1 = m)\,\mathbb{P}_i(T_1 = n)
\end{aligned}$$

for all $n,m \in \mathbb{N}$ and then also, after summation over $m,n \in \mathbb{N}$, that $\mathbb{P}_i(T_2 < \infty) = \beta^2$. This shows conditional independence and identical distribution (under $\mathbb{P}_i$) of T_1 and $T_2 - T_1$ given $T_2 < \infty$. For arbitrary $n \in \mathbb{N}$, it follows by an inductive argument that $T_1, T_2 - T_1, ..., T_n - T_{n-1}$ are conditionally iid given $T_n < \infty$ as well as $\mathbb{P}_i(T_n < \infty) = \beta^n$. The assertions of the lemma are now easily concluded. □

As an immediate consequence, we now obtain the following zero-one law.

Lemma 1.3.2. *If $i \in \mathscr{S}$ is recurrent, then*

$$\mathbb{P}_i(M_n = i \text{ infinitely often}) = 1,$$

while

$$\mathbb{P}_j(M_n = i \text{ infinitely often}) = 0$$

for all $j \in \mathscr{S}$ if i is transient.

Proof. Observe that $\{M_n = i \text{ infinitely often}\} = \{T_n(i) < \infty \text{ for all } n \in \mathbb{N}\}$. Now use the previous lemma to infer that

$$\mathbb{P}_i(T_n(i) < \infty \text{ for all } n \in \mathbb{N}) \;=\; \lim_{n\to\infty} \mathbb{P}_i(T(i) < \infty)^n$$

which clearly equals 1 if i is recurrent and 0 otherwise. If i is transient, the strong Markov property further implies

$$\mathbb{P}_j(M_n = i \text{ infinitely often}) \;=\; \mathbb{P}_j(T(i) < \infty)\,\mathbb{P}_i(M_n = i \text{ infinitely often}) \;=\; 0$$

for all $j \in \mathscr{S}$. □

We see from the previous result that any finite Markov chain has at least one recurrent state because otherwise all $i \in \mathscr{S}$ would be visited only finitely often which is clearly impossible if $|\mathscr{S}| < \infty$. Adding irreducibility as a further assumption, solidarity now leads to the following important conclusion:

Theorem 1.3.3. *Every irreducible finite Markov chain is recurrent.*

We can now turn to the most interesting question about the long run behavior of an irreducible finite Markov chain $(M_n)_{n\ge 0}$. Since M_n moves around in $\mathscr{S}$ visiting every state infinitely often and since, by the Markov property, the chain has no memory, one can expect M_n to converge in distribution to some limit law $\pi = \{\pi_i : i \in \mathscr{S}\}$ which does not depend on the initial distribution $\mathbb{P}(M_0 \in \cdot)$. An important invariance property of any such limit law π is stated in the following lemma. Put $\mathbb{P}_\lambda := \sum_{i\in\mathscr{S}} \lambda_i \mathbb{P}_i$ for any distribution $\lambda = (\lambda_i)_{i\in\mathscr{S}}$ on $\mathscr{S}$ and notice that $(M_n)_{n\ge 0}$ has initial distribution λ under $\mathbb{P}_\lambda$.

Lemma 1.3.4. *Suppose that, for some initial distribution $\lambda = (\lambda_i)_{i\in\mathscr{S}}$ on $\mathscr{S}$,*

$$\pi_i \;:=\; \lim_{n\to\infty} \mathbb{P}_\lambda(M_n = i) \tag{1.14}$$

exists for all $i \in \mathscr{S}$, i.e., $M_n \xrightarrow{d} \pi = (\pi_i)_{i\in\mathscr{S}}$. Then π is a left eigenvector of $\mathbf{P}$ for the eigenvalue 1, i.e. $\pi = \pi\mathbf{P}$, and

$$\mathbb{P}_\pi(M_n \in \cdot) \;=\; \pi \quad \textit{for all } n \in \mathbb{N}_0.$$

Any such π is called ***invariant*** *or* ***stationary distribution of*** *$(M_n)_{n\ge 0}$. If the chain is irreducible then all π_i are positive.*

Proof. Since $\pi\mathbf{P}^n$ equals the distribution of M_n under $\mathbb{P}_\pi$ as stated earlier, it suffices to prove the first assertion and the positivity of π in the irreducible case. But with $\mathscr{S}$ being finite condition (1.14) implies

$$\pi_j = \lim_{n\to\infty} \mathbb{P}_\lambda(M_{n+1}=j) = \sum_{i\in\mathscr{S}} \lim_{n\to\infty} \mathbb{P}_\lambda(M_n=i)p_{ij} = \sum_{i\in\mathscr{S}} \pi_i p_{ij}$$

for all $j\in\mathscr{S}$ which is the same as $\pi=\pi\mathbf{P}$. Now suppose that $(M_n)_{n\ge 0}$ is irreducible. As $\pi=\pi\mathbf{P}^n$ for all $n\in\mathbb{N}_0$, we see that, if $\pi_j=0$ for some $j\in\mathscr{S}$, then

$$0 = \pi_j = \sum_{i\in\mathscr{S}} \pi_i p_{ij}^{(n)} \quad \text{for all } n\in\mathbb{N}$$

and thus $p_{ij}^{(n)}=0$ for all π-positive i and all $n\in\mathbb{N}$. But this means that j cannot be reached from any π-positive i which is impossible by irredubility. □

In view of the previous lemma we are now facing two questions for a given irreducible finite Markov chain $(M_n)_{n\ge 0}$:

(Q1) Does $(M_n)_{n\ge 0}$ always have a stationary distribution π?
(Q2) Does (1.14) hold true for any choice of λ and with the same limit π?

Of course, a positive answer to (Q1) follows from a positive answer to (Q2) which, however, is not generally true and brings in fact aperiodicity into play. Namely, if the chain is not aperiodic, then it can be shown that (by solidarity) it has a unique period $d\ge 2$ in the sense that d is the maximal integer such that $\mathbb{P}_i(T(i)\in d\mathbb{N})=1$ for all $i\in\mathscr{S}$. This further entails $p_{ii}^{(n)}=0$ for all $i\in\mathscr{S}$ and all $n\in\mathbb{N}\backslash d\mathbb{N}$ [☞ ??????? for further details]. On the other hand, if (1.14) held true for any λ, we inferred upon choosing $\lambda=\delta_i$ that

$$\pi_i = \lim_{n\to\infty} p_{ii}^{(n)} = \liminf_{n\to\infty} p_{ii}^{(n)} = 0$$

which contradicts that all π_i must be positive.

Based on the previous observations we further confine ourselves now to aperiodic finite Markov chains $(M_n)_{n\ge 0}$. Then, for (Q2) to be answered affirmatively, it suffices to show that

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} \quad \text{for all } i,j\in\mathscr{S}.$$

But with the help of (1.13) and the dominated convergence theorem, this reduces to

$$\pi_j = \lim_{n\to\infty} \sum_{k\ge 1} \mathbf{1}_{\{1,\ldots,n\}}(k) f_{ij}^{(k)} p_{jj}^{(n-k)} = \lim_{n\to\infty} p_{jj}^{(n)} \quad \text{for all } j\in\mathscr{S}$$

and finally makes us return to renewal theory via the following observation: Since $\{M_n=j\}=\sum_{k\ge 1}\{T_k(j)=n\}$, where the summation indicates as usual the union of pairwise disjoint events, we infer that

$$p_{jj}^{(n)} = \sum_{k\ge 1} \mathbb{P}_j(T_k(j)=n) = \mathbb{U}_j(\{n\}), \quad \text{for all } j\in\mathscr{S} \text{ and } n\in\mathbb{N} \tag{1.15}$$

where $\mathbb{U}_j$ denotes the renewal measure of the discrete renewal process $(T_k(j))_{k\geq 1}$. Consequently, in order to find the limiting behavior of $p_{jj}^{(n)}$ we must find the limiting behavior of the renewal measure $\mathbb{U}_j$ along singleton sets tending towards infinity. This is in perfect accordance with (Q3) of Section 1.1 once observing that, due to the fact that the $T_k(j)$ are integer-valued, $\mathbb{U}_j(\{n\}) = \mathbb{U}_j(n) - \mathbb{U}_j(n-1)$ for all $n \in \mathbb{N}$.

1.4 Branching processes: a surprising connection

In this section, we will take a look at a very simple branching model of cell division. It may be surprising at first glance that branching as a typically exponential-type phenomenon can be studied with the help of renewal theory which rather deals with stochastic phenomena of linear type. Let it be said to all weisenheimers that this is not accomplished by just using a logarithmic transformation.

We consider a population of cells having independent lifetimes with a standard exponential distribution. At the end of its lifetime, each cell either splits into two new cells with probability p or dies with probability $1-p$ independent of all other cells alive. Suppose that at time $t=0$ the evolution starts with one cell having lifetime T and number of offspring Y, thus $\mathbb{P}(Y=2) = p = 1-\mathbb{P}(Y=0)$, and Y is independent of T. Let $Z(t)$ be the number of cells alive at time $t \in \mathbb{R}_{\geq}$. Then

$$Z(t) \ = \ \mathbf{1}_{\{T>t\}} \ + \ \mathbf{1}_{\{T\leq t, X=2\}}(Z_1(t-T)+Z_2(t-T)) \tag{1.16}$$

where $Z_i(t-T)$ denotes the size at time t of the subpopulation of cells stemming from the i^{th} daughter cell born at $T \leq t$ $(i=1,2)$. As following from the model assumptions, the $(Z_i(t))_{t\geq 0}$ are mutually independent copies of $(Z(t))_{t\geq 0}$ and further independent of (T,Y).

Our goal is to find the expected population size $M(t) := \mathbb{E}Z(t)$. Since T is independent of Y and $T \stackrel{d}{=} Exp(1)$, we infer from (1.16)

$$\begin{aligned} M(t) \ &= \ \mathbb{P}(T>t) + \int_{[0,t]} M(t-s)\, 2p\, \mathbb{P}(T \in ds) \\ &= \ e^{-t} + \int_0^t M(t-s)\, 2pe^{-s}\, ds \quad \text{for all } t \geq 0 \end{aligned}$$

which is an integral equation of convolutional type that may be rewritten as

$$M \ = \ \overline{F} + M * Q, \tag{1.17}$$

here with $\overline{F}(t) := e^{-t}$ for $t \geq 0$ and $Q(ds) := 2pe^{-s}\mathbf{1}_{(0,\infty)}(s)\, ds$. An equation of this type can be solved with the help of renewal theory as we will see in a moment and is therefore called *renewal equation*. Notice first that Q has total mass

$$\|Q\| \ = \ 2p \int_0^\infty e^{-s}\, ds \ = \ 2p \ = \ \mathbb{E}Y$$

and is thus a probability distribution only if $\mathbb{E}Y = 1$. On the other hand, a look at the mgf of Q, viz.

$$\phi_Q(t) \;=\; \int e^{st}\, Q(ds) \;=\; 2p\int_0^\infty e^{(t-1)s}\, ds \;=\; \frac{2p}{1-t} \quad (-\infty < t < 1),$$

shows the existence of a unique θ such that $\phi_Q(\theta) = 1$, namely $\theta = 1-2p$. Now observe that

$$e^{\theta t}M(t) \;=\; e^{(\theta-1)t} + \int_0^t e^{\theta(t-s)}M_\theta(t-s)\,2pe^{(\theta-1)s}\, ds \quad \text{for all } t\ge 0$$

which may be rewritten as

$$M_\theta \;=\; \overline{F}_\theta + M_\theta * Q_\theta, \tag{1.18}$$

where $M_\theta(t) := e^{\theta t}M(t)$, $\overline{F}_\theta(t) := e^{-2pt}$ and $Q_\theta(ds) := 2pe^{-2ps}\mathbf{1}_{(0,\infty)}(s)\,ds$. Since $\|Q_\theta\| = \phi_Q(\theta) = 1$ (by choice of θ), we see that a change of measure turns the original renewal equation (1.17) into an equivalent so-called *proper renewal equation* with a probability distribution as convolution measure, in fact $Q_\theta = Exp(2p)$. Let $(S_n)_{n\ge 1}$ be a renewal process with this increment distribution. Then (1.18) becomes

$$M_\theta(t) \;=\; \overline{F}_\theta(t) \;+\; \mathbb{E}M_\theta(t-S_1) \quad \text{for all } t\ge 0$$

and upon n-fold iteration

$$M_\theta(t) \;=\; \sum_{k=0}^{n-1}\mathbb{E}\overline{F}_\theta(t-S_k) \;+\; \mathbb{E}M_\theta(t-S_n) \quad \text{for all } t\ge 0 \text{ and } n\in\mathbb{N}.$$

It is intuitively clear and taken for granted here that M is continuous on $\mathbb{R}_\ge$ and thus bounded on compact subintervals. Consequently,

$$\lim_{n\to\infty}\mathbb{E}M_\theta(t-S_n) \;\le\; \lim_{n\to\infty}\mathbb{P}(S_n\le t)\,\max_{s\in[0,t]}M_\theta(s) \;=\; 0$$

because $S_n\to\infty$ a.s., and we therefore conclude

$$M_\theta(t) \;=\; \sum_{k\ge 0}\mathbb{E}\overline{F}_\theta(t-S_k) \;=\; \overline{F}_\theta(t) + \overline{F}_\theta * \mathbb{U}(t),$$

where $\mathbb{U} := \sum_{n\ge 1}\mathbb{P}(S_n\in\cdot)$ denotes the renewal measure of $(S_n)_{n\ge 1}$. But the latter has exponentially distributed increments with parameter $2p$ and so $\mathbb{U} = 2p\lambda\!\!\lambda_0$ on $\mathbb{R}_>$ by Thm. 1.2.1. This finally allows us to compute $\overline{F}_\theta * \mathbb{U}(t)$ explicitly leading to

$$M_\theta(t) \;=\; e^{-2pt} \;+\; \int_0^t 2pe^{-2p(t-s)}\, ds \;=\; 1 \quad \text{for all } t\ge 0$$

and therefore

$$M(t) \;=\; e^{(2p-1)t} \quad \text{for all } t\ge 0 \tag{1.19}$$

The parameter $2p-1$ thus giving the exponential rate of mean growth or decay of the population is called its *Malthusian parameter*.

Astute readers will have noticed that for this particularly nice example of a cell splitting model (1.19) could have been obtained far more easily by showing with the help of the memoryless property of the exponential distribution that $(e^{\theta t}Z(t))_{t\ge 0}$ is a martingale with $Z(0)=1$ and thus having constant expectation equal to one. However, if we replace the exponential lifetime distribution with an arbitrary distribution F with finite mean μ, then this latter argument breaks down while the renewal argument still works. In fact, we may even additionally assume an arbitrary offspring distribution $\{p_k\}$ to arrive at the following general renewal equation for $M(t)=\mathbb{E}Z(t)$:

$$M(t) \;=\; \overline{F}(t) \;+\; \int_{[0,t]} M(t-s)\;Q(ds) \quad \text{for all } t\ge 0 \tag{1.20}$$

where $Q(ds)=\mu F(ds)$. If $\mu\neq 1$ and thus Q is not a probability distribution, then a transformation of (1.20) into a proper one requires the existence of a (necessarily unique) θ such that $\phi_Q(\theta)=\mu\phi_F(\theta)=1$ which may fail if $\mu<1$. In the case where θ exists the result is as before that

$$M_\theta(t) \;=\; \overline{F}_\theta(t)+\overline{F}_\theta * \mathbb{U}(t) \tag{1.21}$$

where $\mathbb{U}$ denotes the renewal measure of a renewal process with increment distribution $Q_\theta(ds)=\mu e^{\theta s}F(ds)$. Unlike the exponential case, however, this does not generally lead to an explicit formula for $M(t)$ because $\mathbb{U}$ is not known explicitly. Instead, one must resort once again to asymptotic considerations as $t\to\infty$. For a further discussion of these aspects in this more general situation, we refer to Chapter 5.

1.5 Collective risk theory: a classical application

This application is a relative of the previous one in that it eventually leads to a renewal equation that must be solved in order to gain information on the quantity of interest.

In collective risk theory, a part of nonlife insurance mathematics, the following problem is of fundamental interest: An insurance company earns premiums at a constant rate $c\in\mathbb{R}_>$ from a portfolio of insurance policies and faces negative claims from these of absolute sizes $X_1,X_2,\ldots$ at successive random epochs $0<T_1<T_2<\ldots$ Given an initial risk reserve $R(0)$, the risk reserve $R(t)$ at time t, i.e., the available capital at t to cover incurred future claims, is given by

$$R(t) \;=\; R(0)+ct-\sum_{k=1}^{N(t)} X_k \quad \text{for all } t\ge 0,$$

where $N(t) := \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$ denotes the number of claims up to time t. If $R(t)$ becomes negative, so-called *technical ruin* occurs. It is therefore a main concern of the insurance company to choose $R(0)$ and c in such a way that the probability for this event, called *ruin probability*, is small. Plainly, this requires a computation of this probability after the specification of a stochastic model for the bivariate sequence $(T_n, X_n)_{n\geq 1}$. Again, we do not strive for greatest generality in this introductory section but will instead discuss the problem in the framework of what is known today as the *Cramér-Lundberg model* which has its origin in a dissertation by F. LUNDBERG [34]:

(CL1) $(N(t))_{t\geq 0}$ is a homogeneous Poisson process with intensity λ or, equivalently, $T_1, T_2 - T_1, \ldots$ are iid with $T_1 \stackrel{d}{=} Exp(\lambda)$.
(CL2) $X_1, X_2, \ldots$ are iid with common distribution F and finite positive mean μ.
(CL3) $(T_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ are independent.

Put $Y_n := T_n - T_{n-1}$ for $n \in \mathbb{N}$ (with $T_0 = 0$) and let (X, Y) denote a generic copy of (X_n, Y_n) hereafter. Defining the epoch of technical ruin, viz.

$$\Lambda \;:=\; \inf\{t \geq 0 : R(t) < 0\} \quad (\inf \emptyset := \infty),$$

the task is to compute for a fixed premium rate c

$$\Psi(r) \;:=\; \mathbb{P}(\Lambda < \infty | R(0) = r) \quad \text{for } r > 0.$$

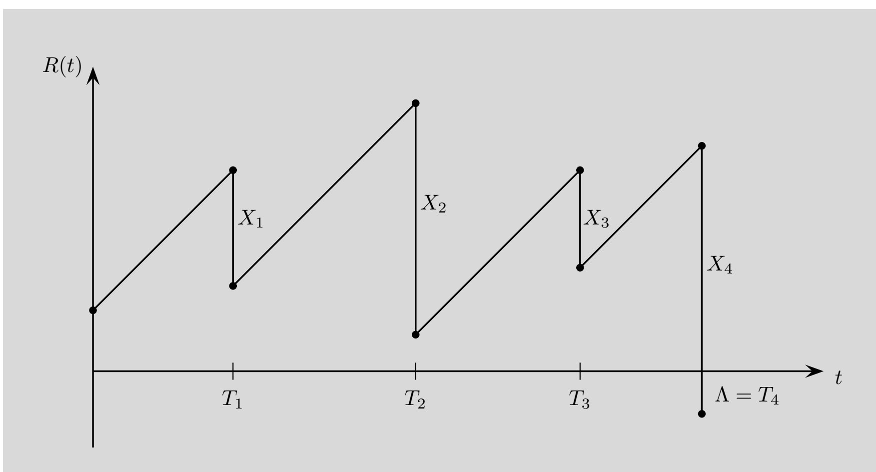

Fig. 1.2 The risk reserve process $(R(t))_{t\geq 0}$ with ruin epoch Λ

Let us begin with the observation that technical ruin can only occur at the epochs T_n, that is (given $R(0) = r$)

$$\Lambda \;=\; T_\tau, \quad \text{where } \tau \;:=\; \inf\{n \geq 1 : r + cT_n - S_n < 0\}$$

and $(S_n)_{n\geq 1}$ denotes the renewal process with increments $X_1, X_2, \ldots$ Hence

$$\Psi(r) \;=\; \mathbb{P}(\tau < \infty | R(0) = r) \quad \text{for } r \geq 0.$$

In the following considerations we keep $R(0) = r$ fixed and simply write $\mathbb{P}$ instead of $\mathbb{P}(\cdot|R(0) = r)$. Rewriting τ as

$$\tau \;=\; \inf\{n \geq 1 : S_n - cT_n > r\}$$

we see that ν is a so-called *first passage time* for the random walk $(S_n - cT_n)_{n\geq 1}$ with drift $\nu := \mathbb{E}(X - cY) = \mu - c\lambda^{-1}$. Hence

$$\Psi(r) = 1 \quad \text{for all } r > 0$$

if $\nu \geq 0$, because $S_n - cT_n \to \infty$ a.s. by the SLLN if $\nu > 0$, and $\limsup_{n\to\infty} S_n - cT_n = \infty$ a.s. by the Chung-Fuchs theorem [☞ Thm. 2.2.11] in the case $\nu = 0$. The interesting case to be discussed hereafter is therefore when

(CL4) $$\nu = \mathbb{E}(X - cY) = \mu - \frac{c}{\lambda} < 0$$

which means that the mean premium earned between two claim epochs is larger than the expected claim size. We first prove a renewal equation for $\overline{\Psi} := 1 - \Psi$.

Lemma 1.5.1. *Assuming (CL1–4), $\overline{\Psi}$ satisfies the renwal equation*

$$\overline{\Psi}(r) \;=\; \overline{\Psi}(0) \;+\; \int_0^r \overline{\Psi}(r-x)\, Q(dx) \quad \textit{for all } r \geq 0. \tag{1.22}$$

where $Q(dx) := \frac{\lambda}{c}\,\mathbb{P}(X > x)\,dx$ on $\mathbb{R}_{\geq}$.

Proof. Since X and Y are independent with $X \stackrel{d}{=} F$ and $Y \stackrel{d}{=} Exp(\lambda)$, a conditioning argument leads to

$$\begin{aligned}
\overline{\Psi}(r) \;&=\; \mathbb{P}(S_n \leq r + cT_n \text{ for all } n \geq 1) \\
&=\; \int_{\{(x,y):x\leq r+cy\}} \mathbb{P}(x + S_n \leq r + c(y + T_n) \text{ for all } n \geq 1)\; \mathbb{P}(X \in dx, Y \in dy) \\
&=\; \int_0^\infty \int_{[0,r+cy]} \overline{\Psi}(r - x + cy)\, \lambda e^{-\lambda y}\; F(dx)\; dy \\
&=\; \int_r^\infty \frac{\lambda}{c} e^{-(\lambda/c)(y-r)} \int_{[0,y]} \overline{\Psi}(y-x)\; F(dx)\; dy
\end{aligned}$$

and thus shows the differentiability of $\overline{\Psi}$ with

$$\overline{\Psi}'(r) = -\frac{\lambda}{c}\int_{[0,r]}\overline{\Psi}(r-x)\,F(dx) + \int_r^\infty \frac{\lambda^2}{c^2}e^{-(\lambda/c)(y-r)}\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\,dy$$
$$= -\frac{\lambda}{c}\int_{[0,r]}\overline{\Psi}(r-x)\,F(dx) + \frac{\lambda}{c}\overline{\Psi}(r)$$

for all $r \ge 0$. Consequently, we obtain upon integration

$$\overline{\Psi}(r)-\overline{\Psi}(0) = \frac{\lambda}{c}\left(\int_0^r\left(\overline{\Psi}(y)-\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\right)\,dy\right)$$

which leaves us with the verification of

$$\int_0^r\left(\overline{\Psi}(y)-\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\right)\,dy = \int_0^r\overline{\Psi}(r-x)\,\mathbb{P}(X>x)\,dx$$

for $r \ge 0$. But this follows from

$$\int_0^r\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\,dy = \int_{[0,r]}\int_0^{r-x}\overline{\Psi}(y)\,dy\,F(dx)$$
$$= \int_0^r\overline{\Psi}(y)\,\mathbb{P}(X\le r-y)\,dy = \int_0^r\overline{\Psi}(r-y)\,\mathbb{P}(X\le y)\,dy$$

and $\int_0^r\overline{\Psi}(y)\,dy = \int_0^r\overline{\Psi}(r-y)\,dy$. □

Notice that Q as defined in the lemma has total mass $\|Q\| = \frac{\lambda}{c}\,\mathbb{E}X = \frac{\lambda\mu}{c} < 1$ because $\nu < 0$. This means that (1.22) is a so-called *defective renewal equation*. Since $\|Q^{*n}\| = \|Q\|^n \to 0$, we infer that the renewal measure $\mathbb{U}_Q := \sum_{n\ge 1}Q^{*n}$ associated with Q is a finite measure with total mass $(1-\|Q\|)^{-1}\|Q\|$ and

$$\overline{\Psi}*Q^{*n}(r) = \int_{[0,r]}\overline{\Psi}(r-x)\,Q^{*n}(dx) \le \|Q\|^n \to 0 \quad \text{as } n\to 1$$

Hence, by a similar iteration argument as in the previous section we find that

$$\overline{\Psi}(r) = \overline{\Psi}(0)+\overline{\Psi}(0)*\mathbb{U}_Q(r) = \overline{\Psi}(0)+\int_0^r\overline{\Psi}(0)\,\mathbb{U}_Q(dx) = \overline{\Psi}(0)(1+\mathbb{U}_Q(r))$$

and thereupon that

$$\lim_{r\to\infty}\overline{\Psi}(r) = \overline{\Psi}(0)(1+\|U_Q\|) = \frac{\overline{\Psi}(0)}{1-\|Q\|} = \frac{c\overline{\Psi}(0)}{c-\lambda\mu}. \tag{1.23}$$

On the other hand, we have $S_n - cT_n \to -\infty$ a.s. if $\nu < 0$ and therefore $\overline{\Psi}(r) \to 1$ as $r\to\infty$. By combining this with (1.23) and solving for $\overline{\Psi}(0)$ yields

$$\overline{\Psi}(0) = \frac{c-\lambda\mu}{c} = 1-\frac{\lambda\mu}{c} = 1-\|Q\|. \tag{1.24}$$

Naturally, this is not the end of the story when striving for the asymptotic behavior of $\Psi(r)$ beyond the quite trivial statement that $\lim_{r\to\infty}\Psi(r)=0$ if $\nu<0$. As in the branching example of the previous section, we will now make use of a change of measure argument which, however, requires an additional condition on the distribution F of X. As before, let ϕ_F be the mgf of F. Note that $Q_\theta(dx)=e^{\theta x}Q(dx)$ has total mass [☞ formula ????? in Appendix ?]

$$\|Q_\theta\| \;=\; \frac{\lambda}{c}\int_0^\infty e^{\theta x}\,\mathbb{P}(X>x)\,dx \;=\; \frac{\lambda}{c\theta}(\phi_F(\theta)-1)$$

which is less than $\|Q\|<1$ for all $\theta<0$ because ϕ_F is increasing on $\mathbb{R}_\leq$. In order for finding a θ such that $\|Q_\theta\|=1$, this θ must therefore be positive if it exists at all. Therefore we introduce as a further condition:

(CL5) There exists $\theta>0$ such that $\phi_F(\theta) \;=\; 1+\dfrac{c\theta}{\lambda}$.

With the extra condition (CL5) the ruin probability $\Psi(r)$ after multiplication with $e^{\theta r}$ satisfies a proper renewal equation as stated in the next theorem. In the case where X is also exponentially distributed this can be converted into an explicit formula for $\Psi(r)$, while in the general case a statement on its asymptotic behavior as $r\to\infty$ is possible but must wait until Section ???

Theorem 1.5.2. *Assuming (CL1–5), the ruin probability $\Psi_\theta(r):=e^{\theta r}\Psi(r)$ satisfies the proper renewal equation*

$$\Psi_\theta(r) \;=\; e^{\theta r}Q((r,\infty)) \;+\; \int_{[0,r]}\Psi_\theta(r-x)\,Q_\theta(dx) \quad \textit{for all } r\geq 0 \qquad (1.25)$$

and thus

$$\Psi_\theta(r) \;=\; e^{\theta r}Q((r,\infty)) \;+\; \int_{[0,r]} e^{\theta(r-x)}Q((r-x,\infty))\,\mathbb{U}_\theta(dx), \qquad (1.26)$$

where Q is as in Lemma 1.5.1 and $\mathbb{U}_\theta:=\sum_{n\geq 1}Q_\theta^{*n}$ denotes the renewal measure associated with Q_θ.

Proof. Rewriting (1.22) with the help of (1.24), we obtain

$$\begin{aligned}\Psi(r) \;&=\; 1-\left(\overline{\Psi}(0) \;+\; \int_0^r \overline{\Psi}(r-x)\,Q(dx)\right)\\ &=\; \Psi(0)-Q(r) \;+\; \int_{[0,r]}\Psi(r-x)\,Q(dx)\\ &=\; Q((r,\infty)) \;+\;+\; \int_{[0,r]}\Psi(r-x)\,Q(dx) \quad \text{for all } r\geq 0\end{aligned}$$

and then (1.25) after multiplication with $e^{\theta r}$. Validity of (1.26) follows now in the same manner as in the previous section when using that Ψ_θ is bounded on compact subintervals of $\mathbb{R}_{\geq}$ in combination with $Q_\theta^{*n}(r) = \mathbb{P}(S_n \leq r) \to 0$ as $n \to \infty$, where $(S_n)_{n\geq 1}$ denotes a renewal process with increment distribution Q_θ. $\square$

Finally looking at the special case where $F = Exp(1/\mu)$, we first note that then

$$Q_\theta(dx) \;=\; \frac{\lambda}{c}\, e^{-((1/\mu)-\theta)x}\, dx$$

and thus equals an exponential distribution (with parameter λ/c) if

$$\theta \;=\; \frac{1}{\mu} - \frac{\lambda}{c} \;>\; 0$$

which is easily seen to be an equivalent version of (CL4). Hence (CL5) does automatically hold here once (CL4) is valid. Now use $\mathbb{U}_\theta = \frac{\lambda}{c}\lambda\!\!\lambda_0$ and $Q = \frac{\lambda\mu}{c} Exp(1/\mu)$ to infer with the help of Thm. 1.5.2 that

$$\Psi_\theta(r) \;=\; \frac{\lambda\mu}{c} e^{-(\lambda/c)r} \;+\; \frac{\lambda}{c}\int_0^r \frac{\lambda\mu}{c} e^{-(\lambda/c)x}\, dx \;=\; \frac{\lambda\mu}{c} \quad \text{for all } r \geq 0$$

and therefore

$$\Psi(r) \;=\; \frac{\lambda\mu}{c}\, e^{-((1/\mu)-(\lambda/c))r} \quad \text{for all } r \geq 0 \tag{1.27}$$

1.6 Queuing theory: a typical application

Queuing theory as an important branch of Applied Probability deals with the performance analysis of service facilities which are subject to random input. Here we consider a single server station who is facing (beginning at time $T_0 = 0$) arrivals of customers at random epochs $0 < T_1 < T_2 < \ldots$ with service requests of (temporal) size $B_1, B_2, \ldots$ Customers who find the server busy join a queue and are served in the order they have arrived (first in, first out). Typical performance measures are quantities like workload, queue length or sojourn times of customers in the system. They may be studied over time (transient analysis) or in the long run (steady state analysis). Typically, the complexity of queuing systems does not allow a transient analysis whence one usually resorts to a steady state analysis. Like for finite Markov chains, the idea is that after a relaxation period the system is approximately in stochastic equilibrium so that relevant quantities may be approximated by their value under the stationary distribution. The computation of such approximations often requires the use of renewal theory as we will briefly demonstrate in this section.

We consider the so-called *M/G/1-queue* specified by the following assumptions:

(M/G/1-1) The arrival process $(N(t))_{t\geq 0}$, where $N(t) := \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$ for $t \geq 0$, is a homogeneous Poisson process with intensity λ (Poisson input).

(M/G/1-2) The service times $B_1, B_2, \ldots$ are iid with common distribution G and finite positive mean μ.
(M/G/1-3) The sequences $(T_n)_{n\ge 1}$ and $(B_n)_{n\ge 1}$ are independent.
(M/G/1-4) There is one server and a waiting room of infinite capacity.
(M/G/1-5) The queue discipline is FIFO ("first in, first out").

The *Kendall notation* "M/G/1", which may be expanded by further symbols when referring to more complex systems, has the following meaning:

"M": The first letter refers to the arrival pattern, and "M" stands for "Markovian". This means that the interarrival times $A_n := T_n - T_{n-1}$ are iid with an exponential distribution which renders $(N(t))_{t\ge 0}$ a homogeneous Poisson process and, in particular, a continuous time Markov process.
"G": The second letter refers to the sequence of service times, and "G" stands for "general". This means that the service times B_n are iid with an arbitrary distribution on $\mathbb{R}_{\ge}$ with positive mean.
"1": The number in the third position refers to the number of servers (or counters).

In the following, we will focus on the analysis of the queue length in steady state. Suppose for simplicity that at time $T_0 = 0$ the system is empty. For $t \ge 0$, let $Q(t)$ be the queue length at t, i.e. the number of waiting customers in the system at this time including the one currently in service. This is a stochastic process with $Q(0) = 0$ (due to our previous assumption) and trajectories that are right continuous with left hand limits. It is also a pure jump process with jumps being of size $+1$ at an arrival epoch and -1 at a departure epoch. On the other hand, it is *not* a Markov process unless the B_n are also exponentially distributed because the future evolution of the process at any time t does not only depend on the past through $Q(t)$ but also the time the customer currently in service already spent at the counter. The classical way out of this dilemma is a resort to an embedded discrete Markov chain in the sense defined in Section 1.3 but with countable state space $\mathbb{N}_0$. It is obtained by looking at

$$Q_n \;:=\; Q(D_n) \quad (n \in \mathbb{N}_0)$$

where $D_0 := 0$ and the D_n for $n \ge 1$ denote the successive departure epochs of customers in the system (thus $D_1 = T_1 + B_1$). As one can easily see, the Q_n and D_n satisfy the recursive relations

$$Q_n \;=\; (Q_{n-1} - 1)^+ + K_n \quad \text{and} \tag{1.28}$$

$$D_n \;=\; (D_{n-1} \vee T_n) + B_n \quad \text{for } n \in \mathbb{N}, \tag{1.29}$$

where K_n is the number of customers that enter the system during the service of the n^{th} customer (with service time B_n), thus

$$K_n \;=\; N(D_n) - N(D_{n-1} \vee T_n) \quad \text{for } n \in \mathbb{N}.$$

By using the model assumptions, notably (M/G/1-3) and the properties of a Poisson process, it is not difficult to verify that the K_n are iid with distribution $\{\kappa_j : j \in \mathbb{N}_0\}$

given by

$$\kappa_j \;:=\; \int_{\mathbb{R}_>} \mathbb{P}(N(y)=j)\; G(dy) \;=\; \int_{\mathbb{R}_>} e^{-\lambda y}\frac{(\lambda y)^j}{j!}\; G(dy).$$

In particular,

$$\mathbb{E}K_1 \;=\; \sum_{j\ge 1} j\kappa_j \;=\; \int_{\mathbb{R}_>} \mathbb{E}N(y)\; G(dy) \;=\; \int_{\mathbb{R}_>} \lambda y\; G(dy) \;=\; \lambda\mu \;=:\; \rho. \tag{1.30}$$

Moreover, K_n is independent of $Q_0,\dots,Q_{n-1}$ which in combination with (1.28) easily proves:

Lemma 1.6.1. *The queue length process $(Q_n)_{n\ge 0}$ at departure epochs constitutes an irreducible and aperiodic discrete Markov chain with state space $\mathbb{N}_0$ and transition matrix*

$$\mathbf{P} \;=\; (p_{ij})_{i,j\in\mathbb{N}_0} \;=\; \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \dots \\ \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \dots \\ 0 & \kappa_0 & \kappa_1 & \kappa_2 & \dots \\ 0 & 0 & \kappa_0 & \kappa_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us continue by finding the necessary and sufficient condition under which the queuing system is stable or, equivalently, the chain $(Q_n)_{n\ge 0}$ is positive recurrent. Here we should mention that all notions introduced in Section 1.3 for finite Markov chains like irreducibility, recurrence and aperiodicity carry over without changes to the case of countable state space. Observe that ρ as defined in (1.30) satisfies

$$\rho \;=\; \frac{\mathbb{E}B_1}{\mathbb{E}T_1} \;=\; \frac{\text{mean service time}}{\text{mean interarrival time}}. \tag{1.31}$$

It is called the *traffic intensity* of the system because it provides a measure of its throughput rate. Intuitively, one can expect stability of the system if ρ is less than 1 because then, on the average, the server works at a faster rate than customers enter the system. The following renewal theoretic analysis will confirm this assertion.

For the random walk $S_n := \sum_{j=1}^n (K_j - 1)$ $(n\in\mathbb{N}_0)$, we define the associated sequence $\{\sigma_n\}$ of so-called *descending ladder epochs* by $\sigma_0 := 0$ and, recursively,

$$\sigma_n \;:=\; \begin{cases} \inf\{k>\sigma_{n-1} : S_k < S_{\sigma_{n-1}}\}, & \text{if } \sigma_{n-1}<\infty, \\ \infty, & \text{otherwise} \end{cases} \qquad \text{for } n\ge 1.$$

Since $(S_n)_{n\ge 1}$ can obviously make downward jumps of size -1 only, we infer $S_{\sigma_n} = -n$ if $\sigma_n < \infty$. Furthermore, as $\mathbb{E}S_1 = \mathbb{E}K_1 - 1 = \rho - 1$, we see that $\sigma_n < \infty$ a.s.

for all $n \in \mathbb{N}$ and thus $\liminf_{n\to\infty} S_n = -\infty$ a.s. requires $\rho \le 1$ [by the SLLN or the Chung-Fuchs theorem 2.2.11]. More precisely, we will show in Thm. 2.2.9 that $\rho < 1$ implies $\mathbb{E}\sigma_1 < \infty$, while $\rho = 1$ implies $\sigma_1 < \infty$ a.s. and $\mathbb{E}\sigma_1 = \infty$. Furthermore, in any of these two cases, the σ_n have iid increments [☞ Thm. 2.2.7] and hence form a discrete renewal process. Let $\mathbb{U} = \sum_{n\ge 1} \mathbb{P}(\sigma_n \in \cdot)$ denote its renewal measure with *renewal counting density*

$$u_n \;:=\; \mathbb{U}(\{n\}) \quad \text{for } n \in \mathbb{N}_0.$$

With the help of the previous facts we are now able to prove the following result about $\{Q_n\}$.

Theorem 1.6.2. *The queue length process at departure epochs $(Q_n)_{n\ge 0}$ is a recurrent Markov chain iff $\rho \le 1$, and it is positive recurrent iff $\rho < 1$. Furthermore $\rho \le 1$ implies*

$$\mathbb{P}(Q_n = j) \;=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_k = j, \sigma_1 > k)\, u_{n-k} \tag{1.32}$$

for all $j,n \in \mathbb{N}_0$, in particular $\mathbb{P}(Q_n = 0) = u_n$.

Proof. Recall that $Q_0 = 0$ is assumed, i.e. $\mathbb{P} = \mathbb{P}(\cdot|Q_0 = 0)$. It suffices to study the recurrence of state 0 as $(Q_n)_{n\ge 0}$ is irreducible. The crucial observation is that $\sigma_1 = \inf\{n \ge 1 : Q_n = 0\}$ and that $Q_n = S_n$ for $0 \le n < \sigma_1$. Indeed, if $\rho \le 1$ and thus $\mathbb{P}(\sigma_1 < \infty) = 1$, the recurrence of 0 follows just by definition. On the other hand, if $\rho > 1$ and thus $\mathbb{P}(\sigma_1 = \infty) > 1$, then there is a positive chance of never hitting 0 again after time 0 so that 0 must be transient. As mentioned above, $\rho < 1$ further ensures $\mathbb{E}\sigma_1 < \infty$ and thus the positive recurrence of the chain. Finally,

$$\begin{aligned}
\mathbb{P}(Q_n = j) \;&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{m\ge 1} \mathbb{P}(Q_n = j, \sigma_m \le n < \sigma_{m+1}) \\
&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \sum_{m\ge 1} \mathbb{P}(Q_n = j, \sigma_m = n-k, \sigma_{m+1} - \sigma_m > k) \\
&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_k = j, \sigma_1 > k) \sum_{m\ge 1} \mathbb{P}(\sigma_m = n-k)
\end{aligned}$$

for all $j,n \in \mathbb{N}_0$ shows (1.32). Here we have used for the last line that, with $\mathscr{F}_n := \sigma((Q_k,S_k) : 0 \le k \le n)$,

$$\begin{aligned}
&\mathbb{P}(Q_n = j, \sigma_m = n-k, \sigma_{m+1} - \sigma_m > k) \\
&\quad = \int_{\{\sigma_m = n-k\}} \mathbb{P}(Q_n = j, \sigma_{m+1} - \sigma_m > k | \mathscr{F}_{n-k})\, d\mathbb{P} \\
&\quad = \int_{\{\sigma_m = n-k\}} \mathbb{P}(Q_k = j, \sigma_1 > k | Q_0)\, d\mathbb{P} \\
&\quad = \mathbb{P}(Q_k = j, \sigma_1 > k)\, \mathbb{P}(\sigma_m = n-k)
\end{aligned}$$

for all $j, m, n \in \mathbb{N}_0$ and $0 \le k \le n$. □

It should be observed that (1.32) may be restated as $\mathbb{P}(Q_n = j) = g_j * \mathbb{U}(n)$, where

$$g_j(k) := \mathbb{P}(Q_k = j, \sigma_1 > k) \quad \text{for } j,k \in \mathbb{N}_0,$$

and that this could have also been deduced as in the previous two examples from the fact that $\mathbb{P}(Q_n = j)$ (as a function of n for fixed j) satisfies the *discrete renewal equation*

$$\mathbb{P}(Q_n = j) \;=\; g_j(n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_{n-k} = j)\, \mathbb{P}(\sigma_1 = k) \quad \text{for all } n \in \mathbb{N}_0.$$

So we have a similar result for a quantity of interest as in the previous two sections [☞ (1.21) and (1.26)], here in a discrete setup because the renewal process pertaining to $\mathbb{U}$ is integer-valued. Owing to this fact an application of the dominated convergence theorem to (1.32) immediately leads to the following result on the asymptotic behavior of Q_n once the convergence of u_n as $n \to \infty$ has been proved, the limit actually being $(\mathbb{E}\sigma_1)^{-1}$.

Theorem 1.6.3. *If $\rho < 1$, then*

$$\pi_j \;:=\; \lim_{n\to\infty} \mathbb{P}(Q_n = j) \;=\; \frac{1}{\mathbb{E}\sigma_1} \sum_{k \ge 0} \mathbb{P}(Q_k = j, \sigma_1 > k) \tag{1.33}$$

for all $j \in \mathbb{N}_0$.

It should not take by surprise that $\pi = (\pi_j)_{j \ge 0}$, which obviously forms a probability distribution, is the unique stationary distribution of $(Q_n)_{n \ge 0}$. Using

$$\mathbb{P}(Q_k = j, \sigma_1 > k) = \sum_{n > k} \mathbb{P}(Q_n = j, \sigma_1 = n)$$

in (1.33), we further obtain after interchanging the order of summation that

$$\pi_j \;=\; \frac{1}{\mathbb{E}\sigma_1} \mathbb{E}\left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_{\{Q_n = j\}} \right) \quad \text{for all } j \in \mathbb{N}_0, \tag{1.34}$$

which is the *occupation measure* representation of π having the very intuitive interpretation that the stationary probability for a queue length of j (at departures of customers) is just the expected number of epochs this value is attained during a cycle, defined as a (discrete) time interval between two epochs where the system becomes idle (busy period).

1.7 Record values: yet another surprise

Consider a sequence $(X_n)_{n\geq 1}$ of iid nonnegative random variables with a continuous distribution F. We say that a *record* occurs at time n if X_n exceeds all preceding values of the sequence, i.e., if $X_n > \max_{1\leq k<n} X_k$. In this case n is a *record epoch* and X_n a *record value*. More formally, define $\sigma_1 := 1, R_1 := X_1$ and then, recursively,

$$\sigma_n := \inf\{k > \sigma_{n-1} : X_k > R_{n-1}\} \quad \text{and} \quad R_n := X_{\sigma_n} \quad \text{for } n \geq 2.$$

Clearly, the σ_n are the record epochs and the R_n the record values of the sequence $(X_n)_{n\geq 1}$. Our main concern here is to get information on how the R_n spread on the nonnegative halfline. At first glance this seems to be quite unrelated to renewal theory, for record values do not generally show the pattern of a renewal process. For instance, if F is the uniform distribution on $(0,1)$ so that all X_n take values in this interval, then it is quite clear that the R_n accumulate at 1 from below. On the other hand, and this is the crucial observation, information on the R_n may be gained also after the application of a function G to the X_n that leaves their order unchanged.

Lemma 1.7.1. *Under the stated assumptions, let $G : \mathbb{R}_\geq \to \mathbb{R}_\geq$ be a nondecreasing function such that*

$$\mathbb{P}(G(X_1) < G(X_2)|X_1 < X_2) = 1.$$

Then the record epochs for $(X_n)_{n\geq 1}$ and $(G(X_n))_{n\geq 1}$ are a.s. the same, and $(G(R_n))_{n\geq 1}$ is the sequence of record values associated with $(G(X_n))_{n\geq 1}$.

Proof. Easy. □

As on can easily see, the lemma particularly applies to F itself (viewed as a cdf), and since F is continuous, the iid $F(X_n)$ are uniformly distributed on $(0,1)$. By then applying the transformation $x \mapsto -\log(1-x)$, we arrive at the sequence

$$Y_n := -\log(1-F(X_n)) \quad (n \in \mathbb{N})$$

of iid standard exponentials, for

$$\mathbb{P}(Y_1 > t) \;=\; \mathbb{P}(F(X_1) > 1-e^{-t}) \;=\; e^{-1} \quad \text{for all } t > 0.$$

Consequently, the problem of studying record epochs and values of iid continuous random variables may be reduced to the study of the corresponding variables for iid standard exponentials, for which we have the following result based on the lack of memory property of the exponential distribution.

Theorem 1.7.2. *Let $(X_n)_{n\geq 1}$ be a sequence of iid standard exponentials with canonical filtration $(\mathscr{F}_n)_{n\geq 1}$, associated record epochs σ_n and record values R_n for $n \in \mathbb{N}$. Then the following assertions hold true:*

(a) $(R_n)_{n\geq 1}$ *is a renewal process with standard exponential increments.*
(b) *For each $n \geq 2$, the conditional distribution of the n^{th} interrecord time $\tau_n := \sigma_n - \sigma_{n-1}$ given $\mathscr{F}_{\sigma_{n-1}}$ is geometric on $\mathbb{N}$ with parameter e^{-r} if $R_{n-1} = r$. Moreover,*

$$\mathbb{P}(\tau_n = k) \ = \ \int_0^\infty (1-e^{-r})^{k-1} e^{-2r} \frac{r^{n-2}}{(n-2)!}\, dr$$

for all $k \in \mathbb{N}$.

Proof. Put $Y_n := R_n - R_{n-1}$ for $n \in \mathbb{N}$, where $R_0 := 0$.

(a) Clearly, $T_1 = Y_1 \stackrel{d}{=} Exp(1)$. Suppose now that we have already shown that $Y_1,...,Y_n$ are iid with a standard exponential distribution. As one can easily verify, the sequence $(X_{\sigma_n+k})_{k\geq 1}$ is independent of $\mathscr{F}_{\sigma_n}$ and a copy of $(X_k)_{k\geq 1}$ [☞ Thm. 2.2.1]. Putting $E_k := \{\tau_{n+1} = k\}$ for $k \in \mathbb{N}$, we infer for each $t \geq 0$

$$\begin{aligned} \mathbb{P}(Y_{n+1} > t|\mathscr{F}_{\sigma_n}) \ &= \ \sum_{k\geq 1} \mathbb{P}(X_{\sigma_n+k} > R_n + t, X_{\sigma_n+j} \leq R_n \text{ for } 1 \leq j < k | R_n) \\ &= \ \sum_{k\geq 1} \mathbb{E}(\mathbb{P}(X_{\sigma_n+k} > R_n + t | R_n, E_k)\, \mathbf{1}_{E_k} | R_n) \quad \text{a.s.} \end{aligned}$$

and since X_{σ_n+k} is independent of R_n and E_k for each k, we further obtain by invoking Lemma A.1.1 and using $X_{\sigma_n+k} \stackrel{d}{=} Exp(1)$ that

$$\mathbb{P}(X_{\sigma_n+k} > R_n + t | R_n, E_k) \ = \ e^{-t}\, \mathbf{1}_{E_k} \quad \text{a.s.}$$

Consequently,

$$\mathbb{P}(Y_{n+1} > t|\mathscr{F}_{\sigma_n}) \ = \ e^{-t} \sum_{k\geq 1} \mathbb{P}(E_k|R_n) \ = \ e^{-t} \quad \text{a.s.}$$

for all $t \geq 0$ which proves that Y_{n+1} is independent of $\mathscr{F}_{\sigma_n}$, particularly of $R_1,...,R_n$, and has a standard exponential distribution. Hence the assertion follows by induction over n.

(b) As $(X_{\sigma_{n-1}+k})_{k\ge 1}$ is independent of $\mathscr{F}_{\sigma_{n-1}}$, it is clear that the conditional distribution of τ_n given $\mathscr{F}_{\sigma_{n-1}}$ only depends on the current record at σ_{n-1}, i.e. R_{n-1}, and has a geometric distribution with parameter e^{-r} if $R_{n-1}=r$. The formula for the unconditional probabilities $\mathbb{P}(\tau_n=k)$ then follows by integrating agains the distribution of R_{n-1} which, by (a), is a Gamma law with parameters $n-1$ and 1. □

With the help of the previous result we infer from Theorem 1.2.2 that the *record counting process*

$$N(t) \;:=\; \sum_{n\ge 1}\mathbf{1}_{\{R_n\le t\}}\quad (t\ge 0)$$

forms a homogeneous Poisson process with intensity 1, also called *standard Poisson process*, if the underlying X_n are standard exponentials. It is now straightforward to conclude a similar result in the general case by drawing on Lemma 1.7.1 and the subsequent discussion.

Theorem 1.7.3. *Let $(X_n)_{n\ge 1}$ be a sequence of iid nonnegtive random variables with continuous cdf F and associated record values $(R_n)_{n\ge 1}$. Let further $b:=\inf\{x\in\mathbb{R}:F(x)=1\}$ and ν be the measure on $\mathbb{R}_\ge$, defined by*

$$\Lambda((s,t]) := \log\left(\frac{1-F(s)}{1-F(t)}\right)\quad \textit{for } 0\le s<t<b,$$

and vanishing outside $[0,b]$. Then the record counting process $(N(t))_{t\ge 0}$ forms a ***nonhomogeneous Poisson process with intensity measure*** *Λ, that is*

(NPP1) $N(0)=0$.
(NPP2) $(N(t))_{t\ge 0}$ *has independent increments.*
(NPP3) $N(t)-N(s)\overset{d}{=}$ *Poisson*$(\Lambda((s,t]))$ *for all* $0\le s<t<\infty$, *where* *Poisson*$(0):=\delta_0$.

If F has $\lambda\!\!\lambda_0$-density f, then so does Λ, viz. $\lambda(t)=\frac{f(t)}{1-F(t)}\mathbf{1}_{(0,b)}(t)$, which is then called the ***intensity function*** *of $(N(t))_{t\ge 0}$.*

Proof. As before, put $G(t):=-\log(1-F(t))$ and let $\{\widehat{N}(t)\}$ be the record counting process of the iid standard exponentials $(G(X_n))_{n\ge 1}$, hence a standard Poisson process by Thm. 1.7.2. Now all assertions are immediate consequences of this fact together with the observation that

$$N(t)=\widehat{N}(G(t))\quad \text{for all } t\ge 0.$$

All further details can therefore be left to the reader. □

So we have arrived at the very explicit result that the record counting process of a sequence of iid nonnegtive random variables with continuous cdf F is always a nonhomogeneous Poisson process, and its "renewal function" $\mathbb{U}(t)=\mathbb{E}N(t)$ just equals $\Lambda([0,t]) = -\log(1-F(t))$ for $t \geq 0$. Notice that in the case where F has a $\lambda\!\!\lambda_0$-density f the intensity function $\lambda(t)$ of $(N(t))_{t\geq 0}$ is nothing but the *failure* or *hazard rate* of the sequence $(X_n)_{n\geq 1}$, that is

$$\lambda(t) \;=\; \lim_{h\downarrow 0} \mathbb{P}(X_1 \in (t,t+h]|X_1 > t) \quad \text{for } \lambda\!\!\lambda_0\text{-almost all } t \geq 0.$$

Chapter 2
Random walks and stopping times: classifications and preliminary results

Our motivating examples have shown that renewal theory is typically concerned with sums of iid *nonnegative* random variables, i.e. renewal processes, the goal being to describe the implications of their regenerative properties. On the other hand, it then appears to be quite natural and useful to choose a more general framework by considering also sums of iid *real-valued* random variables, called *random walks*. Doing so, the regenerative structure remains unchanged while monotonicity is lost. However, this will be overcome by providing the concept of *ladder variables*, therefore introduced and studied in Subsection 2.2.3. The more general idea behind this concept is to sample a random walk along a sequence of stopping times that renders a renewal process as a subsequence. This in turn is based on an even more general result which, roughly speaking, states that any finite stopping time for a random walk may be formally copied indefinitely over time and thus always leads to another imbedded random walk. The exact result is stated as Theorem 2.2.3 in Subsection 2.2.2. Further preliminary results besides some basic terminology and classification include the (topological) recurrence of random walks with zero-mean increments, some basic properties of renewal measures of random walks with positive drift, the connection of the renewal measure with certain first passage times including ladder epochs, and the definition of the *stationary delay distribution*.

2.1 Preliminaries and classification of random walks

2.1.1 Lattice-type of random walks

As already mentioned, any sequence $(S_n)_{n\ge 0}$ of real-valued random variables with iid increments $X_1, X_2, \ldots$ and initial value S_0 independent of these is called *random walk (RW)* hereafter, and S_0 its *delay*. If $\mu := \mathbb{E}X_1$ exists, then μ is called the *drift* of $(S_n)_{n\ge 0}$. In the case where all X_n as well as S_0 are nonnegative and μ is positive (possibly infinite), $(S_n)_{n\ge 0}$ is also called a *renewal process (RP)*. Finally, a RW or RP $(S_n)_{n\ge0}$ is called *trivial* if $X_1 = 0$ a.s., *zero-delayed* or *standard* if $S_0 = 0$ a.s. (abbreviated as SRW, respectively SRP), and *delayed* otherwise.

An important characteristic of a RW $(S_n)_{n\ge0}$ in the context of renewal theory is its lattice-type. Let F denote the distribution of the X_n and F_0 the distribution of S_0. Since $(S_n)_{n\ge0}$ forms an additive sequence with state space $\mathbb{G}_0 := \mathbb{R}$, and since $\mathbb{G}_0$ has closed subgroups $\mathbb{G}_d := d\mathbb{Z} := \{dn : n \in \mathbb{Z}\}$ for $d \in \mathbb{R}_>$ and $\mathbb{G}_\infty := \{0\}$, it is natural to ask for the smallest closed subgroup on which the RW is concentrated. Namely, if all the X_n as well as S_0 take only values in a proper closed subgroup $\mathbb{G}$ of $\mathbb{R}$, that is, $F(\mathbb{G}) = F_0(\mathbb{G}) = 1$, then the same holds true for all S_n and its accumulation points. The following classifications of F and $(S_n)_{n\ge0}$ reflect this observation.

Definition 2.1.1. For a distribution F on $\mathbb{R}$, its *lattice-span* $d(F)$ is defined as

$$d(F) \ := \ \sup\{d \in [0,\infty] : F(\mathbb{G}_d) = 1\}.$$

Let $\{F_x : x \in \mathbb{R}\}$ denote the translation family associated with F, i.e., $F_x(B) := F(x+B)$ for all Borel subsets B of $\mathbb{R}$. Then F is called

– *nonarithmetic*, if $d(F) = 0$ and thus $F(\mathbb{G}_d) < 1$ for all $d > 0$.
– *completely nonarithmetic*, if $d(F_x) = 0$ for all $x \in \mathbb{R}$.
– *d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F) = d$.
– *completely d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F_x) = d$ for all $x \in \mathbb{G}_d$.

If X denotes any random variable with distribution F, thus $X - x \stackrel{d}{=} F_x$ for each $x \in \mathbb{R}$, then the previous attributes are also used for X, and we also write $d(X)$ instead of $d(F)$ and call it the lattice-span of X.

For our convenience, a nonarithmetic distribution is sometimes referred to as *0-arithmetic* hereafter, for example in the lemma below. A random variable X is nonarithmetic iff it is not a.s. taking values only in a lattice $\mathbb{G}_d$, and it is completely nonarithmetic if this is not the case for any shifted lattice $x + \mathbb{G}_d$, i.e. any affine closed subgroup of $\mathbb{R}$, either. As an example of a nonarithmetic, but not completely nonarithmetic random variable we mention $X = \pi + Y$ with a standard Poisson variable Y. Then $d(X - \pi) = d(Y) = 1$. If $X = \frac{1}{2} + Y$, then $d(X) = \frac{1}{2}$ and $d(X - \frac{1}{2}) = 1$. In this case, X is $\frac{1}{2}$-arithmetic, but not completely $\frac{1}{2}$-arithmetic. The following simple lemma provides the essential property of a completely d-arithmetic random variable $(d \ge 0)$.

Lemma 2.1.2. *Let X,Y be two iid random variables with lattice-span $d \ge 0$. Then $d \le d(X - Y)$ with equality holding iff X is completely d-arithmetic.*

Proof. Let F denote the distribution of X,Y. The inequality $d \le d(X - Y)$ is trivial, and since $(X + z) - (Y + z) = X - Y$, we also have $d(X + z) \le d(X - Y)$ for all $z \in \mathbb{R}$.

Suppose X is *not* completely d-arithmetic. Then $d(X+z) > d$ for some $z \in \mathbb{G}_d$ and hence also $c := d(X-Y) > d$. Conversely, if the last inequality holds true, then

$$1 \;=\; \mathbb{P}(X - Y \in \mathbb{G}_c) \;=\; \int_{\mathbb{G}_d} \mathbb{P}(X - y \in \mathbb{G}_c)\, F(dy)$$

implies

$$\mathbb{P}(X - y \in \mathbb{G}_c) \;=\; 1 \quad \text{for all } F\text{-almost all } y \in \mathbb{G}_d$$

and thus $d(X-y) \geq c > d$ for F-almost all $y \in \mathbb{G}_d$. Therefore, X cannot be completely d arithmetic. ☐

We continue with a classification of RW's based on Definition 2.1.1.

Definition 2.1.3. A RW $(S_n)_{n\geq 0}$ with increments $X_1, X_2, \ldots$ is called

– *(completely) nonarithmetic* if X_1 is (completely) nonarithmetic.
– *(completely) d-arithmetic* if $d > 0$, $\mathbb{P}(S_0 \in \mathbb{G}_d) = 1$, and X_1 is (completely) d-arithmetic.

Furthermore, the lattice-span of X_1 is also called the lattice-span of $(S_n)_{n\geq 0}$ in any of these cases.

The additional condition on the delay in the d-arithmetic case, which may be restated as $d(S_0) = kd$ for some $k \in \mathbb{N} \cup \{\infty\}$, is needed to ensure that $(S_n)_{n\geq 0}$ is really concentrated on the lattice $\mathbb{G}_d$. The unconsidered case where $(S_n)_{n\geq 0}$ has d-arithmetic increments but non- or c-arithmetic delay for some $c \notin \mathbb{G}_d \cup \{\infty\}$ will not play any role in our subsequent analysis.

2.1.2 *Making life easier: the standard model of a random walk*

Quite common in the theory of Markov chains and also possible and useful here in various situations (as any RW is also a Markov chain with state space $\mathbb{R}$) is the use of a *standard model* for a RW $(S_n)_{n\geq 0}$ with a given increment distribution. In such a model we may vary the delay (initial) distribution by only changing the underlying probability measure and not the delay or the whole sequence itself.

Definition 2.1.4. Let $\mathscr{P}(\mathbb{R})$ be the set of all probability distributions on $\mathbb{R}$. We say that a RW $(S_n)_{n\geq 0}$ is given in a *standard model*

$$(\Omega, \mathfrak{A}, (\mathbb{P}_\lambda)_{\lambda \in \mathscr{P}(\mathbb{R})}, (S_n))$$

if each S_n is defined on $(\Omega,\mathfrak{A})$ and, for each $\mathbb{P}_\lambda$, has delay (initial) distribution λ and increment distribution F, that is $\mathbb{P}_\lambda(S_n \in \cdot) = \lambda * F^{*n}$ for each $n \in \mathbb{N}_0$. If $\lambda = \delta_x$, we also write $\mathbb{P}_x$ for $\mathbb{P}_\lambda$.

A standard model always exists: Let $\mathscr{B}(\mathbb{R})$ denote the Borel σ-field over $\mathbb{R}$ and fix any distribution F on $\mathbb{R}$. By choosing the coordinate space $\Omega := \mathbb{R}^{\mathbb{N}_0}$ with coordinate mappings $S_0, X_1, X_2, \ldots$, infinite product Borel σ-field $\mathfrak{A} = \mathscr{B}(\mathbb{R})^{\mathbb{N}_0}$ and product measure $\mathbb{P}_\lambda = \lambda \otimes F^{\mathbb{N}_0}$ for $\lambda \in \mathscr{P}(\mathbb{R})$, we see that $(S_n)_{n\geq 0}$ forms a RW with delay distribution λ and increment distribution F under each $\mathbb{P}_\lambda$ and is thus indeed given in a standard model, called *canonical* or *coordinate model*.

2.1.3 Classification of renewal processes: persistence vs. termination

Some applications of renewal theory lead to RP's having an increment distribution that puts mass on ∞. This has been encountered in Section 1.3 when defining the sequence of successive hitting times of a transient state of a finite Markov chain. From an abstract point of view, this means nothing but to consider sums of iid random variables taking values in the extended semigroup $\mathbb{R}_\geq \cup \{\infty\}$. Following FELLER[☞ [21, footnote on p. 115]], any distribution F on this set with $F(\{\infty\}) > 0$ and thus $\lim_{x\to\infty} F(x) < 1$ is called *defective*. A classification of RP's that accounts for this possibility is next.

Definition 2.1.5. A RP $(S_n)_{n\geq 0}$ with almost surely finite delay S_0 and increments $X_1, X_2, \ldots$ having distribution F and mean μ is called

– *proper* (or *persistent*, or *recurrent*) if F is nondefective, i.e. $F(\{\infty\}) = 0$.
– *terminating* (or *transient*) if F is defective.

In the proper case, $(S_n)_{n\geq 0}$ is further called

– *strongly persistent* (or *positive recurrent*) if $\mu < \infty$.
– *weakly persistent* (or *null recurrent*) if $\mu = \infty$.

Clearly, the terms "recurrent" and "transience" reflect the idea that any renewal process may be interpreted as sequence of occurrence epochs for a certain event which is then recurrent in the first, and transient in the second case. In the transient case, all S_n have a defective distribution and

$$\mathbb{P}(S_n < \infty) \;=\; \mathbb{P}(X_1 < \infty, \ldots, X_n < \infty) \;=\; \mathbb{P}(X_1 < \infty)^n \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there is a *last renewal epoch*, defined as S_T with

$$T := \sup\{n \geq 0 : S_n < \infty\} \tag{2.1}$$

giving the number of finite renewal epochs. For the latter variable the distribution is easily determined.

Lemma 2.1.6. *Let $(S_n)_{n\geq 0}$ be a terminating renewal process with increment distribution F. Then the number of finite renewal epochs T has a geometric distribution with parameter $p := F(\{\infty\})$, that is, $\mathbb{P}(T = n) = p(1-p)^n$ for $n \in \mathbb{N}_0$.*

Proof. This follows immediately from $\mathbb{P}(T = 0) = \mathbb{P}(X_1 = \infty) = p$ and

$$\mathbb{P}(T = n) \;=\; \mathbb{P}(X_1 < \infty, \ldots, X_n < \infty, X_{n+1} = \infty) \;=\; (1-p)^n p$$

for all $n \in \mathbb{N}$. □

2.1.4 Random walk and renewal measure: the point process view

Although this is not a text on point processes, we will briefly adopt this viewpoint because it appears to be quite natural, in particular for a general definition of the renewal measure of a RW to be presented below. So let $(S_n)_{n\geq 0}$ be a RW given in a standard model and consider the associated *random counting measure* on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$

$$N \;:=\; \sum_{n\geq 0} \delta_{S_n}$$

which more explicitly means that

$$N(\omega, B) \;:=\; \sum_{n\geq 0} \delta_{S_n(\omega)}(B) \quad \text{for all } \omega \in \Omega \text{ and } B \in \mathscr{B}(\mathbb{R}).$$

By endowing the set $\mathscr{M}$ of counting measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with the smallest σ-field $\mathfrak{M}$ that renders measurability of all projection mappings

$$\pi_B : \mathscr{M} \to \mathbb{N}_0 \cup \{\infty\}, \quad \mu \mapsto \mu(B),$$

i.e. $\mathfrak{M} := \sigma(\pi_B : B \in \mathscr{B}(\mathbb{R}))$, we have that $N : (\Omega, \mathfrak{A}, \mathbb{P}_\lambda) \to (\mathscr{M}, \mathfrak{M})$ defines a measurable map and thus a random element in $\mathscr{M}$, called *point process*. For each $B \in \mathscr{B}(\mathbb{R})$, $N(B)$ is an ordinary random variable which counts the number of points S_n in the set B. Taking expectations we arrive at the so-called *intensity measure* of the point process N under $\mathbb{P}_\lambda$, namely

$$\mathbb{U}_\lambda(B) := \mathbb{E}_\lambda N(B) = \sum_{n\ge 0} \mathbb{P}_\lambda(S_n \in B) \quad \text{for } B \in \mathscr{B}(\mathbb{R}).$$

The following definition extends those already given in the Introduction to general RW's on the real line.

Definition 2.1.7. Given a RW $(S_n)_{n\ge 0}$ in a standard model, the intensity measure $\mathbb{U}_\lambda$ of the associated point process N under $\mathbb{P}_\lambda$ is called *renewal measure of* $(S_n)_{n\ge 0}$ *under* $\mathbb{P}_\lambda$ and

$$\mathbb{U}_\lambda(t) := \mathbb{U}_\lambda((-\infty,t]) \quad \text{for } t \in \mathbb{R}$$

the pertinent *renewal function*. If $\lambda = \delta_x$ for some $x \in \mathbb{R}$, we write $\mathbb{U}_x$ for $\mathbb{U}_{\delta_x}$. Finally, the stochastic process $(N(t))_{t\in\mathbb{R}}$ defined by $N(t) := N((-\infty,t])$ is called *renewal counting process of* $(S_n)_{n\ge 0}$.

A detail we should comment on at this point is the following: In all examples of the Introduction we have defined the renewal measure on the basis of the epochs S_n for $n \ge 1$, whereas here we also account for S_0 even if $S_0 = 0$. The reason is that both definitions have their advantages. On the other hand, the general definition above forces us from now on to clearly state that $\mathbb{U}$ is the renewal measure of $(S_n)_{n\ge 1}$ and not $(S_n)_{n\ge 0}$ in those instances where S_0 is not to be accounted for. Fortunately, this will only be necessary occasionally, one example being the Poisson process which by definition has $N(0) = 0$.

Owing to the fact that $\mathbb{P}_\lambda = \lambda * F^{*n}$ for each $n \ge 0$ and $\lambda \in \mathscr{P}(\mathbb{R})$, where F is as usual the distribution of the X_n and $F^{*0} := \delta_0$, we have that

$$\mathbb{U}_\lambda = \sum_{n\ge 0} \lambda * F^{*n} = \lambda * \sum_{n\ge 0} F^{*n} = \lambda * \mathbb{U}_0 \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}). \tag{2.2}$$

Natural questions on $\mathbb{U}_\lambda$ to be investigated are:

(Q1) Is $\mathbb{U}_\lambda$ *locally finite*, i.e. $\mathbb{U}_\lambda(B) < \infty$ for all bounded $B \in \mathscr{B}(\mathbb{R})$?
(Q2) Is $\mathbb{U}_\lambda(t) < \infty$ for all $t \in \mathbb{R}$ if $S_n \to \infty$ a.s.?

For a RP $(S_n)_{n\ge 0}$, the two questions are equivalent as $\mathbb{U}_\lambda(t) = 0$ for $t < 0$, and we have already given a positive answer in Section 1.1 [☞ Thm. 1.1.1]. In the situation of a RW, it may therefore take by surprise that the answer to both questions is not positive in general.

2.2 Random walks and stopping times: the basic stuff

This section is devoted to some fundamental results on RW's and stopping times including the important concept of ladder variables that will allow us to study RW's with the help of embedded RP's. Since stopping often involves more than just a given RW $(S_n)_{n\ge 0}$, due to additionally observed processes or randomizations, it is reasonable to consider stopping times with respect to filtrations that are larger than the one generated by $(S_n)_{n\ge 0}$ itself.

2.2.1 Filtrations, stopping times and some fundamental results

In the following, let $(S_n)_{n\ge 0}$ be a RW in a standard model with increments $X_1,X_2,\ldots$ and increment distribution F. For convenience, it may take values in any $\mathbb{R}^d$, $d\ge 1$. We will use $\mathbb{P}$ for probabilities that do not depend on the distribution of S_0. Let further $(\mathscr{F}_n)_{n\ge 0}$ be a filtration such that

(F1) $(S_n)_{n\ge 0}$ is adapted to $(\mathscr{F}_n)_{n\ge 0}$, i.e., $\sigma(S_0,\ldots,S_n)\subset\mathscr{F}_n$ for all $n\in\mathbb{N}_0$.
(F2) $\mathscr{F}_n$ is independent of $(X_{n+k})_{k\ge 1}$ for each $n\in\mathbb{N}_0$.

Let also $\mathscr{F}_\infty$ be the smallest σ-field containing all $\mathscr{F}_n$. Condition (F2) ensures that $(S_n)_{n\ge 0}$ is a temporally homogeneous Markov chain with respect to $(\mathscr{F}_n)_{n\ge 0}$, viz.

$$\mathbb{P}(S_{n+1}\in B|\mathscr{F}_n)\ =\ \mathbb{P}(S_{n+1}\in B|S_n)\ =\ F(B-S_n)\quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n\in\mathbb{N}_0$, $\lambda\in\mathscr{P}(\mathbb{R}^d)$ and $B\in\mathscr{B}(\mathbb{R}^d)$. A more general, but in fact equivalent statement is that

$$\mathbb{P}((S_{n+k})_{k\ge 0}\in C|\mathscr{F}_n)\ =\ \mathbb{P}((S_{n+k})_{k\ge 0}\in C|S_n)\ =\ \mathbf{P}(S_n,C)\quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n\in\mathbb{N}_0$, $\lambda\in\mathscr{P}(\mathbb{R})$ and $C\in\mathscr{B}(\mathbb{R}^d)^{\mathbb{N}_0}$, where

$$\mathbf{P}(x,C)\ :=\ \mathbb{P}_x((S_k)_{k\ge 0}\in C)\ =\ \mathbb{P}_0((S_k)_{k\ge 0}\in C-x)\quad \text{for } x\in\mathbb{R}^d.$$

Let us recall that, if τ is any stopping time with respect to $(\mathscr{F}_n)_{n\ge 0}$, also called $(\mathscr{F}_n)$-*time* hereafter, then

$$\mathscr{F}_\tau\ =\ \{A\in\mathscr{F}_\infty : A\cap\{\tau\le n\}\in\mathscr{F}_n \text{ for all } n\in\mathbb{N}_0\},$$

and the random vector $(\tau,S_0,\ldots,S_\tau)\mathbf{1}_{\{\tau<\infty\}}$ is $\mathscr{F}_\tau$-measurable. The following basic result combines the strong Markov property and temporal homogeneity of $(S_n)_{n\ge 0}$ as a Markov chain with its additional *spatial homogeneity* owing to its iid increments.

Theorem 2.2.1. *Under the stated assumptions, let τ be a $(\mathscr{F}_n)$-time. Then, for all $\lambda\in\mathscr{P}(\mathbb{R}^d)$, the following equalities hold $\mathbb{P}_\lambda$-a.s. on $\{\tau<\infty\}$:*

$$\mathbb{P}((S_{\tau+n}-S_\tau)_{n\ge 0}\in\cdot|\mathscr{F}_\tau)=\mathbb{P}((S_n-S_0)_{n\ge 0}\in\cdot)=\mathbb{P}_0((S_n)_{n\ge 0}\in\cdot). \quad (2.3)$$
$$\mathbb{P}((X_{\tau+n})_{n\ge 1}\in\cdot|\mathscr{F}_\tau)=\mathbb{P}((X_n)_{n\ge 1}\in\cdot). \quad (2.4)$$

If $\mathbb{P}_\lambda(\tau<\infty)=1$, *then furthermore (under* $\mathbb{P}_\lambda$*)*

(a) $(S_{\tau+n}-S_\tau)_{n\ge 0}$ *and* $\mathscr{F}_\tau$ *are independent.*
(b) $(S_{\tau+n}-S_\tau)_{n\ge 0}\overset{d}{=}(S_n-S_0)_{n\ge 0}$.
(c) $X_{\tau+1},X_{\tau+2},\ldots$ *are iid with the same distribution as* X_1.

Proof. It suffices to prove (2.4) for which we pick any $k\in\mathbb{N}_0$, $n\in\mathbb{N}$, $B_1,\ldots,B_n\in\mathscr{B}(\mathbb{R}^d)$ and $A\in\mathscr{F}_\tau$. Using $A\cap\{\tau=k\}\in\mathscr{F}_k$ and (F2), it follows for each $\lambda\in\mathscr{P}(\mathbb{R}^d)$ that

$$\begin{aligned}&\mathbb{P}_\lambda(A\cap\{\tau=k,X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n\})\\ &\quad=\ \mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n)\\ &\quad=\ \mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_1\in B_1,\ldots,X_n\in B_n),\end{aligned}$$

and this clearly yields the desired conclusion. □

Assuming $S_0=0$ hereafter, let us now turn to the concept of formally copying a stopping time τ for $(S_n)_{n\ge 1}$. The latter means that there exist $B_n\in\mathscr{B}(\mathbb{R}^{nd})$ for $n\ge 1$ such that

$$\tau\ =\ \inf\{n\ge 1:(S_1,\ldots,S_n)\in B_n\}, \quad (2.5)$$

where as usual $\inf\emptyset:=\infty$. With the help of the B_n we can copy this stopping rule to the *post-*τ *process* $(S_{\tau+n}-S_\tau)_{n\ge 1}$ if $\tau<\infty$. For this purpose put $S_{n,k}:=S_{n+k}-S_n$,

$$\begin{aligned}\mathbf{S}_{n,k}\ &:=\ (S_{n+1}-S_n,\ldots,S_{n+k}-S_n)\ =\ (S_{n,1},\ldots,S_{n,k})\quad\text{and}\\ \mathbf{X}_{n,k}\ &:=\ (X_{n+1},\ldots,X_{n+k})\end{aligned}$$

for $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$.

Definition 2.2.2. Let τ be a stopping time for $(S_n)_{n\ge 1}$ as in (2.5). Then the sequences $(\tau_n)_{n\ge 1}$ and $(\sigma_n)_{n\ge 0}$, defined by $\sigma_0:=0$ and

$$\tau_n\ :=\ \begin{cases}\inf\{k\ge 1:\mathbf{S}_{\sigma_{n-1},k}\in B_k\}, & \text{if } \sigma_{n-1}<\infty\\ \infty, & \text{if } \sigma_{n-1}=1\end{cases}\qquad\text{and}\qquad \sigma_n\ :=\ \sum_{k=1}^n\tau_k$$

for $n\ge 1$ (thus $\tau_1=\tau$) are called the *sequence of formal copies of* τ and its associated *sequence of copy sums*, respectively.

The following theorem summarizes the most important properties of the τ_n, σ_n and $S_{\sigma_n}\mathbf{1}_{\{\sigma_n<\infty\}}$.

Theorem 2.2.3. *Given the previous notation, put further* $\beta := \mathbb{P}(\tau<\infty)$ *and* $\mathbf{Z}_n := (\tau_n, \mathbf{X}_{\sigma_{n-1},\tau_n})$ *for* $n\in\mathbb{N}$. *Then the following assertions hold true:*

(a) $\sigma_0, \sigma_1, \ldots$ *are stopping times for* $(S_n)_{n\ge 0}$.
(b) τ_n *is a stopping time with respect to* $(\mathscr{F}_{\sigma_{n-1}+k})_{k\ge 0}$ *and* $\mathscr{F}_{\sigma_{n-1}}$-*measurable for each* $n\in\mathbb{N}$.
(c) $\mathbb{P}(\tau_n\in\cdot|\mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\tau<\infty)$ *a.s. on* $\{\sigma_{n-1}<\infty\}$ *for each* $n\in\mathbb{N}$.
(d) $\mathbb{P}(\tau_n<\infty) = \mathbb{P}(\sigma_n<\infty) = \beta^n$ *for all* $n\in\mathbb{N}$.
(e) $\mathbb{P}(\mathbf{Z}_n\in\cdot, \tau_n<\infty|\mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\mathbf{Z}_1\in\cdot,\tau_1<\infty)$ *a.s. on* $\{\sigma_{n-1}<\infty\}$ *for all* $n\in\mathbb{N}$.
(f) *Given* $\sigma_n<\infty$, *the random vectors* $\mathbf{Z}_1,\ldots,\mathbf{Z}_n$ *are conditionally iid with the same distribution as* $\mathbf{Z}_1$ *conditioned upon* $\tau_1<\infty$.
(g) *If* $G := \mathbb{P}((\tau,S_\tau)\in\cdot|\tau<\infty)$, *then* $\mathbb{P}((\sigma_n,S_{\sigma_n})\in\cdot|\sigma_n<\infty) = G^{*n}$ *a.s. for all* $n\in\mathbb{N}$.

In the case where τ *is a.s. finite* ($\beta=1$), *this implies further:*

(h) $\mathbf{Z}_n$ *and* $\mathscr{F}_{\sigma_{n-1}}$ *are independent for each* $n\in\mathbb{N}$.
(i) $\mathbf{Z}_1,\mathbf{Z}_2,\ldots$ *are iid.*
(j) $(\sigma_n,S_{\sigma_n})_{n\ge 0}$ *forms a SRW taking values in* $\mathbb{N}_0\times\mathbb{R}^d$.

Proof. The simple proof of (a) and (b) is left to the reader. Assertion (c) and (e) follow from (2.3) when observing that, on $\{\sigma_{n-1}<\infty\}$,

$$\tau_n \;=\; \sum_{k\ge 0}\mathbf{1}_{\{\tau_n>k\}} \;=\; \sum_{k\ge 0}\prod_{j=1}^{k}\mathbf{1}_{B_j^c}(\mathbf{S}_{\sigma_{n-1},j}) \quad\text{and}\quad \mathbf{Z}_n\mathbf{1}_{\{\tau_n<\infty\}}$$

are measurable functions of $(S_{\sigma_{n-1},k})_{k\ge 0}$. Since $\mathbb{P}(\tau_n<\infty) = \mathbb{P}(\tau_1<\infty,\ldots,\tau_n<\infty)$, we infer (d) by an induction over n and use of (c). Another induction in combination with (d) gives assertion (f) once we have proved that

$$\begin{aligned}&\mathbb{P}((\mathbf{Z}_1,\ldots,\mathbf{Z}_n)\in A_n, \mathbf{Z}_{n+1}\in B, \sigma_{n+1}<\infty)\\ &\qquad\qquad = \mathbb{P}((\mathbf{Z}_1,\ldots,\mathbf{Z}_n)\in A_n,\sigma_n<\infty)\,\mathbb{P}(\mathbf{Z}_1\in B,\tau<\infty)\end{aligned}$$

for all $n\in\mathbb{N}$ and A_n, B from the σ-fields obviously to be chosen here. But with the help of (e), this is inferred as follows:

$$
\begin{aligned}
&\mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\mathbf{Z}_{n+1}\in B,\sigma_{n+1}<\infty)\\
&\quad=\ \mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\mathbf{Z}_{n+1}\in B,\sigma_n<\infty,\tau_{n+1}<\infty)\\
&\quad=\ \int_{\{(\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\sigma_n<\infty\}}\mathbb{P}(\mathbf{Z}_{n+1}\in B,\tau_{n+1}<\infty|\mathscr{F}_{\sigma_n})\,d\mathbb{P}\\
&\quad=\ \mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\sigma_n<\infty)\,\mathbb{P}(\mathbf{Z}_1\in B,\tau<\infty).
\end{aligned}
$$

Assertion (g) is a direct consequence of (f), and the remaining assertion (h),(i) and (j) in the case $\beta=1$ are just the specializations of (e),(f) and (g) to this case. □

Notice that, for a finite Markov chain, any sequence $(T_n(i))_{n\ge 0}$ of successive return times to a recurrent state i forms a sequence of copy sums, namely the one associated with $T_1(i)$, and it provides us with a cyclic decomposition of the chain as demonstrated in Section 1.3. For a RW $(S_n)_{n\ge 0}$ even more is true owing to its temporal and spatial homogeneity, namely, that *every* finite stopping time τ and its associated sequence of copy sums leads to a cyclic decomposition. This allows us to analyze intrinsic features of the RW by choosing τ in an appropriate manner.

2.2.2 Wald's identities for stopped random walks

Returning to the situation where $(S_n)_{n\ge 0}$ is a *real-valued* zero-delayed random walk, the purpose of this section is to provide two very useful identities originally due to A. WALD [46] for the first and second moment of stopped sums S_τ for finite mean stopping times τ.

Theorem 2.2.4. [Wald's identity] *Let $(S_n)_{n\ge 0}$ be a SRW adapted to a filtration $(\mathscr{F}_n)_{n\ge 0}$ satisfying (F1) and (F2). Let further τ be an a.s. finite $(\mathscr{F}_n)$-time and suppose that $\mu:=\mathbb{E}X_1$ exists. Then*

$$\mathbb{E}S_\tau\ =\ \mu\,\mathbb{E}\tau$$

provided that either the X_n are a.s. nonnegative, or τ has finite mean.

Proof. Since τ is a.s. finite, we have that

$$S_\tau\ =\ \sum_{n\ge 1}X_n\mathbf{1}_{\{\tau\ge n\}}\quad\text{a.s.}\tag{2.6}$$

Observe that, by (F2), X_n and $\{\tau\ge n\}\in\mathscr{F}_{n-1}$ are independent for all $n\in\mathbb{N}$. Hence, if the X_n are nonnegative, we infer

$$\mathbb{E}S_\tau\ =\ \sum_{n\ge 1}\mathbb{E}X_n\mathbf{1}_{\{\tau\ge n\}}\ =\ \mathbb{E}X_1\sum_{n\ge 1}\mathbb{P}(\tau\ge n)\ =\ \mu\,\mathbb{E}\tau$$

as claimed. If τ has finite mean then so does at least one of $\sum_{n=1}^{\tau} X_n^-$ or $\sum_{n=1}^{\tau} X_n^+$. Consequently, under the latter assumption we get the assertion for real-valued X_n as well when observing that $S_\tau = \sum_{n=1}^{\tau} X_n^+ - \sum_{n=1}^{\tau} X_1^-$. □

A little more difficult, though based on the same formula (2.6) for S_τ, is the derivation of Wald's second identity for $\mathbb{E}(S_\tau - \mu\tau)^2$.

Theorem 2.2.5. [Wald's second identity] *Let $(S_n, \mathscr{F}_n)_{n\ge 0}$ be as in Theorem 2.2.4 and suppose that the X_n have finite variance σ^2. Then (with μ as before)*

$$\mathbb{E}(S_\tau - \mu\tau)^2 \;=\; \sigma^2\,\mathbb{E}\tau$$

for any $(\mathscr{F}_n)$-time τ with finite mean.

Proof. W.l.o.g. let $\mu = 0$. By (2.6), we have $S_{\tau\wedge n} = \sum_{k=1}^{n} X_k \mathbf{1}_{\{\tau\ge k\}}$ and therefore

$$S_{\tau\wedge n}^2 \;=\; \left(\sum_{k=1}^{n} X_k \mathbf{1}_{\{\tau\ge k\}}\right)^2 \;=\; \sum_{k=1}^{n} X_k^2 \mathbf{1}_{\{\tau\ge k\}} + \sum_{k=1}^{n} X_k S_{k-1} \mathbf{1}_{\{\tau\ge k\}}$$

for each $n\in\mathbb{N}$. Now use the independence of X_k and $S_{k-1}, \mathbf{1}_{\{\tau\ge k\}}$ together with $\mu = 0$ to infer $\mathbb{E}X_k S_{k-1}\mathbf{1}_{\{\tau\ge k\}} = 0$ and $\mathbb{E}X_k^2 \mathbf{1}_{\{\tau\ge k\}} = \sigma^2 \mathbb{P}(\tau\ge k)$ which provides us with

$$\mathbb{E}S_{\tau\wedge n}^2 \;=\; \sigma^2\,\mathbb{E}(\tau\wedge n) \quad \text{for all } n\in\mathbb{N}.$$

As $S_{\tau\wedge n}\to S_\tau$ a.s., it follows with Fatou's lemma that

$$\mathbb{E}S_\tau^2 \;\le\; \liminf_{n\to\infty} \mathbb{E}S_{\tau\wedge n}^2 \;=\; \sigma^2\mathbb{E}\tau \;<\; \infty. \tag{2.7}$$

This proves integrability of S_τ and hence uniform integrability of the Doob-type martingale $(\mathbb{E}(S_\tau^2|\mathscr{F}_{\tau\wedge n}))_{n\ge 0}$. Moreover,

$$\begin{aligned}
\mathbb{E}(S_\tau^2|\mathscr{F}_{\tau\wedge n}) - S_{\tau\wedge n}^2 \;&=\; \mathbb{E}(S_\tau^2 - S_{\tau\wedge n}^2|\mathscr{F}_{\tau\wedge n})\\
&=\; \mathbb{E}((S_\tau - S_{\tau\wedge n})(S_\tau + S_{\tau\wedge n})|\mathscr{F}_{\tau\wedge n})\\
&=\; 2S_{\tau\wedge n}\,\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) \;+\; \mathbb{E}((S_\tau - S_{\tau\wedge n})^2|\mathscr{F}_{\tau\wedge n})\\
&\ge\; 0 \quad \text{a.s.},
\end{aligned}$$

because $\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) = 0$ a.s. for all $\in\mathbb{N}$ [☞ Lemma A.1.3]. But this implies that $(S_{\tau\wedge n}^2)_{n\ge 0}$ is also uniformly integrable and thus equality must hold in (2.7). □

We remark for the interested reader that the proof just given differs slightly from the one usually found in the literature [☞ e.g. GUT [23]] which shows that $(S_{\tau\wedge n})_{n\ge 0}$ forms a Cauchy sequence in the space L_2 of square integrable random variables and thus converges in square mean to S_τ. Let us further note that Wald-type identities for higher integral moments of S_τ may also be given. We refer to [12].

2.2.3 Ladder variables, a fundamental trichotomy, and the Chung-Fuchs theorem

We will now define the most prominent sequences of copy sums in the theory of RW's which are obtained by looking at the record epochs and record values of a RW $(S_n)_{n\ge 0}$, or its reflection $(-S_n)_{n\ge 0}$.

Definition 2.2.6. Given a SRW $(S_n)_{n\ge 0}$, the stopping times

$$\sigma^> := \inf\{n\ge 1 : S_n > 0\},\quad \sigma^\ge := \inf\{n\ge 1 : S_n \ge 0\},$$
$$\sigma^< := \inf\{n\ge 1 : S_n < 0\},\quad \sigma^\le := \inf\{n\ge 1 : S_n \le 0\},$$

are called *first strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epoch*, respectively, and

$$S_1^> := S_{\sigma^>}\mathbf{1}_{\{\sigma^> <\infty\}},\quad S_1^\ge := S_{\sigma^\ge}\mathbf{1}_{\{\sigma^\ge <\infty\}},$$
$$S_1^< := S_{\sigma^<}\mathbf{1}_{\{\sigma^< <\infty\}},\quad S_1^\le := S_{\sigma^\le}\mathbf{1}_{\{\sigma^\le <\infty\}}$$

their respective *ladder heights*. The associated sequences of copy sums $(\sigma_n^>)_{n\ge 0}$, $(\sigma_n^\ge)_{n\ge 0}$, $(\sigma_n^<)_{n\ge 0}$ and $(\sigma_n^\le)_{n\ge 0}$ are called *sequences of strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epochs*, respectively, and

$$S_n^> := S_{\sigma_n^>}\mathbf{1}_{\{\sigma_n^> <\infty\}},\ n\ge 0,\quad S_n^\ge := S_{\sigma_n^\ge}\mathbf{1}_{\{\sigma_n^\ge <\infty\}},\ n\ge 0,$$
$$S_n^< := S_{\sigma_n^<}\mathbf{1}_{\{\sigma_n^< <\infty\}},\ n\ge 0,\quad S_n^\le := S_{\sigma_n^\le}\mathbf{1}_{\{\sigma_n^\le <\infty\}},\ n\ge 0$$

the respective *sequences of ladder heights*.

Plainly, if $(S_n)_{n\ge 0}$ has nonnegative (positive) increments, then $\sigma_n^\ge = n$ $(\sigma_n^> = n)$ for all $n\in\mathbb{N}$. Moreover, $\sigma_n^\ge = \sigma_n^>$ and $\sigma_n^\le = \sigma_n^<$ a.s. for all $n\in\mathbb{N}$ in the case where the increment distribution is continuous, for then $\mathbb{P}(S_m = S_n) = 0$ for all $m,n\in\mathbb{N}$.

The following theorem provides some basic information on the ladder variables and is a consequence of the SLLN and Thm. 2.2.3.

Theorem 2.2.7. *Let $(S_n)_{n\ge 0}$ be a nontrivial SRW. Then the following assertions are equivalent:*

(a) $\left(\sigma_n^\alpha, S_{\sigma_n^\alpha}\right)_{n\ge 0}$ *is a SRW taking values in $\mathbb{N}_0\times\mathbb{R}$ for any $\alpha\in\{>,\ge\}$ (resp. $\{<,\le\}$).*

(b) $\sigma^\alpha < \infty$ *a.s. for $\alpha\in\{>,\ge\}$ (resp. $\{<,\le\}$).*

(c) $\limsup_{n\to\infty} S_n = \infty$ *a.s.* *(resp.* $\liminf_{n\to\infty} S_n = -\infty$ *a.s.)*

Proof. It clearly suffices to prove equivalence of the assertions outside parentheses. The implications "(a)⇒(b)" and "(c)⇒(b)" are trivial, while "(b)⇒(a)" follows from Thm. 2.2.3(j). This leaves us with a proof of "(a),(b)⇒(c)". But $\mathbb{E}S^> > 0$ in combination with the SLLN applied to $(S_n^>)_{n\geq 0}$ implies

$$\limsup_{n\to\infty} S_n \;\geq\; \lim_{n\to\infty} S_n^> = \infty \quad \text{a.s.}$$

and thus the assertion. □

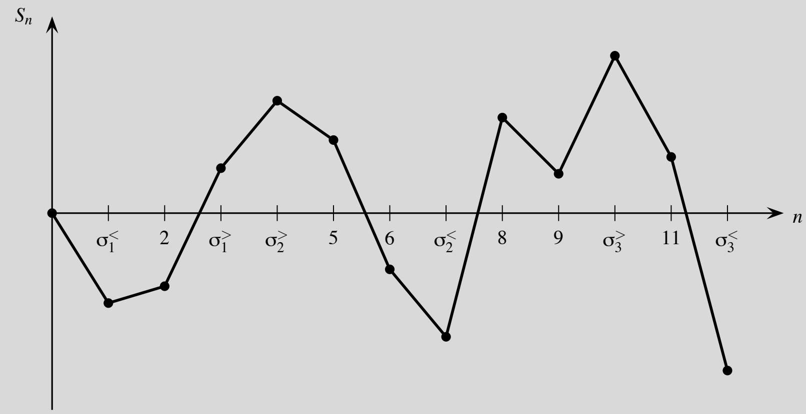

Fig. 2.1 Path of a RW with strictly ascending ladder epochs $\sigma_1^> = 3$, $\sigma_2^> = 4$ and $\sigma_3^> = 10$, and strictly descending ladder epochs $\sigma_1^< = 1$, $\sigma_2^< = 5$ and $\sigma_3^< = 12$.

If $\mathbb{E}X_1 > 0$ (resp. < 0) we thus have that $\sigma^>, \sigma^\geq$ (resp. $\sigma^<, \sigma^\leq$) are a.s. finite whence the associated sequences of ladder epochs and ladder heights each constitute nondecreasing (resp. nonincreasing) zero-delayed RW's. Much deeper information, however, is provided by the next result disclosing a quite unexpected duality between ascending and descending ladder epochs that will enable us to derive a further classification of RW's as to their asymptotic behavior including the *Chung-Fuchs theorem* on the asymptotic behavior of a RW with drift zero.

Theorem 2.2.8. *Given a SRW $(S_n)_{n\ge 0}$ with first ladder epochs $\sigma^{\ge}$, $\sigma^{>}$, $\sigma^{\le}$, $\sigma^{<}$, the following assertions hold true:*

$$\mathbb{E}\sigma^{\ge} \;=\; \frac{1}{\mathbb{P}(\sigma^{<}=\infty)} \quad \textit{and} \quad \mathbb{E}\sigma^{>} \;=\; \frac{1}{\mathbb{P}(\sigma^{\le}=\infty)}, \tag{2.8}$$

$$\mathbb{P}(\sigma^{\le}=\infty) \;=\; (1-\kappa)\,\mathbb{P}(\sigma^{<}=\infty), \tag{2.9}$$

where

$$\kappa \;:=\; \sum_{n\ge 1}\mathbb{P}(S_1>0,\dots,S_{n-1}>0,S_n=0) \;=\; \sum_{n\ge 1}\mathbb{P}(\sigma^{\le}=n,S_1^{\le}=0).$$

Proof. Define

$$\begin{aligned} A_{n1} \;&=\; \{S_1\le S_2,\dots,S_1\le S_n\},\\ A_{nk} \;&=\; \{S_1>S_k,\dots,S_{k-1}>S_k,S_k\le S_{k+1},\dots,S_k\le S_n\} \quad \text{for } 2\le k\le n-1\\ \text{and}\quad A_{nn} \;&=\; \{S_1>S_n,\dots,S_{n-1}>S_n\}. \end{aligned}$$

Using $(S_k-S_i)_{1\le i<k} \stackrel{d}{=} (S_{k-i})_{1\le i<k}$ and $(S_j-S_k)_{k<j\le n} \stackrel{d}{=} (S_j)_{1\le j\le n-k}$ for each $1<k<n$, we then obtain

$$\begin{aligned} \mathbb{P}(A_{nk}) \;&=\; \mathbb{P}(S_k-S_i<0,1\le i<k,S_j-S_k\ge 0,k<j\le n)\\ &=\; \mathbb{P}(S_k-S_i<0,1\le i<k)\,\mathbb{P}(S_j-S_k\ge 0,k<j\le n)\\ &=\; \mathbb{P}(S_i<0,1\le i<k)\,\mathbb{P}(S_j\ge 0,1\le j\le n-k)\\ &=\; \mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k), \end{aligned} \tag{2.10}$$

and a similar result for $k=1$ and $k=n$. It follows for all $n\ge 2$

$$1 \;=\; \sum_{k=1}^{n}\mathbb{P}(A_{nk}) \;=\; \sum_{k=1}^{n}\mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k) \;\ge\; \mathbb{P}(\sigma^{<}=\infty)\sum_{k=1}^{n}\mathbb{P}(\sigma^{\ge}\ge k).$$

and thereupon by letting n tend to ∞ $(\infty\cdot 0=0\cdot\imath:=0)$.

$$\mathbb{P}(\sigma^{<}=\infty)\,\mathbb{E}\sigma^{\ge} \;=\; \mathbb{P}(\sigma^{<}=\infty)\sum_{k\ge 1}\mathbb{P}(\sigma^{\ge}\ge k) \;\le\; 1.$$

Hence $\mathbb{P}(\sigma^{<}=\infty)>0$ entails $\mathbb{E}\sigma^{\ge}<\infty$.

Assuming conversely $\mathbb{E}\sigma^{\ge}<\infty$, we infer with the help of (2.10)

$$1 \;\le\; \sum_{k=1}^{m}\mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k) \;+\; \sum_{k=m}^{n}\mathbb{P}(\sigma^{\ge}\ge k),$$

and then upon letting n tend to ∞

$$1 \le \mathbb{P}(\sigma^< = \infty) \sum_{k=1}^{m} \mathbb{P}(\sigma^\ge \ge k) + \sum_{k\ge m} \mathbb{P}(\sigma^\ge \ge k).$$

By finally taking the limit $m \to \infty$, we arrive at

$$1 \le \mathbb{P}(\sigma^< = \infty)\,\mathbb{E}\sigma^\ge$$

which proves the equivalence of $\mathbb{E}\sigma^\ge < \infty$ and $\mathbb{P}(\sigma^< = \infty) > 0$, and also the first part of (2.8). But the second part follows analogously when replacing each "$>$" with "$\ge$" and each "$\le$" with "$<$" in the definition of theA_{nk}. For the proof of (2.9), we note that

$$\begin{aligned}
\mathbb{P}(\sigma^< = \infty) - \mathbb{P}(\sigma^\le = \infty) &= \mathbb{P}(\sigma^< = \infty, \sigma^\le < \infty) \\
&= \sum_{n\ge 1} \mathbb{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0, S_j \ge 0, j > n) \\
&= \sum_{n\ge 1} \mathbb{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0)\,\mathbb{P}(S_j \ge 0, j \ge 1) \\
&= \mathbb{P}(\sigma^< = \infty) \sum_{n\ge 1} \mathbb{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0) \\
&= \kappa\,\mathbb{P}(\sigma^< = \infty),
\end{aligned}$$

which obviously gives the desired result. □

By considering $(-S_n)_{n\ge 0}$ in the previous result, the relations (2.8) and (2.9) with the roles of $\sigma^>, \sigma^\le$ and $\sigma^<, \sigma^\ge$ interchanged are immediate consequences. This in turn justifies to call each of $(\sigma^>, \sigma^\le)$ and $(\sigma^<, \sigma^\ge)$ a *dual pair*.

The previous result leads also to the following trichotomy that is fundamental for a deeper analysis of RW's.

Theorem 2.2.9. *Let $(S_n)_{n\ge 0}$ be a nontrivial SRW. Then exactly one of the following three cases holds true:*

(i) $\sigma^\le, \sigma^<$ *are both defective and* $\mathbb{E}\sigma^\ge, \mathbb{E}\sigma^>$ *are both finite.*
(ii) $\sigma^\ge, \sigma^>$ *are both defective and* $E\sigma^\le, E\sigma^<$ *are both finite.*
(iii) $\sigma^\ge, \sigma^>, \sigma^\le, \sigma^<$ *are all a.s. finite with infinite expectation.*

In terms of the asymptotic behavior of S_n as $n \to \infty$, these three alternatives are characterized as follows:

(i) $\lim_{n\to\infty} S_n = \infty$ *a.s.*
(ii) $\lim_{n\to\infty} S_n = -\infty$ *a.s.*
(iii) $\liminf_{n\to\infty} S_n = -\infty$ *and* $\limsup_{n\to\infty} S_n = \infty$ *a.s.*

Finally, if $\mu := EX_1$ exists, thus $EX^+ < \infty$ or $EX^- < \infty$, then (i), (ii), and (iii) are equivalent to $\mu > 0$, $\mu < 0$, and $\mu = 0$, respectively.

Proof. Notice first that $\mathbb{P}(X_1 = 0) < 1$ is equivalent to $\kappa < 1$, whence (2.9) ensures that $\sigma^>, \sigma^\geq$ as well as $\sigma^<, \sigma^\leq$ are always defective simultaneously in which case the respective dual ladder epochs have finite expectation by (2.8). Hence, if neither (a) nor (b) holds true, the only remaining alternative is that all four ladder epochs are a.s. finite with infinite expectation. By combining the three alternatives for the ladder epochs just proved with Thm. 2.2.7, the respective characterizations of the behavior of S_n for $n \to \infty$ are immediate.

Suppose now that $\mu = \mathbb{E}X_1$ exists. In view of Thm. 2.2.7 it then only remains to verify that (iii) holds true in the case $\mu = 0$. But any of the alternatives (i) or (ii) would lead to the existence of a ladder epoch σ such that $\mathbb{E}\sigma < \infty$ and S_σ is a.s. positive or negative. On the other hand, $\mathbb{E}S_\sigma = \mu\,\mathbb{E}\sigma = 0$ would follow by an appeal to Wald's identity 2.2.4 which is impossible. Hence $\mu = 0$ entails (iii). □

The previous result calls for a further definition that classifies a RW $(S_n)_{n\geq 0}$ with regard to the three alternatives (i), (ii) and (iii).

Definition 2.2.10. A RW $(S_n)_{n\geq 0}$ is called

– *positive divergent* if $\lim_{n\to\infty} S_n = \infty$ a.s.
– *negative divergent* if $\lim_{n\to\infty} S_n = -\infty$ a.s.
– *oscillating* if $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$ a.s.

While for nontrivial RW's with finite drift Theorem 2.2.9 provides a satisfactory answer in terms of a simple condition on the increment distribution as to when each of the three alternatives occurs, this is not possible for general RW's. The case of zero mean increments is the *Chung-Fuchs theorem* and stated as a corollary below. Another simple criterion for being in the oscillating case is obviously that the increment distribution is symmetric, for then $\sigma^>$ and $\sigma^<$ are identically distributed and hence both a.s. finite.

Corollary 2.2.11. [Chung-Fuchs theorem] *Any nontrivial RW $(S_n)_{n\geq 0}$ with drift zero is oscillating.*

2.3 Recurrence and transience of random walks

Every nontrivial arithmetic RW $(S_n)_{n\ge 0}$ with lattice-span d is also a discrete Markov chain on $\mathbb{G}_d = d\mathbb{Z}$. It is therefore natural to ask under which condition on the increment distribution it also recurrent, i.e.

$$\mathbb{P}(S_n = x \text{ infinitely often}) = 1 \quad \text{for all } x \in \mathbb{G}_d. \tag{2.11}$$

Plainly, this is possible only in the oscillating case, but the question remains whether this is already sufficient. Moreover, the same question may be posed for the case $d = 0$, i.e. nonarithmetic $(S_n)_{n\ge 0}$, however, with the adjustment that condition (2.11) must weakened to

$$\mathbb{P}(|S_n - x| < \varepsilon \text{ infinitely often}) = 1 \quad \text{for all } x \in \mathbb{G}_d \text{ and } \varepsilon > 0, \tag{2.12}$$

because otherwise continuous RW's satisfying $\mathbb{P}(S_n = x) = 0$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ would be excluded right away. The more general condition (2.12) is called *topological recurrence* because it means that open neighborhoods of any $x \in \mathbb{G}_d$ are visited infinitely often. The purpose of this section, which follows the presentation in [9], is to show that any RW $(S_n)_{n\ge 0}$ satisfying $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ is topologically recurrent [☞ Thm. 2.3.5]. We start by defining topological recurrence for individual states.

Definition 2.3.1. Given a sequence $(S_n)_{n\ge 0}$ of real-valued random variables, a state $x \in \mathbb{R}$ is called *(topologically) recurrent* for $(S_n)_{n\ge 0}$ if

$$\mathbb{P}(|S_n - x| < \varepsilon \text{ infinitely often}) = 1 \quad \text{for all } \varepsilon > 0$$

and *transient* otherwise. If $\sup_{n\ge 0}\mathbb{P}(|S_n - x| < \varepsilon) > 0$ for all $\varepsilon > 0$, then the state x is called *possible*.

Plainly, every recurrent state is possible, and in the case where $(S_n)_{n\ge 0}$ is concentrated on a lattice $d\mathbb{Z}$ we may take $\varepsilon = 0$ in the definition of a recurrent state thus leading to the usual one also used for discrete Markov chains.

In the following, let $(S_n)_{n\ge 0}$ be a nontrivial SRW,

$$\mathscr{R} := \{x \in \mathbb{R} : x \text{ is recurrent for } (S_n)_{n\ge 0}\}$$

its *recurrence set* and $\mathscr{E}$ its set of possible states. If $(S_n)_{n\ge 0}$ is d-arithmetic $(d > 0)$, then naturally $\mathscr{R} \subset \mathscr{E} \subset d\mathbb{Z}$ holds true. In fact, if $(S_n)_{n\ge 0}$ has lattice-span $d \ge 0$, then $\mathbb{G}_d$ is the smallest closed subgroup of $\mathbb{R}$ containing $\mathscr{E}$ as one can easily verify. Further information on the structure of $\mathscr{R}$ is provided by the next theorem.

Theorem 2.3.2. *The recurrence set $\mathscr{R}$ of a nontrivial SRW $(S_n)_{n\ge0}$ is either empty or a closed subgroup of $\mathbb{R}$. In the second case, $\mathscr{R} = \mathbb{G}_d$ if $(S_n)_{n\ge0}$ has lattice-span d, thus $\mathscr{R} = \mathbb{R}$ in the nonarithmetic case and $\mathscr{R} = d\mathbb{Z}$ in the d-arithmetic case.*

Proof. Suppose that $\mathscr{R} \ne \emptyset$. Let $(x_k)_{k\ge1}$ be a sequence in $\mathscr{R}$ with limit x and pick $m \in \mathbb{N}$ such that $|x_m - x| < \varepsilon$ for any fixed $\varepsilon > 0$. Then

$$\mathbb{P}(|S_n - x| < 2\varepsilon \text{ infinitely often}) \ \ge\ \mathbb{P}|S_n - x_m| < \varepsilon \text{ infinitely often}) \ =\ 1.$$

Hence $x \in \mathscr{R}$, i.e., $\mathscr{R}$ is closed. The next step is to verify that $x - y \in \mathscr{R}$ whenever $x \in \mathscr{R}$ and $y \in \mathscr{E}$. To this end, fix any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\mathbb{P}(|S_m - y| < \varepsilon) > 0$. Then

$$\begin{aligned}
&\mathbb{P}(|S_m - y| < \varepsilon)\,\mathbb{P}(|S_n - (x-y)| < 2\varepsilon \text{ finitely often})\\
&\quad=\ \mathbb{P}(|S_m - y| < \varepsilon, |S_{m+n} - S_m - (x-y)| < 2\varepsilon \text{ finitely often})\\
&\quad\le\ \mathbb{P}(|S_n - x| < \varepsilon \text{ finitely often}) \ =\ 0
\end{aligned}$$

showing $\mathbb{P}(|S_n - (x-y)| < 2\varepsilon \text{ finitely often}) = 0$ and thus $x - y \in \mathscr{R}$. In particluar, $x - y \in \mathscr{R}$ for $x, y \in \mathscr{R}$ so that $\mathscr{R}$ is a closed additive subgroup of $\mathbb{R}$. Moreover, $\mathscr{R} = \mathscr{E}$, for $0 \in \mathscr{R}, x \in \mathscr{E}$ implies $-x = 0 - x \in \mathscr{R}$ and thus $x \in \mathscr{R}$. By what has been pointed out before this theorem, we finally conclude $\mathscr{R} = \mathscr{E} = \mathbb{G}_d$ if d is the lattice-span of $(S_n)_{n\ge0}$. □

Although Thm. 2.3.2 provides complete information about the structure of the recurrence set $\mathscr{R}$ of a RW $(S_n)_{n\ge0}$, the more difficult problem of finding sufficient conditions ensuring $\mathscr{R} \ne \emptyset$ remains. A crucial step in this direction is to relate the problem to the behavior of the renewal measure $\mathbb{U}$ of $(S_n)_{n\ge0}$ which in the present context should be viewed as an occupation measure. Notice that $x \in \mathscr{R}$ implies $N(I) = \sum_{n\ge0} \mathbf{1}_I(S_n) = \infty$, in particular $\mathbb{U}(I) = \mathbb{E}N(I) = \infty$ for any open interval I containing x. That the converse is also true, will be shown next.

Theorem 2.3.3. *Let $(S_n)_{n\ge0}$ be a nontrivial SRW. If there exists an open interval I such that*

$$0 \ <\ \sum_{n\ge0} \mathbb{P}(S_n \in I) \ =\ \mathbb{U}(I) \ <\ \infty,$$

then $(S_n)_{n\ge0}$ is transient, that is $\mathscr{R} = \emptyset$. Conversely, if $\mathbb{U}(I) = \infty$ for some bounded interval I, then $(S_n)_{n\ge0}$ is recurrent.

Proof. Clearly, $\mathbb{U}(I) > 0$ implies $I \ne \emptyset$, while $\mathbb{U}(I) < \infty$ in combination with the Borel-Cantelli lemma implies $\mathbb{P}(S_n \in I \text{ infintely often}) = 0$ and thus $I \subset \mathscr{R}^c$ as I is open. Consequently, $\mathscr{R} = \emptyset$ if the RW is nonarithmetic. In the d-arithmetic case, the same follows from $I \cap d\mathbb{Z} \neq \emptyset$ which in turn is a consequence of $\mathbb{U}(I) > 0$.

Now suppose that $\mathbb{U}(I) = \infty$ for some bounded and w.l.o.g. open interval I. It suffices to prove $0 \in \mathscr{R}$. Fix an arbitrary $\varepsilon > 0$. Since I may be covered by finitely many intervals of length 2ε, we have $\mathbb{U}(J) = \infty$ for some $J = (x-\varepsilon, x+\varepsilon)$. Let

$$T(J) \; := \; \sup\{n \geq 1 : S_n \in J\} \quad [\sup\emptyset := 0].$$

denote the last time ≥ 1 where the RW visits J and put $A_n := \{T(J) = n\}$ for $n \in \mathbb{N}_0$. It follows that

$$\{S_n \in J \text{ finitely often}\} \; = \; \{T(J) < \infty\} \; = \; \sum_{k \geq 0} A_k$$

and, furthermore,

$$A_k \; \supset \; \{S_k \in J, |S_{k+n} - S_k| \geq 2\varepsilon \text{ for all } n \geq 1\} \quad \text{for all } k \in \mathbb{N}_0.$$

Consequently, by using the independence of S_k and $(S_{k+n} - S_k)_{n \geq 0}$ in combination with $(S_{k+n} - S_k)_{n \geq 0} \stackrel{d}{=} (S_n)_{n \geq 0}$, we infer

$$\mathbb{P}(A_k) \; \geq \; \mathbb{P}(S_k \in J)\,\mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1) \quad \text{for all } k \in \mathbb{N}_0$$

and thereby via summation over $k \geq 0$

$$\mathbb{P}(S_n \in J \text{ finitely often}) \; \geq \; \mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1)\,\mathbb{U}(J)$$

which shows

$$\mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1) = 0, \tag{2.13}$$

because $\mathbb{U}(J) = \infty$. Notice that this holds for any $\varepsilon > 0$.

Defining $\widehat{J}_\delta := (-\delta, \delta)$, $\widehat{J} := \widehat{J}_\varepsilon$ and $\widehat{A}_n := \{T(\widehat{J}) = n\}$ for $n \in \mathbb{N}_0$, we have

$$\mathbb{P}(\widehat{A}_k) \; = \; \lim_{\delta \uparrow \varepsilon} \mathbb{P}(S_k \in \widehat{J}_\delta, S_{k+n} \notin \widehat{J} \text{ for all } n \geq 1) \quad \text{for all } k \in \mathbb{N}.$$

Now, (2.13) implies

$$\mathbb{P}(\widehat{A}_0) \; = \; \mathbb{P}(S_n \notin \widehat{J} \text{ for all } n \geq 1) \; = \; 0$$

and for $k \geq 1$, $\delta < \varepsilon$

$$\begin{aligned} &\mathbb{P}(S_k \in \widehat{J}_\delta, S_{k+n} \notin \widehat{J} \text{ for all } n \geq 1) \\ &\quad \leq \; \mathbb{P}(S_k \in \widehat{J}_\delta, |S_{k+n} - S_k| \geq \varepsilon - \delta \text{ for all } n \geq 1) \\ &\quad \leq \; \mathbb{P}(S_k \in \widehat{J}_\delta)\,\mathbb{P}(|S_{k+n} - S_k| \geq \varepsilon - \delta \text{ for all } n \geq 1) \; = \; 0 \end{aligned}$$

so that $\mathbb{P}(\widehat{A}_k) = 0$ for all $k \in \mathbb{N}_0$. We have thus proved that

$$\mathbb{P}(|S_n| < \varepsilon \text{ finitely often}) \;=\; \mathbb{P}(T(\widehat{J}) = \infty) \;=\; \sum_{k\geq 0} \mathbb{P}(\widehat{A}_k) \;=\; 0$$

for all $\varepsilon > 0$, that is $0 \in \mathscr{R}$. □

The following equivalences are direct consequences of the result just shown.

Corollary 2.3.4. *Given a nontrivial SRW $(S_n)_{n\geq 0}$, the following assertions are equivalent:*

(a1) $(S_n)_{n\geq 0}$ *is recurrent.*
(a2) $N(I) = \infty$ *a.s. for all open intervals I such that $I \cap \mathfrak{R} \neq \emptyset$.*
(a3) $\mathbb{U}(I) = \infty$ *for all open intervals I such that $I \cap \mathfrak{R} \neq \emptyset$.*
(a4) $\mathbb{U}(I) = \infty$ *for some finite interval I.*

By contraposition, equivalence of

(b1) $(S_n)_{n\geq 0}$ *is transient.*
(b2) $N(I) < \infty$ *P-f.s. for all finite intervals I.*
(b3) $\mathbb{U}(I) < \infty$ *for all finite intervals I.*
(b4) $0 < \mathbb{U}(I) < \infty$ *for some finite open interval I.*

holds true.

We are now ready to prove the main result of this section.

Theorem 2.3.5. *Any nontrivial SRW $(S_n)_{n\geq 0}$ which satisfies $n^{-1}S_n \stackrel{\mathbb{P}}{\to} 0$ or, a fortiori, has drift zero is recurrent.*

Proof. In view of the previous result it suffices to show $\mathbb{U}([-1,1]) = \infty$. Define $\tau(x) := \inf\{n \geq 0 : S_n \in [x, x+1]\}$ for $x \in \mathbb{R}$. Since τ is a stopping time for $(S_n)_{n\geq 0}$, we infer with the help of Thm. 2.2.1 that

$$\begin{aligned} \mathbb{U}([x,x+1]) \;&=\; \int_{\{\tau(x)<\infty\}} \sum_{n\geq 0} \mathbf{1}_{[x,x+1]}(S_{\tau+n})\, d\mathbb{P} \\ &\leq\; \int_{\{\tau(x)<\infty\}} \sum_{n\geq 0} \mathbf{1}_{[-1,1]}(S_{\tau+n} - S_\tau)\, d\mathbb{P} \\ &=\; \mathbb{P}(\tau(x) < \infty) \sum_{n\geq 0} \mathbb{P}(S_n \in [-1,1]) \\ &=\; \mathbb{P}(\tau(x) < \infty)\, \mathbb{U}([-1,1]). \end{aligned} \tag{2.14}$$

Therefore

$$\mathbb{U}([-n,n]) \;\le\; \sum_{k=-n}^{n-1} \mathbb{U}([k,k+1]) \;\le\; 2n\,\mathbb{U}([-1,1]) \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Fix now any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ so large that $\mathbb{P}(|S_k| \le \varepsilon k) \ge \frac{1}{2}$ for all $k > m$, which is possible by our assumption $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$. As a consequence,

$$\mathbb{P}(|S_k| \le n) \;\ge\; \frac{1}{2} \quad \text{for all } m < k \le \frac{n}{\varepsilon}$$

and therefore

$$\mathbb{U}([-n,n]) \;\ge\; \sum_{m<k\le n/\varepsilon} \mathbb{P}(|S_k| \le n) \;\ge\; \frac{1}{2}\left(\frac{n}{\varepsilon} - m - 1\right).$$

This yields in combination with (2.15)

$$\mathbb{U}([-1,1]) \;\ge\; \frac{1}{2n}\,\mathbb{U}([-n,n]) \;\ge\; \frac{1}{4\varepsilon} - \frac{m+1}{4n} \quad \text{for all } n \in \mathbb{N}$$

so that, by letting n tend to infinity, we finally obtain

$$\mathbb{U}([-1,1]) \;\ge\; \limsup_{n\to\infty} \frac{1}{2n}\,\mathbb{U}([-n,n]) \;\ge\; \frac{1}{4\varepsilon}$$

which gives the desired conclusion, for $\varepsilon > 0$ was arbitrary. □

Remark 2.3.6. (a) Since any recurrent RW is oscillating, the previous result particularly provides an extension of the Chung-Fuchs theorem in that $\mathbb{E}X_1 = 0$ may be replaced with the weaker assumption that the WLLN $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ holds true.

(b) All previous results are easily extended to the case of arbitrary delay distribution λ as long as $\lambda(\mathbb{G}_d) = 1$ in the d-arithmetic case. In fact, assuming a standard model $(\Omega, \mathfrak{A}, (\mathbb{P}_\lambda)_{\lambda\in\mathscr{P}(\mathbb{R})}, (S_n)_{n\ge0})$ and letting $\mathscr{R}(\lambda)$ denote the recurrence set of $(S_n)_{n\ge0}$ under $\mathbb{P}_\lambda$, it suffices to observe for $\mathscr{R}(\lambda) = \mathscr{R}(\delta_0)$ that

$$\mathbb{P}_\lambda(|S_n - x| < \varepsilon \text{ infinitely often}) = \int_{\mathbb{G}_d} \mathbb{P}_0(|S_n - (x-y)| < \varepsilon \text{ infinitely often})\, \lambda(dy)$$

holds true if $\lambda(\mathbb{G}_d) = 1$. For a transient RW the local finiteness of the renewal measure $\mathbb{U}_\lambda$ as stated in Cor. 2.3.4 for $\lambda = \delta_0$ even extends to all initial distributions λ as is readily seen.

(c) Dealing with the recurrence of RW's as Markov chains it is worth mentioning that Lebesgue measure $\lambda\!\!\lambda_0$ in the nonarithmetic case and d times counting measure $\lambda\!\!\lambda_d$ on $\mathbb{G}_d$ in the d-arithmetic case provides a *stationary measure* for a RW $(S_n)_{n\ge0}$, no matter whether it is recurrent or not. This follows from

$$
\begin{aligned}
\mathbb{P}_{\lambda\!\!\lambda_d}(S_n \in B) \;:=\; & \int_{\mathbb{G}_d} \mathbb{P}_x(S_n \in B)\; \lambda\!\!\lambda_d(dx) \\
=\; & \int \mathbb{P}_0(S_n \in B - x)\; \lambda\!\!\lambda_d(ds) \\
=\; & \iint \mathbf{1}_B(x+s)\; \lambda\!\!\lambda_d(dx)\; \mathbb{P}_0(S_n \in ds) \\
=\; & \int \lambda\!\!\lambda_d(B-s)\; \mathbb{P}_0(S_n \in ds) \;=\; \lambda\!\!\lambda_d(B) \quad \text{for all } B \in \mathscr{B}(\mathbb{R})
\end{aligned}
$$

where the translation invariance of $\lambda\!\!\lambda_d$ on $\mathbb{G}_d$ has been utilized.

(d) The condition $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ in Thm. 2.3.5 is not sharp for the recurrence. Independent work by ORNSTEIN [37] and STONE [44] showed that, if $\Re(z)$ denotes the real part of a complex number z and φ is the Fourier transform of the increments of the RW, then

$$\int_{-\varepsilon}^{\varepsilon} \Re\left(\frac{1}{1-\varphi(t)}\right)\, dt \;=\; 1$$

provides a necessary and sufficient condition for the recurrence of $(S_n)_{n\ge 0}$. We will return to this in ????????

(e) In view of the fact that any recurrent RW is oscillating, it might be tempting to believe that these two properties are actually equivalent. However, this holds true only if $\mathbb{E}X_1$ exists [☞ final part of Thm. 2.2.9]. In the case where $\mathbb{E}X_1^- = \mathbb{E}X_1^+ = \infty$ the RW may be oscillating , so that every finite interval is crossed infinitely often, and yet be transient. KESTEN [26] further showed the following trichotomy which is stated here without proof.

Theorem 2.3.7. [Kesten] *Given a RW $(S_n)_{n\ge 0}$ with $\mathbb{E}X_1^- = \mathbb{E}X_1^+ = \infty$, exactly one of the following three cases holds true:*

(a) $\lim_{n\to\infty} n^{-1}S_n = \infty$ *a.s.*
(b) $\lim_{n\to\infty} n^{-1}S_n = -\infty$ *a.s.*
(c) $\liminf_{n\to\infty} n^{-1}S_n = -\infty$ *and* $\limsup_{n\to\infty} n^{-1}S_n = \infty$ *a.s.*

2.4 The renewal measure in the transient case: cyclic decompositions and basic properties

As shown in the previous section, the renewal measure $\mathbb{U}_\lambda$ of a transient RW $(S_n)_{n\ge 0}$ is locally finite for any initial distribution λ [☞ Cor. 2.3.4 and Rem. 2.3.6(b)]. This section is devoted to further investigations in the transient case that are also in preparation of one of the principal results in renewal theory, viz. Blackwell's renewal theorem to be derived in the next chapter.

2.4.1 Uniform local boundedness

We have seen in the Introduction that the renewal measure of a RW with exponential increments with mean μ equals exactly μ^{-1} times Lebesgue measure if the initial distribution is the same as for the increments. In view of this result and also intuitively, it is quite plausibel that for a general RW with drift μ and renewal measure $\mathbb{U}_\lambda$ the behavior of the renewal measure should at least be of a similar kind in the sense that

$$\mathbb{U}_\lambda(I) \approx \frac{\lambda\!\!\lambda_0(I)}{\mu}$$

for any bounded interval I and any initial distribution λ. What is indeed shown next is *uniform local boundedness* of $\mathbb{U}_\lambda$.

Lemma 2.4.1. *Let $(S_n)_{n\ge 0}$ be a transient RW in a standard model. Then*

$$\sup_{t\in\mathbb{R}} \mathbb{P}_\lambda(N([t,t+a]) \ge n) \le \mathbb{P}_0(N([-a,a]) \ge n) \tag{2.16}$$

for all $a > 0$, $n \in \mathbb{N}_0$ and $\lambda \in \mathscr{P}(\mathbb{R})$. In particular,

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda([t,t+a]) \le \mathbb{U}_0([-a,a]) \tag{2.17}$$

and $\{N([t,t+a]) : t \in \mathbb{R}\}$ is uniformly integrable under each $\mathbb{P}_\lambda$ for all $a > 0$.

Proof. If (2.16) holds true, then the uniform integrability of $\{N([t,t+a]) : t \in \mathbb{R}\}$ is a direct consequence, while (2.17) follows by summation over n. So (2.16) is the only assertion to be proved. Fix $t \in \mathbb{R}$, $a > 0$, and define $\tau := \inf\{n \ge 0 : S_n \in [t,t+a]\}$. Then

$$N([t,t+a]) = \begin{cases} \sum_{k\ge 0} \mathbf{1}_{[t,t+a]}(S_{\tau+k}), & \text{if } \tau < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The desired estimate now follows by a similar argument as in (2.14), viz.

$$\begin{aligned} \mathbb{P}_\lambda(N([t,t+a]) \ge n) &= \mathbb{P}_\lambda\left(\tau < \infty, \sum_{k\ge 0} \mathbf{1}_{[t,t+a]}(S_{\tau+k}) \ge n\right) \\ &\le \mathbb{P}_\lambda\left(\tau < \infty, \sum_{k\ge 0} \mathbf{1}_{[-a,a]}(S_{\tau+k} - S_\tau) \ge n\right) \\ &= \mathbb{P}_\lambda(\tau < \infty)\mathbb{P}_0(N([-a,a]) \ge n) \end{aligned}$$

for all $n \in \mathbb{N}$ and $\lambda \in \mathscr{P}(\mathbb{R})$. □

2.4.2 A useful connection with first passage times

In the case of a RP $(S_n)_{n\ge 0}$, there is a very useful connection between its renewal function $t\mapsto \mathbb{U}_\lambda(t)$ and the *first passage times*

$$\tau(t) := \inf\{n\ge 0: S_n>t\}\quad \text{for } t\in\mathbb{R}_\ge.$$

These are clearly stopping times for $(S_n)_{n\ge 0}$, arise in many applications and will be studied in greater detail in ???????? Furthermore, as the RP has a.s. nondecreasing trajectories, we see that, for all $t\in\mathbb{R}_\ge$ and $n\in\mathbb{N}_0$,

$$\{\tau(t)>n\} = \left\{\max_{0\le k\le n} S_k\le t\right\} = \{S_n\le t\}$$

and further

$$\tau(t) = \sum_{n=0}^{\tau(t)-1}\mathbf{1}_{[0,t]}(S_n) = \sum_{n\ge 0}\mathbf{1}_{[0,t]}(S_n) = N(t)\quad \text{a.s.,} \tag{2.18}$$

that is, $(\tau(t))_{t\ge 0}$ also equals the renewal counting process associated with $(S_n)_{n\ge 0}$. The following result is now immediate.

Lemma 2.4.2. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model. Then its renewal function satisfies*

$$\mathbb{U}_\lambda(t) = \mathbb{E}_\lambda\tau(t)\quad \textit{for all } t\in\mathbb{R}_{\ge 0} \textit{ and } \lambda\in\mathscr{P}(\mathbb{R}_\ge).$$

Provided that $(S_n)_{n\ge 0}$ has also finite drift μ, the previous result may be combined with Wald's identity 2.2.4 to infer

$$\infty > \mathbb{U}_0(t) = \mathbb{E}_0\tau(t) = \frac{1}{\mu}\mathbb{E}_0 S_{\tau(t)} = \frac{t}{\mu}+\frac{\mathbb{E}_0(S_{\tau(t)}-t)}{\mu} \tag{2.19}$$

for all $t\in\mathbb{R}_\ge$, the finiteness of $\mathbb{U}_0(t)$ following from Thm. 1.1.1 or Cor. 2.3.4. The nonnegative random variable

$$R(t) := S_{\tau(t)}-t\quad \text{for } t\in\mathbb{R}_\ge$$

provides the amount by which the RP exceeds the level t at the first passage epoch $\tau(t)$ [☞ Fig. 2.2]. It plays an important role in renewal theory and has been given a number of different names, depending on the context in which it is discussed: *overshoot, excess (over the boundary), forward recurrence time*, or *residual waiting time*. Clearly, the first two names express the afore-mentioned exceedance property

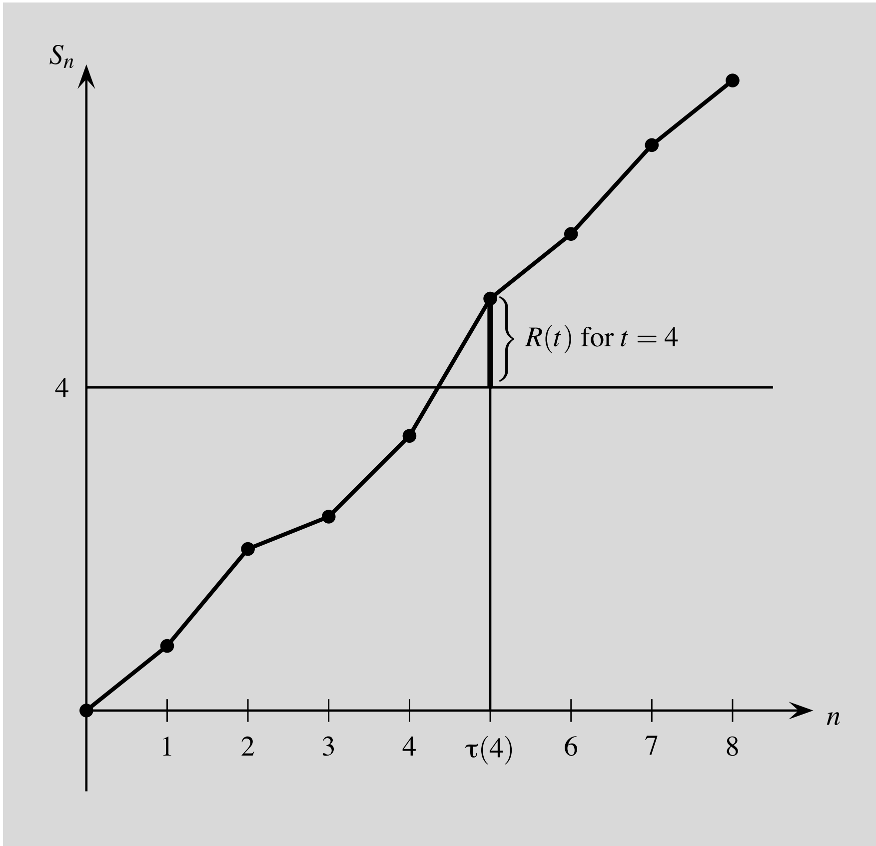

Fig. 2.2 Path of a RP with first pasage time $\tau(4) = 5$ and associated overshoot $R(4)$.

of $R(t)$, whereas the last two names refer to its meaning in a renewal scheme of being the residual time at t one has to wait until the next renewal will occur. The crucial question with regard to (2.19) is now about the asymptotic behavior of $R(t)$ under $\mathbb{P}_0$ and its expectation $\mathbb{E}_0 R(t)$ as $t \to \infty$. Since $R(t) \le X_{\tau(t)}$ and the X_n are identically distributed with finite mean, it seems plausible that $R(t)$ converges in distribution and that $\sup_{t\ge 0} \mathbb{E}_0 R(t)$ is finite. We will see already soon that the first assertion is indeed correct (modulo an adjustment in the arithmetic case), whereas the second one requires the nonobvious additional condition that the X_n have finite variances.

For general $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$ use $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ [☞ (2.2)] to obtain

$$\begin{aligned} \mathbb{U}_\lambda(t) \ &= \ \int_{[0,t]} \mathbb{U}_0(t-x)\,\lambda(dx) \ = \ \int_{[0,t]} \mathbb{E}_0\tau(t-x)\,\lambda(dx) \\ &= \ \frac{1}{\mu}\int_{[0,t]} (t-x+\mathbb{E}_0 R(t-x))\ \lambda(dx) \end{aligned} \tag{2.20}$$

A combination of (2.19) and (2.20) now leads to the following result, called *elementary renewal theorem*, which complements Thm. 1.1.2 in Section 1.1 and completes the answer to question (Q2) posed there. It was first proved by FELLER [20] by making use of a Tauberian theorem for Laplace integrals [☞ also his textbook [21]] and later by DOOB [18] via probabilistic arguments.

Theorem 2.4.3. [Elementary renewal theorem] *Let $(S_n)_{n\ge 0}$ be a RP in a standard model. Then*

$$\lim_{t\to\infty}\frac{\mathbb{U}_\lambda(t)}{t} \ = \ \frac{1}{\mu} \tag{2.21}$$

$$\text{and}\quad \lim_{t\to\infty}\frac{\tau(t)}{t} \ = \ \frac{1}{\mu}\quad \mathbb{P}_\lambda\text{-a.s.} \tag{2.22}$$

for all $\lambda\in\mathscr{P}(\mathbb{R}_\ge)$.

Strictly speaking, the elementary renewal theorem is just (2.21), but we have included (2.22) because it can be viewed as its pathwise analog and also provides the extension of Thm. 1.1.2 to arbitrary initial distributions λ when recalling from (2.18) that $\tau(t)=N(t)$ for all $t\in\mathbb{R}_\ge$.

Proof. Since $(S_n)_{n\ge 0}$ satisfies the SLLN under each $\mathbb{P}_\lambda$ and $S_{\tau(t)-1}\le t<S_{\tau(t)}$, Doob's argument used in the proof of Thm. 1.1.2 works here as well to give (2.22). Left with the proof of (2.21), we infer from (2.20) with the help of Fatou's lemma

$$\liminf_{t\to\infty}\ \frac{\mathbb{U}_\lambda(t)}{t} \ \ge \ \frac{1}{\mu}\int \lim_{t\to\infty}\left(\frac{t-x}{t}\,\mathbf{1}_{[0,t]}(x)\right)\,\lambda(dx) \ = \ \frac{1}{\mu}.$$

for any $\lambda\in\mathscr{P}(\mathbb{R}_\ge)$. For the reverse inequality we must work a little harder and consider the RP $(S_{c,n})_{n\ge 0}$ with truncated increments $X_n\wedge c$ for some $c>0$. Denote by $\tau_c(t)$ the corresponding first passage time and note that $S_{c,n}\le S_n$ for all $n\in\mathbb{N}_0$ implies $\tau(t)\le\tau_c(t)$. Also, $R_c(t):=S_{c,\tau_c(t)}-t\le c$ for all $t\in\mathbb{R}_\ge$. Consequently, by another appeal to (2.19),

$$\mathbb{E}_0\tau(t)\ \le\ \mathbb{E}_0\tau_c(t)\ =\ \frac{t+\mathbb{E}_0 R_c(t)}{\mu_c}\ \le\ \frac{t+c}{\mu_c}\quad \text{for all } t\in\mathbb{R}_\ge$$

where $\mu_c:=\mathbb{E}(X_1\wedge c)$. By using this in (2.20), the dominated convergence theorem ensures that, for any $\lambda\in\mathscr{P}(\mathbb{R}_\ge)$,

$$\limsup_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \ \le\ \int \lim_{t\to\infty} \left(\frac{t-x+c}{\mu_c t}\,\mathbf{1}_{[0,t]}(x)\right)\ \lambda(dx)\ =\ \frac{1}{\mu_c}.$$

But this holds true for all $c>0$ and $\lim_{c\to\infty}\mu_c=\mu$. Therefore we finally obtain

$$\limsup_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \ \le\ \frac{1}{\mu}$$

which completes the proof. □

It is to be emphasized that all results and identities in this subsection have been obtained only for the case of RP's, that is for RW's with nonnegative positive mean increments. This may also be taken as a transitional remark so as to motivate the consideration of general RW's $(S_n)_{n\ge 0}$ with a.s. finite ladder epoch $\sigma^>$ in the next subsection, the main focus being on the case where the RW has positive drift.

2.4.3 Cyclic decomposition via ladder epochs

Here we consider a nontrivial SRW $(S_n)_{n\ge 0}$ with renewal measure $\mathbb{U}$. Let σ be any a.s. finite stopping time for $(S_n)_{n\ge 0}$ with associated sequence $(\sigma_n)_{n\ge 0}$ of copy sums. Denote by $\mathbb{U}^{(\sigma)}$ the renewal measure of the RW $(S_{\sigma_n})_{n\ge 0}$ and define the *pre-σ occupation measure of* $(S_n)_{n\ge 0}$ by

$$\mathbb{V}^{(\sigma)}(A)\ :=\ \mathbb{E}\left(\sum_{n=0}^{\sigma-1}\mathbf{1}_A(S_n)\right)\quad \text{for } A\in\mathscr{B}(\mathbb{R}), \tag{2.23}$$

which has total mass $\|V^{(\sigma)}\|=\mathbb{E}\sigma$ and is hence finite if σ has finite mean. The next lemma provides us with a useful relation between $\mathbb{U}$ and $\mathbb{U}^{(\sigma)},\mathbb{V}^{(\sigma)}$.

Lemma 2.4.4. *Under the stated assumptions,*

$$\mathbb{U}\ =\ \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}$$

for any a.s. finite stopping time σ for $(S_n)_{n\ge 0}$.

Proof. Using cyclic decomposition with the help of the σ_n, we obtain

$$\begin{aligned}\mathbb{U}(A) &= \mathbb{E}\left(\sum_{k\ge 0}\mathbf{1}_A(S_k\right) = \sum_{n\ge 0}\mathbb{E}\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1}\mathbf{1}_A(S_k)\right)\\ &= \sum_{n\ge 0}\int_{\mathbb{R}} E\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1}\mathbf{1}_{A-x}(S_k-S_{\sigma_n})\middle| S_{\sigma_n}=x\right)\,\mathbb{P}(S_{\sigma_n}\in dx)\\ &= \sum_{n\ge 0}\int_{\mathbb{R}}\mathbb{V}^{(\sigma)}(A-x)\,\mathbb{P}(S_{\sigma_n}\in dx)\\ &= \mathbb{V}^{(\sigma)}*\mathbb{U}^{(\sigma)}(A)\quad\text{for all } A\in\mathscr{B}(\mathbb{R}),\end{aligned}$$

where (2.3) of Thm. 2.2.1 has been utilized in the penultimate line. □

Now, if $(S_n)_{n\ge 0}$ has a.s. finite ladder epoch $\sigma=\sigma^\alpha$ for some $\alpha\in\{>,\ge,<,\le\}$, we write $\mathbb{U}^\alpha$ and V^α for $\mathbb{U}^{(\sigma)}$ and $\mathbb{V}^{(\sigma)}$, respectively. Note that U^α is the renewal measure of the ladder height process $(S_n^\alpha)_{n\ge 0}$. As a trivial but important consequence of the previous lemma, we note that

$$\mathbb{U} \;=\; V^\alpha*\mathbb{U}^\alpha\quad \text{if } \mathbb{P}(\sigma^\alpha<\infty)=1. \tag{2.24}$$

In the case where $\sigma^> <\infty$ a.s. (and thus also $\sigma^\ge<\infty$ a.s.), we thus have a convolution formula for the renewal measure $\mathbb{U}$ that involves the renewal measure of a RP, namely $\mathbb{U}^>$ or $\mathbb{U}^\ge$. This will allow us in various places to reduce a problem for RW's with positive drift to those that have a.s. nonnegative or even positive increments.

If $(S_n)_{n\ge 0}$, given in a standard model, has arbitrary initial distribution λ, then Lemma 2.4.4 in combination with $\mathbb{U}_\lambda=\lambda*\mathbb{U}_0$ immediately implies

$$\mathbb{U}_\lambda \;=\; \lambda*\mathbb{V}^{(\sigma)}*\mathbb{U}^{(\sigma)} \;=\; \mathbb{V}^{(\sigma)}*\mathbb{U}_\lambda^{(\sigma)} \tag{2.25}$$

where $\mathbb{V}^{(\sigma)},\mathbb{U}^{(\sigma)}$ are defined as before under $\mathbb{P}_0$.

Returning to the zero-delayed situation, let us finally note that a simple computation shows that

$$\mathbb{V}^{(\sigma)} \;=\; \sum_{n\ge 0}\mathbb{P}(\sigma>n, S_n\in\cdot), \tag{2.26}$$

and that, for any real- or complex-valued function f

$$\int f\,d\mathbb{V}^{(\sigma)} \;=\; \sum_{n\ge 0}\int_{\{\sigma>n\}} f(S_n)\,d\mathbb{P} \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma-1} f(S_n)\right) \tag{2.27}$$

whenever one of the three expressions exist.

2.5 The stationary delay distribution

2.5.1 What are we looking for and why?

In order to motivate the derivation of what is called the *stationary delay distribution of a RP* $(S_n)_{n\ge 0}$, let us first dwell on some heuristic considerations that embark on the results in Section 1.2. We saw there that, if $(S_n)_{n\ge 0}$ has exponential increment distribution with parameter θ and the same holds true for S_0, then the renewal measure $\mathbb{U}$ equals $\theta\lambda\!\!\lambda_0$ on $\mathbb{R}_\ge$ which means that $\mathbb{U}([t,t+h])$ for any fixed h is temporally invariant. In fact, we further found that the associated renewal counting process $N := (N(t))_{t\ge 0}$ is a homogeneous Poisson process and as such a *stationary point process* in the sense that

$$\Theta_s N \;:=\; (N(s+t)-N(s))_{t\ge 0} \;\stackrel{d}{=}\; N \quad \text{for all } s\in\mathbb{R}_\ge. \tag{2.28}$$

This follows immediately from the fact that a Poisson process has stationary and independent increments [☞ Thm. 1.2.2]. Cleary, N is a functional of the renewal process $(S_n)_{n\ge 0}$ or, equivalently, of $S_0, X_1, ...$, i.e. $N = h(S_0, X_1, \ldots)$ for some measurable function h. For the same reason

$$\Theta_s N \;=\; h(S_{\tau(s)}-s, X_{\tau(s)+1},\ldots) \;=\; h(R(s), X_{\tau(s)+1},\ldots) \quad \text{for all } s\in\mathbb{R}_\ge,$$

where $R(s)$ is the overshoot at s [see Fig. 2.3]. After this observation, the validity of (2.28) can be explained as follows. Even without providing a rigorous argument, the memoryless property of the exponential distribution is readily seen to imply that $R(s)\stackrel{d}{=} Exp(\theta)$ for all $s\in\mathbb{R}_\ge$. By Thm. 2.2.1, $R(s)$ is further independent of the iid sequence $(X_{\tau(s)+n})_{n\ge 1}$ with $X_{\tau(t)+1}\stackrel{d}{=} X_1$. But this is all it takes to conclude (2.28).

Now let us turn to the general case, the question being whether an appropriate choice of the delay distribution F^s of a RP $(S_n)_{n\ge 0}$ with increment distribution F renders $R(t)\stackrel{d}{=} F^s$ *for all* $t\in\mathbb{R}_\ge$. If yes then by the same reasoning as before we should obtain a stationary associated renewal counting process $N = (N(t))_{t\ge 0}$ under $\mathbb{P}_{F^s}$ in the sense of (2.28) and in particular a translation invariant renewal measure $\mathbb{U}_{F^s}$ on $\mathbb{R}_\ge$. But the translation invariance entails that $\mathbb{U}_{F^s}$ equals $\theta\lambda\!\!\lambda_0$ on $\mathbb{R}_\ge$ for some $\theta>0$, while the elementary renewal theorem 2.4.3 gives $\theta = 1/\mathbb{E}X_1$. It will be shown in ??????? that $(R(t))_{t\ge 0}$ constitutes a continuous time Markov process for which F^s, if it exists, forms a stationary distribution. However, the way F^s is derived hereafter, does not use this fact.

The question not addressed so far is why we should strive for F^s. The short reply is that its existence provides strong evidence that for arbitrary delay distributions λ the behavior of $\Theta_t N$ under $\mathbb{P}_\lambda$ and particularly of $\mathbb{U}_\lambda([t,t+h])$ is expected to be the same as in the stationary situation if t tends to infinity. However, what we are really hoping for and in fact going to deliver later is to turn this into a rigorous result by a suitable construction of two RP's on the same probability space with the same

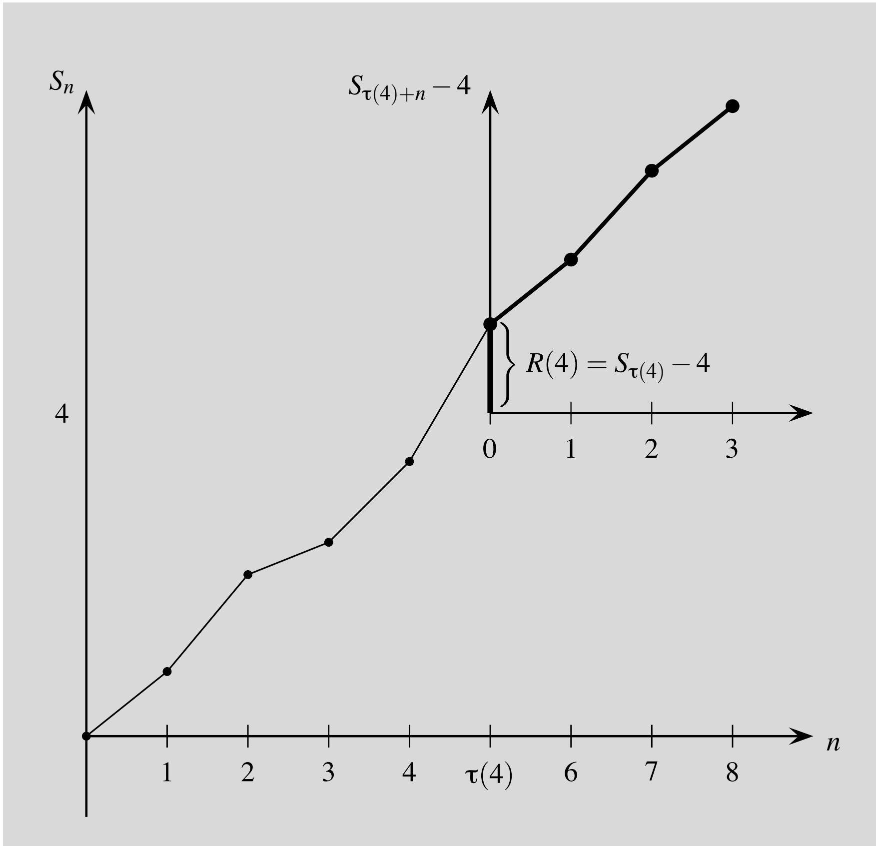

Fig. 2.3 Path of a RP with first pasage time $\tau(4) = 5$ and associated overshoot $R(4)$.

increment distribution but different delay distributions λ and F^s, such that their renewal counting processes N and N', say, are asymptotically equal in the sense that $\lim_{t\to\infty} \rho(\Theta_t N, \Theta_t N') = 0$, where ρ denotes a suitable distance function. The technique behind this idea is well-known under the name *coupling* in the theory of stochastic processes.

2.5.2 The derivation

If λ denotes any measure on $\mathbb{R}$, let $\lambda^+ := \lambda(\cdot \cap \mathbb{R}_>)$ be its restriction to $\mathbb{R}_>$ hereafter in the sense that its mass on $\mathbb{R}_\leq$ has been removed. Having outlined what we are looking for, let us now turn to the derivation of the stationary delay distribution F^s for a RP $(S_n)_{n\geq 0}$ with increment distribution F, finite mean μ and given as usual in a standard model. The first thing to note is that $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ satisfies the convolution equation

$$\mathbb{U}_\lambda \;=\; \lambda \;+\; F * \mathbb{U}_\lambda \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}_\geq)$$

which in terms of the renewal function becomes a *renewal equation* as encountered in Section 1.4, namely

$$\mathbb{U}_\lambda(t) \;=\; \lambda(t) \;+\; \int_{[0,t]} F(t-x)\,\mathbb{U}_\lambda(dx) \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}_\geq) \tag{2.29}$$

The goal is to find a λ such that $\mathbb{U}_\lambda(t) = \mu^{-1}t$ for all $t \in \mathbb{R}_\geq$ (thus $\mathbb{U}_\lambda = \mu^{-1}\lambda\!\!\lambda_0^+$) and we will now do so by simply plugging the result into (2.29) and solving for $\lambda(t)$. Then, with $\overline{F} := 1 - F$

$$\begin{aligned}\lambda(t) \;&=\; \frac{t}{\mu} \;-\; \frac{1}{\mu}\int_0^t F(t-x)\,dx\\ &=\; \frac{1}{\mu}\int_0^t \overline{F}(t-x)\,dx \;=\; \frac{1}{\mu}\int_0^t \overline{F}(x)\,dx \quad \text{for all } t \geq 0.\end{aligned}$$

We thus see that there is only one λ, now called F^s, that gives the desired property of $\mathbb{U}_\lambda$, viz.

$$F^s(t) \;:=\; \frac{1}{\mu}\int_0^t \overline{F}(x)\,dx \;=\; \frac{1}{\mu}\int_0^t \mathbb{P}(X_1 > x)\,dx \quad \text{for all } t \geq 0,$$

which is continuous and requires that μ is finite. To all those who prematurely lean back now let is be said that this is not yet the end of the story because there are questions still open like "What about the infinite mean case?" and "Is this really the answer we are looking for if the RP is arithmetic?"

If $(S_n)_{n\geq 0}$ is d-arithmetic, w.l.o.g. suppose $d = 1$, then indeed a continuous delay distribution gives a continuous renewal measure which is in sharp contrast to the zero-delayed situation where the renewal measure is concentrated on $\mathbb{N}_0$. The point to be made here is that an appropriate definition of stationarity of the associated renewal counting process N must be restricted to those times at which renewals naturally occur in the 1-arithmetic case, namely integer epochs. In other words, the stationary delay distribution F^s to be defined must now be concentrated on $\mathbb{N}$, but only give (2.28) for $t \in \mathbb{N}_0$ under $\mathbb{P}_{F^s}$. This particularly implies $\mathbb{U}_{F^s}^+ = \mu^{-1}\lambda\!\!\lambda_1^+$ and thus $\mathbb{U}_{F^s}^+(n) = \mathbb{U}_{F^s}(\{1,...,n\}) = \mu^{-1}n$ for $n \in \mathbb{N}$, where $\lambda\!\!\lambda_1$ is counting measure on the set of integers $\mathbb{Z}$. By pursuing the same argument as above, but for $t \in \mathbb{N}_0$ only, we then find that F^s must satisfy

$$F^s(n) \;=\; \frac{n}{\mu} \;-\; \frac{1}{\mu}\sum_{k=1}^n F(n-k) \quad \text{for all } n \in \mathbb{N}$$

and therefore

$$F^s(n) \;=\; \frac{1}{\mu}\sum_{k=0}^{n-1}\overline{F}(k)\,dx \;=\; \frac{1}{\mu}\sum_{k=1}^{n}\mathbb{P}(X_1 \ge k) \quad \text{for all } n\in\mathbb{N}$$

as the unique solution among all distributions concentrated on $\mathbb{N}$. We summarize our findings as follows.

Theorem 2.5.1. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with finite drift μ and lattice-span $d\in\{0,1\}$. Define its* ***stationary delay distribution*** *F^s on $\mathbb{R}_\ge$ by*

$$F^s(t) \;:=\; \begin{cases} \frac{1}{\mu}\int_0^t \mathbb{P}(X_1 > x)\,dx, & \text{if } d=0,\\ \frac{1}{\mu}\sum_{k=1}^{n(t)}\mathbb{P}(X_1\ge k), & \text{if } d=1\end{cases} \tag{2.30}$$

for $t\in\mathbb{R}_\ge$, where $n(t):=\lfloor t\rfloor=\sup\{n\in\mathbb{Z}: n\le t\}$. Then $\mathbb{U}_{F^s}=\mu^{-1}\lambda\!\!\lambda_d^+$.

Now observe that the integral equation (2.29) remains valid if λ is any locally finite measure on $\mathbb{R}_\ge$ and $\mathbb{U}_\lambda$ is still defined as $\lambda * \mathbb{U}_0$. This follows because (2.29) is linear in λ. Hence, if we drop the normalization μ^{-1} in the definition of F^s, we obtain without further ado the following extension of the previous theorem.

Corollary 2.5.2. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with lattice-span $d\in\{0,1\}$. Define the locally finite measure ξ on $\mathbb{R}_\ge$ by*

$$\xi(t) \;:=\; \begin{cases} \int_0^t \mathbb{P}(X_1 > x)\,dx, & \text{if } d=0,\\ \sum_{k=1}^{n(t)}\mathbb{P}(X_1\ge k), & \text{if } d=1\end{cases} \tag{2.31}$$

for $t\in\mathbb{R}_\ge$ and $n(t)$ as in Thm. 2.5.1. Then $\mathbb{U}_\xi=\lambda\!\!\lambda_d^+$.

2.5.3 The infinite mean case: restricting to finite horizons

There is no stationary delay distribution if $(S_n)_{n\ge 0}$ has infinite mean μ, but Cor. 2.5.2 helps us to provide a family of delay distributions for which stationarity still yields when restricting to finite horizons, that is to time sets $[0,a]$ for $a\in\mathbb{R}_>$. As a further ingredient we need the observation that the renewal epochs in $[0,a]$ of $(S_n)_{n\ge 0}$ and $(S_{a,n})_{n\ge 0}$, where $S_{a,n}:=S_0+\sum_{k=1}^n (X_k\wedge a)$, are the same. As a trivial consequence they also have the same renewal measure on $[0,a]$, whatever the delay distribution is. But by choosing the latter appropriately, we also have a domination result on (a,∞) as the next result shows.

Theorem 2.5.3. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model with drift $\mu=\infty$ and lattice-span $d\in\{0,1\}$. With ξ given by (2.31) and for $a>0$, define distributions F_a^s on $\mathbb{R}_\ge$ by*

$$F_a^s(t) \;:=\; \frac{\xi(t\wedge a)}{\xi(a)} \quad \textit{for } t\in\mathbb{R}_\ge. \tag{2.32}$$

Then, for all $a\in\mathbb{R}_>$, $\mathbb{U}_{F_a^s}\le \xi(a)^{-1}\lambda\!\!\lambda_d^+$ with equality holding on $[0,a]$.

Proof. Noting that F_a^s can be written as $F_a^s=\xi(a)^{-1}\xi-\lambda_a$, where $\lambda_a\in\mathscr{P}(\mathbb{R}_\ge)$ is given by

$$\lambda_a(t) \;:=\; \frac{\xi(t)-\xi(a\wedge t)}{\xi(a)} \;=\; \mathbf{1}_{(a,\infty)}(t)\,\frac{\xi(t)-\xi(a)}{\xi(a)} \quad \text{for all } t\in\mathbb{R}_\ge,$$

we infer with the help of Cor. 2.5.2 that

$$\mathbb{U}_{F_a^s} \;=\; \xi(a)^{-1}\mathbb{U}_\xi-\lambda_a*\mathbb{U}_0 \;\le\; \xi(a)^{-1}\mathbb{U}_\xi \;=\; \xi(a)^{-1}\lambda\!\!\lambda_d \quad \text{on } \mathbb{R}_\ge$$

as claimed. □

2.5.4 And finally random walks with positive drift via ladder epochs

A combination of the previous results with a cyclic decomposition via ladder epochs allows us to further extend these results to the situation where $(S_n)_{n\ge 0}$ is a RW with positive drift μ and lattice-span $d\in\{0,1\}$ and thus may also take on negative values. This is accomplished by considering F^s, F_a^s and ξ as defined before, but for the associated ladder height RP $(S_n^>)_{n\ge 0}$. Hence we put

$$\xi(t) \;:=\; \begin{cases} \int_0^t \mathbb{P}(S_1^> > x)\,dx, & \text{if } d=0,\\ \sum_{k=1}^{n(t)} \mathbb{P}(S_1^>\ge k), & \text{if } d=1 \end{cases} \tag{2.33}$$

for $t\in\mathbb{R}_\ge$ (with $n(t)$ as in Thm. 2.5.1) and then again F_a^s by (2.32) for $a\in\mathbb{R}_>$. If $S_1^>$ has finite mean $\mu^>$ and hence ξ is finite, then let F^s be its normalization, i.e. $F^s=(\mu^>)^{-1}\xi$. In order to be able to use the results from the previous section we must first verify that $S_1^>$ and X_1 are of the same lattice-type.

Lemma 2.5.4. *Let $(S_n)_{n\ge 0}$ be a nontrivial SRW and σ an a.s. finite first ladder epoch. Then $d(X_1)=d(S_\sigma)$.*

Proof. If X_1 is nonarithmetic the assertion follows directly from the obvious inequality $d(S_\sigma) \le d(X_1) = 0$. Hence it suffices to consider the case when $d(X_1) > 0$. W.l.o.g. suppose $d(S_\sigma) = 1$ and $\sigma = \sigma^>$, so that $d(X_1) = 1$ must be verified. If $d(X_1) < 1$, then $p := \mathbb{P}(X_1 = c) > 0$ for some $c \notin \mathbb{Z}$. Define $W_n := S_{n+1} - S_1$ for $n \in \mathbb{N}_0$, clearly a copy of $(S_n)_{n\ge0}$ and independent of X_1. Then

$$S_1^> \;=\; X_1 \mathbf{1}_{\{X_1>0\}} + (X_1 + W_{\tau(-X_1)})\mathbf{1}_{\{X_1\le 0\}},$$

where $\tau(t) := \inf\{n \ge 0 : W_n > t\}$. Consequently, if $c > 0$, then $\mathbb{P}(S_1^> = c) \ge p > 0$ which is clearly impossible as $c \notin \mathbb{Z}$. If $c < 0$, then use that $W_{\tau(t)}$ is a strictly ascending ladder height for $(W_n)_{n\ge0}$ and thus integer-valued for any $t \in \mathbb{R}_\ge$ to infer that

$$\mathbb{P}(S_1^> \in c+\mathbb{Z}) \;\ge\; p\,\mathbb{P}(c + W_{\tau(-c)} \in c+\mathbb{Z}) \;=\; p\,\mathbb{P}(W_{\tau(-c)} \in \mathbb{Z}) \;>\; 0$$

which again contradicts our assumption $d(S_1^>) = 1$, for $(c+\mathbb{Z}) \cap \mathbb{Z} = \emptyset$. ◻

Theorem 2.5.5. *Let $(S_n)_{n\ge0}$ be a RW in a standard model with positive drift μ and lattice-span $d \in \{0,1\}$. Then the following assertions hold with ξ, F_a^s and F^s as defined in* (2.33) *and thereafter.*

(a) $\mathbb{U}_\xi^+ = \mathbb{E}\sigma^> \lambda\!\!\lambda_d^+$.
(b) $\mathbb{U}_{F_a^s}^+ \le \xi(a)^{-1}\mathbb{E}\sigma^> \lambda\!\!\lambda_d^+$ *for all* $a \in \mathbb{R}_>$.
(c) *If μ is finite, then* $\mathbb{U}_{F^s}^+ = \mu^{-1}\lambda\!\!\lambda_d^+$.

Proof. First note that $\mu > 0$ implies $\mathbb{E}\sigma^> < \infty$ [☞ Thm. 2.2.9] and $\mu^> = \mathbb{E}S_1^> = \mu\,\mathbb{E}\sigma^>$ by Wald's identity. By (2.25), $\mathbb{U}_\lambda = \mathbb{V}^> * \mathbb{U}_\lambda^>$ for any distribution λ, where

$$\mathbb{V}^>(A) \;=\; \mathbb{E}_0\left(\sum_{n=0}^{\sigma^>-1} \mathbf{1}_A(S_n)\right) \quad \text{for } A \in \mathscr{B}(\mathbb{R})$$

is the pre-$\sigma^>$ occupation measure and thus concentrated on $\mathbb{R}_\le$ with $\|\mathbb{V}^>\| = \mathbb{E}\sigma^>$. Of course, (2.25) extends to arbitrary locally finite measures λ. Therefore,

$$\begin{aligned}
\mathbb{U}_\xi(A) \;&=\; \mathbb{V}^> * \mathbb{U}_\xi^>(A) \;=\; \int_{\mathbb{R}_\le} \mathbb{U}_\xi^>(A-x)\,\mathbb{V}^>(dx)\\
&=\; \int_{\mathbb{R}_\le} \lambda\!\!\lambda_d^+(A-x)\,\mathbb{V}^>(dx) \;=\; \mathbb{V}^>(\mathbb{R}_\le)\,\lambda\!\!\lambda_d^+(A)\\
&=\; \mathbb{E}\sigma^>\,\lambda\!\!\lambda_d^+(A) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}_\ge)
\end{aligned}$$

where Cor. 2.5.2, the translation invariance of $\lambda\!\!\lambda_d$ and $A - x \subset \mathbb{R}_\ge$ for all $x \in \mathbb{R}_\le$ have been utilized. Hence assertion (a) is proved. As (b) and (c) are shown in a similar manner, we omit supplying the details again and only note for (c) that, if $\mu < \infty$,

$\mathbb{U}_{F^s}^+ = (\mu^>)^{-1}\,\mathbb{E}\sigma^>\,\lambda\!\!\!\lambda_d^+$ really equals $\mu^{-1}\lambda\!\!\!\lambda_d^+$ because $\mu^> = \mu\,\mathbb{E}\sigma^>$ as mentioned above. $\square$

Remark 2.5.6. (a) Replacing the strictly ascending ladder height $S_1^>$ with the weakly ascending one $S_1^\geq$ in the definition (2.33) of ξ leads to $\mathbb{U}_\xi = \mathbb{E}\sigma^\geq\,\lambda\!\!\!\lambda_d$ and thus to a proportional, but different result if $S_1^>$ and $S_1^\geq$ are different. On the other hand, it is clear that, for some $\beta \in (0,1]$,

$$\mathbb{P}(S_1^\geq \in \cdot) \;=\; (1-\beta)\delta_0 \;+\; \beta\,\mathbb{P}(S_1^> \in \cdot) \tag{2.34}$$

where $\mu^\geq = \beta\mu^>$ gives $\beta = \mu^\geq/\mu^>$. Consequently, a substitution of $S_1^>$ for $S_1^\geq$ in (2.33) changes ξ merely by a factor and thus has the same normalization in the case $\mu < \infty$, namely F^s. In other words, the stationary delay distributions of $(S_n^>)_{n\geq 0}$ and $(S_n^\geq)_{n\geq 0}$ are identical.

(b) Given a RP $(S_n)_{n\geq 0}$ with finite drift μ, we have already mentioned that the forward recurrence times $R(t)$ form a continuous time Markov process with stationary distribution $\pi(dt) := \mu^{-1}\mathbb{P}(X_1 > t)\,dt$, and we have found that $\mathbb{U}_\pi = \mu^{-1}\lambda\!\!\!\lambda_0^+$, no matter what the lattice-type of $(S_n)_{n\geq 0}$ is. On the other hand, if the latter is 1-arithmetic, then the stationary delay distribution F^s that gives $\mathbb{U}_{F^s} = \mu^{-1}\lambda\!\!\!\lambda_1^+$ is discrete and thus differs from π. In terms of the forward recurrence times this can be explained as follows: The Markov process $(R(t))_{t\geq 0}$ has continuous state space $\mathbb{R}_>$ regardless of the lattice-type of the underlying RP and as such the continuous stationary distribution π. As a consequence, its subsequence $(R(n))_{n\geq 0}$ forms a Markov chain with the same state space and the same stationary distribution. However, in the 1-arithmetic case $\mathbb{N}$ forms a closed subclass of states in the sense that $\mathbb{P}(R_n \in \mathbb{N} \text{ for all } n \geq 0) = 1$ if $\mathbb{P}(R_0 \in \mathbb{N}) = 1$. Hence $(R(n))_{n\geq 0}$ may also be considered as a *discrete* Markov chain on $\mathbb{N}$ and as such has stationary distribution F^s.

(c) The *positive* forward recurrence time $R(t) = S_{\tau(t)} - t$, $\tau(t) = \inf\{n : S_n > t\}$, has been used in the previous derivations as a motivation for the stationary delay distribution. As an alternative, one may also consider the *nonnegative* variant $V(t) := S_{\nu(t)} - t$ with $\nu(t) := \inf\{n : S_n \geq t\}$. In the nonarithmetic case, this does not lead to anything different because stopping on the boundary, i.e. $V(t) = 0$, has asymptotic probability zero. However, if $(S_n)_{n\geq 0}$ is 1-arithmetic, then the stationary delay distribution on $\mathbb{R}_\geq$ instead of $\mathbb{R}_>$ is obtained, namely

$$\widehat{F}^s(t) \;:=\; \frac{1}{\mu^>}\sum_{k=0}^{n}\mathbb{P}(S_1^> > k)\mathbf{1}_{[n,n+1)}(t) \quad \text{for } t \in \mathbb{R}_\geq.$$

As a consequence, $\mathbb{U}_{\widehat{F}^s}$ equals $\mu^{-1}\lambda\!\!\!\lambda_1$ on $\mathbb{R}_\geq$ instead of $\mathbb{R}_>$ only. Similar adjustments apply to the definitions of ξ and F_a^s when dealing with $V(t)$, but we refrain from a further discussion.

Chapter 3
Blackwell's renewal theorem

Blackwell's renewal theorem may be rightfully called the mother of all deeper results in renewal theory. Not only it provides an answer to question (Q3) stated in the first section on the expected number of renewals in a finite remote interval but is also the simpler, yet equivalent version of the *key renewal theorem* discussed in the next chapter that allows us to determine the asymptotic behavior of many interesting quantities in applied stochastic models some of which we have already seen in the Introduction. The present chapter is devoted exclusively to this result and its derivation via four quite different proofs, the first and most recent one being based on probabilistic arguments, notably the coupling method, the second one on an analytic argument in combination with the Choquet-Deny equation, the third one on Fourier analysis, and the last one on Blackwell's original 1948 proof paying obeisance to the eponymous mathematician.

3.1 Statement of the result and historical account

We are now going to state Blackwell's theorem for RW's with positive drift which is often called the *two-sided* version of the original result that only considers RP's (nonnegative or one-sided case). But with the concept of ladder variables at hand, a proof of the general version is easily reduced to the nonnegative case as we will demonstrate below. The following notation is introduced so as to facilitate a unified formulation of subsequent results for the arithmetic and the nonarithmetic case. For $d \geq 0$, define

$$d\text{-}\lim_{t\to\infty} f(t) \ := \ \begin{cases} \lim_{t\to\infty} f(t), & \text{if } d = 0, \\ \lim_{n\to\infty} f(nd), & \text{if } d > 0. \end{cases}$$

Recall that $\lambda\!\!\lambda_0$ denotes Lebesgue measure, while $\lambda\!\!\lambda_d$ for $d > 0$ is d times counting measure on $d\mathbb{Z}$. The factor d is actually chosen for convenience in the following theorem.

Theorem 3.1.1. [Blackwell's renewal theorem] *Let $(S_n)_{n\ge 0}$ be a RW in a standard model with lattice-span $d \ge 0$ and positive drift μ. Then*

$$d\text{-}\lim_{t\to\infty} \mathbb{U}_\lambda([t,t+h]) = \mu^{-1}\lambda\!\!\lambda_d([0,h]) \quad \text{and} \tag{3.1}$$

$$\lim_{t\to-\infty} \mathbb{U}_\lambda([t,t+h]) = 0 \tag{3.2}$$

for all $\lambda \in \mathscr{P}(\mathbb{R})$, where $\mu^{-1} := 0$ if $\mu = \infty$.

The result has a long history dating back, as pointed out in [14], [15] and [6], to old work from 1936 by KOLMOGOROV [28] on discrete Markov chains from which the theorem for arithmetic RP's is deducible. In 1949 this case was explicitly settled by ERDÖS, FELLER & POLLARD [19], while BLACKWELL [6] in 1948 provided the first proof for nonarithmetic RP's. The latter case was also treated by DOOB [18, Thm. 12] under the additional restriction that some S_n is nonsingular with respect to $\lambda\!\!\lambda_0$ (such distributions will later be called *spread out*, ☞ Section 4.3). Proofs in the general case of RW's with positive drift had not to wait long. The arithmetic case was settled in 1952 by CHUNG & WOLFOWITZ [15], the nonarithmetic one again by BLACKWELL [7] in 1953, although a version under the afore-mentioned spread out assumption was given one year earlier by CHUNG & POLLARD [14]. BLACKWELL's work, which mentions a further unpublished proof by HARRIS, is also remarkable for the fact that the concept of ladder variables is introduced and used there for the first time. Nevertheless all proofs that had been published until the mid fifties appeared to be quite technical and did not meet the intuitive appeal of the result itself. The following quote of SMITH [41] from his 1958 survey on renewal theory and its ramifications describes it accurately:

> Blackwell's renewal theorem has a strong intuitive appeal and it is surprising that its proof should be so difficult. We suspect that no one has yet thought on the "right" mode but that when this mode is ultimately realized a simple proof will emerge.

After further proofs by FELLER & OREY [22] in 1961 via Fourier analysis [☞ Section 3.5], by FELLER in his 1971 textbook [21] using the Helly-Bray theorem in connection with the Choquet-Deny equation [☞ Section 3.4], by MCDONALD [35] in 1975 via harmonic analysis of the forward recurrence time process, and by sc Chan [10] using a constructive method, a breakthrough towards a more natural and really probabilistic approach had to wait until the advent of the coupling method, first introduced as early as 1938 by the ingenious mathematician DOEBLIN [16], [17] in his study of discrete Markov chains but unnoticed or forgotten afterwards for more than 30 years due to his untimely and tragic death in 1940. Until today, four coupling proofs of Blackwell's renewal theorem have been published, the first ones by LINDVALL [30] in 1977 for RP's with finite drift and by ATHREYA, MCDONALD & NEY [4] one year later for RW's with finite positive drift. In 1987, THORISSON [45] provided a third proof that "improved" on the former two ones by giving a self-contained argument for successful coupling where those had utilized the Hewitt-Savage zero-one law or the topological recurrence of RW's with drift zero. The coupling proof we are going to present in Section 3.3 may be viewed as a blend of these three contributions. Yet another, very short proof was finally given in 1996 by LINDVALL & ROGERS [32] and is inspired by a special coupling that had been christened *Mineka coupling* by the first author in his monography [31] on the coupling method (referring to [36]). We will describe it in Section 4.4 including a short account of how it may be used to prove Blackwell's theorem [☞ Subsection 4.4.3].

We close this section with a brief classification of Blackwell's renewal theorem in a measure-theoretic context. Recall that a sequence $(Q_n)_{n\ge 1}$ of locally finite measures in $\mathbb{R}$ converges *vaguely* ($\stackrel{v}{\to}$) to a locally finite measure Q, in short $Q_n \stackrel{v}{\to} Q$, if for every continuous function $g:\mathbb{R}\to\mathbb{R}$ with compact support ($g\in\mathscr{C}_0(\mathbb{R})$)

$$\lim_{n\to\infty}\int g\,dQ_n \;=\; \int g\,dQ$$

holds true. Since any $g\in\mathscr{C}_0(\mathbb{R})$ may be approximated pointwise by primitive functions of the form $h=\sum_{j=1}^k a_j\mathbf{1}_{I_j}$ with bounded intervals I_j and conversely indicators of such intervals are pointwise limits of functions in $\mathscr{C}_0(\mathbb{R})$, an equivalent condition is

$$\lim_{n\to\infty} Q_n(I) \;=\; Q(I)$$

for all Q-continuous intervals I, that is, whose endpoints have no mass under Q. Now it is easily seen that the two assertions in Blackwell's renewal theorem may also be stated as

$$\mathbb{U}_\lambda(t+\cdot) \;\stackrel{v}{\to}\; \mu^{-1}\lambda\!\!\lambda_d \quad \text{as } t\to\infty \text{ (through } d\mathbb{Z} \text{ if } d>0) \quad \text{and} \tag{3.3}$$

$$\mathbb{U}_\lambda(t+\cdot) \;\stackrel{v}{\to}\; 0 \quad \text{as } t\to -\infty \tag{3.4}$$

for every $\lambda\in\mathscr{P}(\mathbb{R})$. It is worth mentioning that, once the convergence of $\mathbb{U}_\lambda(t+\cdot)$ has been established, the translation invariance of the limiting measure Λ, say, on $\mathbb{G}_d$ immediately follows from

$$\Lambda(I) \;=\; d\text{-}\lim_{t\to\infty}\mathbb{U}_\lambda(t+I) \;=\; d\text{-}\lim_{t\to\infty}\mathbb{U}_\lambda(s+t+I) \;=\; \Lambda(s+I)$$

for all bounded intervals I and all $s\in\mathbb{G}_d$. Hence, Λ must be a multiple of $\lambda\!\!\lambda_d$, for this is the unique (up to scaling) translation invariant measure (Haar measure) on $\mathbb{G}_d$.

3.2 The easy part first: the behavior of $\mathbb{U}$ at $-\infty$

The renewal measure of any RW with positive drift vanishes at $-\infty$. This is the second and less interesting assertion (3.2) of Blackwell's theorem. Since it is also easily verified with the help of Lemma 2.4.1, we treat this part of the theorem first so as to focus exclusively on the nontrivial part thereafter, i.e. on (3.1).

Given a RW $(S_n)_{n\ge 0}$ with drift $\mu > 0$ and arbitrary delay distribution, it follows from $S_n \to \infty$ a.s. that $N(0) = N((-\infty,0]) < \infty$ a.s. and thus

$$\lim_{t\to-\infty} N([t,t+h]) \;=\; \lim_{t\to-\infty}\sum_{n\ge 0}\mathbf{1}_{[t,t+h]}(S_n) \;=\; 0 \quad \text{a.s.}$$

for each $h > 0$. This yields the desired conclusion (3.2) because, by Lemma 2.4.1, the family $\{N([t,t+h] : t \in \mathbb{R}\}$ is further uniformly integrable.

Plainly, (3.2) is equivalent to

$$\lim_{t\to-\infty} \mathbb{U}(t+B) = 0 \quad \text{for all bounded } B \in \mathscr{B}(\mathscr{R}), \tag{3.5}$$

but even more can be said without further ado. As $N(0) < \infty$ a.s. further implies

$$\lim_{t\to-\infty} N(t) = 0 \quad \text{a.s.}$$

and this convergence is monotone, we infer with the help of the monotone convergence theorem that

$$\lim_{t\to-\infty} \mathbb{U}(t) = 0 \quad \text{if} \quad \mathbb{U}(0) = \mathbb{E}N(0) < \infty. \tag{3.6}$$

However, the question remains and will so for a further while under which condition on the RW we have that $\mathbb{U}(0) < \infty$. We refer to ???????

3.3 A coupling proof

We now proceed with a presentation of the first proof of (3.1) which is purely probabilistic and based on the coupling technique. To disenthrall the ensuing arguments from distracting technicalities we begin with a couple of preliminary considerations.

3.3.1 Shaking off technicalities

1st reduction: $S_0 = 0$.
It is no loss of generality to prove (3.1) for zero-delayed RW's only. Indeed, if S_0 has distribution λ, then

$$\mathbb{U}_\lambda([t,t+h]) \;=\; \int \mathbb{U}_0([t-x,t-x+h])\,\lambda(dx)$$

together with $\sup_{t\in\mathbb{R}} \mathbb{U}_0([t,t+h]) \le \mathbb{U}_0([-h,h]) < \infty$ [☞ Lemma 2.4.1] implies by an appeal to the dominated convergence theorem that (3.1) is valid for $\mathbb{U}_\lambda$ if so for $\mathbb{U}_0$.

2nd reduction: $(S_n)_{n\ge 0}$ is a RP.
The cyclic decomposition via ladder variables described in Subsection 2.4.3 further allows us to reduce the proof to the case where $(S_n)_{n\ge 0}$ has nonnegative increments. To see this recall first that, by Lemma 2.5.4, the ladder height process $(S_n^>)_{n\ge 0}$ is of same lattice-type as $(S_n)_{n\ge 0}$ and suppose that its renewal measure $\mathbb{U}^>$ satisfies (3.1). Then $\mathbb{U} = \mathbb{V}^> * \mathbb{U}^>$, $\|\mathbb{V}^>\| = \mathbb{E}\sigma^> < \infty$, $\mu^> = \mu\mathbb{E}\sigma^>$ and another appeal to the dominated convergence theorem yield

$$\begin{aligned} d\text{-}\lim_{t\to\infty} \mathbb{U}([t,t+h]) \ &= \ \int d\text{-}\lim_{t\to\infty} \mathbb{U}^>([t-x,t-x+h])\, \mathbb{V}^>(dx) \\ &= \ (\mu^>)^{-1}\lambda\!\!\lambda_d([0,h])\,\|\mathbb{V}^>\| \\ &= \ \begin{cases} \mu^{-1}\lambda\!\!\lambda_d([0,h]), & \text{if } \mu<\infty, \\ \qquad\qquad 0, & \text{if } \mu=\infty \end{cases} \end{aligned} \tag{3.7}$$

for all $h>0$, i.e. validity of (3.1) for $\mathbb{U}$ as well.

3rd reduction: $(S_n)_{n\ge 0}$ is completely d-arithmetic $(d\ge 0)$.
The final reduction that will be useful hereafter is to assume that the increment distribution is completely d-arithmetic so that, by Lemma 2.1.2, its symmetrization has the the same lattice-span.

Lemma 3.3.1. *Let $(S_n)_{n\ge 0}$ be a nontrivial SRW with lattice-span $d\ge 0$ and renewal measure $\mathbb{U}$. Let $(\rho_n)_{n\ge 0}$ a SRP independent of $(S_n)_{n\ge 0}$ and with geometric increments, viz. $\mathbb{P}(\rho_1=n) = (1-\theta)^{n-1}\theta$ for some $\theta\in(0,1)$ and $n\in\mathbb{N}$. Then $(S_{\rho_n})_{n\ge 0}$ is a completely d-arithmetic SRW with renewal measure $\mathbb{U}^{(\rho)}$ satisfying $\mathbb{U}^{(\rho)} = (1-\theta)\delta_0 + \theta\,\mathbb{U}$.*

Proof. First of all, let $(I_n)_{n\ge 1}$ be a sequence of iid Bernoulli variables with parameter θ independent of $(S_n)_{n\ge 0}$. Each I_n may be interpreted as the outcome of a coin tossing performed at time n. Let $(J_n)_{n\ge 0}$ be the SRP associated with $(I_n)_{n\ge 1}$ and let $(\rho_n)_{n\ge 0}$ be the sequence of copy sums associated with $\rho=\rho_1 := \inf\{n\ge 1 : I_n = 1\}$. Then $(\rho_n)_{n\ge 0}$ satisfies the assumptions of the lemma, and since ρ is a stopping time for the bivariate SRW $(S_n,J_n)_{n\ge 0}$, we infer by invoking Thm. 2.2.3(j) that $(S_{\rho_n})_{n\ge 0}$ forms a SRW. Next observe that, for each $A\in\mathscr{B}(\mathbb{R})$,

$$\mathbb{U}^{(\rho)}(A) - \delta_0(A) \ = \ \mathbb{E}\left(\sum_{n\ge 1} I_n \mathbf{1}_A(S_n)\right) \ = \ \mathbb{E}I_1\left(\mathbb{U}(A)-\delta_0(A)\right)$$

which proves the relation between $\mathbb{U}^{(\rho)}$ and $\mathbb{U}$, for $\mathbb{E}I_1 = \theta$.

It remains to show that $(S_{\rho_n})_{n\ge 0}$ is completely d-arithmetic. Let $(S_n',\rho_n')_{n\ge 0}$ be an independent copy of $(S_n,\rho_n)_{n\ge 0}$ and put $\rho := \rho_1$ and $\rho' := \rho_1'$. By Lemma 2.1.2, it suffices to show that the symmetrization $S_\rho - S_{\rho'}'$ is d-arithmetic. Since $c := d(S_\rho - S_{\rho'}') \ge d$, we must only consider the case $c > 0$ and then show that $\mathbb{P}(X_1 \in c\mathbb{Z}) = 1$. But

$$1 \;=\; \mathbb{P}(S_\rho - S_{\rho'}' \in c\mathbb{Z}) \;=\; \sum_{m,n\ge 1} \theta^2(1-\theta)^{m+n-2}\,\mathbb{P}(S_m - S_n' \in c\mathbb{Z})$$

clearly implies $\mathbb{P}(S_m - S_n' \in c\mathbb{Z}) = 1$ for all $m,n \in \mathbb{N}$. Hence

$$0 < \mathbb{P}(S_1 - S_1' = 0) = \mathbb{P}(S_2 - S_1' \in c\mathbb{Z}, S_1 - S_1' = 0) = \mathbb{P}(X_1 \in c\mathbb{Z})\,\mathbb{P}(S_1 - S_1' = 0)$$

giving $\mathbb{P}(X_1 \in c\mathbb{Z}) = 1$ as asserted. □

With the relation between $\mathbb{U}^{(\rho)}$ and $\mathbb{U}$ at hand it is clear that it suffices to show (3.1) for the first of these measures. Alternatively, we could rely on cyclic decomposition as in the 2^{nd} reduction, viz. $\mathbb{U} = \mathbb{V}^{\rho} * \mathbb{U}^{(\rho)}$ with $\mathbb{V}^{(\rho)}$ denoting the pre-ρ occupation measure.

3.3.2 Setting up the stage: the coupling model

Based on the previous considerations, we now assume that $(S_n)_{n\ge 0}$ is a zero-delayed completely d-arithmetic RP with drift μ. As usual, the increment distribution is denoted by F. The starting point of the coupling construction is to consider this sequence together with a second one $(S_n')_{n\ge 0}$ such that the following conditions are satisfied:

(C1) $(S_n,S_n')_{n\ge 0}$ is a bivariate RW with iid increments (X_n,X_n'), $n \ge 1$.

(C2) $(S_n' - S_0')_{n\ge 0} \stackrel{d}{=} (S_n)_{n\ge 0}$ and thus $X_1' \stackrel{d}{=} X_1$.

(C3) $S_0' \stackrel{d}{=} F^s$ if $\mu < \infty$, and $S_0' \stackrel{d}{=} F_a^s$ for some $a \in \mathbb{R}_>$ if $\mu = \infty$.

Here F^s and F_a^s denote the stationary delay distribution and its truncated variant defined in (2.30) and (2.32), respectively. By the results in Section 2.5, the renewal measure $\mathbb{U}'$ of $(S_n')_{n\ge 0}$ satisfies $\mathbb{U}'([t,t+h]) = \mu^{-1}\lambda\!\!\lambda_d([0,h])$ for all $t,h \in \mathbb{R}_>$ if $\mu < \infty$, and $\mathbb{U}'([t,t+h]) \le \xi(a)^{-1}\lambda\!\!\lambda_d([0,h])$ for all $t,h \in \mathbb{R}_>$ if $\mu = \infty$ where $\xi(a)$ tends to ∞ as $a \to \infty$. Hence $\mathbb{U}'$ satisfies (3.1) in the finite mean case and does so approximately for sufficiently large a if $\mu = \infty$. The idea is now to construct a third RP $(S_n'')_{n\ge 0}$ from the given two which is a copy of $(S_n')_{n\ge 0}$ and such that S_n'' is equal or at least almost equal to S_n for all $n \ge T$, T an a.s. finite stopping time for $(S_n,S_n')_{n\ge 0}$, called *coupling time*. This entails that the *coupling process* $(S_n'')_{n\ge 0}$ has renewal measure $\mathbb{U}'$ while simultaneously being close to $\mathbb{U}$ on remote intervals because with high probability such intervals contain only renewal epochs S_n'' for $n \ge T$.

Having outlined the path towards the asserted result we must now complete the specification of the above bivariate model so as to facilitate a successful coupling.

But the only unspecified component of the model is the joint distribution of (X_1, X_1') for which the following two alternatives will be considered:

(C4a) X_1, X_1' are independent or, equivalently, $(S_n)_{n\ge 0}$ and $(S_n')_{n\ge 0}$ are independent.

(C4b) $X_1' = Y_1 \mathbf{1}_{[0,b]}(|X_1 - Y_1|) + X_1 \mathbf{1}_{(b,\infty)}(|X_1 - Y_1|)$, where Y_1 is an independent copy of X_1 an b is chosen so large that $G_b := \mathbb{P}(X_1 - Y_1 \in \cdot||X_1 - Y_1| \le b)$ is d-arithmetic (and thus nontrivial).

The existence of b with $d(G_b) = d$ follows from the fact that $G := \mathbb{P}(X_1 - Y_1 \in \cdot)$ is d-arithmetic together with $G_b \xrightarrow{w} G$.

Condition (C4a) is clearly simpler than (C4b) and will serve our needs in the finite mean case because then it is sufficient to guarantee that the symmetrization $X_1 - X_1'$ is integrable with mean zero. Since the latter is also d-arithmetic, we infer from Thm. 2.3.5 that $(S_n - S_n')_{n\ge 0}$ is (topologically) recurrent on $\mathbb{G}_d$.

On the other hand, if $\mu = \infty$, the difference of two independent X_1, X_1' fails to be integrable, while under (C4b) we have $X_1 - X_1' = (X_1 - Y_1)\mathbf{1}_{[-b,b]}(X_1 - Y_1)$ which is again symmetric with mean zero and d-arithmetic by choice of b. Once again we hence infer the recurrence of the symmetric RW $(S_n - S_n')_{n\ge 0}$ on $\mathbb{G}_d$.

A coupling of two independent processes with identical transition mechanism but different initial distributions is what DOEBLIN realized in [16] for Markov chains and is therefore called *Doeblin coupling* hereafter. The construction with (X_1, X_1') as specified in (C4b) goes back to ORNSTEIN [37] and is thus named *Ornstein coupling*.

3.3.3 Getting to the point: the coupling process

In the following suppose that (C1–3) and (C4a) are valid if $\mu < \infty$, while (C4a) is replaced with (C4b) if $\mu = \infty$. Fix any $\varepsilon > 0$ if F is nonarithmetic, while $\varepsilon = 0$ if F has lattice-span $d > 0$. Since $(S_n - S_n')_{n\ge 0}$ is recurrent on $\mathbb{G}_d$ (recall that the delay distribution of S_0' is also concentrated on $\mathbb{G}_d$) we infer the a.s. finiteness of the ε*-coupling time*

$$T \;:=\; \inf\{n \ge 0 : |S_n - S_n'| \le \varepsilon\}$$

and define the *coupling process* $(S_n'')_{n\ge 0}$ by

$$S_n'' \;:=\; \begin{cases} S_n', & \text{if } n \le T, \\ S_n - (S_T - S_T'), & \text{if } n \ge T \end{cases} \quad \text{for } n \in \mathbb{N}_0, \tag{3.8}$$

which may also be stated as

$$S_n'' \;:=\; \begin{cases} S_n', & \text{if } n \le T, \\ S_T' + \sum_{k=T+1}^{n} X_k, & \text{if } n > T \end{cases} \quad \text{for } n \in \mathbb{N}_0. \tag{3.9}$$

The subsequent lemma accounts for the intrinsic properties of this construction.

Lemma 3.3.2. *Under the stated assumptions, the following assertions hold true for the coupling process* $(S_n'')_{n\ge 0}$*:*

(a) $(S_n'')_{n\ge 0} \stackrel{d}{=} (S_n')_{n\ge 0}$.
(b) $|S_n''-S_n|\le\varepsilon$ *for all* $n\ge T$.

Proof. We only need to show (a) because (b) is obvious from the definition of the coupling process and the coupling time. Since T is a stopping time for the bivariate RW $(S_n,S_n')_{n\ge 0}$, we infer from Thm. 2.2.1 that $X_{T+1},X_{T+2},\ldots$ are iid with the same distribution as X_1 and further independent of $T,(S_n,S_n')_{0\le n\le T}$. But this easily seen to imply assertion (a), namely

$$\begin{aligned}
&\mathbb{P}(S_0''\in B_0,X_j''\in B_j \text{ for } 1\le j\le n)\\
&= \sum_{k=0}^n \mathbb{P}(T=k,S_0'\in B_0,X_j'\in B_j \text{ for } 1\le j\le k)\,\mathbb{P}(X_j\in B_j \text{ for } k<j\le n)\\
&\quad +\mathbb{P}(T>n,S_0'\in B_0,X_j'\in B_j \text{ for } 1\le j\le n)\\
&= \sum_{k=0}^n \mathbb{P}(T=k,S_0'\in B_0,X_j'\in B_j \text{ for } 1\le j\le k)\,\mathbb{P}(X_j'\in B_j \text{ for } k<j\le n)\\
&\quad +\mathbb{P}(T>n,S_0'\in B_0,X_j'\in B_j \text{ for } 1\le j\le n)\\
&= \mathbb{P}(S_0'\in B_0,X_j'\in B_j \text{ for } 1\le j\le n)
\end{aligned}$$

for all $n\in\mathbb{N}$ and $B_1,\ldots,B_n\in\mathscr{B}(\mathbb{R}_\ge)$. □

Before moving on to the finishing argument, let us note that a coupling with a.s. finite 0-coupling time is called *exact coupling*, while we refer to an ε*-coupling* otherwise.

3.3.4 The final touch

As usual, let $N(I)$ denote the number of renewals S_n in I, and let $N''(I)$ be the corresponding variable for the coupling process $(S_n'')_{n\ge 0}$. Define further $N_k(I):=\sum_{j=0}^k \mathbf{1}_I(S_n)$ and $N_k''(I)$ in a similar manner. Fix any $h>0$, $\varepsilon\in(0,h/2)$, and put $I:=[0,h]$, $I_\varepsilon:=[\varepsilon,h-\varepsilon]$, and $I^\varepsilon:=[-\varepsilon,h+\varepsilon]$. The following proof of (3.1) focusses on the slightly more difficult nonarithmetic case, i.e. $d=0$ hereafter. We first treat the case $\mu<\infty$.

A. The finite mean case. By Lemma 3.3.2(a), $(S_n'')_{n\ge 0}$ has renewal measure $\mathbb{U}'$ which in turn equals $\mu^{-1}\lambda\!\!\!\lambda_0^+$ by our model assumption (C3). It follows from the coupling construction that

$$\{S_n'' \in t+I_\varepsilon\} \subset \{S_n \in t+I\} \subset \{S_n'' \in t+I^\varepsilon\}$$

for all $t \in \mathbb{R}_\geq$ and $n \geq T$. Consequently,

$$N''(t+I_\varepsilon) - N_T(t+I) \leq N(t+I) \leq N''(t+I^\varepsilon) + N_T(t+I)$$

and therefore, by taking expectations,

$$\mathbb{U}'(t+I_\varepsilon) - \mathbb{E}N_T(t+I) \leq \mathbb{U}(t+I) \leq \mathbb{U}'(t+I^\varepsilon) + \mathbb{E}N_T(t+I) \qquad (3.10)$$

for all $t \in \mathbb{R}_\geq$. But $\mathbb{U}'(t+I_\varepsilon) = \mu^{-1}(h-2\varepsilon)$ and $\mathbb{U}'(t+I^\varepsilon) = \mu^{-1}(h+2\varepsilon)$ for all $t > \varepsilon$. Moreover, the uniform integrability of $\{N(t+I) : t \in \mathbb{R}\}$ [☞ Lemma 2.4.1] in combination with $N_T(t+I) \leq N(t+I)$ and $\lim_{t\to\infty} N_T(t+I) = 0$ a.s. entails

$$\lim_{t\to\infty} \mathbb{E}N_T(t+I) = 0.$$

Therefore, upon letting t tend to infinity in (3.10), we finally arrive at

$$\frac{h-2\varepsilon}{\mu} \leq \liminf_{t\to 1} \mathbb{U}(t+I) \leq \limsup_{t\to 1} \mathbb{U}(t+I) \leq \frac{h+2\varepsilon}{\mu}.$$

As ε can be made arbitrarily small, we have proved (3.1).

A. The infinite mean case. Here we have $\mathbb{U}' \leq \xi(a)^{-1}\lambda\!\!\lambda_0^+$ where a may be chosen so large that $\xi(a)^{-1} \leq \varepsilon$. Since validity of (3.10) remains unaffected by the drift assumption, we infer by just using the upper bound

$$\limsup_{t\to 1} \mathbb{U}(t+I) \leq \xi(a)^{-1}(h+2\varepsilon) \leq \varepsilon(h+2\varepsilon)$$

and thus again the assertion, for ε can be made arbitrarily small. This completes our coupling proof of Blackwell's theorem.

3.4 Feller's analytic proof

The following analytic proof of (3.1), given for nonarithmetic SRW with *finite* mean, is largely due to FELLER as given in his textbook [21, p. 264–266] and embarks on the following lemma by CHOQUET & DENY [11]. An adaptation to the arithmetic case is easily provided and left to the interested reader.

Lemma 3.4.1. [Choquet-Deny] *Given a nonarithmetic distribution F on $\mathbb{R}$ with finite mean, any bounded continuous function $\varphi:\mathbb{R}\to\mathbb{R}$ satisfying $\varphi=\varphi*F$ is necessarily constant.*

Proof. Start by observing that, for any $t\in\mathbb{R}$, the sequence $M_n(t):=\varphi(t-S_n)$, $n\in\mathbb{N}_0$, constitutes a martingale w.r.t. the natural filtration $(\mathscr{F}_n)_{n\ge 0}$ of $(S_n)_{n\ge 0}$ because

$$\begin{aligned}\mathbb{E}(\varphi(t-S_n)|\mathscr{F}_{n-1}) \ &= \ \mathbb{E}(\varphi(t-S_n)|S_{n-1})\\ &= \ \int \varphi(t-x-S_{n-1})\,F(dx)\\ &= \ \varphi*F(t-S_{n-1}) \ = \ \varphi(t-S_{n-1}) \quad \text{f.s.}\end{aligned}$$

for all $n\in\mathbb{N}$. As $(M_n(t))_{n\ge 0}$ is also bounded, we infer that $M_n(t)\to M_\infty(t)$ a.s. and in second mean, and the limit is a.s. constant by an appeal to the Hewitt-Savage zero-one law, the constant being $\mathbb{E}M_\infty(t)=\mathbb{E}M_0(t)=\varphi(t)$. But as an L_2-martingale it further satisfies

$$0 \ = \ \mathbb{E}(M_\infty(t)-M_0(t))^2 \ = \ \sum_{n\ge 1}\mathbb{E}(M_n(t)-M_{n-1}(t))^2$$

and thus $\varphi(t)=\varphi(t-S_n)$ a.s. for all $n\in\mathbb{N}_0$. Now let $(W_n)_{n\ge 0}:=(S_{\rho_n})_{n\ge 0}$ be the embedded completely nonarithmetic RW according to Lemma 3.3.1 and denote by ν its finite drift. Then

$$\varphi(t) \ = \ \varphi(t-W_n) \ = \ \varphi(t+n\nu-W_n) \quad \text{a.s. for all } n\in\mathbb{N}_0 \text{ and } t\in\mathbb{R}. \tag{3.11}$$

Use the recurrence of the zero-mean RW $(n\nu-W_n)_{n\ge 0}$ on $\mathbb{R}$ to infer

$$\overline{\{n\nu-W_n : n\in\mathbb{N}_0\}}=\mathbb{R} \quad \text{a.s.}$$

Combining this with the continuity of φ in (3.11) finally shows $\varphi\equiv\varphi(0)$. □

With the help of this lemma, we will now proceed to show the vague convergence of $\mathbb{U}(t+\cdot)$ to $\mu^{-1}\lambda\!\!\lambda_0$ as $t\to\infty$ which is equivalent to (3.1) as pointed out at the end of the first section of the present chapter. The uniform local boundedness of $\mathbb{U}$, stated as $\sup_{t\in\mathbb{R}}\mathbb{U}([t,t+a])\le\mathbb{U}([-a,a])<\infty$ for all $a\in\mathbb{R}_>$ in (2.17), obviously ensures that $\sup_{t\in\mathbb{R}}\mathbb{U}(t+K)<\infty$ for all compact $K\subset\mathbb{R}$ and thus the vague compactness of the family $\{\Theta_t\mathbb{U}:t\in\mathbb{R}\}$ where $\Theta_t\mathbb{U}:=\mathbb{U}(t+\cdot)$. Hence, by invoking the Helly-Bray theorem [☞ e.g. [9, p. 160f]], every sequence $(\Theta_{t_k}\mathbb{U})_{k\ge 1}$ with $t_k\to\infty$ contains a vaguely convergent subsequence. In order to prove $\Theta_t\mathbb{U}\xrightarrow{v}\mu^{-1}\lambda\!\!\lambda_0$ as $t\to\infty$, it therefore suffices to verify that any vaguely convergent subsequence has the same limit $\mu^{-1}\lambda\!\!\lambda_0$.

To this end, suppose that $t_k\to\infty$ and $\Theta_{t_k}\mathbb{U}\xrightarrow{v}\lambda$, so that

$$\varphi(x) \ := \ \lim_{k\to\infty} g*\Theta_{t_k}\mathbb{U}(x) \ = \ \lim_{k\to\infty}\sum_{n\ge 0}\mathbb{E}g(x+t_k-S_n) \ = \ g*\lambda(x)$$

for every $g \in \mathscr{C}_0(\mathbb{R})$. Plainly, $\Theta_{t_k}\mathbb{U} \stackrel{v}{\to} \lambda$ implies $\Theta_{t_k+x}\mathbb{U} \stackrel{v}{\to} \Theta_x\lambda$ for all $x \in \mathbb{R}$. We will show next that $\Theta_{t_k+x}\mathbb{U} \stackrel{v}{\to} \lambda$ for all $x \in \mathbb{R}$ holds true as well whence the uniqueness of the vague limit implies the translation invariance of λ and thus $\lambda = c\lambda\!\!\lambda_0$ for some $c \in \mathbb{R}_\geq$.

Lemma 3.4.2. *Under the stated assumptions,* $\Theta_{t_k+x}\mathbb{U} \stackrel{v}{\to} \lambda$ *for all* $x \in \mathbb{R}$.

Proof. Fix any such g, let $a > 0$ be such that $supp(g) \subset [-a,a]$, and put

$$\omega(g,\varepsilon) := \sup_{x,y \in \mathbb{R}:|x-y| \le \varepsilon} |g(x) - g(y)|$$

As g is uniformly continuous, $\lim_{\varepsilon \downarrow 0} \omega(g,\varepsilon) = 0$. With the help of the dominated convergence theorem, we infer

$$\begin{aligned}\varphi(x) &= \lim_{k\to\infty}\left(g(x+t_k) + \int g * \mathbb{U}(x-y+t_k)\, F(dy)\right)\\ &= \int \lim_{k\to\infty} g * \mathbb{U}(x-y+t_k)\, F(dy)\\ &= \varphi * F(x) \quad \text{for all } x \in \mathbb{R},\end{aligned}$$

that is, φ satisfies the Choquet-Deny equation. Furthermore, by another use of (2.17) of Lemma 2.4.1, we find that φ is a uniformly continuous function because

$$\begin{aligned}|\varphi(x) - \varphi(y)| &\le \sup_{k\ge 1} \int_{[x+t_k-a-\varepsilon, x+t_k+a+\varepsilon]} |g(x-u+t_k) - g(y-u+t_k)|\, \mathbb{U}(du)\\ &\le \omega(g,|x-y|) \sup_{t\in\mathbb{R}} \mathbb{U}([t-a-\varepsilon, t+a+\varepsilon])\\ &\le \omega(g,|x-y|)\, \mathbb{U}([-2a-2\varepsilon, 2a+2\varepsilon]) \quad \text{for all } x,y \in \mathbb{R}.\end{aligned}$$

Hence the Choquet-Deny lemma implies that $\varphi \equiv \varphi(0)$. Now we have that

$$\begin{aligned}g * \Theta_x\lambda(y) &= \lim_{k\to\infty} g * \Theta_{t_k+x}\mathbb{U}(y)\\ &= \lim_{k\to\infty} g * \Theta_{t_k}\mathbb{U}(x+y) = \varphi(x+y) = \varphi(0) \quad \text{for all } x \in \mathbb{R}\end{aligned}$$

and since this is true for all $g \in \mathscr{C}_0(\mathbb{R})$, we infer $\Theta_x\lambda = \lambda$ for all x and thereby the assertion of the lemma. ☐

It remains to verify that $\Theta_t\mathbb{U} \stackrel{v}{\to} c\lambda\!\!\lambda_0$ holds with $c = \mu^{-1}$. But the the fact that $\mathbb{U}_{F^s}([t,t+h]) = \mu^{-1}h$ for all $t,h \in \mathbb{R}_>$ if F^s denotes the stationary delay distribution of $(S_n)_{n\ge 0}$ yields by the dominated convergence theorem

$$\mu^{-1}h = \lim_{t\to\infty} \int \mathbb{U}([t-x, t-x+h])\, F^s(dx) = ch$$

and thus $c=\mu^{-1}$.

3.5 The Fourier analytic proof by Feller and Orey

In this section, we will give a proof of Blackwell's theorem under the assumption that $(S_n)_{n\ge 0}$ is a SRW with increment distribution F, lattice-span $d\in\{0,1\}$, and finite positive drift μ. Lévy's continuity theorem is a powerful tool to show the weak convergence of *finite* measures on $\mathbb{R}$ but fails to work in the present context where the vague convergence must be verified for the *locally finite* measures $\Theta_t\mathbb{U}=\mathbb{U}(t+\cdot)$ as $t\to\pm\infty$. The subsequent Fourier-analytic approach that embarks on a study of the finite measures

$$\mathbb{V}_s(B) \;:=\; \sum_{n\ge 0} s^n\big(F^{*n}(B)+F^{*n}(-B)\big)\quad\text{for } B\in\mathscr{B}(\mathbb{R})\text{ and } s\in(0,1),$$

obtained from $\mathbb{U}$ by *symmetrization* and *discounting*, is essentially due to FELLER & OREY [22]. Here we follow closely the nice adaptation by BREIMAN [9, Section 10.3] [☞ also [47]]. His textbook and the one by CHOW & TEICHER [13] should also be consulted for a number of Fourier-analytic results that are taken hereafter as known facts. Since we are going to make use of complex numbers, let $i:=\sqrt{-1}$ and $\Re z,\Im z,\bar z=\Re z-i\Im z$ denote the real part, imaginary part and conjugate-complex of a complex number z, respectively.

3.5.1 Preliminaries

Let $\phi(t)=\mathbb{E}e^{itX_1}$ denote the Fourier transform (FT) of X_1. Since S_n has FT ϕ^n for all $n\in\mathbb{N}_0$, the finite *discounted renewal measure* $\mathbb{U}_s=\sum_{n\ge 0}s^nF^{*n}$ has FT

$$\Psi_s \;:=\; \sum_{n\ge 0} s^n\phi^n \;=\; \frac{1}{1-s\phi}$$

which converges to $\Psi:=(1-\phi)^{-1}$ on $\mathbb{D}_\phi:=\{t\in\mathbb{R}:\phi(t)\ne 1\}$. The latter set equals $\mathbb{R}\backslash\{0\}$ in the nonarithmetic case ($d=0$) and $\mathbb{R}\backslash 2\pi\mathbb{Z}$ in the 1-arithmetic case. Note further that $\mathbb{U}_s\uparrow\mathbb{U}$ and $\mathbb{V}_s\uparrow\mathbb{V}:=\mathbb{V}_1$ as $s\uparrow 1$. Using the already proved fact $\lim_{t\to-\infty}\mathbb{U}(t+I)\to 0$ for all bounded intervals I, we see that

$$\lim_{t\to\infty}\big(\mathbb{U}(t+I)-\mathbb{V}(t+I)\big) \;=\; \lim_{t\to\infty}\mathbb{U}(-t-I) \;=\; 0$$

and thus that it suffices to prove

$$\Theta_t\mathbb{V} \;\stackrel{v}{\to}\; \mu^{-1}\lambda\!\!\lambda_d\quad\text{for } t\to\infty\text{ through }\mathbb{G}_d. \tag{3.12}$$

Since $-X_1$ has distribution $F(-\cdot)$ and FT $\overline{\phi} = \Re\phi - i\Im\phi$, we see that $F^{*n} + F^{*n}(-\cdot)$ has FT $\Re(\phi^n)$ for all $n \in \mathbb{N}_0$. Hence, the FT of V_s equals

$$\Re\Psi_s \;=\; \Re\left(\frac{1}{1-s\phi}\right) \;=\; \frac{1-s\Re\phi}{|1-s\phi|^2} \;\geq\; 0.$$

As $s \uparrow 1$, it converges pointwise on $\mathbb{D}_\phi$ to

$$\Re\Psi \;=\; \Re\left(\frac{1}{1-\phi}\right) \;=\; \frac{1-\Re\phi}{|1-\phi|^2}. \tag{3.13}$$

The following lemma provides a key to our attempted result and describes the behavior of $\Re\Psi_s$ as $s \uparrow 1$.

Lemma 3.5.1. *Let $b > 0$ be such that $\phi(t) \neq 1$ for all $t \in [-b,b]\backslash\{0\}$. Then $\Re\Psi$ is $\lambda\!\!\lambda_0$-integrable on $[-b,b]$, and*

$$\lim_{s\uparrow 1} \int_{-b}^{b} f(t)\,\Re\Psi_s(t)\;dt \;=\; \frac{\pi f(0)}{\mu} \;+\; \int_{-b}^{b} f(t)\,\Re\Psi(t)\;dt \tag{3.14}$$

for all $f \in \mathscr{C}_0(\mathbb{R})$, that is

$$\Re\Psi_s\,\mathbf{1}_{[-b,b]}\,\lambda\!\!\lambda_0 \;\stackrel{v}{\to}\; \frac{\pi g(0)}{\mu}\,\delta_0 \;+\; \Re\Psi\,\mathbf{1}_{[-b,b]}\,\lambda\!\!\lambda_0 \quad (s \uparrow 1).$$

Proof. We begin with the proof of the $\lambda\!\!\lambda_0$-integrability of $\Re\Psi$ on $[-b,b]$. Since $\Re\Psi$ is continuous on $[-b,b]\backslash\{0\}$, it suffices to prove local integrability at 0. We choose $\varepsilon \in (0,b]$ so small that $|1-\phi(t)| \geq \mu t/2$ for $t \in [-\varepsilon,\varepsilon]$, which is possible because $\phi'(0) = \mu$. After these observations and recalling (3.13) it suffices to verify that

$$\int_{-\varepsilon}^{\varepsilon} \frac{1-\Re\phi(t)}{t^2}\;dt \;=\; 2\int_0^{\varepsilon} \frac{1-\mathbb{E}\cos(tX_1)}{t^2}\;dt \;<\; \infty.$$

But with the help of Fubini's theorem, we obtain

$$\begin{aligned}\int_0^{\varepsilon} \frac{1-\mathbb{E}\cos(tX_1)}{t^2}\;dt \;&=\; \mathbb{E}\left(\int_0^{\varepsilon} \frac{1-\cos(tX_1)}{t^2}\;dt\right)\\ &=\; \mathbb{E}\left(|X_1|\int_0^{\varepsilon|X_1|} \frac{1-\cos t}{t^2}\;dt\right) \;\leq\; \mathbb{E}|X_1|\int_0^{\infty} \frac{1-\cos t}{t^2}\;dt\end{aligned}$$

with the last estimate clearly being finite.

Turning to the proof of (3.14), we first write, for any given $f \in \mathscr{C}_0(\mathbb{R})$,

$$\begin{aligned}\int_{-b}^{b} f(t)\Big(\Re\Psi_s(t)-\Re\Psi(t)\Big)\,dt \;&=\; \int_{-b}^{b} f(t)\,\Re\left(\frac{1}{1-s\phi(t)}-\frac{1}{1-\phi(t)}\right)\,dt\\ &=\; \int_{-b}^{b} f(t)\,\frac{(s-1)\phi(t)}{|1-s\phi(t)|^2}\,\Re\left(\frac{\phi(t)(1-s\overline{\phi(t)})}{1-\phi(t)}\right)\,dt \;=:\; I(s,b).\end{aligned}$$

On $[-b,b]\backslash[-\varepsilon,\varepsilon]$, the integrand of $I(s,b)$ remains bounded and converges to 0 as $s\uparrow 1$. Hence we must examine $I(s,\varepsilon)$ for sufficiently small $\varepsilon>0$. To this end write

$$\begin{aligned} I(s,\varepsilon) \;&=\; I_1(s,\varepsilon)-I_2(s,\varepsilon) \quad \text{with}\\ I_1(s,\varepsilon) \;&:=\; \int_{-\varepsilon}^{\varepsilon} f(t)\,\frac{(s-1)\phi(t)}{|1-\phi(t)|^2}\,\Re\left(\frac{\phi(t)(1-s\overline{\phi(t)})}{1-\phi(t)}\right)\,dt \quad \text{and}\\ I_2(s,\varepsilon) \;&:=\; \int_{-\varepsilon}^{\varepsilon} f(t)\,\frac{(s-1)^2\phi(t)}{|1-\phi(t)|^2}\,|\phi(t)|^2\Re\Psi(t)\,dt\end{aligned}$$

and notice that the integrand of $I_2(s,\varepsilon)$ is bounded by a multiple of $\Re\Psi$, for $|1-s\phi|\geq 1-s$. Hence, by the dominated convergence theorem,

$$\lim_{s\uparrow 1} I_2(s,\varepsilon) \;=\; 0 \quad \text{for all } 0<\varepsilon\leq b.$$

Left with the critical term $I_1(s,\varepsilon)$, fix any $\eta\in(0,1)$ and $\varepsilon\in(0,b]$ sufficiently small such that, for $\phi(t)=1+\phi'(t^*)t$, $t^*\in[-t,t]$, we have

$$(1-\eta)\left((1-s)^2+\mu^2t^2\right) \;\leq\; |1-s\phi(t)|^2 \;\leq\; (1+\eta)\left((1-s)^2+\mu^2t^2\right)$$

and therefore the continuity of

$$g(s,t) \;:=\; \frac{|1-s\phi(t)|^2}{(1-s)^2+\mu^2t^2}$$

for $(s,t)\in(0,1]\times[-\varepsilon,\varepsilon]$ with $1-\eta\leq g(s,t)\leq 1+\eta$. Furthermore,

$$g(t) \;:=\; \lim_{s\uparrow 1} g(s,t) \;=\; \frac{|1-\phi(t)|^2}{\mu^2t^2} \quad \text{for all } t\neq 0$$

with continuous extension at 0, defined by

$$g(0) \;:=\; \lim_{t\to 0}\frac{|1-\phi(t)|^2}{\mu^2t^2} \;=\; \frac{|\phi'(0)|^2}{\mu} \;=\; 1.$$

In a similar manner, we find that

$$h(t) \;:=\; -f(t)\,\Re\left(\frac{\phi(t)(1-\overline{\phi(t)})}{1-\phi(t)}\right) \quad \text{for } t\neq 0$$

is continuous on $[-\varepsilon,\varepsilon]$ after setting $h(0):=f(0)$. Just note that

$$\lim_{t\to 0}\frac{1-\overline{\phi(t)}}{1-\phi(t)} \;=\; \frac{\overline{\phi'(0)}}{\phi'(0)} \;=\; -1.$$

Possibly after further diminishing ε such that

$$\frac{1-\eta}{1+\eta}\,h(0) \;\le\; \frac{h(t)}{g(s,t)} \;\le\; \frac{1+\eta}{1-\eta}\,h(0) \quad \text{for all } (s,t)\in[1-\varepsilon,1)\times[-\varepsilon,\varepsilon],$$

we finally conclude

$$\begin{aligned}
\lim_{s\uparrow 1} I_1(s,\varepsilon) \;&=\; \lim_{s\uparrow 1}\int_{-\varepsilon}^{\varepsilon}\frac{h(t)}{g(s,t)}\,\frac{(1-s)}{(1-s)^2+\mu^2t^2}\,dt\\
&=\; \lim_{s\uparrow 1}\int_{-\varepsilon/(1-s)}^{\varepsilon/(1-s)}\frac{h(t(1-s))}{g(s,t(1-s))}\,\frac{1}{1+\mu^2t^2}\,dt\\
&\lesseqgtr\; \frac{1\pm\eta}{1\mp\eta}\,\frac{h(0)}{\mu}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}\\
&=\; \frac{1\pm\eta}{1\mp\eta}\,\frac{\pi h(0)}{\mu}
\end{aligned}$$

and thereupon (3.14) as $h(0)=f(0)$ and η can be made arbitrarily small. □

Instead of proving (3.12) we will prove the equivalent assertion [☞ Thm. **??**] that

$$\widehat{h}(x)\,\Theta_a\mathbb{V}(dx) \;\xrightarrow{v}\; \frac{\widehat{h}(x)}{\mu}\,\lambda\!\!\lambda_d(dx) \quad \text{for } a\to\infty \text{ through } \mathbb{G}_d \tag{3.15}$$

for some positive continuous and $\lambda\!\!\lambda_d$-integrable function $\widehat{h}$. As will be seen soon, a convenient choice is a function $\widehat{h}$ that is the FT of a symmetric distribution with $\lambda\!\!\lambda_0$-density $h\in\mathscr{C}_0(\mathbb{R})$. The next lemma shows that such a choice is actually possible.

Lemma 3.5.2. *For each $\alpha>0$, the triangular distribution G_α with continuous $\lambda\!\!\lambda_0$-density $h_\alpha(x):=(2\alpha-|x|)\,\mathbf{1}_{(-2\alpha,2\alpha)}(x)$ has the FT*

$$\widehat{h}_\alpha(t) \;=\; \frac{\sin^2\alpha t}{\alpha^2t^2},$$

and for any $\alpha,\beta>0$ with $\alpha/\beta\notin\mathbb{Q}$ the FT $\widehat{h}_\alpha+\widehat{h}_\beta$ of $G_\alpha+G_\beta$ is everywhere positive.

Proof. The first assertion follows from the well-known fact that G_α is the convolution square of a uniform distribution on $(-\alpha,\alpha)$, the latter having real FT $\frac{\sin\alpha t}{\alpha t}=\widehat{h}_\alpha(t)^{1/2}$. For the second assertion it is to be observed that $\{\widehat{h}_\alpha=0\}$ equals $\alpha^{-1}\pi\mathbb{Z}$ and is disjoint from $\beta^{-1}\pi\mathbb{Z}$ whenever α,β have irrational ratio. □

3.5.2 Proof of the nonarithmetic case

Assuming the nonarithmetic case, we will show now (3.15) for a positive function $\widehat{h} = \widehat{h}_\alpha + \widehat{h}_\beta$ as in the above lemma. In fact, we will even show weak convergence there which, by Lévy's continuity theorem, amounts to the verification of

$$\lim_{a\to\infty} \int e^{itx}\widehat{h}(x)\ \Theta_a\mathbb{V}(dx)\ =\ \frac{1}{\mu}\int e^{itx}\widehat{h}(x)\ dx \quad \text{for all } t\in\mathbb{R}. \tag{3.16}$$

Note that

$$\Theta_a\mathbb{V}_s\ =\ \sum_{n\ge 0} s^n\left((F^{*n}(a+\cdot)+F^{*n}(a-\cdot)\right))$$

and has FT $2e^{-iat}\,\Re\Psi_s(t)$ for $a\in\mathbb{R}$ and $s\in(0,1)$. Since, for a suitable constant $c\in\mathbb{R}_>$,

$$\widehat{h}_n^*\ :=\ \max_{n-1\le x\le n}\widehat{h}(|x|)\ \le\ cn^{-2} \quad \text{for all } n\in\mathbb{N}$$

and $\mathbb{V}$ is uniformly locally bounded, we also have

$$\sup_{s\in(0,1]}\int \widehat{h}\ d\Theta_a\mathbb{V}_s\ =\ \int \widehat{h}\ d\Theta_a\mathbb{V}\ \le\ 2c\sup_x \mathbb{V}([x,x+1])\sum_{n\ge 1}\frac{1}{n^2}\ <\ \infty$$

and

$$\lim_{s\uparrow 1}\int e^{itx}\widehat{h}(x)\ \Theta_a\mathbb{V}_s(dx)\ =\ \int e^{itx}\widehat{h}(x)\ \Theta_a\mathbb{V}(dx) \tag{3.17}$$

for all $a\in\mathbb{R}$. Now use that $e^{itx}\widehat{h}(x)$ is the FT (at x) of $h(y-t)\,\lambda\!\!\lambda_0(dy)$ (with $h = h_\alpha + h_\beta$ as in the above lemma) together with Fubini's theorem to obtain the crucial equation

$$\begin{aligned}\int e^{itx}\widehat{h}(x)\ \Theta_a\mathbb{V}_s(dx)\ &=\ \iint e^{ixy}h(y-t)\ dy\ \Theta_a\mathbb{V}_s(dx)\\ &=\ \iint e^{ixy}h(y-t)\Theta_a\mathbb{V}_s(dx)\ dy\\ &=\ \int_{-b}^{b} 2e^{-iay}\,\Re\Psi_s(y)\,h(y-t)\ dy\end{aligned} \tag{3.18}$$

for all $t\in\mathbb{R}$, where $b=b(t)$ is such that $supp(h)(\cdot - t)\subset[-b,b]$. The last integral is perfectly tailored to an application of Lemma 3.5.1, which yields

$$\lim_{s\uparrow 1}\int_{-b}^{b} 2e^{-iay}\,\Re\Psi_s(y)\,h(y-t)\ dy\ =\ \frac{2\pi}{\mu}\,h(-t)\ +\ \int_{-b}^{b} 2e^{-iay}\,\Re\Psi(y)\,h(y-t)\ dy$$

for all $t\in\mathbb{R}$ if $\phi(t)\ne 1$ for $t\in[-b,b]\backslash\{0\}$. But the latter holds because F is nonarithmetic. Moreover, the Riemann-Lebesgue lemma [☞ [9, p. 216]] implies

$$\lim_{a\to\infty}\int_{-b}^{b} 2e^{-iay}\,\Re\Psi(y)\,h(y-t)\ dy\ =\ 0$$

while Fourier inversion [☞ [9, Thm. 8.39]] gives

$$h(-t) \;=\; \frac{1}{2\pi}\int e^{itx}\,\widehat{h}(x)\,dx.$$

By combining these facts with (3.17) and (3.18), we finally arrive at the desired conclusion (3.16).

3.5.3 The arithmetic case: taking care of periodicity

Now assume that F is 1-arithmetic and thus its FT 2π-periodic which necessitates some adjustments of the previous arguments because $\Re\Psi$ has singularities at all integral multiples of 2π. Instead of (3.16), we must verify now

$$\text{1-}\lim_{a\to\infty}\int e^{itx}\,\widehat{h}(x)\,\Theta_a\mathbb{V}(dx) \;=\; \frac{1}{\mu}\sum_{n\in\mathbb{Z}} e^{itn}\,\widehat{h}(n)\quad\text{for all } t\in\mathbb{R}. \tag{3.19}$$

As one can readily see, eqs. (3.17) and (3.18) remain valid in the present situation but an application of Lemma 3.5.1 requires to write the last integral in (3.18) in the form

$$\int_{-\pi}^{\pi} 2e^{-iky}\,\Re\Psi_s(y)\,g(y-t)\,dy,\quad \text{where } g(y) \;:=\; \sum_{n\in\mathbb{Z}} h(y+2\pi n)$$

and the 2π-periodicity of $\Re\Psi_s$ has been utilized. Note also that the defining sum for g always ranges over finitely many n only because h has compact support. We then infer

$$\lim_{s\uparrow 1}\int_{-\pi}^{\pi} 2e^{-iky}\,\Re\Psi_s(y)\,g(y-t)\,dy \;=\; \frac{2\pi}{\mu}\,g(-t) \;+\; \int_{-\pi}^{\pi} 2e^{-iky}\,\Re\Psi(y)\,g(y-t)\,dy$$

for all $t\in\mathbb{R}$ and $k\in\mathbb{Z}$. As $k\to\infty$, the last integral converges to 0 by another appeal to the Riemann-Lebesgue lemma. Moreover, the 2π-periodic continuous function g has Fourier coefficients

$$\begin{aligned} g_n^* \;&=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,g(x)\,dx \\ &=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\sum_{m\in\mathbb{Z}} h(x+2\pi m)\,dx \\ &=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-inx}h(x)\,dx \;=\; \frac{\widehat{h}(-n)}{2\pi}\quad\text{for } n\in\mathbb{Z} \end{aligned}$$

which are absolutely summable so that [☞ [29, Thm. 9.1]]

$$g(-t) \;=\; \sum_{n\in\mathbb{Z}} e^{-int}g_n^* \;=\; \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{int}\widehat{h}(n)\quad\text{for all } t\in\mathbb{R}.$$

Putting the stated facts together we obtain (3.19).

3.6 Back to the beginning: Blackwell's original proof

Let us finally present the proof of (3.1) that was originally provided by the eponymous mathematician DAVID BLACKWELL in 1948 for a nonarithmetic zero-delayed RP $(S_n)_{n\ge0}$ with drift μ. Albeit keeping the order of his arguments, we take the freedom of streamlining some of them and of using our notation from the previous sections. As a particular difference we should note that BLACKWELL considers the renewal measure that does *not* include the renewal S_0.

3.6.1 Preliminary lemmata

Referring to STEIN [42] for a proof, BLACKWELL embarks on the result [Lemma 1 in [6]] that $\mathbb{U}((t,t+h])$ is finite for all $t,h\in\mathbb{R}_\ge$, followed by two further lemmata stated next.

Lemma 3.6.1. *Let* $N_k((t,t+h]) := \sum_{n\ge k}\mathbf{1}_{(t,t+h]}(S_{k+n}-S_k)$ *and* $\mathscr{F}_k := \sigma(S_0,...,S_k)$ *for* $k\in\mathbb{N}_0$. *Then*

$$\int_A N_k((T,T+h])\,d\mathbb{P} \;=\; \int_A \mathbb{U}((T,T+h])\,d\mathbb{P}$$

for all $A\in\mathscr{F}_k$, $h,t\in\mathbb{R}_\ge$, *and any nonnegative* $\mathscr{F}_k$*-measurable random variable* T.

Proof. This is immediate from

$$\int_A N_k((T,T+h])\,d\mathbb{P} \;=\; \int_A \mathbb{E}(N_k((T,T+h])|\mathscr{F}_k)\,d\mathbb{P}$$

and the fact that the point process N_k is independent of $\mathscr{F}_k$ and thus of T, whence

$$\mathbb{E}(N_k((T,T+h])|\mathscr{F}_k) \;=\; \mathbb{E}(N_k((T,T+h])|T) \;=\; \mathbb{U}((T,T+h]) \quad \text{a.s.}$$

for $h,t\in\mathbb{R}_\ge$. □

Lemma 3.6.2. *We have* $\mathbb{U}((t,t+h]) \le \mathbb{U}((0,h])$ *or all* $h,t\in\mathbb{R}_>$.

Proof. Fix positive h,t and put $A_k := \{S_{k-1} \le t < S_k\}$ for $k \in \mathbb{N}$. Then $N((t,t+h]) = N_0((t,t+h]) \le N_k((0,h]) + 1$ on A_k. Hence

$$\begin{aligned}\mathbb{U}((t,t+h]) \;&=\; \sum_{k\ge 1}\int_{A_k} N((t,t+h])\,d\mathbb{P} \;\le\; \sum_{k\ge 1}\int_{A_k} N_k((0,h])\,d\mathbb{P} \;+\;1\\ &=\; \sum_{k\ge 1}\int_{A_k} \mathbb{U}((0,h])\,d\mathbb{P} \;+\;1 \;=\; \mathbb{U}((0,h])+1,\end{aligned}$$

where Lemma 3.6.1 has been used for the last line. □

The reader will have noticed that the previous lemma is just a slightly weaker variant of (2.17) the proof of which avoids the definition of a stopping time. Of course, it still shows uniform local boundedness of $\mathbb{U}$. Defining

$$d(h) \;:=\; \liminf_{t\to\infty}\mathbb{U}((t,t+h]) \quad\text{and}\quad D(h) \;:=\; \limsup_{t\to\infty}\mathbb{U}((t,t+h]),$$

and choosing $a_n, b_n \to \infty$ such that

$$\lim_{n\to\infty}\mathbb{U}((a_n,a_n+h]) \;=\; d(h) \quad\text{and}\quad \lim_{n\to\infty}\mathbb{U}((b_n,b_n+h]) \;=\; D(h),$$

BLACKWELL takes the next step by proving the following result for any $h > 0$.

Lemma 3.6.3. *For all $k \in \mathbb{N}_0$,*

$$\begin{aligned} Y_{k,n} \;&:=\; \mathbb{U}((a_n - S_k, a_n + h - S_k]) \;\xrightarrow{\mathbb{P}}\; d(h) \quad \textit{and}\\ Z_{k,n} \;&:=\; \mathbb{U}((b_n - S_k, b_n + h - S_k]) \;\xrightarrow{\mathbb{P}}\; D(h)\end{aligned}$$

as $n \to \infty$.

Proof. Put $I := (0,h]$ for any fixed $h > 0$. By observing

$$N(t+I)\begin{cases} =\; N_k(t-S_k+I), & \text{if } S_k \le t,\\ \le\; k + N_k(I), & \text{if } S_k > t\end{cases}$$

we infer with the help of Lemma 3.6.2 (first inequality)

$$\begin{aligned}\int \mathbb{U}(t-S_k+I)\,d\mathbb{P} \;-\; (1+\mathbb{U}(I))\,\mathbb{P}(S_k>t) \;&\le\; \int_{\{S_k\le t\}}\mathbb{U}(t-S_k+I)\,d\mathbb{P}\\ \le\; \mathbb{U}(t+I) \;\le\; \int_{\{S_k\le t\}}\mathbb{U}(t-S_k+I)\,d\mathbb{P} \;&+\; (k+\mathbb{U}(I))\,\mathbb{P}(S_k>t).\end{aligned}$$

and thus in particular

$$\int Y_{k,n}\,d\mathbb{P} \;+\; (1+\mathbb{U}(I))\,\mathbb{P}(S_k > a_n) \;\geq\; \mathbb{U}(a_n+I), \tag{3.20}$$

$$\int Z_{k,n}\,d\mathbb{P} \;+\; (1+\mathbb{U}(I))\,\mathbb{P}(S_k > b_n) \;\leq\; \mathbb{U}(b_n+I). \tag{3.21}$$

Now (3.20) and (3.21) imply

$$\limsup_{n\to\infty} \mathbb{E}Y_{k,n} \;\geq\; d(h) \quad\text{and}\quad \liminf_{n\to\infty} \mathbb{E}Z_{k,n} \;\leq\; D(h)$$

for all $k \in \mathbb{N}_0$. Furthermore,

$$\liminf_{n\to\infty} Y_{k,n} \;\leq\; d(h) \quad\text{and}\quad \limsup_{n\to\infty} Z_{k,n} \;\leq\; D(h) \quad\text{a.s.}$$

A combination of these facts easily leads to the desired conclusions. □

Corollary 3.6.4. *The sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ can be chosen in such a way that $Y_{k,n} \to d(h)$ and $Z_{k,n} \to D(h)$ a.s. for all $k \in \mathbb{N}_0$.*

Proof. For each k, the sequences $(Y_{k,n})_{n\geq 1}$ and $(Z_{k,n})_{n\geq 1}$ as convergent in probability contain an a.s. convergent subsequence which amounts to picking suitable subsequences $(a_n(k))_{n\geq 1}$ and $(b_n(k))_{n\geq 1}$. By replacing the original a_n, b_n with the diagonal sequence $a_k(k), b_k(k)$ gives the result. □

The fact that $(S_n)_{n\geq 0}$ is nonarithmetic enters in the next lemma.

Lemma 3.6.5. *For each $\varepsilon > 0$ there is a number t_ε such that for all $t \geq t_\varepsilon$ there exists $c \in (t-\varepsilon, t+\varepsilon)$ such that $\mathbb{U}((a_n - c, a_n - c + h]) \to d(h)$ and $\mathbb{U}((b_n - c, b_n - c + h]) \to D(h)$ as $n \to \infty$.*

Proof. The crucial ingredient following from the lattice-type of $(S_n)_{n\geq 0}$ is that, if $\mathbb{H}$ denotes the set of possible states of $(S_n)_{n\geq 0}$ [☞ Def. 2.3.1], viz.

$$\mathbb{H} \;=\; \{x \in \mathbb{R} : \sup_{n\geq 0} \mathbb{P}(|S_n - x| < \varepsilon) > 0 \text{ for all } \varepsilon > 0\},$$

then $\mathbb{H}$ is asymptotically dense at ∞ in the sense that for all $\varepsilon > 0$ there exists t_0 such that $(t-\varepsilon, t+\varepsilon) \cap \mathbb{H} \neq \emptyset$ for all $t \geq t_0$. A proof of this will be given at the end of this subsection. Now fix any ε and then any $t \geq t_\varepsilon$. The definition of $\mathbb{H}$ ensures that $\mathbb{P}(|S_k - t| < \varepsilon) > 0$ for some $k \in \mathbb{N}_0$. Hence, by invoking the previous corollary, we can pick any $c = S_k(\omega)$ with ω from

$$\{|S_k - t| < \varepsilon, Y_{k,n} \to d(h), Z_{k,n} \to D(h)\}$$

to obtain the desired conclusion. □

Lemma 3.6.6. *For all $s,t,h \in \mathbb{R}_>$, the inequalities*

$$\frac{\mathbb{P}(N((t,t+h]>0)}{\mathbb{P}(X_1>0)} \;\le\; \mathbb{U}((t,t+h]) \;\le\; \frac{\mathbb{P}(N((t,t+h]>0)}{\mathbb{P}(X_1>h)}, \tag{3.22}$$

$$\begin{aligned}\frac{\mathbb{P}(X_1>s+h)}{\mathbb{P}(X_1>0)} &\le \frac{\mathbb{P}(N((t,t+h]>0,N((t+h,t+h+s]=0)}{\mathbb{P}(N((t,t+h]>0)}\\ &\le \frac{\mathbb{P}(X_1>s)}{\mathbb{P}(X_1>h)}.\end{aligned} \tag{3.23}$$

hold true.

Proof. For brevity we write $N := N((t,t+h])$ and $N' := N((t+h,t+h+s])$. As earlier, let $\tau(t)$ denote the first passage time of $(S_n)_{n\ge 0}$ beyond t and $R(t) = S_{\tau(t)} - t$ the associated overshoot. Put $E_k := \{\tau(t)=k, R(t)\le h\}$ for $k\in\mathbb{N}_0$. These are clearly pairwise disjoint with $\sum_{k\ge 0} E_k = \{N>0\}$. Furthermore,

$$E_i \cap \{X_{i+j}=0,\, j<k\} \;\subset\; E_i\cap\{N\ge k\} \;\subset\; E_i\cap\{X_{i+j}\le h,\, j<k\}$$

so that

$$\mathbb{P}(E_i)\,\mathbb{P}(X_1=0)^{k-1} \;\le\; \mathbb{P}(E_i\cap\{N\ge k\}) \;\le\; \mathbb{P}(E_i)\,\mathbb{P}(X_1\le h)^{k-1}.$$

Summing over $i\ge 0$ we obtain

$$\mathbb{P}(N>0)\,\mathbb{P}(X_1=0)^{k-1} \;\le\; \mathbb{P}(N\ge k) \;\le\; \mathbb{P}(N>0)\,\mathbb{P}(X_1\le h)^{k-1},$$

and a summation over k yields (3.22) as $\mathbb{U}((t,t+h]) = \mathbb{E}N = \sum_{k\ge 1}\mathbb{P}(N\ge k)$.

Turning to the proof of (3.23), define

$$\sigma_j := \inf\{k>j : X_k>h\} \quad\text{and}\quad \tau_j := \inf\{k>j : X_k>0\}$$

and $A_j := \{X_{\sigma_j}>s\}$, $B_j := \{X_{\tau_j}>h+s\}$ for $j\in\mathbb{N}_0$. Then $E_j\cap B_j \subset E_j\cap\{N'=0\} \subset E_j\cap A_j$, and

$$\mathbb{P}(A_j) \;=\; \frac{\mathbb{P}(X_1>s)}{\mathbb{P}(X_1>h)}, \quad \mathbb{P}(B_j) \;=\; \frac{\mathbb{P}(X_1>h+s)}{\mathbb{P}(X_1>0)}.$$

Using the independence of E_j from A_j, B_j, we infer

$$\mathbb{P}(E_j)\,\mathbb{P}(B_j) \;\le\; \mathbb{P}(E_j\cap\{N'=0\}) \;\le\; \mathbb{P}(E_j)\,\mathbb{P}(A_j) \quad \text{for all } j\in\mathbb{N}_0$$

and thereupon (3.23) when summing over j. ∎

Lemma 3.6.7. *Let $(S_n)_{n\ge 0}$ be a nonarithmetic RW with $\limsup_{n\to\infty} S_n = \infty$ a.s. and associated set of possible states*

$$\mathbb{H} := \left\{x \in \mathbb{R} : \sup_{n\ge 0} \mathbb{P}(|S_n - x| < \varepsilon) > 0 \textit{ for all } \varepsilon > 0\right\}.$$

Then $\mathbb{H}$ is a closed semigroup and asymptotically dense at ∞ in the sense that $\lim_{x\to\infty} \rho(x,\mathbb{H}) = 0$ where $\rho(x,\mathbb{H}) := \min_{y\in\mathbb{H}} |x-y|$.

Proof. To verify that $\mathbb{H}$ forms a closed semigroup of $\mathbb{R}$ is an easy exercise left to the reader. Possibly after switching to the ladder height process $(S_n^>)_{n\ge 0}$ of the same lattice-type [☞ Lemma 2.5.4], we may assume w.l.o.g. that $(S_n)_{n\ge 0}$ is a RP and thus $\mathbb{H} \subset \mathbb{R}_\ge$. The lattice-type asumption implies that, for any fixed $\varepsilon > 0$, there exist positive $\theta_1,...,\theta_m \in \mathbb{H}$ and $\alpha_1,...,\alpha_m \in \mathbb{Z}\backslash\{0\}$ satisfying $\kappa := \sum_{i=1}^m \alpha_i\theta_i \in (0,\varepsilon]$. Denote by $\mathbb{H}_\varepsilon$ the subsemigroup of $\mathbb{H}$ generated $\theta_1,...,\theta_m$ and put $z := \sum_{i=1}^m |\alpha_i|\theta_i$. Then $z\mathbb{N} \subset \mathbb{H}_\varepsilon$. Now, if $n \in \mathbb{N}$ is such that $n\kappa \le z < (n+1)\kappa$, then for all $k \ge n$ and $1 \le j \le n$

$$kz + j\kappa \;=\; \sum_{i=1}^m \underbrace{(k|\theta_i| + j\theta_i)}_{\in\mathbb{N}}\theta_i \;\in\; \mathbb{H}_\varepsilon$$

holds true and thus $\rho(x,\mathbb{H}_\varepsilon) \le \kappa \le \varepsilon$ for all $x \ge nz$. □

3.6.2 Getting the work done

To finish the proof of (3.1), we will show now that $D(h) \le \mu^{-1}h \le d(h)$ or, equivalently,

$$\overline{D}(h) \;:=\; \frac{D(h)}{h} \;\le\; \frac{1}{\mu} \;\le\; \frac{d(h)}{h} \;=:\overline{d}(h) \quad \text{for all } h > 0.$$

It follows from

$$\mathbb{U}((t,t+h]) \;=\; \sum_{k=1}^n \mathbb{U}\left(\left(t + \frac{(k-1)h}{n}, t + \frac{kh}{n}\right]\right)$$

that $nd(h/n) \le d(h)$ and $nD(h/n) \ge D(h)$. Hence $\overline{D}(h/n) \ge \overline{D}(h)$ and $\overline{d}(h/n) \le \overline{d}(h)$. It is therefore sufficient to show

$$\limsup_{h\downarrow 0} \overline{D}(h) \;\le\; \frac{1}{\mu} \;\le\; \liminf_{h\downarrow 0} \overline{d}(h). \tag{3.24}$$

To this end, fix any $h > 0$ and any $\varepsilon \in (0,h)$. By Lemma 3.6.5, we can pick numbers c_i, d_i for which

$$h-\varepsilon < c_{i+1}-c_i < h < d_{i+1}-d_i < h+\varepsilon,$$
$$\mathbb{U}((a_n-d_i,a_n-d_i+h]) \to D(h), \quad \mathbb{U}((b_n-c_i,b_n-c_i+h]) \to d(h)$$

for all $i \in \mathbb{N}_0$. Notice that this implies $c_i \uparrow \infty$ as well as $d_i \uparrow \infty$.

BLACKWELL starts by investigating $\overline{d}(h)$. Fix $n \in \mathbb{N}$ and then m such that $c_m < a_n \le c_{m+1}$. Put $N_m := N((0,a_n-c_m])$ and $N_i := N((a_n-c_{i+1},a_n-c_i])$ for $i=0,\ldots,m-1$ with expectations $\mathbb{U}_i$. Then $N((0,a_n-c_0]) = \sum_{i=0}^m N_i$ and

$$\begin{aligned}
1 &= \mathbb{P}(N((0,a_n-c_0])=0) + \mathbb{P}(N((0,a_n-c_0])>0) \\
&= \mathbb{P}(X_1>a_n-c_0) + \sum_{i=0}^m \mathbb{P}(N_i>0, N((a_n-c_i,a_n-c_0]=0) \\
&\le \mathbb{P}(X_1>a_n-c_0) + \sum_{i=0}^m \frac{\mathbb{P}(X_1>c_i-c_0)}{\mathbb{P}(X_1>c_{i+1}-c_i)}\mathbb{P}(N_i>0) && \text{(by (3.23))} \\
&\le \mathbb{P}(X_1>a_n-c_0) + \sum_{i=0}^m \frac{\mathbb{P}(X_1>c_i-c_0)}{\mathbb{P}(X_1>c_{i+1}-c_i)}\mathbb{P}(X_1>0)\,\mathbb{U}_i && \text{(by (3.22))} \\
&\le \mathbb{P}(X_1>a_n-c_0) + \sum_{i=0}^m \frac{\mathbb{P}(X_1>i(h-\varepsilon))}{\mathbb{P}(X_1>h)}\mathbb{P}(X_1>0)\,\mathbb{U}_i.
\end{aligned}$$

By letting $n \to \infty$ (entailing $m \to \infty$) and rearranging terms, we arrive at

$$\frac{\mathbb{P}(X_1>h)}{\mathbb{P}(X_1>0)} \le h\overline{d}(h)\sum_{i\ge 0}\mathbb{P}(X_1>i(h-\varepsilon)) \le h\overline{d}(h)\left(\frac{\mu}{h-\varepsilon}+1\right)$$

having utilized the standard inequality $\mathbb{E}X \le \sum_{i\ge 0}\mathbb{P}(X>i) \le \mathbb{E}X+1$ for any random variable $X \ge 0$. Consequently, if μ is finite, then by first letting ε and then h tend to 0, we obtain $1 \le \mu \liminf_{h\downarrow 0}\overline{d}(h)$ as desired.

A similar argument works for $\overline{D}(h)$. Let $N_i', \mathbb{U}_i$ be the analogs of $N_i, \mathbb{U}_i$ when substituting a_n for b_n and the c_i for d_i. Let m this time be such that $d_m < b_n \le d_{m+1}$. Then $N((0,b_n-c_0]) = \sum_{i=0}^q N_i$ and, by another use of Lemma 3.6.6,

$$\begin{aligned}
1 &\ge \sum_{i=0}^m \mathbb{P}(N_i'>0, N((b_n-d_i,a_n-d_0]=0) \\
&\ge \sum_{i=0}^m \frac{\mathbb{P}(X_1>d_{i+1}-d_0)}{\mathbb{P}(X_1>0)}\mathbb{P}(N_i'>0) \\
&\ge \sum_{i=0}^m \frac{\mathbb{P}(X_1>d_{i+1}-d_0)}{\mathbb{P}(X_1>0)}\mathbb{P}(X_1>d_{i+1}-d_i)\,\mathbb{U}_i' \\
&\ge \sum_{i=0}^m \frac{\mathbb{P}(X_1>(i+1)(h+\varepsilon))}{\mathbb{P}(X_1>0)}\mathbb{P}(X_1>h)\,\mathbb{U}_i'.
\end{aligned}$$

Letting n (and thus m) go to infinity and making rearrangements as before, we obtain

$$\frac{\mathbb{P}(X_1 > 0)}{\mathbb{P}(X_1 > h)} \;\ge\; h\overline{D}(h)\sum_{i=0}^{l}\mathbb{P}(X_1 > (i+1)(h+\varepsilon)) \quad \text{for all } l \in \mathbb{N} \tag{3.25}$$

and therefore in the case $\mu < \infty$

$$\frac{\mathbb{P}(X_1 > 0)}{\mathbb{P}(X_1 > h)} \;\ge\; h\overline{D}(h)\left(\frac{\mu}{h+\varepsilon} - 1\right).$$

With ε and then h going to 0, we arrive at $1 \ge \mu \,\limsup_{h\downarrow 0}\overline{D}(h)$ which is the desired conclusion.

If $\mu = \infty$, the sum on the right-hand side of (3.25) approaches ∞ as $l \to \infty$ and so $\overline{D}(h)$ must be 0 for all $h > 0$. This shows the desired result in this case as well.

3.6.3 The two-sided case: a glance at Blackwell's second paper

As BLACKWELL [7] provided also the first complete proof of (3.1) for nonarithmetic RW's with positive drift μ, we close this section with a brief account of its major ingredients. Probably the most interesting one is the use of ladder variables which occur here for the very first time in the literature. Given a SRW $(S_n)_{n\ge 0}$ with positive drift, the first result shown is that

$$\mathbb{U}((t,t+h]) \;=\; \int_{[0,\infty)}(H(x-t)-H(x-t-h))\,\mathbb{U}^{>}(dx) \tag{3.26}$$

[☞ [7, Thm. 2]], where $\mathbb{U}^{>}$ is the renewal measure of the ladder height process $(S_n^{>})_{n\ge 0}$ and $H(t) := \sum_{n\ge 0}P(\sigma^{>} > n, -t < S_n \le 0)$ for $t \in \mathbb{R}_{\ge}$. Further ingredients are (3.1) for $\mathbb{U}^{>}$, i.e., $\mathbb{U}^{>}((t,t+h]) \to h/\mu^{>}$ as $t \to \infty$, and Wald's equation which gives $\mu^{>} = \mu\,\mathbb{E}\sigma^{>}$. The proof itself mainly consists of a careful analysis of the integral on the right-hand side of (3.26), however, without observing that it actually coincides with $\mathbb{U}^{>} * \mathbb{V}^{>}(t) = \int_{(-\infty,0]}\mathbb{U}^{>}((t-x,t+h-x])\,\mathbb{V}^{>}(dx)$, the cyclic decomposition formula stated in (2.24). As seen in 3.3.1, this observation would have led directly to the desired conclusion by an appeal to the dominated convergence theorem, for $\mathbb{U}^{>}$ is uniformly locally bounded and satisfying (3.1).

Chapter 4
The key renewal theorem and refinements

Given a RW $(S_n)_{n\geq 0}$ in a standard model with positive drift μ and lattice-span d, the simple observation

$$\mathbb{U}_\lambda([t-h,t]) \;=\; \int \mathbf{1}_{[0,h]}(t-x)\,\mathbb{U}_\lambda(dx) \;=\; \mathbf{1}_{[0,h]} * \mathbb{U}_\lambda(t)$$

for all $t\in\mathbb{R}$, $h\in\mathbb{R}_>$ and $\lambda\in\mathscr{P}(\mathbb{R})$ shows that the nontrivial part of Blackwell's renewal theorem may also be stated as

$$d\text{-}\lim_{t\to\infty} \mathbf{1}_{[0,h]} * \mathbb{U}_\lambda(t) \;=\; \frac{1}{\mu}\int \mathbf{1}_{[0,h]}\,d\lambda\!\!\lambda_d \tag{4.1}$$

for all $h\in\mathbb{R}_>$ and $\lambda\in\mathscr{P}(\mathbb{G}_d)$, in other words, as a limiting result for convolutions of indicators of compact intervals with the renewal measure. This raises the question, supported further by numerous applications [☞ e.g. (1.21) (1.26)], to which class $\mathscr{R}$ of functions $g:\mathbb{R}\to\mathbb{R}$ an extension of (4.1) in the sense that

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}_\lambda(t) \;=\; \frac{1}{\mu}\int g\,d\lambda\!\!\lambda_d \quad \text{for all } g\in\mathscr{R} \tag{4.2}$$

is possible. Obviously, all finite linear combinations of indicators of compact intervals are elements of $\mathscr{R}$. By taking monotone limits of such step functions, one can further easily verify that $\mathscr{R}$ contains any g that vanishes outside a compact interval I and is Riemann integrable on I. On the other hand, in view of applications a restriction to functions with compact support appears to be undesirable and calls for appropriate conditions on g that are not too difficult to check in concrete examples. In the nonarithmetic case one would naturally hope for $\lambda\!\!\lambda_0$-integrability as being a sufficient condition, but unfortunately this is not generally true. The next section specifies the notion of *direct Riemann integrability*, first introduced and thus named by Feller [21], and provides also a discussion of necessary and sufficient conditions for this property to hold. Assertion (4.2) for functions g of this kind, called *key renewal theorem*, is proved in Section 4.2. We then proceed with a longer treatment of an important special class of nonarithmetic RW's which are called *spread out* and are characterized by having a renewal measure which is nonsingular with respect to Lebesgue measure $\lambda\!\!\!\lambda_0$. *Stone's decomposition* of such renewal measures, stated and proved in Section 4.3, will provide us with the key to enlarge the class of functions satisfying (4.2) and to even infer uniform convergence [☞ Section 4.6]. Naturally related to these improvements is exact coupling of spread out RW's which is therefore studied in Section 4.4. A further discussion of this problem for general RW's forms the topic of Section 4.5 and leads to the notion of the *minimal subgroup* of a RW.

4.1 Direct Riemann integrability

Definition 4.1.1. Let g be a real-valued function on $\mathbb{R}$ and define, for $\delta > 0$ and $n \in \mathbb{Z}$,

$$\begin{aligned} I_{n,\delta} &:= (\delta n, \delta(n+1)], \\ m_{n,\delta} &:= \inf\{g(x) : x \in I_{n,\delta}\}, \quad M_{n,\delta} := \sup\{g(x) : x \in I_{n,\delta}\} \\ \underline{\sigma}(\delta) &:= \delta \sum_{n\in\mathbb{Z}} m_{n,\delta} \quad \text{and} \quad \overline{\sigma}(\delta) := \delta \sum_{n\in\mathbb{Z}} M_{n,\delta}. \end{aligned}$$

The function g is called *directly Riemann integrable (dRi)* if $\underline{\sigma}(\delta)$ and $\overline{\sigma}(\delta)$ are both absolutely convergent for all $\delta > 0$ and

$$\lim_{\delta\to 0}(\overline{\sigma}(\delta) - \underline{\sigma}(\delta)) = 0.$$

The definition reduces to ordinary Riemann integrability if the domain of g is only a compact interval instead of the whole line. In the case where $\int_{-\infty}^{\infty} g(x)\,dx$ may be defined as the limit of such ordinary Riemann integrals $\int_{-a}^{b} g(x)\,dx$ with a, b tending to infinity, the function g is called *improperly Riemann integrable*. An approximation of g by upper and lower step functions having integrals converging to a common value is then still only taken over compact intervals which are made bigger and bigger. However, in the above definition such an approximation is required to be possible *directly* over the whole line and therefore of a more restrictive type than improper Riemann integrability.

The following lemma, partly taken from [3, Prop. V.4.1], collects a whole bunch of necessary and sufficient criteria for direct Riemann integrability.

Lemma 4.1.2. *Let g be an arbitrary real-valued function on $\mathbb{R}$. Then the following two conditions are necessary for direct Riemann integrability:*

(dRi-1) g *is bounded and* $\lambda\!\!\lambda_0$*-a.e. continuous.*
(dRi-2) g *is* $\lambda\!\!\lambda_d$*-integrable for all* $d \geq 0$.

Conversely, any of the following conditions is sufficient for g *to be dRi:*

(dRi-3) *For some* $\delta > 0$, $\underline{\sigma}(\delta)$ *and* $\overline{\sigma}(\delta)$ *are absolutely convergent, and* g *satisfies (dRi-1).*
(dRi-4) g *has compact support and satisfies (dRi-1).*
(dRi-5) g *satisfies (dRi-1) and* $f \leq g \leq h$ *for dRi functions* f, h.
(dRi-6) g *vanishes on* $\mathbb{R}_<$, *is nonincreasing on* $\mathbb{R}_\geq$ *and* $\lambda\!\!\lambda_0$*-integrable.*
(dRi-7) $g = g_1 - g_2$ *for nondecreasing functions* g_1, g_2 *and* $f \leq g \leq h$ *for dRi functions* f, h.
(dRi-8) g^+ *and* g^- *are dRi.*

Proof. (a) Suppose that g is dRi. Then the absolute convergence of $\underline{\sigma}(1)$ and $\overline{\sigma}(1)$ ensures that g is bounded, for

$$\sup_{x\in\mathbb{R}} |g(x)| \;\leq\; \sup_{n\in\mathbb{Z}} (|m_n^1| + |M_n^1|) \;<\; \infty.$$

That g must also be $\lambda\!\!\lambda_0$-a.e. continuous is a standard fact from Lebesgue integration theory but may also be quickly assessed as follows: If g fails to have this property then, with $g_*(x) := \liminf_{y\to x} g(y)$ and $g^*(x) := \limsup_{y\to x} g(y)$, we have

$$\alpha \;:=\; \lambda\!\!\lambda_0(\{g^* \geq g_* + \varepsilon\}) > 0 \quad \text{for some } \varepsilon > 0.$$

As $m_{n,\delta} \leq g_*(x) \leq g^*(x) \leq M_{n,\delta}$ for all $x \in (n\delta, (n+1)\delta)$, $n \in \mathbb{Z}$ and $\delta > 0$, it follows that

$$\overline{\sigma}(\delta) - \underline{\sigma}(\delta) \;\geq\; \int (g^*(x) - g_*(x))\, \lambda\!\!\lambda_0(dx) \;\geq\; \varepsilon\alpha \quad \text{for all } \delta > 0$$

which contradicts direct Riemann integrability. We have thus proved necessity of (dRi-1).

As for (dRi-2), it suffices to note that, with

$$\underline{\phi}(\delta) \;:=\; \delta \sum_{n\in\mathbb{Z}} |m_{n,\delta}| \quad \text{and} \quad \delta\overline{\phi}(\delta) \;:=\; \sum_{n\in\mathbb{Z}} |M_{n,\delta}|,$$

we have $\int |g(x)|\, \lambda\!\!\lambda_0(dx) \leq \underline{\phi}(1) + \overline{\phi}(1)$ and $\int |g(x)|\, \lambda\!\!\lambda_d(dx) \leq \underline{\phi}(d) + \overline{\phi}(d)$ for each $d > 0$.

(b) Turning to the sufficient criteria, put

$$g_\delta \;:=\; \sum_{n\in\mathbb{Z}} m_{n,\delta} \mathbf{1}_{I_{n,\delta}} \quad \text{and} \quad g^\delta \;:=\; \sum_{n\in\mathbb{Z}} M_{n,\delta} \mathbf{1}_{I_{n,\delta}} \quad \text{for } \delta > 0. \tag{4.3}$$

If (dRi-3) holds true, then $g_\delta \uparrow g$ and $g^\delta \downarrow g$ $\lambda\!\!\!\lambda_0$-a.e. as $\delta \downarrow 0$ by the $\lambda\!\!\!\lambda_0$-a.e. continuity of g. Hence the monotone convergence theorem implies (using $-\infty < \underline{\sigma}(\delta) \le \overline{\sigma}(\delta) < \infty$)

$$\underline{\sigma}(\delta) \;=\; \int g_\delta \, d\lambda\!\!\!\lambda_0 \uparrow \int g \, d\lambda\!\!\!\lambda_0 \quad \text{and} \quad \overline{\sigma}(\delta) \;=\; \int g^\delta \, d\lambda\!\!\!\lambda_0 \uparrow \int g \, d\lambda\!\!\!\lambda_0$$

proving that g is dRi.

Since each of (dRi-4) and (dRi-5) implies (dRi-3), there is nothing to prove under these conditions.

Assuming (dRi-6), the monotonicity of g on $\mathbb{R}_\ge$ gives

$$M_{n,\delta} = g(n\delta+) \quad \text{and} \quad m_{n,\delta} = g((n+1)\delta) \ge M_{n,\delta} \quad \text{for all } n \in \mathbb{N}_0,\ \delta > 0.$$

Consequently,

$$\begin{aligned} 0 \;\le\; \underline{\sigma}(\delta) \;\le\; \int_0^\infty g(x)\,dx \;\le\; \overline{\sigma}(\delta) \\ \le\; \delta g(0) + \underline{\sigma}(\delta) \;\le\; \int_0^\infty g(x)\,dx \;+\; \delta g(0) \;<\; \infty \end{aligned}$$

and therefore $\overline{\sigma}(\delta) - \underline{\sigma}(\delta) \le \delta g(0) \to 0$ as $\delta \to 0$.

Assuming (dRi-7) the monotonicity of g_1 and g_2 ensures that g has at most countably many discontinuities and is thus $\lambda\!\!\!\lambda_0$-a.e. continuous. g is also bounded because $f \le g \le h$ for dRi function f, h. Hence (dRi-5) holds true.

Finally assuming (dRi-8), note first that g^-, g^+ both satisfy (dRi-1) because this is true for g. Moreover,

$$0 \;\le\; g^\pm \;\le\; (g^\delta)^+ + (g_\delta)^- \;\le\; \sum_{n\in\mathbb{Z}} \big(|M_{n,\delta}| + |m_{n,\delta}|\big)\,\mathbf{1}_{I_{n,\delta}} \quad \text{for all } \delta > 0$$

whence g^-, g^+ both satisfy (dRi-5). □

For later purposes, we give one further criterion for direct Riemann integrability.

Lemma 4.1.3. *Let g be a function on $\mathbb{R}$ that vanishes on $\mathbb{R}_<$ and is nondecreasing on $\mathbb{R}_\ge$. Then $g_\theta(x) := e^{\theta x} g(x)$ is dRi for any $\theta \in \mathbb{R}$ such that g_θ is $\lambda\!\!\!\lambda_0$-integrable.*

Proof. Clearly, g must be nonnegative. For $\theta \le 0$ the assertion follows directly from (dRi-6) because g_θ is nondecreasing on $\mathbb{R}_\ge$. Hence suppose $\theta > 0$ and let $\delta > 0$ be arbitrary. Then we have

$$\begin{aligned} M_{n,\delta} \;&:=\; \sup\{g_\theta(x) : x \in I_{n,\delta}\} \;\le\; e^{\theta\delta(n+1)} g(\delta n) \\ &=\; e^{2\theta\delta}(e^{\theta\delta(n-1)} g(\delta n)) \;\le\; \frac{e^{2\theta\delta}}{\delta} \int_{\delta(n-1)}^{\delta n} g_\theta(y)\,dy \end{aligned}$$

for all $n \in \mathbb{N}$ and similarly

$$m_{n,\delta} := \inf\{g_\theta(x) : x \in I_{n,\delta}\} \geq e^{\theta\delta n} g(\delta(n+1)) \geq \frac{e^{-2\theta\delta}}{\delta} \int_{\delta(n+1)}^{\delta(n+2)} g_\theta(y)\, dy$$

for all $n \in \mathbb{N}_0$. Consequently,

$$\begin{aligned} \overline{\sigma}(\delta) - \underline{\sigma}(\delta) &= \delta M_{0,\delta} + \delta \sum_{n \geq 1} (M_{n,\delta} - m_{n-1,\delta}) \\ &\leq \delta g_\theta(0) + (e^{2\theta\delta} - e^{-2\theta\delta}) \int_{\mathbb{R}_\geq} g_\theta(y)\, dy \to 0 \quad \text{as } \delta \to 0 \end{aligned}$$

which proves the assertion. □

4.2 The key renewal theorem

We are now ready to formulate the announced extension of Blackwell's renewal theorem, called *key renewal theorem*. The name was given by Smith [41] in allusion to its eminent importance in applications.

Theorem 4.2.1. [Key renewal theorem] *Let $(S_n)_{n\geq 0}$ be a RW with positive drift μ, lattice-span d and renewal measure $\mathbb{U}$. Then*

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}(t) = \frac{1}{\mu} \int g\, d\lambda_d \quad \text{and} \tag{4.4}$$

$$\lim_{t\to-\infty} g * \mathbb{U}(t) = 0 \tag{4.5}$$

for every dRi function $g : \mathbb{R} \to \mathbb{R}$.

Listing non- and d-arithmetic case separately, (4.4) takes the form

$$d\text{-}\lim_{t\to\infty} g * \mathbb{U}(t) = \frac{1}{\mu} \int_{-\infty}^{\infty} g(x)\, dx \tag{4.6}$$

if $d = 0$ where the right-hand integral is meant as an improper Riemann integral. In the case $d > 0$, we have accordingly

$$\lim_{n\to\infty} g * \mathbb{U}(nd) = \frac{d}{\mu} \sum_{n\in\mathbb{Z}} g(nd) \tag{4.7}$$

and, furthermore, for any $a \in \mathbb{R}$,

$$d\text{-}\lim_{t\to\infty} g*\mathbb{U}(nd+a) \;=\; \frac{d}{\mu}\sum_{n\in\mathbb{Z}} g(nd+a), \tag{4.8}$$

because $g(\cdot+a)$ is clearly dRi as well.

Proof. We restrict ourselves to the more difficult nonarithmetic case. Given a dRi function g, let g_δ, g^δ be as in (4.3) for $\delta>0$ and recall that

$$g_\delta \le g \le g^\delta, \quad \underline{\sigma}(\delta)=\int_{-\infty}^{\infty} g_\delta(x)\,dx \quad\text{and}\quad \overline{\sigma}(\delta)=\int_{-\infty}^{\infty} g^\delta(x)\,dx.$$

Fix any $\delta\in(0,1)$ and $m\in\mathbb{N}$ large enough such that $\sum_{|n|>m}|M_{n,\delta}|<\delta$. Then, using inequality (2.17), we infer

$$g^\delta*\mathbb{U}(t) \;=\; \sum_{n\in\mathbb{Z}} M_{n,\delta}\mathbb{U}(t-I_{n,\delta}) \;\le\; \sum_{|n|\le m} M_{n,\delta}\,\mathbb{U}(t-n\delta-I_{0,\delta}) \;+\; \delta\,\mathbb{U}([-1,1])$$

and therefore with Blackwell's theorem

$$\begin{aligned}
\limsup_{t\to\infty} g^\delta*\mathbb{U}(t) \;&\le\; \sum_{|n|\le m} M_{n,\delta}\lim_{t\to\infty}\mathbb{U}(t-n\delta-I_{0,\delta}) \;+\; \delta\,\mathbb{U}([-1,1])\\
&=\; \frac{\delta}{\mu}\sum_{|n|\le m} M_{n,\delta} \;+\; \delta\,\mathbb{U}([-1,1])\\
&\le\; \frac{1}{\mu}\int_{-\infty}^{\infty} g^\delta(x)\,dx \;+\; \frac{\delta^2}{\mu} \;+\; \delta\,\mathbb{U}([-1,1])\\
&=\; \frac{1}{\mu}\overline{\sigma}(\delta) \;+\; \frac{\delta^2}{\mu} \;+\; \delta\,\mathbb{U}([-1,1]).
\end{aligned} \tag{4.9}$$

Consequently, as $g*\mathbb{U}\le g^\delta*\mathbb{U}$ for all $\delta>0$,

$$\limsup_{t\to\infty} g*\mathbb{U}(t) \;\le\; \lim_{\delta\downarrow 0}\limsup_{t\to\infty} g^\delta*\mathbb{U}(t) \;\le\; \frac{1}{\mu}\int_{-\infty}^{\infty} g(x)\,dx.$$

Replace g with $-g$ in the above estimation to obtain

$$\liminf_{t\to\infty} g*\mathbb{U}(t) \;\ge\; \frac{1}{\mu}\int_{-\infty}^{\infty} g(x)\,dx.$$

This completes the proof of (4.4). But if we let t tend to $-\infty$ in (4.9), use the trivial half of Blackwell's theorem and let then δ go to 0, we arrive at

$$\limsup_{t\to-\infty} g*\mathbb{U}(t) \;\le\; 0.$$

The reverse inequality for the lower limit is again obtained by substituting g for $-g$ in this derivation. Hence (4.5) is proved completing the proof of the theorem. □

The first proof of the key renewal theorem was given by Smith [38, Thm. 1 and Cor. 1.1] for the case of nonarithmetic RP's and $\lambda\!\!\lambda_0$-integrable functions that are either zero on $\mathbb{R}_\leq$ and nonincreasing on $\mathbb{R}_>$, or of bounded variation on compact intervals. In a second paper [39], he also pointed out how this result may be deduced from Blackwell's theorem, as did KESTEN & RUNNENBURG [27]. The proof presented here is essentially due to FELLER [21] for nonarithmetic RP's and dRi functions and was later extended to the case of nonarithmetic RW's with positive drift by ATHREYA, MCDONALD & NEY [5], however using a different argument.

Remark 4.2.2. In the d-arithmetic and thus discrete case, the convolution $g * \mathbb{U}$ may actually be considered as a function on the discrete group $d\mathbb{Z}$ and thus requires g to be considered on this set only which reduces it to a sequence $(g_n)_{n\in\mathbb{Z}}$. Doing so merely absolute summability, i.e. $\sum_{n\in\mathbb{Z}} |g_n| < \infty$, is needed instead of direct Riemann integrability. With this observation the result reduces to a straightforward consequence of Blackwell's renewal theorem and explains that much less attention has been paid to it in the literature.

Remark 4.2.3. The following counterexample shows that in the nonarithmetic case $\lambda\!\!\!\lambda_0$-integrability of g does not suffice to ensure (4.4). Consider a distribution F on $\mathbb{R}$ with positive mean $\mu = \int x F(dx)$ and renewal measure $\mathbb{U} = \sum_{n\geq 0} F^{*n}$. The function $g := \sum_{n\geq 1} n^{1/2} \mathbf{1}_{[n.n+n^{-2})}$ is obviously $\lambda\!\!\!\lambda_0$-integrable, but

$$g * \mathbb{U}(n) \;=\; \sum_{k\geq 0} g * F^{*k}(n) \;\geq\; g * F^{*0}(n) \;=\; g(n) \;=\; n^{1/2}$$

diverges to ∞ as $n \to \infty$. Here the atom at 0 that any renewal measure of a SRW possesses already suffices to demonstrate that $g(x)$ must not have unbounded oscillations as $x \to \infty$. But there are also examples of renewal measures with no atom at 0 (thus pertaining to a delayed RW) such that the key renewal theorem fails to hold for $\lambda\!\!\!\lambda_0$-integrable g. FELLER [21, p. 368] provides an example of a $\lambda\!\!\!\lambda_0$-continuous distribution F with finite positive mean such that $\mathbb{U} = \sum_{n\geq 1} F^{*n}$ satisfies $\limsup_{t\to\infty} g * \mathbb{U}(t) = \infty$ for some $\lambda\!\!\!\lambda_0$-integrable g.

4.3 Spread out random walks and Stone's decomposition

4.3.1 Nonarithmetic distributions: Getting to know the good ones

Whenever striving for limit theorems for RW's as time goes to infinity, there is a fundamental difference between the arithmetic and the nonarithmetic case as for the way these processes are forgetting their initial conditions. This becomes apparent most clearly when trying to couple two completely d-arithmetic $(d \geq 0)$ RW's $(S_n)_{n\geq 0}$ and $(S'_n)_{n\geq 0}$ with the same increment distribution but different delays S_0 and S'_0. As has been seen in the coupling proof of Blackwell's theorem in Section 3.3, such RW's, if arithmetic, may be defined on a common probability space in such a way that *exact coupling* holds true, i.e., the coupling time

$$T\ :=\ \inf\{n \geq 0 : S_n = S'_n\}$$

is finite with probability one. In other words, after a nonanticipative random time T the two RW's meet exactly. Hence, the observation of any of the two RW's after T does not provide information on where they came from (complete loss of memory). This property does not generally persist when moving to the (completely) nonarithmetic case. More precisely, there are situations, and these will be characterized soon, where T is infinite with positive probability however the joint distribution of $(S_n, S'_n)_{n\geq 0}$ is chosen. As a consequence we had to resort in Section 3.3 to a weaker coupling, called ε-*coupling*, where only

$$T(\varepsilon)\ :=\ \inf\{n \geq 0 : |S_n - S_n| \leq \varepsilon'\}$$

is finite with probability one. In such a situation a loss of memory merely holds true up to a (small) error ε. On the other hand, it is natural to ask whether there exist nonarithmetic increment distributions that do allow exact coupling of two associated RW's. The good news is: Yes, there are and they can be precisely defined.

Definition 4.3.1. A distribution F on $\mathbb{R}$ is called *spread out* if F^{*n} is nonsingular with respect to $\lambda\!\!\lambda_0$ for some $n \in \mathbb{N}$, that is, if there are a $\lambda\!\!\lambda_0$-continuous measure $F_1 \neq 0$ and a measure F_2 such that $F^{*n} = F_1 + F_2$ for some $n \in \mathbb{N}$. A random variable X (RW $(S_n)_{n\geq 0}$) is called spread out if this holds true for its distribution (increment distribution).

Plainly, we have the following implications:

$\lambda\!\!\lambda_0$-continuous $\Rightarrow$ spread out $\Rightarrow$ completely nonarithmetic $\Rightarrow$ nonarithmetic.

With regard to the renewal measure of a spread out RW the basic observation is the following.

Lemma 4.3.2. *A SRW $(S_n)_{n\geq 0}$ is spread out if, and only if, its renewal measure $\mathbb{U}$ is nonsingular with respect to $\lambda\!\!\lambda_0$.*

Proof. Obvious. □

4.3.2 Stone's decomposition

In view of the last lemma it is natural to ask for more information on the renewal measure of a spread out SRW $(S_n)_{n\geq 0}$. The result to be presented hereafter goes back to STONE [43] [☞ also [3, Section VII.1]] but will be derived here in a different and more probabilistic manner based on cyclic decomposition.

Let us start by considering the particularly simple case first, when the increment distribution F has a continuous $\lambda\!\!\lambda_0$-density f with compact support ($f \in \mathscr{C}_0(\mathbb{R})$).

Lemma 4.3.3. *Let $(S_n)_{n\ge 0}$ be a SRW with positive drift μ and $\lambda\!\!\lambda_0$-continuous increment F having a density of the form $f*\lambda$ for some $f \in \mathscr{C}_0(\mathbb{R})$ and $\lambda \in \mathscr{P}(\mathbb{R})$. Then its renewal measure $\mathbb{U}$ can be written as $\mathbb{U} = \delta_0 + u\lambda\!\!\lambda_0$, where the renewal density u is bounded and continuous ($u \in \mathscr{C}_b(\mathbb{R})$) and satisfies $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to\infty} u(x) = \mu^{-1}$.*

Proof. Suppose first that $F = f\lambda\!\!\lambda_0$ for some $f \in \mathscr{C}_0(\mathbb{R})$, that is $\lambda = \delta_0$. Then $F*\mathbb{U}$ has density $u := f*\mathbb{U}$. Hence the asserted decomposition of $\mathbb{U}$ follows from $\mathbb{U} = \delta_0 + F*\mathbb{U}$, which leaves us with the proof of the asserted properties of u. Choose $a > 0$ so large that $[-a,a] \supset supp(f)$. Then $\|u\|_\infty = \|f\|_\infty \sup_{x\in\mathbb{R}} \mathbb{U}([-2a,2a) < \infty$, where $\|\cdot\|_\infty$ denotes the supremum norm and (2.17) has been utilized. Moreover, we infer with the help of the dominated convergence theorem

$$\lim_{h\to 0} |u(x+h) - u(x)| \;\le\; \int_{[x-2a,x+2a]} \lim_{h\to 0} |f(x+h-y) - f(x-y)|\; \mathbb{U}(dx) \;=\; 0$$

and thus $u \in \mathscr{C}_b(\mathbb{R})$. Finally, the limiting assertions follow from the key renewal theorem, for f is dRi by (dRi-4) of Lemma 4.1.2.

Now suppose the general case that $F = (f*\lambda)\lambda\!\!\lambda_0$. Then the same decomposition $\mathbb{U} = \delta_0 + u\lambda\!\!\lambda_0$ holds true with $u = (f*\lambda)*\mathbb{U} = (f*\mathbb{U})*\lambda$. But the asserted properties for u are then easily inferred from the fact fact that $f*\mathbb{U}$ has these properties (by the first part). □

Proceeding to the general case, we first need the following smoothing lemma for convolutions of functions also stated in [3, Section VII.1].

Lemma 4.3.4. *If F is spread out, there exists a convolution power F^{*m} having a nontrivial $\lambda\!\!\lambda_0$-continuous component with density in $\mathscr{C}_0(\mathbb{R})$.*

Proof. As F is spread out, we have $F^{*n} \ge f\lambda\!\!\lambda_0$ for some $f \in \mathscr{L}_1(\mathbb{R})$ and $n \in \mathbb{N}$. We may further assume that f is bounded with compact support. Now choose functions $f_k \in \mathscr{C}_b(\mathbb{R})$ such that $\|f - f_k\|_1 \to 0$. Then $f*f_k \in \mathscr{C}_b(\mathbb{R})$ for all $k \in \mathbb{N}$ follows by a similar argument as in the previous lemma using the dominated convergence theorem. Furthermore,

$$\|f^{*2} - f*f_k\|_\infty \;\le\; \|f\|_\infty \|f-f_k\|_1 \;\to\; 0 \quad \text{as } k\to\infty$$

showing the continuity of f^{*2} as a uniform limit of continuous functions. Finally, since $F^{*2n} \ge f^{*2}\,\lambda\!\!\lambda_0$ the proof is complete (with $m = 2n$) because f^* maybe replaced with a minorant in $\mathscr{C}_0(\mathbb{R})$. □

After these preparations we are ready to present the general decomposition result for renewal measures of spread out RW's.

Theorem 4.3.5. [Stone's decomposition] *Let $(S_n)_{n\ge0}$ be a spread out RW in a standard model with positive drift μ. Then, for each $\lambda \in \mathscr{P}(\mathbb{R})$, its renewal measure $\mathbb{U}_\lambda$ under $\mathbb{P}_\lambda$ has the decomposition $\mathbb{U}_\lambda = \mathbb{U}'_\lambda + \mathbb{U}''_\lambda$, where $\mathbb{U}'_\lambda$ is a finite measure and $\mathbb{U}''_\lambda = u_\lambda \lambda\!\!\lambda_0$ with $u_\lambda \in \mathscr{C}_b(\mathbb{R})$ satisfying $\lim_{x\to-\infty} u_\lambda(x) = 0$ and $\lim_{x\to\infty} u_\lambda(x) = \mu^{-1}$.*

Proof. Since $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ it suffices to prove the result for the zero-delayed case as one can easily verify. So let $(S_n)_{n\ge0}$ be a spread out SRW and suppose first that its increment distribution F satisfies $F = \alpha f\,\lambda\!\!\lambda_0 + (1-\alpha)G$ for some $\alpha > 0$, some density function $f \in \mathscr{C}_0(\mathbb{R})$, and some $G \in \mathscr{P}(\mathbb{R})$. In this case we may assume that $X_n = \eta_n Y_n + (1-\eta_n)X'_n$ for each $n \in \mathbb{N}$, where $(\eta_n)_{n\ge1}$, $(Y_n)_{n\ge1}$ and $(X'_n)_{n\ge1}$ are independent sequences of iid random variables with

(i) $\eta_1 \stackrel{d}{=} B(1,\alpha)$, i.e. $\mathbb{P}(\eta_1 = 1) = 1 - \mathbb{P}(\eta_1 = 0) = \alpha$,

(ii) $Y_1 \stackrel{d}{=} f\,\lambda\!\!\lambda_0$, and

(iii) $X'_1 \stackrel{d}{=} G$.

Put $S'_n := \sum_{j=1}^n X'_j$ for $n \in \mathbb{N}_0$ and let $\overline{\mathbb{U}}'_{1-\alpha} = \sum_{n\ge0} \alpha(1-\alpha)^n G^{*n} \in \mathscr{P}(\mathbb{R})$ denote its normalized $(1-\alpha)$-discounted renewal measure. Let $(\sigma_n)_{n\ge0}$ be the SRP defined by the successive times when the η_k's equal 1. Plainly, this is the sequence of copy sums associated with $\sigma := \inf\{n \ge 1 : \eta_n = 1\}$ and has geometrically distributed increments with parameter α. As usual let $\mathbb{V}^{(\sigma)}$ be the pre-σ occupation measure and $\mathbb{U}^{(\sigma)}$ be the renewal measure of $(S_{\sigma_n})_{n\ge0}$. By Lemma 2.4.4 (cyclic decomposition), we have $\mathbb{U} = \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}$. Observe that

$$\mathbb{P}(S_\sigma \in \cdot) \;=\; \sum_{n\ge0} \alpha(1-\alpha)^n\, \mathbb{P}(S'_n + Y_{n+1} \in \cdot) \;=\; \overline{\mathbb{U}}'_{1-\alpha} * (f\,\lambda\!\!\lambda_0) \;=\; (f * \overline{\mathbb{U}}'_{1-\alpha})\,\lambda\!\!\lambda_0$$

to infer from Lemma 4.3.3 that $\mathbb{U}^{(\sigma)} = \delta_0 + u^{(\sigma)}\lambda\!\!\lambda_0$ for some $u^{(\sigma)} \in \mathscr{C}_b(\mathbb{R})$ with $\lim_{x\to-\infty} u^{(\sigma)}(x) = 0$ and $\lim_{x\to\infty} u^{(\sigma)}(x) = (\mathbb{E}S_\sigma)^{-1} = (\mu\,\mathbb{E}\sigma)^{-1}$. Consequently,

$$\mathbb{U} \;=\; \mathbb{V}^{(\sigma)} * (\delta_0 + u^{(\sigma)}\lambda\!\!\lambda_0) \;=\; \mathbb{V}^{(\sigma)} + (u^{(\sigma)} * \mathbb{V}^{(\sigma)})\,\lambda\!\!\lambda_0.$$

Using $\|\mathbb{V}^{(\sigma)}\| = \mathbb{E}\sigma < \infty$ it is easily verified that $u := u^{(\sigma)} * \mathbb{V}^{(\sigma)}$ has all asserted properties.

To finish the proof is now easy use Lemma 4.3.4 to infer for general spread out F that F^{*m} satisfies the condition of the previous part. Then, with $\mathbb{V}^{(m)} := \sum_{k=0}^{m-1} F^{*k}$ and $\mathbb{U}^{(m)} := \sum_{n\ge0} F^{*mn}$, we infer that

$$\mathbb{U} \;=\; \mathbb{V}^{(m)} * \mathbb{U}^{(m)} \;=\; \mathbb{V}^{(m)} * \left(\mathbb{U}^{(m)\prime} + u^{(m)}\lambda\!\!\lambda_0\right) \;=\; \mathbb{V}^{(m)} * \mathbb{U}^{(m)\prime} + (u^{(m)} * \mathbb{V}^{(m)})\lambda\!\!\lambda_0$$

where $\|\mathbb{U}^{(m)\prime}\| < \infty$ and $u^{(m)} \in \mathscr{C}_b(\mathbb{R})$ with $\lim_{x\to-\infty} u^{(m)}(x) = 0$ and $\lim_{x\to\infty} u^{(m)}(x) = (m\mu)^{-1}$. Hence, $\mathbb{U}' := \mathbb{V}^{(m)} * \mathbb{U}^{(m)\prime}$ and $\mathbb{U}'' := (u^{(m)} * \mathbb{V}^{(m)})\lambda\!\!\lambda_0$ are easily seen to have all required properties. □

4.4 Exact coupling of spread out random walks

Returning to the motivating discussion at the beginning of the previous section we will now proceed to show that two spread out random walks with common increment distribution do always allow exact coupling. Our derivation is based on the work by LINDVALL & ROGERS [32] but confines to the case of one-dimenisonal RW's which is our main concern here.

4.4.1 A clever device: Mineka coupling

Let $(S_n)_{n\ge0}$ be an arbitrary RW in a standard model with increment distribution F. For the moment we do not assume that F is spread out. The reader is referred to A.3.2 for a collection of some basic facts on the total variation distance and the maximal coupling of probability measures F,G on a measurable space $(\mathfrak{X},\mathscr{A})$ including the definition of $F\wedge G$.

Lemma 4.4.1. [Mineka coupling] *Let F be a distribution on $\mathbb{R}$ such that $1-\alpha := \|F-\delta_x * F\| < 1$. Then there exist random variables X,Y on a common probability space $(\Omega,\mathfrak{A},\mathbb{P})$ such that $X \stackrel{d}{=} F$, $Y \stackrel{d}{=} F$, and $X-Y$ has a symmetric distribution on $\{-x,0,x\}$ with $\mathbb{P}(X-Y=x) = \alpha/2$.*

Proof. Notice that $\alpha = \|F-\delta_x * F\| = \|F-\delta_{-x} * F\|$. Hence, by (A.6) in A.3.2, we have

$$\|F\wedge\delta_x * F\| \;=\; \|F\wedge\delta_{-x} * F\| \;=\; \alpha.$$

Now let (X,η) be a pair of random variables on some probability space $(\Omega,\mathfrak{A},\mathbb{P})$ with distribution defined by

$$\mathbb{P}(X\in A,\eta=i) \;=\; \begin{cases} (F\wedge\delta_x * F)(A)/2, & \text{if } i=1,\\ (F\wedge\delta_{-x} * F)(A)/2, & \text{if } i=-1,\\ F(A)-((F\wedge\delta_x * F)(A)+(F\wedge\delta_{-x} * F)(A))/2, & \text{if } i=0\end{cases}$$

for $A \in \mathscr{B}(\mathbb{R})$. Then $X \stackrel{d}{=} F$ and η is symmetric on $\{-1,0,1\}$ with $\mathbb{P}(\eta = 1) = \alpha/2$ as one can easily verify. Hence, putting $Y := X - \eta x$, we see that $X - Y$ has the asserted distribution and leaves us with a proof of $Y \stackrel{d}{=} F$. To this end, we obtain for any $A \in \mathscr{B}(\mathbb{R})$

$$\begin{aligned}
\mathbb{P}(Y \in A) &= \mathbb{P}(X - x \in A, \eta = 1) + \mathbb{P}(X + x \in A, \eta = -1) + \mathbb{P}(X \in A, \eta = 0) \\
&= \frac{1}{2}\left(\delta_{-x} * (F \wedge \delta_x * F)(A) + \delta_x * (F \wedge \delta_{-x} * F)(A)\right) \\
&\quad + F(A) - \frac{1}{2}((F \wedge \delta_x * F)(A) + (F \wedge \delta_{-x} * F)(A)) \\
&= F(A)
\end{aligned}$$

because $\delta_{\pm x} * (F \wedge \delta_{\pm x} * F) = F_{\pm x} \wedge F$. □

4.4.2 A zero-one law and exact coupling in the spread out case

With the help of a Mineka coupling we are now able to prove the following zero-one law about the total variation distance between two random walks with equal increment distributions but different delays.

Theorem 4.4.2. *Let $(S_n)_{n \geq 0}$ be a RW in a standard model with increment distribution F and, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$\Delta_n(x) := \|\mathbb{P}_x(S_n \in \cdot) - \mathbb{P}_0(S_n \in \cdot)\| = \|\delta_x * F^{*n} - F^{*n}\|.$$

Then exactly one of the following two alternatives holds true for any $x \in \mathbb{R}$:

(i) $\Delta_n(x) = 1$ *for all* $n \in \mathbb{N}$.
(ii) $\Delta_n(x) \to 0$ *as* $n \to \infty$.

Proof. Fix $x \in \mathbb{R} \setminus \{0\}$ and suppose that (i) does not hold, thus $\|F^{*m} - \delta_x * F^{*m}\| < 1$ for some $m \in \mathbb{N}$. By Lemma 4.4.1, we can define iid random vectors (Y_n', Y_n''), $n \geq 1$, such that $Y_1' \stackrel{d}{=} Y_1'' \stackrel{d}{=} S_m$ and the distribution of $Y_1' - Y_1''$ is symmetric on $\{-x, 0, x\}$ with $\mathbb{P}(Y_1' - Y_1'' = x) = (1 - \Delta_m(x))/2$. Hence, putting $(U_n', U_n'') := \sum_{k=1}^n (Y_k', Y_k'')$ for $n \in \mathbb{N}_0$, the sequence $(U_n' - U_n'')_{n \geq 0}$ forms a symmetric SRW on $x\mathbb{Z}$ and is thus recurrent on this set. As a consequence, the stopping time

$$T_x := \inf\{n \geq 0 : U_n' = x + U_n''\}$$

is a.s. finite whence by the coupling inequality [☞ Lemma A.3.8]

$$\|\mathbb{P}(U_n' \in \cdot) - \mathbb{P}(x + U_n'' \in \cdot)\| \leq \mathbb{P}(T_x > n) \to 0 \quad \text{as } n \to \infty.$$

Since $U_n' \stackrel{d}{=} F^{*mn}$ and $x + U_n'' \stackrel{d}{=} \delta_x * F^{*mn}$, this shows $\Delta_{mn}(x) \to 0$ as $n \to \infty$. But $\Delta_n(x)$ is nonincreasing in n by the Contraction lemma A.3.7 and thus converges to 0 as well. □

Turning to the question of exact coupling of two RW's with increment distribution F, the following result is now straightforward. For our convenience, an exact coupling $(S_n', S_n'')_{n\ge 0}$ of two RW's $(S_n')_{n\ge 0}$ and $(S_n'')_{n\ge 0}$ with increment distribution F and $(S_0', S_0'') = (0,x)$ is called *exact* (F,x)*-coupling* hereafter. Here it should be noticed that any such coupling with $(S_0', S_0'') = (x,y)$ for $(x,y) \in \mathbb{R}^2$ may be viewed as an exact $(F, y-x)$-coupling by redefining $(S_0', S_0'') := (0, y-x)$.

Corollary 4.4.3. *In the situation of Thm. 4.4.2, an exact (F,x)-coupling exists if $\Delta_m(x) < 1$ for some $m \in \mathbb{N}$.*

Proof. Given a RW $(S_n)_{n\ge 0}$ with increment distribution F, we have for any $m \ge 1$ that

$$\mathbb{P}((X_{mn+1},\ldots,X_{m(n+1)}) \in \cdot | (S_{mk})_{k\ge 0}) \;=\; Q_m(S_{m(n+1)} - S_{mn}, \cdot),$$

where $Q_m(s,\cdot)$ denotes the conditional distribution of $(X_1,\ldots,X_m)$ given $S_m = s$. Hence, if $x \in \mathbb{R}$ is such that $\Delta_m(x) < 1$ for some $m \in \mathbb{N}$, then $(U_n', x + U_n'')_{n\ge 0}$ as defined in the proof of the previous result provides an exact (F,x)-coupling if $m = 1$. In the case $m > 1$ we must first define $\mathbf{X}_n' := (X_{mn+1}',\ldots,X_{m(n+1)}')$ and $\mathbf{X}_n'' := (X_{mn+1}'',\ldots,X_{m(n+1)}'')$ (possibly on an enlarged probability space) such that

$$\mathbb{P}((\mathbf{X}_n', \mathbf{X}_n'') \in \cdot | (U_k', U_k'')_{k\ge 0}) \;=\; Q(U_{n+1}' - U_n', \cdot) \otimes Q(U_{n+1}'' - U_n'', \cdot)$$

for each $n \in \mathbb{N}_0$. An exact (F,x)-coupling of is then obtained by setting $(S_0', S_0'') := (0,x)$ and

$$(S_n', S_n'') \;:=\; (X_1' + \ldots + X_n', x + X_1'' + \ldots + X_n'') \quad \text{for } n \in \mathbb{N},$$

hence $(S_{mn}', S_{mn}'') = (U_n', x + U_n'')$ for all $n \in \mathbb{N}_0$. □

Finally, we will show that in the spread out case a coupling of the above type exists for every $x \in \mathbb{R}$.

Theorem 4.4.4. *If F is spread out, then only case (ii) in Thm. 4.4.2 occurs, i.e. $\Delta_n(x) \to 0$ for all $x \in \mathbb{R}$. As a consequence, an exact (F,x)-coupling exists for all $x \in \mathbb{R}$.*

Proof. It suffices to verify that, for any $x \in \mathbb{R}$, we have $\|F^{*m} - \delta_x * F^{*m}\| < 1$ for some $m \in \mathbb{N}$. By Lemma 4.3.4, there exists $m \in \mathbb{N}$ and a nontrivial $g \in \mathscr{C}_0(\mathbb{R})$

such that $F^{*m} \geq g\lambda\!\!\lambda_0$. Since every such g is bounded from below by $\alpha \mathbf{1}_{(a,b)}$ for suitable $\alpha \in (0,1]$ and $a < b$, we infer $F^{*m} \geq \alpha \mathbf{1}_{(a,b)}\lambda\!\!\lambda_0$, in other words, F^{*m} contains the uniform distribution on (a,b) as a $\lambda\!\!\lambda_0$-continuous component, that is $F^{*m} \geq \beta\, Unif(a,b)$ for some $\beta \in (0,1]$ (in fact $\beta := \alpha(b-a)$ will do). As a consequence, $F^{*mn} \geq \beta^n Unif(a,b)^{*n}$ for each $n \in \mathbb{N}$, and since $Unif(a,b)^{*n}$ has a positive density on (na,nb) it is clear that $\|Unif(a,b)^{*n} - \delta_x * Unif(a,b)^{*n}\| < 1$ for some $n = n(x) \in \mathbb{N}$ implying

$$\|F^{*mn} - \delta_x * F^{*mn}\| \;\leq\; (1-\beta^n) + \beta^n \|Unif(a,b)^{*n} - \delta_x * Unif(a,b)^{*n}\| \;<\; 1$$

as asserted. □

Remark 4.4.5. Let us point out that the previous result is easily used to show that in the spread out case an exact coupling exists for any initial distribution λ, i.e., we may construct a bivariate RW $(S_n', S_n'')_{n\geq 0}$ such that both components are RW's with a given spread out increment distribution F and $(S_0', S_0'') \stackrel{d}{=} \delta_0 \otimes \lambda$. This is accomplished by a Mineka coupling conditioned upon $(S_0', S_0'') = (0,x)$ for $x \in \mathbb{R}$. In this case the X_n' as well as the X_n'' are still independent of (S_0', S_0''), but the pairs (X_n', X_n'') are not. Further details are left to the reader.

4.4.3 Blackwell's theorem once again: sketch of Lindvall & Rogers' proof

Let $(S_n)_{n\geq 0}$ be a nonarithmetic SRW with finite positive drift μ, increment distribution F and renewal measure $\mathbb{U}$. In order for a proof of (3.1), Lindvall & Rogers confine themselves to an argument that proves a successful ε-coupling (for any $\varepsilon > 0$) with a delayed version $(S_n')_{n\geq 0}$ having the stationary delay distribution F^s and thus renewal measure $\mu^{-1}\lambda\!\!\lambda_0$ on $\mathbb{R}_>$. Their point is to avoid making use of any more advanced result like the Chung-Fuchs theorem or the Hewitt-Savage zero-one law. What they do need is the fact that the support of the completely nonarithmetic $\widehat{F} := \sum_{n\geq 1} 2^{-n} F^{*n}$, given by

$$supp(\widehat{F}) \;:=\; \{x \in \mathbb{R} : \widehat{F}((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$$

contains elements that are arbitarily close, i.e.

$$\forall\, \varepsilon > 0 \quad \exists\, x,y \in \mathrm{supp}(\widehat{F}): \quad |x-y| < \varepsilon. \tag{4.10}$$

But this follows directly from Lemma 3.6.7. Furthermore, by Lemma 3.3.1, the renewal measure $\widehat{\mathbb{U}} := \sum_{n\geq 0} \widehat{F}^{*n}$ is related to $\mathbb{U}$ through the relation $\widehat{\mathbb{U}} = (\delta_0 + \mathbb{U})/2$. Hence it is no loss of generality to assume that F itself satisfies the support condition (4.10).

Turning to the coupling argument, let $(X_n'')_{n\geq 1}$ be an independent copy of $(X_n)_{n\geq 1}$ and S_0' be a further independent random variable with distribution F^s. For any fixed

$\varepsilon > 0$, define

$$X'_n \;:=\; X''_n \mathbf{1}_{\{|X_n - X''_n| \le \varepsilon\}} \;+\; X_n \mathbf{1}_{\{|X_n - X''_n| > \varepsilon\}} \quad \text{for } n \in \mathbb{N},$$

where (4.10) ensures that $\mathbb{P}(X_1 - X'_1 \neq 0) > 0$. This is an Ornstein coupling as introduced in Subsection 3.3.2, however with a small ε instead of a large b. As a consequence, $(S_n - S'_n)_{n \ge 0}$ constitutes a RW with symmetric increments that are further bounded by $\varepsilon > 0$. Any $x \in \mathbb{R}$ that is crossed infinitely often is thus missed at each crossing time by at most ε. Here is a simple argument that shows that $(S_n - S'_n)_{n \ge 0}$ is oscillating and hence crosses every $x \in \mathbb{R}$ infinitely often. By symmetry, we have that its first strictly ascending and descending ladder epochs are identically distributed and thus either both a.s. finite or both defective. Since the last possibility is excluded by the fact that $\limsup_{n\to\infty} |S_n - S'_n| = \infty$ a.s., we arrive at the desired conclusion. Finally noting that now the ε-coupling time

$$T(\varepsilon) \;:=\; \inf\{n \ge 0 : |S_n - S'_n| \le \varepsilon\}$$

has been shown to be a.s. finite we can refer to Subsection 3.3.4 for the remaining argument that proves Blackwell's theorem with the help of this coupling.

Albeit an Ornstein rather than a Mineka coupling, the previous construction is more inspired by the latter one in that the RW $(S_n - S'_n)_{n \ge 0}$ is "almost simple" by having ε-bounded increments.

4.5 The minimal subgroup of a random walk

This section may be skipped upon first reading. Returning to Thm. 4.4.2, a natural question to be addressed hereafter is the following: Given a nontrivial distribution F on $\mathbb{R}$ with lattice-span $d \ge 0$, what can be said about the sets

$$\mathbb{G}(F) \;:=\; \{x \in \mathbb{R} : \Delta_n(x) < 1 \text{ for some } n \in \mathbb{N}\} \;=\; \{x \in \mathbb{R} : \Delta_n(x) \to 0\}$$

and $\mathbb{G}(\widehat{F})$, where as before $\Delta_n(x) = \|\delta_x * F^{*n} - F^{*n}\|$ and $\widehat{F} = \sum_{n\ge 1} 2^{-n} F^{*n}$? Obviously, $\{0\} \subset \mathbb{G}(F) \subset \mathbb{G}(\widehat{F}) \subset \mathbb{G}_d$ and, by what has been shown in 4.4.2, $\mathbb{G}(F) = \mathbb{G}(\widehat{F}) = \mathbb{R}$ in the spread out case. Since all $\widehat{F}^{*n}$ are equivalent as one can easily verify we have $\widehat{\Delta}_n(x) := \|\delta_x * \widehat{F}^{*n} - \widehat{F}^{*n}\| < 1$ for some $n \in \mathbb{N}$ iff $\widehat{\Delta}_1(x) < 1$. Consequently,

$$\mathbb{G}(\widehat{F}) \;=\; \{x \in \mathbb{R} : \widehat{\Delta}_1(x) < 1\}. \tag{4.11}$$

Lemma 4.5.1. $\mathbb{G}(F)$ *is a measurable subgroup of* $\mathbb{R}$ *for any* $F \in \mathscr{P}(\mathbb{R})$.

Proof. We show first that $\mathbb{G}(F)$ is a group for which we must verify that $x, y \in \mathbb{G}(F)$ implies $x - y \in \mathbb{G}(F)$. But this follows immediately from the general fact that the total variation distance is shift-invariant on $\mathscr{P}(\mathbb{R})$, i.e. $\|\lambda-\mu\| = \|\delta_x*\lambda-\delta_x*\mu\|$ for all $x\in\mathbb{R}$. Namely, if $x,y\in\mathbb{G}(F)$, it yields

$$\Delta_n(x-y) \;=\; \|\delta_x*F^{*n}-\delta_y*F^{*n}\| \;\le\; \Delta_n(x)+\Delta_n(y) \;\to\; 0$$

and thus the required $x-y\in\mathbb{G}(F)$.

In order to see that $\mathbb{G}(F)$ is measurable, it suffices to prove measurability of $\mathbb{G}_n(F):=\{x\in\mathbb{R}:\Delta_n(x)<1\}$ for $n\in\mathbb{N}$, for $\mathbb{G}(F)=\bigcup_{n\ge 1}\mathbb{G}_n(F)$. We will do so for $n=1$ only because the general case is treated analogously. Let $\mathscr{B}_n$ be the σ-field generated by the pairwise disjoint dyadic intervals $I_{n,k}:=(k/2^n,(k+1)2^n]$, $k\in\mathbb{Z}$. Then $(\mathscr{B}_n)_{n\ge 0}$ forms a filtration with $\mathscr{B}(\mathbb{R})=\sigma(\mathscr{B}_n:n\ge 0)$. Furthermore, defining the function $f_n:(\mathbb{R}^2,\mathscr{B}(\mathbb{R}^2)\to([0,1],\mathscr{B}([0,1]))$

$$f_n(x,y) \;:=\; \sum_{k\in\mathbb{Z}} \mathbf{1}_{I_{n,k}}(y)\left(\frac{F(I_{n,k})}{F(I_{n,k})+\delta_x*F(I_{n,k})}\right)$$

(with the convention $\frac{0}{0+0}:=0$), it is easily seen that $f_n(x,\cdot)$ is a density of $F|\mathscr{B}_n$, the restriction of F to $\mathscr{B}_n$, with respect to $(F+F_x)|\mathscr{B}_n$. Next observe that $f_n(x,\cdot)$ also equals the conditional expectation of $dF/d(F+F_x)$ with respect to $\mathscr{B}_n$ under $\nu_x:=(F+\delta_x*F)/2$ and thus defines a bounded martingale which converges to

$$f(x,y):=\liminf_{n\to\infty} f_n(x,y)$$

outside the ν_x-null set

$$N_x \;:=\; \left\{\liminf_{n\to\infty} f_n(x,y)<\limsup_{n\to\infty} f_n(x,y)\right\}$$

for each $x\in\mathbb{R}$. By the dominated convergence theorem,

$$\begin{aligned}
F(I_{n,k}) \;&=\; \lim_{m\to\infty}\int_{I_{n,k}\cap N_x^c} f_{m+n}(x,y)\;(2\nu_x)(dy)\\
&=\; \int_{I_{n,k}\cap N_x^c} f(x,y)\;(2\nu_x)(dy) \;=\; \int_{I_{n,k}} f(x,y)\;(2\nu_x)(dy)
\end{aligned}$$

for all $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $x\in\mathbb{R}$ showing that

$$f(x,\cdot) \;=\; \frac{dF}{d(2\nu_x)} \quad\text{and thus}\quad 1-f(x,\cdot) \;=\; \frac{dF_x}{d(2\nu_x)}$$

for all $x\in\mathbb{R}$ because the $I_{n,k}$ generate $\mathscr{B}(\mathbb{R})$ and are closed under intersections. Now observe that $\Delta_1(x)<1$ iff $F(\{x:f(x,\cdot)(1-f(x,\cdot))>0\}>0$. Moreover, $f(x,\cdot)>0$ F-a.s. for all $x\in\mathbb{R}$. Consequently,

$$\mathbb{G}_1(F)=\{x\in\mathbb{R}:F(\{f(x,\cdot)<1\})>0\}, \tag{4.12}$$

which together with the joint measurability of $f(x,y)$ in x and y implies the measurability of $\mathbb{G}_1(F)$. $\square$

The previous measurability argument is from [1] where the groups $\mathbb{G}(F)$ and $\mathbb{G}(\widehat{F})$ have emerged from the attempt to solve a different problem, namely, to define the smallest measurable subgroup $\mathbb{G}^*$ of $\mathbb{R}$ (not necessarily closed) on which a SRW $(S_n)_{n\ge 0}$ with increment distribution F is concentrated in the sense that

(G1) $\mathbb{P}(S_n \in \mathbb{G}^* \text{ for all } n \ge 0) = 1$.
(G2) If $\mathbb{P}(S_n \in \mathbb{G} \text{ for all } n \ge 0) = 1$ for a measurable subgroup $\mathbb{G}$ of $\mathbb{R}$, then $\mathbb{G}^* \subset \mathbb{G}$.

It is immediate from the definition of lattice-type that, if $(S_n)_{n\ge 0}$ has lattice-span $d \ge 0$, then $\overline{\mathbb{G}^*}$, the closure of $\mathbb{G}^*$, equals $\mathbb{G}_d$. However, in general $\mathbb{G}^* \subsetneq \mathbb{G}_d$. Take, for example, a SRW on the rationals such that its increment distribution puts positive probability on every $q \in \mathbb{Q}$. Then $\mathbb{G}^* = \mathbb{Q}$, whereas $\overline{\mathbb{G}^*} = \mathbb{R}$.

Definition 4.5.2. The measurable subgroup $\mathbb{G}^*$ of $\mathbb{R}$ satisfying (G1) and (G2) is called the *minimal subgroup of* $(S_n)_{n\ge 0}$.

Two theorems derived hereafter collect all relevant information about $\mathbb{G} := \mathbb{G}(F)$ and $\widehat{\mathbb{G}} := \mathbb{G}(\widehat{F})$. As usual, let $d(F)$ denote the lattice-span of F, and denote by F° the symmetrization of F, that is the distribution of $X - Y$ if X, Y are two independent random variables with distribution F. We also write F_n for F^{*n}, and $F_{n,x}$ for $\delta_x * F^{*n}$.

Theorem 4.5.3. *Let $(S_n)_{n\ge 0}$ be a nontrivial SRW with increment distribution F and $\widehat{F} = \sum_{n\ge 1} 2^{-n} F^{*n}$. Then the following assertions hold true:*

(a) $\widehat{\mathbb{G}}$ *is the minimal subgroup of* $(S_n)_{n\ge 0}$.
(b) $\widehat{\mathbb{G}} = \{x \in \mathbb{R} : \|F_{m+n} - F_{m,x}\| < 1 \text{ for some } m \in \mathbb{N}, n \in \mathbb{N}_0\}$.
(c) *If* $\mathbb{G} \ne \widehat{\mathbb{G}}$, *then* $\widehat{F}(\widehat{\mathbb{G}}) = 0$ *or, equivalently,* $\mathbb{P}(S_n \in \mathbb{G}) = 0$ *for all* $n \in \mathbb{N}$.

Proof. (a) In order to show that $\widehat{\mathbb{G}} := \mathbb{G}(\widehat{F})$ satisfies (G1) we first show $\widehat{F}(\widehat{\mathbb{G}}) > 0$. Put $\widehat{\nu}_x := (\widehat{F} + \delta_x * \widehat{F})/2$ and infer from (4.11) and (4.12) that

$$\widehat{\mathbb{G}} = \{x \in \mathbb{R} : \widehat{F}(\{\widehat{f}(x,\cdot) < 1\}) > 0\}, \quad \text{where } \widehat{f}(x,\cdot) := \frac{d\widehat{F}}{d(2\widehat{\nu}_x)}.$$

It follows that $\widehat{f}(x,\cdot) \equiv 1$ $\widehat{F}$-a.s. for all $x \in \widehat{\mathbb{G}}^c$.

Next, let X be a random variable with distribution $\widehat{F}$. Then the calculation

$$\int_B \mathbb{E}\left(\frac{1}{\widehat{f}(X,y)}\right)\widehat{F}(dy) \;=\; \iint_B \frac{1}{\widehat{f}(x,y)}\,\widehat{F}(dy)\,\widehat{F}(dx)$$
$$=\; \int \Big(\widehat{F}(B)+\delta_x*\widehat{F}(B)\Big)\,\widehat{F}(dx)$$
$$=\; \widehat{F}(B)+\widehat{F}_2(B) \qquad (B\in\mathscr{B}(\mathbb{R}))$$

with

$$\widehat{F}_2 \;=\; \widehat{F}^{*2} \;=\; \sum_{m,n\ge 1} 2^{-(m+n)}F_{m+n}$$

shows that $\widehat{F}+\widehat{F}_2$ has $\widehat{F}$-density $\mathbb{E}(1/\widehat{f}(X,y))$. If $\widehat{F}(\widehat{\mathbb{G}}^c)=1$, we infer $\widehat{f}(X,\cdot)\equiv 1$ a.s. from above and thereby

$$\widehat{F}(\widehat{\mathbb{G}}^c)+\widehat{F}_2(\widehat{\mathbb{G}}^c) \;=\; \widehat{F}(\widehat{\mathbb{G}}^c) \;=\; 1,$$

i.e. $\widehat{F}_2(\widehat{\mathbb{G}})=1$. But this is impossible because $\widehat{F}_2$ is dominated by $\widehat{F}$. Hence $\widehat{F}(\mathbb{G})>0$ must hold.

Next, suppose that $\widehat{F}(\mathbb{G})<1$ or, equivalently, $\alpha := F(\widehat{\mathbb{G}}^c)>0$. Consider the distribution $F' := \alpha^{-1}F(\cdot\cap\mathbb{G}^c)$ and use the previous part to infer that $\widehat{\mathbb{G}}' := \mathbb{G}(\widehat{F'})$ has positive probability under $\widehat{F'}=\sum_{n\ge 1}2^{-n}F_n'$, that is

$$0 \;<\; \widehat{F'}(\widehat{\mathbb{G}}') \;=\; \widehat{F'}(\widehat{\mathbb{G}}'\cap\widehat{\mathbb{G}}^c)$$

and so $\widehat{\mathbb{G}}'\cap\widehat{\mathbb{G}}^c\neq\emptyset$. Pick any x from this set which must be nonzero because $0\notin\widehat{\mathbb{G}}^c$. It follows $\|\widehat{F'}-\delta_x*\widehat{F'}\|<1$. On the other hand, $F_n\ge\alpha^nF_n'$ for all $n\in\mathbb{N}$ implies

$$\widehat{F} \;=\; \sum_{n\ge 1}2^{-n}F_n \;\ge\; \sum_{n\ge 1}(\alpha/2)^nF_n' \;=:\; \frac{\alpha}{2-\alpha}H$$

with $H\in\mathscr{P}(\mathbb{R})$ being equivalent to $\widehat{F'}$. But the last fact clearly implies $\|H-H_x\|<1$ and therefore

$$\|\widehat{F}-\delta_x*\widehat{F}\| \;\le\; \left(1-\frac{\alpha}{2-\alpha}\right)+\frac{\alpha}{2-\alpha}\,\|H-\delta_x*H\| \;<\; 1,$$

which is impossible for $x\in\widehat{\mathbb{G}}^c$. We conclude $\widehat{F}(\widehat{\mathbb{G}})=1$ and thus (G1) for $\widehat{\mathbb{G}}$.

In order to see that $\widehat{\mathbb{G}}$ also satisfies (G2) we must derive $\mathbb{G}^c\subset\widehat{\mathbb{G}}^c$ for any measurable subgroup $\mathbb{G}$ of $\mathbb{R}$ with $\widehat{F}(\mathbb{G})=1$. But $x\in\mathbb{G}^c$ for any such $\mathbb{G}$ entails $-x+\mathbb{G}\subset\mathbb{G}^c$ and therefore

$$\widehat{F}_x(\mathbb{G}) \;=\; \mathbb{P}(X\in -x+\mathbb{G}) \;\le\; \widehat{F}(\mathbb{G}^c) \;=\; 0.$$

As a consequence, $\|\widehat{F}-\delta_x*\widehat{F}\|=1$, that is $x\in\widehat{\mathbb{G}}^c$ as desired.

(b) Here it suffices to note that

$$\widehat{F}\wedge\delta_x*\widehat{F} \;=\; \left(\sum_{n\ge1}2^{-n}F_n\right)\wedge\left(\sum_{n\ge1}2^{-n}F_{n,x}\right) \;\neq\; 0$$

holds iff $F_m\wedge F_{n,x}\neq 0$ for some $m,n\in\mathbb{N}$.

(c) Suppose $F(\mathbb{G})>0$ and thus $F_n(\mathbb{G})>0$ for all $n\in\mathbb{N}$. Pick any $x\in\widehat{\mathbb{G}}$ and then $n\in\mathbb{N}$, $r\in\mathbb{N}_0$ such that $\|F_{n+r,x}-F_n\|<1$. Since

$$\begin{aligned}\|F_{n+r}-F_n\| \;&\le\; \int\|F_n-F_{n,y}\|\;F_r(dy)\\ &\le\; \int_{\mathbb{G}}\|F_n-F_{n,y}\|\;F_r(dy)\;+\;F_r(\mathbb{G}^c)\;<\;1\end{aligned}$$

we infer with the help of Lemma A.3.4 that

$$\Delta_{2n+r}(x) \;=\; \|F_{2n+r}-F_{2n+r,x}\| \;=\; 1-(1-\|F_{n+r}-F_n\|)(1-\|F_{n+r,x}-F_n\|) \;<\; 1$$

and thus $x\in\mathbb{G}$. Hence $\mathbb{G}=\widehat{\mathbb{G}}$ and $F(\mathbb{G})=1$. ☐

Before providing a characterization of $\mathbb{G}:=\mathbb{G}(F)$, the following auxiliary result is needed about the structure of the sets

$$\mathbb{S}_x \;:=\; \left\{r\in\mathbb{Z}:\inf_{n\ge0}\|F_{n+r}-F_{n,x}\|<1\right\} \;=\; \left\{r\in\mathbb{Z}:\lim_{n\to\infty}\|F_{n+r}-F_{n,x}\|<1\right\}$$

for $x\in\mathbb{G}$ and the corresponding ones $\mathbb{S}_x^{\circ}$ for the symmetrization F° and $x\in\mathbb{G}^{\circ}:=\mathbb{G}(F^{\circ})$. Let us also define $\widehat{\mathbb{G}}^{\circ}:=\mathbb{G}(\widehat{F^{\circ}})$ with $\widehat{F^{\circ}}:=\sum_{n\ge1}2^{-n}F_n^{\circ}$. By the previous theorem, $\widehat{\mathbb{G}}^{\circ}$ forms the minimal subgroup of a SRW $(S_n^{\circ})_{n\ge0}$ with increment distribution F°, i.e., a symmerization of $(S_n)_{n\ge0}$.

Lemma 4.5.4. *The following assertions hold for the previously defined sets* $\mathbb{S}_x$ *and* $\mathbb{S}_x^{\circ}$.

- *(a)* $\mathbb{S}_0$ *is a subgroup of* $\mathbb{Z}$, *thus* $\mathbb{S}_0=s_0\mathbb{Z}$ *for some* $s_0\in\mathbb{N}$.
- *(b)* $r+\mathbb{S}_0\subset\mathbb{S}_x$ *for all* $r\in\mathbb{S}_x$ *and all* $x\in\widehat{\mathbb{G}}$.
- *(c)* $F(\mathbb{G})=1$ *and hence* $\mathbb{G}=\widehat{\mathbb{G}}$ *holds iff* $\mathbb{S}_0=\mathbb{Z}$ *(and thus* $\mathbb{S}_x=\mathbb{Z}$ *for all* $x\in\mathbb{G}$ *by (b)).*
- *(d)* *There exists* $x\in\widehat{\mathbb{G}}$ *such that* $1\in\mathbb{S}_x$.
- *(e)* *If* F *is symmetric, i.e.* $F=F^{-}$, *then* $\mathbb{S}_0=\mathbb{Z}$ *if* $F(\mathbb{G})=1$, *and* $\mathbb{S}_0=2\mathbb{Z}$ *otherwise.*
- *(f)* $\mathbb{S}_x^{\circ}=\mathbb{Z}$ *for all* $x\in\mathbb{G}^{\circ}$ *and hence* $\mathbb{G}^{\circ}=\widehat{\mathbb{G}}^{\circ}$ *(by (c)).*

Proof. (a) We have $\mathbb{S}_0\neq\emptyset$ because $0\in\mathbb{S}_0$. Given any $r,s\in\mathbb{S}_0$, pick $m,n\in\mathbb{N}$ such that $\|F_{m+r}-F_m\|<1$ and $\|F_{n+s}-F_n\|<1$. It then follows from Lemma A.3.4 that

$$\begin{aligned}\|F_{m+n+r}-F_{m+n+s}\| &= \|F_{m+r}*F_n-F_m*F_{n+s}\|\\ &= 1-(1-\|F_{m+r}-F_m\|)(1-\|F_{n+s}-F_n\|) \;<\; 1\end{aligned}$$

This shows $r-s\in\mathbb{S}_0$ and thus the asserted group property of $\mathbb{S}_0$.

(b) Given $x\in\widehat{\mathbb{G}}\backslash\{0\}$, Thm. 4.5.3(b) implies $\mathbb{S}_x\neq\emptyset$. Now, given $r\in\mathbb{S}_x$ and $s\in\mathbb{S}_0$, pick $m,n\in\mathbb{N}$ such that $\|F_{m+r}-F_{m,x}\|<1$ and $\|F_{n+s}-F_n\|<1$. By another appeal to Lemma A.3.4, we obtain

$$\|F_{m+n+r+s}-F_{m+n,x}\| \;=\; 1-(1-\|F_{m+r}-F_{m,x}\|)(1-\|F_{n+s}-F_n\|) \;<\; 1$$

and thus $r+s\in\mathbb{S}_x$ as claimed.

(c) If $F(\mathbb{G})=1$, then $1\in\mathbb{S}_0$ and thus $\mathbb{S}_x=\mathbb{S}_0=\mathbb{Z}$ for all $x\in\widehat{\mathbb{G}}$ (by (b)) follows from

$$\begin{aligned}\lim_{n\to\infty}\|F_{n+1}-F_n\| &= \lim_{n\to\infty}\left\|\int_{\mathbb{G}}(F_{n,x}-F_n)\,F(dx)\right\|\\ &\le \int_{\mathbb{G}}\lim_{n\to\infty}\Delta_n(x)\,F(dx) \;=\; 0.\end{aligned}$$

For the reverse conclusion it suffices to note that $\mathbb{S}_x=\mathbb{Z}$ and thus $0\in\mathbb{S}_x$ for all $x\in\widehat{\mathbb{G}}$ implies $\lim_{n\to\infty}\|F_n-F_{n,x}\|<1$ for all $x\in\widehat{\mathbb{G}}$, that is $\widehat{\mathbb{G}}=\mathbb{G}$.

(d) If $F(\mathbb{G})=1$, the assertion follows directly from part (c). Hence suppose $F(\mathbb{G})=0$ and put

$$\mathbb{G}_{k,r} \;:=\; \{x\in\widehat{\mathbb{G}}:\|F_{k+r,x}-F_k\|<1\}\quad\text{for } k\in\mathbb{N}_0,\ r\in\mathbb{Z},\ k\ge -r.$$

Then $\widehat{\mathbb{G}}=\bigcup_{k,r\ge 0}\mathbb{G}_{k,r}$ (by Thm. 4.5.3(b)), $\mathbb{G}=\bigcup_{k\ge 0}\mathbb{G}_{k,0}$ and $\widehat{F}(\widehat{\mathbb{G}})=1$ imply the existence of $k\in\mathbb{N}$ and $r\in\mathbb{Z}\backslash\{0\}$ such that $F(\mathbb{G}_{k,r})>0$. Since $\|F_{k+r,x}-F_k\|=\|F_{k+r}-F_{k,-x}\|$, we have $-r\in\mathbb{S}_x$. Furthermore,

$$\begin{aligned}\|F_{k+r+1}-F_{k+r}\| &\le \int\|F_{k+r,x}-F_k\|\,F(dx)\\ &\le F(\mathbb{G}_{k,r}^c) + \int_{\mathbb{G}_{k,r}}\|F_{k+r,x}-F_k\|\,F(dx) \;<\; 1\end{aligned}\tag{4.13}$$

gives $r+1\in\mathbb{S}_0$, which in combination with $-r\in\mathbb{S}_x$ and (b) shows $1\in\mathbb{S}_{-x}$ for all $x\in\mathbb{G}_{k,r}$.

(e) By (c) it suffices to consider the case when F is symmetric and $F(\mathbb{G})=0$. Use Lemma A.3.6 (first equality below) and the symmetry of F to obtain

$$\begin{aligned}\|F_{k+r}-F_{k,x}\| &= \|F_{k+r}^- -(F_{k,x})^-\| \;=\; \|F_{k+r}^- -F_{k,-x}^-\|\\ &= \|F_{k+r,x}^- -F_k^-\| \;=\; \|F_{k+r,x}-F_k\|\end{aligned}\tag{4.14}$$

for all $k,r\in\mathbb{N}$ and $x\in\mathbb{R}$. Consequently, if $r\in\mathbb{S}_x$ then $-r\in\mathbb{S}_x=-\mathbb{S}_{-x}$ entailing $\mathbb{S}_x=-\mathbb{S}_x=\mathbb{S}_{-x}$ for all $x\in\mathbb{R}$.

Now choose $k\in\mathbb{N}$, $r\in\mathbb{Z}\backslash\{0\}$ as in the previous part. Then, by another appeal to (4.14), $\|F_{k+r}-F_{k,x}\|=\|F_{k+r,x}-F_k\|=\|F_{k+r}-F_{k,-x}\|$ showing $\mathbb{G}_{k,r}=\mathbb{G}_{k+r,-r}=-\mathbb{G}_{k,r}$. It is therefore no loss of generality to assume $r\ge 1$. By (4.13), we have $r+1\in\mathbb{S}_0$ which together with $r\in\mathbb{S}_x$ implies $1\in\mathbb{S}_x$ for all $x\in\mathbb{G}_{k,r}$ and thus $\mathbb{G}_{k,r}\subset\bigcup_{m\ge 1}\mathbb{G}_{m,1}$, in particular $F(\mathbb{G}_{l,1})>0$ for some l. Another application of (4.13), now with $k=l$, $r=1$, shows $2\in\mathbb{S}_0$ and thus $\mathbb{S}_0=2\mathbb{Z}$, for $\mathbb{S}_0\ne\mathbb{Z}$ by what has been shown in (b).

(f) By (d), we can choose $x\in\widehat{\mathbb{G}}$ such that

$$\|F_{k+1}-F_{k,x}\| \;=\; \|F_{k+1}^{-}-(F_{k,x})^{-}\| \;<\; 1 \quad \text{for some } k\in\mathbb{N}.$$

Observe that $(F_{k,x})^{-}=\delta_{-x}*F_k^{-}=F_{k,-x}^{-}$, so that $F_{k,x}*(F_{k,x})^{-}=F_k^{\circ}$. By a further appeal to Lemma A.3.4, we infer

$$\|F_{k+1}^{\circ}-F_k^{\circ}\| \;=\; \|F_{k+1}*F_{k+1}^{-}-F_{k,x}*(F_{k,x})^{-}\| \;<\; 1$$

which shows $1\in\mathbb{S}_0^{\circ}$, i.e. $\mathbb{S}_0=\mathbb{Z}$ since $\mathbb{S}_0^{\circ}$ forms a group. Now use (c) to finally conclude that $F^{\circ}(\mathbb{G}^{\circ})=1$ and $\mathbb{G}^{\circ}=\widehat{\mathbb{G}}^{\circ}$. □

Furnished by the last part of the previous lemma, we are now able to prove quite easily that $\mathbb{G}=\mathbb{G}(F)$ forms the subgroup of the symmetrization $(S_n^{\circ})_{n\ge 0}$ of $(S_n)_{n\ge 0}$.

Theorem 4.5.5. *Under the assumptions of Thm. 4.5.3,* $\mathbb{G}$ *is the minimal subgroup of the symmetrization* $(S_n^{\circ})_{n\ge 0}$ *with increment distribution* F°*, that is* $\mathbb{G}=\widehat{\mathbb{G}}^{\circ}$.

Proof. We keep the notation from before. Since $\widehat{\mathbb{G}}^{\circ}=\mathbb{G}^{\circ}$ by Lemma 4.5.4(f), it remains to verify that $\mathbb{G}=\mathbb{G}^{\circ}$. But Lemma A.3.4 provides us with

$$\|F_k^{\circ}\wedge F_{k,x}^{\circ}\| \;=\; \|(F_k*F_k^{-})\wedge(F_{k,x}\wedge F_k^{-})\| \;=\; \|F_k\wedge F_{k,x}\|\,\|F_k\| \;=\; \|F_k\wedge F_{k,x}\|$$

for all $x\in\mathbb{R}$ and $k\in\mathbb{N}$, whence $x\in\mathbb{G}^{\circ}$, i.e. $\|F_k^{\circ}\wedge F_{k,x}^{\circ}\|>0$ form some k, holds iff $x\in\mathbb{G}$, i.e. $\|F_k\wedge F_{k,x}\|>0$ for some k. □

Our final result in this section provides a characterization of the lattice-type of a distribution F in terms of $\mathbb{G}$ and $\widehat{\mathbb{G}}$.

Theorem 4.5.6. *Let* $(S_n)_{n\ge 0}$ *be a nontrivial SRW with increment distribution* F *and renewal measure* $\mathbb{U}$.

(a) *If F is nonarithmetic, then the following assertions are equivalent:*
   (a1) *F is spread out.*
   (a2) $\mathbb{G} = \mathbb{R}$.
   (a3) $\widehat{\mathbb{G}} = \mathbb{R}$.

(b) *If F is d-arithmetic for some $d > 0$, then the following assertions are equivalent:*
   (b1) *F is completely d-arithmetic.*
   (b2) $\mathbb{G} = d\mathbb{Z}$.
   (b3) $\widehat{\mathbb{G}} = \mathbb{G}$.

Proof. (a) We must only verify "(a3)⇒(a1)", because "(a1)⇒(a2)" has been shown as Thm. 4.4.4 and "(a2)⇒(a3)" is a trivial consequence of $\mathbb{G} \subset \widehat{\mathbb{G}}$. Let $(\widehat{S}_n)_{n\ge 0}$ be a SRW with increment distribution $\widehat{F}$. Then $\widehat{\mathbb{G}} = \mathbb{R}$ ensures that an exact $(\widehat{F}, x)$-coupling exists for each $x \in \mathbb{R}$ [☞ proof of Thm. 4.4.4]. As has been pointed out in Remark 4.4.5, this further entails the existence of an exact coupling $(\widehat{S}_n', \widehat{S}_n'')_{n\ge 0}$ such that $\widehat{S}_0' = 0$ and $\widehat{S}_0'' \stackrel{d}{=} \lambda$ for any $\lambda \in \mathscr{P}(\mathbb{R})$. Here we choose any λ with a $\lambda\!\!\lambda$-density implying that all $\widehat{S}_n''$ have a $\lambda\!\!\lambda$-continuous distribution. Let $T := \inf\{n \ge 0 : \widehat{S}_n' = \widehat{S}_n''\}$ be the a.s. finite coupling time and $n \in \mathbb{N}$ so large that $\mathbb{P}(T \le n) > 0$. Then

$$\widehat{F}^{*n} \ \ge \ \mathbb{P}(S_n' \in \cdot, T \le n) \ = \ \mathbb{P}(S_n'' \in \cdot, T \le n)$$

shows that $\widehat{F}^{*n}$, and thus $\widehat{F}$ itself by equivalence, has a $\lambda\!\!\lambda$-continuous component. Hence F is spread out.

(b) If F is $d$arithmetic for $d > 0$, then $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are necessarily closed subgroups of $d\mathbb{Z}$. Consequently, any d-arithmetic RW has minimal subgroup $d\mathbb{Z}$. By Lemma 2.1.2, F is completely d-arithmetic iff the symmetrization $(S_n^\circ)_{n\ge 0}$ is d-arithmetic and thus has minimal subgroup $d\mathbb{Z}$. But the latter also equals $\mathbb{G}$ by Thm. 4.5.5, hence (b1) implies (b2) which in turn trivially implies (b3) because $\mathbb{G} \subset \widehat{G}$. But (b3) also implies (b1) because it first implies $\mathbb{G} = \widehat{\mathbb{G}} = d\mathbb{Z}$ as pointed out above and thereby that F° is d-arithmetic. By another appeal to Lemma 2.1.2 we finally conclude (b1). □

4.6 Uniform renewal theorems in the spread out case

Returning to renewal theory, we will now derive stronger versions of Blackwell's theorem and the key renewal theorem for spread out random walks by making use of exact coupling and Stone's decomposition.

Theorem 4.6.1. *Let $(S_n)_{n\ge 0}$ be a spread out RW in a standard model with positive drift μ. Then, for all $a\in\mathbb{R}_>$ and $\lambda\in\mathscr{P}(\mathbb{R})$,*

$$\lim_{t\to\infty}\left\|U_\lambda(\cdot\cap[t,t+a])-\mu^{-1}\lambda\!\!\!\lambda_0(\cdot\cap[t,t+a])\right\| \;=\; 0 \tag{4.15}$$

or, equivalently,

$$\lim_{t\to\infty}\sup_{A\in\mathscr{B}([0,a])}\left|U_\lambda(t+A)-\mu^{-1}\lambda\!\!\!\lambda_0(A)\right| \;=\; 0. \tag{4.16}$$

Proof. In view of the results in Section 4.4, notably Thm. 4.4.4 and Remark 4.4.5, we may assume that $(S_n)_{n\ge 0}$ is given together with a second RW $(S_n')_{n\ge 0}$ having the same increment distribution and such that $(S_n,S_n')_{n\ge 0}$ forms a bivariate RW in a standard model with a.s. finite coupling time $T=\inf\{n\ge 0: S_n=S_n'\}$ under each $\mathbb{P}_{\lambda_1,\lambda_2}$, where $\mathbb{P}_{\lambda_1,\lambda_2}((S_0,S_0')\in\cdot)=\lambda_1\otimes\lambda_2$ for any $\lambda_1,\lambda_2\in\mathscr{P}(\mathbb{R})$. Now choose an arbitrary $\lambda_1=\lambda$ and λ_2 such that $\mathbb{U}_{\lambda_2}^+=\mu^{-1}\lambda\!\!\!\lambda_0^+$ if $\mu<\infty$, and $\mathbb{U}_{\lambda_2}^+\le\varepsilon\lambda\!\!\!\lambda_0^+$ for any given $\varepsilon>0$ if $\mu=\infty$. Since exact coupling occurs at T we infer with the help of a standard coupling argument that for any fixed $a\in\mathbb{R}_>$ and all bounded $A\in\mathscr{B}([0,a])$

$$\left|\mathbb{U}_{\lambda_1}(t+A)-\mathbb{U}_{\lambda_2}(t+A)\right| \;\le\; \mathbb{E}_{\lambda_1}\left(\sum_{n=0}^{T-1}\mathbf{1}_{[t,t+a]}(S_n)\right)+\mathbb{E}_{\lambda_2}\left(\sum_{n=0}^{T-1}\mathbf{1}_{[t,t+a]}(S_n')\right).$$

But the right-hand side converges to 0 because $\sum_{n=0}^{T-1}\mathbf{1}_{[t,t+a]}(S_n)\to 0$ a.s. and the family $\{\sum_{n=0}^{T-1}\mathbf{1}_{[t,t+a]}(S_n): t\in\mathbb{R}\}$ is uniformly integrable under every $\mathbb{P}_\lambda$ [☞ Subsection 3.3.4]. By the choice of λ_2, this yields the asserted result (by additionally letting ε tend to 0 in the case $\mu=\infty$). □

Turning to the key renewal theorem, the following version extends the classical one in two ways. The asserted convergence first is uniform within suitable function classes and second does not require direct Riemann integrability but only $\lambda\!\!\!\lambda_0$-integrability together with boundedness and asymptotic evanescence at $\pm\infty$. The crucial ingredient is Stone's decomposition [☞ Thm. 4.3.5].

Theorem 4.6.2. *Under the same assumptions as in the previous theorem, let $h:\mathbb{R}\to\mathbb{R}_\ge$ be a bounded, $\lambda\!\!\!\lambda_0$-integrable function satisfying $\lim_{|t|\to\infty}h(t)=0$. Then*

$$\lim_{t\to\infty}\sup_{|g|\le h}\left|g*\mathbb{U}_\lambda(t)-\mu^{-1}\int_{\mathbb{R}}g(x)\,\lambda\!\!\!\lambda_0(dx)\right| \;=\; 0,\quad \textit{and} \tag{4.17}$$

$$\lim_{t\to\infty}\sup_{|g|\le h}\left|g*\mathbb{U}_\lambda(t)\right| \;=\; 0 \tag{4.18}$$

for all $\lambda \in \mathscr{P}(\mathbb{R})$.

Proof. We restrict ourselves to the proof of (4.17). Stone's decomposition states that, for any fixed $\lambda \in \mathscr{P}(\mathbb{R})$,

$$\mathbb{U}_\lambda \;=\; u_\lambda \lambda\!\!\lambda_0 \;+\; \mathbb{U}'_\lambda,$$

where $\|U'_\lambda\| < \infty$ and $u_\lambda \in \mathscr{C}_b(\mathbb{R})$ satisfies

$$\lim_{x\to\infty} u_\lambda(x) = \mu^{-1} \quad \text{and} \quad \lim_{x\to-\infty} u_\lambda(x) = 0.$$

It follows that, for any g, h as claimed in the theorem,

$$\begin{aligned}
&\left| g * \mathbb{U}_\lambda(t) - \frac{1}{\mu}\int_{\mathbb{R}} g(x)\, \lambda\!\!\lambda_0(dx) \right| \\
&\qquad \le \int_{\mathbb{R}} |g(x)| \left| u_\lambda(t-x) - \mu^{-1} \right| \, \lambda\!\!\lambda_0(dx) \;+\; |g| * \mathbb{U}'_\lambda(t) \\
&\qquad \le \int_{\mathbb{R}} h(x) \left| u_\lambda(t-x) - \mu^{-1} \right| \, \lambda\!\!\lambda_0(dx) \;+\; h * \mathbb{U}'_\lambda(t).
\end{aligned}$$

But the last two expressions converge to 0 as $t \to \infty$ by the dominated convergence theorem. This is easily seen by using the assumptions about h, the properties of u_λ and the finiteness of $\mathbb{U}'_\lambda$. □

The main assertion (4.17) of Thm. 4.6.2 has been obtained by ARJAS, NUMMELIN & TWEEDIE [2] although an earlier version without uniformity was given much earlier by SMITH [38] (with a correction in [40]) but hardly recognized for a long time. Thm. 4.6.1 is also stated in [2]. For $\lambda\!\!\lambda_0$-continuous F it was first proved by BREIMAN [8].

Let us finally note that in the d-arithmetic case the counterparts of the previous two results are easily seen to be equivalent to the seemingly weaker Blackwell's theorem and the key renewal theorem. We therefore refrain from an explicit statement of these extensions. As for key renewal theorem in the d-arithmetic case, recall from Remark 4.2.2 that the condition of direct Riemann integrability for the considered function g may be replaced with the condition $\sum_{n\in\mathbb{Z}} |g(nd)| < \infty$ which particularly implies the boundedness of g on $d\mathbb{Z}$ as well as $\lim_{|n|\to\infty} g(nd) = 0$. In other words, g must exactly satisfy the conditions of Thm. 4.6.2, but on the lattice $d\mathbb{Z}$.

Chapter 5
The renewal equation

This chapter is devoted to a systematic treatment of the so-called *renewal equation*, an integral equation of Volterra-type we have already encountered in various of the introductory examples in Chapter 1. This is not surprising because almost every renewal quantity may in fact be described as the solution Ψ to a renewal equation of the general form

$$\Psi \;=\; \psi \,+\, \Psi * Q, \tag{5.1}$$

where Q is a given locally finite measure and ψ a given locally bounded function on $\mathbb{R}_\geq$ (standard case) or $\mathbb{R}$ (general case). If $\psi = 0$, then (5.1) is also a well-known object in harmonic analysis where its solutions are called *Q-harmonic functions*. It has been studied in the more general framework of Radon measures on separable locally compact Abelian groups by CHOQUET & DENY [11] and is therefore also known as the *Choquet-Deny equation*. It has already been encountered in Section 3.4 as an ingredient to Feller's analytic proof of Blackwell's renewal theorem [☞ Lemma 3.4.1].

The next section provides a short historical account and a classification of (5.1) with regard to the domain of ψ, Q and the total mass of Q. We then study existence, uniqueness and asymptotic behavior of solutions in the various cases in Sections 5.2–**??**. ???????????

5.1 Classification of renewal equations and historical account

The problem of solving a renewal equation of the form

$$\Psi(x) \;=\; \psi(x) \,+\, \int_0^x \Psi(x-y) f(y)\, dy, \quad x \in \mathbb{R}_\geq \tag{5.2}$$

for suitable given functions $\psi, f : \mathbb{R}_\geq \to \mathbb{R}_\geq$ leads back to the origins of renewal theory. As LOTKA, one of the pioneers in this area, writes in his 1939 paper [33]:

> The analysis of problems of industrial replacement forms part of the more general analysis of problems presented by "self-renewing aggregates"[1] ... Historically, the investigation of an actuarial problem came first. L. HERBELOT [24] examined the number of annual accessions required to maintain a body of N policyholders constant, as members drop out by death. He assumes an initial body of N "charter" members at time $t = 0$, all of the same age, which for simplicity may be called age zero, since this merely amounts to fixing an arbitrary origin of the age scale. He further assumes the same uniform age at entry for each "new" member.

Let $q(s)$ denote the rate per head at which members drop out by death at time s, being then immediately replaced by a new member of the fixed age of entry. Let further $P(t)$ be the probability at the age of entry of surviving t years and $p(t)$ its derivative (thus assumed to exist). Note that $P(0) = 1$. Then the expected number of survivors of charter members at time t equals $NP(t)$, while $N\int_0^t q(s)P(t-s)\,ds$ gives the expected number of survivors at time t of "new" members. Hence, the condition for a constant membership N is

$$Np(t) \;+\; N\int_0^t q(s)P(t-s)\,ds \;=\; N \quad \text{for all } t \in \mathbb{R}_{\geq}$$

and leads to the equation

$$p(t) \;+\; \int_0^t q(s)p(t-s)\,ds \;+\; q(t) \;=\; 0 \quad \text{for all } t \in \mathbb{R}_{\geq}$$

after dividing by N and differentiating with respect to t. But the last equation is clearly equivalent to

$$q(t) \;=\; -p(t) \;-\; \int_0^t q(t-s)p(s)\,ds \quad \text{for all } t \in \mathbb{R}_{\geq}$$

and therefore a renewal equation of type (5.2) with $\Psi = q$ and $\psi = f = -p$. As Lotka further points out in [33]

> ... there is nothing in Herbelot's treatment to limit its application to living organisms. It is directly applicable to the problem of industrial replacement of an equipment comprising N original units installed at time $t = 0$, and maintained constant by the replacement of disused units with new.

His paper also contains a good account of the relevant literature between 1907 and 1939. Beginning with a further quote from this source, we finally give an example of (5.2) that arises in a biological context.

> A population of living organisms, unlike industrial equipment, has practically no beginning. We know its existence only as a continuing process. Accordingly, the equation for its development is most naturally framed without *explicit* reference to any 'charter members".
>
> The basis of the analysis is as follows: In a population growing solely by excess of births over deaths (i.e. in the absence of immigration and emigration), the annual female births $B(t)$ at time t are the daughters of mothers a years old, born at time $t-a$ when the annual female births were $B(t-a)$. If fertility and mortality are constant and such that a

[1] chosen by Lotka as a translation of the German phrase "sich erneuernde Gesamtheiten" used by Swiss actuaries.

fraction $p(a)$ of all births survive to age a, and are then reproducing at an average rate $m(a)$ daugthers per head per annum, then, evidently,

$$B(t) \;=\; \int_0^\infty B(t-a)p(a)m(a)\,da.$$

Lotka then points out that this equation, though not referring explicitly to any initial state, may be written as an equation of type (5.2), namely

$$B(t) \;=\; b(t) \;+\; \int_0^t B(t-a)p(a)m(a)\,da$$

where $b(t) := \int_t^\infty B(t-a)p(a)m(a)\,da$ denotes the number of annual births of daughters at t whose mothers where born before time zero.

Until 1941 a fairly large numbers of articles had dealt with the renewal equation in various applied contexts, but as FELLER [20] wrote in that year:

> Unfortunately most of this literature is of a heuristic nature so that the precise conditions for the validity of different methods or statements are seldom known. This literature is, moreover, abundant in controversies and different conjectures which are sometimes supported or disproved by unnecessarily complicated examples.

In the very same paper, he provided, with the help of Laplace transforms, the first rigorous proof of the existence and uniqueness of a nondecreasing solution of (5.1) in the standard case with nondecreasing ψ. His result is contained in our Theorem 5.2.2.

In order for a systematic analysis of equation (5.1), which in more explicit form reads

$$\Psi(x) \;=\; \psi(x) \;+\; \int_{\mathbb{R}} \Psi(x-y)\,Q(dy), \quad x\in\mathbb{R}, \tag{5.3}$$

we first provide a classification with respect to the given data ψ and Q. The goal is to determine all solutions Ψ within a reasonable class of functions. If Q is concentrated on $\mathbb{R}_\geq$, i.e. $Q(0-) = Q((-\infty,0)) = 0$, and only solutions Ψ vanishing on $\mathbb{R}_<$ are considered, then (5.3) simplifies to

$$\Psi(x) \;=\; \psi(x) \;+\; \int_{[0,x]} \Psi(x-y)\,Q(dy), \quad x\in\mathbb{R}. \tag{5.4}$$

Obviously, such solutions can only exist if ψ vanishes on $\mathbb{R}_<$ as well. We call this situation the *standard case* and (5.4) a *standard renewal equation* because it is the one encountered in most applications. For its analysis to be provided in the next section we may restrict right away to the domain $\mathbb{R}_\geq$.

In the case where Q has lattice-span $d>0$, w.l.o.g. $d=1$, it is further to be noted that (5.3) may also be stated as a system of uncoupled *discrete renewal equations* on $\mathbb{Z}$ (or $\mathbb{N}_0$ in the standard case), viz.

$$\Psi_n^{(a)} \;=\; \psi_n^{(a)} \;+\; \sum_{n\in\mathbb{Z}} \Psi_{n-k}^{(a)} q_k, \quad n\in\mathbb{Z} \tag{5.5}$$

for $0 \le a < d$, where $\Psi_n^{(a)} := \Psi(a+n)$, $\psi_n^{(a)} := \psi(a+n)$ and $q_n := Q(\{n\})$. This equation may then be analyzed separately for each a, the given data being the sequence $(\psi_n^{(a)})_{n\in\mathbb{Z}}$ and the measure Q on $\mathbb{Z}$.

Regarding the total mass of Q, a renewal equation is called *defective* if $\|Q\| < 1$, *proper* if $\|Q\| = 1$, and *excessive* if $\|Q\| > 1$. Equation (5.2) constitutes a special case of a proper standard renewal equation where the convolving probability measure Q on $\mathbb{R}_\ge$ has a $\lambda\!\!\!\lambda_0$-density. ????????????????????

5.2 The standard renewal equation

5.2.1 Preliminaries

Some further notation is needed hereafter and therefore introduced first. Recall that Q is assumed to be locally finite, thus $Q(t) = Q([0,t]) < \infty$ for all $t \in \mathbb{R}_\ge$. We denote its mean value by $\mu(Q)$ and its mgf by ϕ_Q, that is

$$\mu(Q) := \int_{\mathbb{R}_\ge} x\, Q(dx)$$

and

$$\phi_Q(\theta) := \int_{\mathbb{R}_\ge} e^{\theta x}\, Q(dx).$$

The latter function is nondecreasing and convex on its natural domain

$$\mathbb{D}_Q := \{\theta \in \mathbb{R} : \phi_Q(\theta) < \infty\}$$

for which one of the four alternatives

$$\mathbb{D}_Q = \emptyset,\ (-\infty,\theta^*),\ (-\infty,\theta^*],\ \text{or } \mathbb{R}$$

with $\theta^* \in \mathbb{R}$ must hold. If $\mathbb{D}_Q$ has interior points, then ϕ_Q is infinitely often differentiable on $\mathrm{int}(\mathbb{D}_Q)$ with n^{th} derivative given by

$$\phi_Q^{(n)}(\theta) = \int_{\mathbb{R}_\ge} x^n e^{\theta x}\, Q(dx) \quad \text{for all } n \in \mathbb{N}.$$

In the following we will focus on measures Q on $\mathbb{R}_\ge$, called *admissible*, for which $\mu(Q) > 0$, $Q(0) < 1$ and $\mathbb{D}_Q \ne \emptyset$ holds true. Note that the last condition is particularly satisfied if $\|Q\| < \infty$ or, more generally, Q is uniformly locally bounded, i.e.

$$\sup_{t\ge 0} Q([t,t+1]) < \infty.$$

Moreover, ϕ_Q is increasing and strictly convex for such Q. Hence, there exists at most one value $\vartheta \in \mathbb{D}_Q$ such that $\phi_Q(\vartheta) = 1$. It is called the *characteristic exponent of Q* hereafter.

Let $\mathbb{U} := \sum_{n\ge 0} Q^{*n}$ with $Q^{*0} := \delta_0$ be the renewal measure of Q. Put further

$$Q_\theta(dx) \;:=\; e^{\theta x} Q(dx)$$

again a locally finite measure for any $\theta \in \mathbb{R}$, and let $\mathbb{U}_\theta$ be its renewal measure. Then

$$\mathbb{U}_\theta(dx) \;=\; \sum_{n\ge 0} Q_\theta^{*n}(dx) \;=\; \sum_{n\ge 0} e^{\theta x} Q^{*n}(dx) \;=\; e^{\theta x}\,\mathbb{U}(dx). \tag{5.6}$$

Moreover, $\phi_{Q_\theta} = \phi_Q(\cdot + \theta)$ and $\phi_{\mathbb{U}_\theta} = \phi_{\mathbb{U}}(\cdot + \theta)$.

Lemma 5.2.1. *Given an admissible measure Q on $\mathbb{R}_\ge$, the following assertions hold true for any $\theta \in \mathbb{R}$:*

(a) Q_θ^{*n} *is admissible for all* $n \in \mathbb{N}$.
(b) $\mathbb{U}_\theta$ *is locally finite, that is* $\mathbb{U}_\theta(t) < \infty$ *for all* $t \in \mathbb{R}_\ge$.
(c) $\lim_{n\to\infty} Q^{*n}(t) = 0$ *for all* $t \in \mathbb{R}_\ge$.

Proof. Assertion (a) is trivial when noting that $Q_\theta^{*n}(0) = Q^{*n}(0) = Q(0)^n$ for all $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. As for (b), it clearly suffices to show that $\mathbb{U}_\theta$ is locally finite for *some* $\theta \in \mathbb{R}$. To this end note that $\mathbb{D}_Q \ne \emptyset$ implies $\phi_Q(t) \to 0$ as $t \to -\infty$ and thus the existence of $\theta \in \mathbb{R}$ such that $\|Q_\theta\| = \phi_Q(\theta) < 1$. Hence $\mathbb{U}_\theta$ is the renewal measure of the defective probability measure Q_θ and thus finite, for

$$\|\mathbb{U}_\theta\| \;=\; \sum_{n\ge 0} \|Q_\theta^{*n}\| \;=\; \sum_{n\ge 0} \|Q_\theta\|^n \;=\; \frac{1}{1-\phi_Q(\beta)} \;<\; \infty.$$

Finally, the local finiteness of $\mathbb{U} = \mathbb{U}_0$ gives $\mathbb{U}(t) = \sum_{n\ge 0} Q^{*n}(t) < \infty$ for all $t \in \mathbb{R}_\ge$ from which (c) directly follows. □

5.2.2 Existence and uniqueness of a locally bounded solution

We are now ready to prove the fundamental theorem about existence and uniqueness of solutions in the standard case (5.4) under the assumption that the measure Q is regular and the function ψ is *locally bounded* on $\mathbb{R}_\ge$, i.e.

$$\sup_{x\in[0,t]} |\psi(x)| \;<\; \infty \quad \text{for all } t \in \mathbb{R}_\ge.$$

Before stating the result let us note that n-fold iteration of equation (5.4) leads to

$$\Psi(x) \;=\; \sum_{k=0}^{n} \psi * Q^{*k}(x) \;+\; \Psi * Q^{*(n+1)}(x)$$

which in view of part (c) of the previous lemma suggests that $\Psi = \psi * \mathbb{U}$ forms the unique solution of (5.4).

Theorem 5.2.2. *Let Q be an admissible measure on $\mathbb{R}_{\geq}$ and $\psi : \mathbb{R}_{\geq} \to \mathbb{R}$ a locally bounded function. Then there exists a unique locally bounded solution Ψ of the renewal equation* (5.4)*, viz.*

$$\Psi(x) \;=\; \psi * \mathbb{U}(x) \;=\; \int_{[0,x]} \psi(x-y)\, \mathbb{U}(dy), \quad x \in \mathbb{R}_{\geq}$$

where $\mathbb{U}$ denotes the renewal measure of Q. Moreover, Ψ is nondecresing if the same holds true for ψ.

Proof. Since $\mathbb{U}$ is locally finite, the local boundedness of ψ entails the same for the function $\psi * \mathbb{U}$, and the latter function satisfies (5.4) as

$$\psi * \mathbb{U} \;=\; \psi * \delta_0 \;+\; \left(\sum_{n\geq 1} \psi * Q^{*(n-1)}\right) * Q \;=\; \psi \;+\; (\psi * \mathbb{U}) * Q.$$

Moreover, $\psi * \mathbb{U}$ is nondecreasing if ψ has this property.

Turning to uniqueness, suppose we have two locally bounded solutions Ψ_1, Ψ_2 of (5.4). Then its difference Δ, say, satisfies the very same equation with $\psi \equiv 0$, that is $\Delta = \Delta * Q$. By iteration,

$$\Delta \;=\; \Delta * Q^{*n} \quad \text{for all } n \in \mathbb{N}.$$

Since Δ is locally bounded, it follows upon setting $\Delta^*(x) := \sup_{y\in[0,x]} |\Delta(x)|$ and an appeal to Lemma 5.2.1(c) that

$$|\Delta(x)| \;=\; \lim_{n\to\infty} |\Delta * Q^{*n}(x)| \;\leq\; \Delta^*(x) \lim_{n\to\infty} Q^{*n}(x) \;=\; 0 \quad \text{for all } x \in \mathbb{R}_{\geq}$$

which proves $\Psi_1 = \Psi_2$. □

The following version of the Choquet-Deny lemma is a direct consequence of the previous result [☞ also Lemma 3.4.1].

Corollary 5.2.3. *If Q is an admissible measure on $\mathbb{R}_{\geq}$, then $\Psi \equiv 0$ is the only locally bounded solution to the Choquet-Deny equation $\Psi = \Psi * Q$.*

5.2.3 Asymptotics

Continuing with a study of the asymptotic behavior of solutions $\psi * \mathbb{U}$ a distinction of the cases $\|Q\| < 1$, $\mathbb{Q}\| = 1$, and $\|Q\| > 1$ is required. Put $I_d := \{0\}$ if $d = 0$, and $I_d := [0, d)$ if $d > 0$.

We begin with the defective case when $\phi_Q(0) = \|Q\| < 1$ and thus $\mathbb{U}$ is finite with total mass $\|U\| = (1 - \phi_Q(0))^{-1}$.

Theorem 5.2.4. *Given a defective renewal equation of the form* (5.4) *with locally bounded ψ such that $\psi(\infty) := \lim_{x\to\infty} \psi(x) \in [-\infty, \infty]$ exists, the same holds true for $\Psi = \psi * \mathbb{U}$, namely*

$$\Psi(\infty) \ = \ \frac{\psi(\infty)}{1 - \phi_Q(0)}.$$

Proof. If $\psi(\infty) = \infty$, then the local boundedness of ψ implies $\inf_{x\ge 0} \psi(x) > -\infty$. Consequently, by an appeal to Fatou's lemma,

$$\begin{aligned}
\liminf_{x\to\infty} \Psi(x) \ &= \ \liminf_{x\to\infty} \int_{[0,x]} \psi(x-y)\, \mathbb{U}(dy) \\
&\ge \ \int_{\mathbb{R}_\ge} \liminf_{x\to\infty} \mathbf{1}_{[0,x]}(y)\psi(x-y)\, \mathbb{U}(dy) \\
&= \ \psi(\infty)\,\|U\| \ = \ \infty.
\end{aligned}$$

A similar argument shows $\limsup_{x\to\infty} \Psi(x) = -\infty$ if $\psi(\infty) = -\infty$. But if $\psi(\infty)$ is finite then ψ necessarily bounded and we obtain by the dominated convergence theorem that

$$\lim_{x\to\infty} \Psi(x) \ = \ \int_{\mathbb{R}_\ge} \lim_{x\to\infty} \mathbf{1}_{[0,x]}(y)\psi(x-y)\, \mathbb{U}(dy) \ = \ \psi(\infty)\,\|U\| \ = \ \frac{\psi(\infty)}{1-\phi_Q(0)}$$

as claimed. □

Turning to the case where $Q \ne \delta_0$ is a probability distribution on $\mathbb{R}_\ge$ (proper case) a statement about the asymptotic behavior of solutions $\psi * \mathbb{U}$ can be directly deduced with the help of the key renewal theorem 4.2.1 or its refinement Thm. 4.6.2 if Q is spread out.

Theorem 5.2.5. *Given a proper renewal equation of the form* (5.4) *with dRi function ψ, it follows for all $a \in I_d$ that*

$$d\text{-}\lim_{x\to\infty} \Psi(x+a) \;=\; \frac{1}{\mu(Q)} \int_{\mathbb{R}_\geq} \psi(x+a)\, \lambda\!\!\lambda_d(dx), \tag{5.7}$$

where d denotes the lattice-span of Q. If Q is spread out, then (5.7) *persists to hold with* $d=0$ *for all bounded,* $\lambda\!\!\lambda_0$*-integrable* ψ *with* $\lim_{x\to\infty}\psi(x)=0$.

Our further investigations will rely on the subsequent lemma which shows that a renewal equation preserves its structure under an exponential transform $Q(dx)\mapsto Q_\theta(dx)=e^{\theta x}Q(dx)$ for any $\theta\in\mathbb{R}$. Plainly, Q_θ is a probability measure iff θ equals the characteristic exponent of Q. This fact has already been used in our introductory examples on branching [☞ Section 1.4] and collective risk theory [☞ Section 1.5] to be reconsidered in the next section. Given a function ψ on $\mathbb{R}_\geq$, put

$$\psi_\theta(x) \;:=\; e^{\theta x}\psi(x), \quad x\in\mathbb{R}_\geq$$

for any $\theta\in\mathbb{R}$.

Lemma 5.2.6. *Let Q be an admissible measure on* $\mathbb{R}_\geq$*,* $\psi:\mathbb{R}_\geq\to\mathbb{R}$ *a locally bounded function and* Ψ *any solution to the pertinent renewal equation* (5.4)*. Then, for any* $\theta\in\mathbb{R}$*,* Ψ_θ *forms a solution to* (5.4) *for the pair* (ψ_θ,Q_θ)*, i.e.*

$$\Psi_\theta \;=\; \psi_\theta+\Psi_\theta * Q_\theta. \tag{5.8}$$

Moreover, if $\Psi=\psi*\mathbb{U}$*, then* $\Psi_\theta=\psi_\theta*\mathbb{U}_\theta$ *is the unique locally bounded solution to* (5.8)*.*

Proof. For the first assertion, it suffices to note that $\Psi=\psi+\Psi*Q$ obviously implies (5.8), for

$$e^{\theta x}\Psi(x) \;=\; e^{\theta x}\psi(x) \;+\; \int_{[0,x]} e^{\theta(x-y)}\,\Psi(x-y)\,e^{\theta y}\,Q(dy)$$

for all $x\in\mathbb{R}_\geq$. Since Q_θ is admissible for any $\theta\in\mathbb{R}$, the second assertion follows by Thm. 5.2.2. □

With the help of this lemma we are now able to derive the following general result on the asymptotic behavior of $\psi*\mathbb{U}$ for a standard renewal equation of the form (5.4). It covers the excessive as well as the defective case.

Theorem 5.2.7. *Given a renewal equation of the form* (5.4) *with admissible Q with lattice-span d and locally bounded function ψ, the following assertions hold true for its unique locally bounded solution $\Psi = \psi * \mathbb{U}$:*

(a) *If $\theta \in \mathbb{R}$ is such that $\|Q_\theta\| < 1$ and $\psi_\theta(\infty)$ exists, then*

$$\lim_{x\to\infty} e^{\theta x}\Psi(x) \;=\; \frac{\psi_\theta(\infty)}{1-\phi_Q(\theta)} \tag{5.9}$$

(b) *If Q possesses a characteristic exponent ϑ, then*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\Psi(x+a) \;=\; \frac{1}{\mu(Q_\vartheta)}\int_{\mathbb{R}_\geq} e^{\vartheta x}\,\psi(x+a)\,\lambda\!\!\lambda_d(dx) \tag{5.10}$$

for all $a \in I_d$ if ψ_ϑ is dRi. If Q and thus Q_ϑ is spread out, then (5.10) *extends to all ψ such that ψ_ϑ is bounded and $\lambda\!\!\lambda_0$-integrable with $\psi_\vartheta(\infty) = 0$.*

Proof. All assertions are direct consequences of the previous results. □

Remark 5.2.8. If $1 < \|Q\| < \infty$ in the previous theorem, then $\mathbb{D}_Q \supset (-\infty, 0]$ and the continuity ϕ_Q together with $\lim_{\theta\to-\infty}\phi_Q(\theta) = 0$ *always* ensures the existence of $\vartheta < 0$ with $\phi_Q(\vartheta) = \|Q_\vartheta\| = 1$ by the intermediate value theorem. On the other hand, if Q is an infinite admissible measure, then it is possible that $\phi_Q(\theta) < 1$ *for all* $\theta \in \mathbb{D}_Q$.

There is yet another situation uncovered so far where further information on the asymptotic behavior of $\psi * \mathbb{U}$ may be obtained. Suppose that, for some $\theta \in \mathbb{R}$, $\psi_\theta(\infty)$ exists but is nonzero and that Q_θ is defective. Then Thm. 5.2.4 provides us with

$$\Psi_\theta(\infty) \;=\; \lim_{x\to\infty} e^{\theta x}\Psi(x) \;=\; \frac{\psi_\theta(\infty)}{1-\phi_Q(\theta)} \;\neq\; 0$$

which in turn raises the question whether the rate of convergence of $\Psi_\theta(x)$ to $\Psi_\theta(\infty)$ may be studied by finding a renewal equation satisfied by the difference $\Psi_\theta^0 := \Psi_\theta(\infty) - \Psi_\theta$. An answer is given by the next theorem for which $\theta = 0$ is assumed without loss of generality. For $d \in \mathbb{R}_\geq$ and $\theta \in \mathbb{R}$, let us define

$$e(d,\theta) \;:=\; \begin{cases} \theta, & \text{if } d = 0,\\ (e^{\theta d}-1)/d, & \text{if } d > 0. \end{cases} \tag{5.11}$$

which is a continuous function on $\mathbb{R}_\geq \times \mathbb{R}$.

Theorem 5.2.9. *Given a defective renewal equation of the form* (5.4) *with locally bounded* ψ *such that* $\psi(\infty)\neq 0$, *it follows that* $\Psi^0 := \Psi(\infty)-\Psi$ *forms the unique locally bounded solution to the renewal equation* $\Psi^0 = \widehat{\psi}+\Psi^0 * Q$ *with*

$$\widehat{\psi}(x) \ := \ \psi^0(x)+\psi(\infty)\,\frac{Q((x,\infty))}{1-\phi_Q(0)}, \quad x\in\mathbb{R}.$$

Furthermore, if Q *has characteristic exponent* ϑ *(necessarily positive) and lattice-span* d*, then*

$$d\text{-}\lim_{x\to\infty} \Psi^0_\vartheta(x+a) \ = \ \frac{e^{\vartheta a}}{\mu(Q_\vartheta)}\left(\frac{\psi(\infty)}{e(d,\vartheta)}+\int_{\mathbb{R}_\geq} e^{\vartheta y}\,\psi^0(y+a)\ \lambda\!\!\lambda_d(dy)\right) \tag{5.12}$$

for any $a\in I_d$ *provided that either* $\widehat{\psi}_\vartheta$ *is dRi, or* Q *is spread out and* $\widehat{\psi}_\vartheta$ *a bounded,* $\lambda\!\!\lambda_0$*-integrable function.*

Proof. A combination of

$$\Psi(\infty)-\int_{[0,x]}\Psi(\infty)\ Q(dx) \ = \ \Psi(\infty)\,Q((x,\infty)) \ = \ \frac{\psi(\infty)\,Q((x,\infty))}{1-\phi_Q(0)} \ = \ \widehat{\psi}(x)-\psi^0(x)$$

and $\Psi = \psi+\Psi * Q$ shows the asserted renewal equation for Ψ^0. By the previous results, we then infer under the stated conditions on $\widehat{\psi}$ and Q that

$$d\text{-}\lim_{x\to\infty} \Psi^0_\vartheta(x+a) \ = \ \frac{1}{\mu(Q_\vartheta)}\int_{\mathbb{R}_\geq}\widehat{\psi}_\vartheta(y+a)\ \lambda\!\!\lambda_d(dy) \quad \text{for any } a\in[0,d).$$

Hence it remains to verify that the right-hand side equals the right-hand side of (5.12).

Let us first consider the case $d=0$: Using $\phi_Q(\vartheta)=1$, we find that

$$\begin{aligned}\int_{\mathbb{R}_\geq} e^{\vartheta y}(\widehat{\psi}(y)-\psi^0(y))\ \lambda\!\!\lambda_0(dy) \ &= \ \frac{\psi(\infty)}{1-\phi_Q(0)}\int_{\mathbb{R}_\geq} e^{\vartheta y} Q((y,\infty))\ \lambda\!\!\lambda_0(dy)\\ &= \ \frac{\psi(\infty)}{\vartheta(1-\phi_Q(0))}\int_{\mathbb{R}_\geq}(e^{\vartheta y}-1)\ Q(dy) \ = \ \frac{\psi(\infty)}{\vartheta}\end{aligned}$$

which is the desired result.

If $d>0$ and $a\in[0,d)$, use $Q((y+a,\infty)) = Q((y,\infty))$ for any $y\in d\mathbb{Z}$ to see that

$$\int_{d\mathbb{N}_0} e^{\vartheta y}(\widehat{\psi}(y+a)-\psi^0(y+a))\,\lambda\!\!\lambda_d(dy) \;=\; \frac{\psi(\infty)}{1-\phi_Q(0)}\int_{d\mathbb{N}_0} e^{\vartheta y}Q((y,\infty))\,\lambda\!\!\lambda_d(dy)$$

$$= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{n\ge 0}\sum_{k>n} e^{\vartheta nd}Q(\{kd\})$$

$$= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{k\ge 1} Q(\{kd\})\sum_{n=0}^{k-1} e^{\vartheta nd}$$

$$= \frac{d\psi(\infty)}{1-\phi_Q(0)}\sum_{k\ge 1}\frac{e^{\vartheta kd}-1}{e^{\vartheta d}-1}Q(\{kd\})$$

$$= \frac{\psi(\infty)}{(1-\phi_Q(0))e(d,\vartheta)}\sum_{k\ge 0}(e^{\vartheta kd}-1)\,Q(\{kd\}) \;=\; \frac{\psi(\infty)}{e(d,\vartheta)}.$$

The proof is herewith complete. □

It is worthwile to give the following corollary that provides information on the behavior of the renewal function $\mathbb{U}(t)$ pertaining to an admissible measure Q that possesses a characteristic exponent $\vartheta \neq 0$. The proper renewal case $\vartheta = 0$ will be considered more carefully later [☞ Section 5.7].

Corollary 5.2.10. *Let Q be an admissible measure on $\mathbb{R}_\geq$ with lattice-span d and characteristic exponent ϑ. Then its renewal function $\mathbb{U}(x)$ satisfies*

(a) *in the defective case ($\vartheta > 0$):*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\left(\frac{1}{1-\phi_Q(0)} - \mathbb{U}(x)\right) = d\text{-}\lim_{x\to\infty} \mathbb{U}((x,\infty)) \;=\; \frac{1}{\mu(Q_\vartheta)e(d,\vartheta)}.$$

(b) *in the excessive case ($\vartheta < 0$):*

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\,\mathbb{U}(x) \;=\; \frac{1}{\mu(Q_\vartheta)|e(d,\vartheta)|}$$

Proof. Since $\mathbb{U}(x) = I(x) + \mathbb{U} * Q(x)$ for $x \in \mathbb{R}_\geq$ with $I := \mathbf{1}_{[0,\infty)}$, we infer from Thm. 5.2.9 that in the defective case $\mathbb{U}^0(x) = \|U\| - \mathbb{U}((x,\infty))$ satisfies the renewal equation

$$\mathbb{U}^0(x) \;=\; \widehat{I}(x) + \mathbb{U}^0 * Q(x) \quad \text{with} \quad \widehat{I}(x) \;:=\; \|\mathbb{U}\|\,Q((x,\infty)).$$

The function $\widehat{I}_\vartheta$ is dRi by Lemma 4.1.3 because $\widehat{I}$ is nondecreasing on $\mathbb{R}_\geq$ and $\int_0^\infty \vartheta e^{\vartheta y} Q((y,\infty))\,dy = \phi_Q(\vartheta) - \phi_Q(0) < \infty$. Hence we obtain the asserted result by an appeal to (5.12) of Thm. 5.2.9.

In the excessive case, $\vartheta < 0$ implies that $I_\vartheta(x) = e^{\vartheta x}\mathbf{1}_{[0,\infty)}(x)$ is dRi so that, by (5.10) of Thm. 5.2.7(b),

$$d\text{-}\lim_{x\to\infty} e^{\vartheta x}\mathbb{U}(x) \;=\; \frac{1}{\mu(Q_\vartheta)}\int_{\mathbb{R}_\geq} e^{\vartheta x}\,\lambda\!\!\!\lambda_d(dx) \;=\; \frac{1}{\mu(Q_\vartheta)|e(d,\vartheta)|}$$

as claimed. □

Example 5.2.11. For $\alpha > 0$, consider the infinite admissible measure

$$Q^{(\alpha)}(dx) \;:=\; x^{\alpha-1}\lambda\!\!\!\lambda_0^+(dx)$$

with associated renewal measure $\mathbb{U}^{(\alpha)}$ and mgf $\phi^{(\alpha)}$. Then

$$\phi^{(\alpha)}(\theta) \;=\; \int_0^\infty x^{\alpha-1}e^{\theta x}\,dx \;=\; \frac{1}{|\theta|^\alpha}\int_0^\infty x^{\alpha-1}e^{-x}\,dx \;=\; \frac{\Gamma(\alpha)}{|\theta|^\alpha}$$

for all $\theta < 0$, where $\Gamma(x)$ denotes Euler's gamma function. This shows in particular that $Q^{(\alpha)}$ has characteristic exponent $\vartheta_\alpha = -\Gamma(\alpha)^{1/\alpha}$. Using the functional equation $\Gamma(x+1) = x\Gamma(x)$, we further obtain

$$\mu(Q^{(\alpha)}_{\vartheta_\alpha}) \;=\; \int_0^\infty x^\alpha e^{\vartheta_\alpha x}\,dx \;=\; \frac{\Gamma(\alpha+1)}{|\vartheta_\alpha|^{\alpha+1}} \;=\; \frac{\alpha}{\Gamma(\alpha)^{1/\alpha}}.$$

Consequently, Cor. 5.2.10 provides us with

$$\lim_{x\to\infty} e^{\vartheta_\alpha x}\,\mathbb{U}^{(\alpha)}(x) \;=\; \frac{\alpha}{\Gamma(\alpha)^{2/\alpha}}. \tag{5.13}$$

The case $\alpha = 1$ where $Q := Q^{(1)} = \lambda\!\!\!\lambda_0^+$ even allows an explicit computation of $\mathbb{U} = \mathbb{U}^{(1)}$: By induction over n we find that

$$Q^{*n}(x) \;=\; \frac{x^n}{n!} \quad \text{for all } x \geq 0 \text{ and } n \in \mathbb{N}_0.$$

Consequently,

$$\mathbb{U}(x) \;=\; \sum_{n\geq 0}\frac{x^n}{n!} \;=\; e^x \quad \text{for all } x \geq 0 \tag{5.14}$$

or, equivalently,

$$\mathbb{U}(dx) \;=\; \delta_0(dx) \;+\; e^x\lambda\!\!\!\lambda_0^+(dx). \tag{5.15}$$

This may alternatively be derived from the fact that, by (5.6), $\mathbb{U}(dx) = e^x\mathbb{V}(dx)$, where $\mathbb{V}$ denotes the renewal measure of the standard exponential distribution $e^{-x}\lambda\!\!\!\lambda_0^+(dx)$ which, by Thm. 1.2.1, equals $\delta_0 + \lambda\!\!\!\lambda_0^+$.

5.3 The renewal equation on the whole line

Proceeding to the general renewal equation (5.3) of the form

$$\Psi(x) \;=\; \psi(x) \;+\; \int_{\mathbb{R}} \Psi(x-y)\, Q(dy), \quad x\in\mathbb{R}$$

the situation becomes more difficult, not least owing to the unbounded range of integration that necessitates a further restriction with regard to the functions ψ for which a solution exists at all. In the following, we will briefly discuss extensions of the results obtained for standard renewal equations but limit ourselves to the cases where Q is a defective or a proper probability distribution on $\mathbb{R}$.

Beginning again with the defective case we see that the expected solution

$$\psi*\mathbb{U}(x) \;=\; \int_{\mathbb{R}} \psi(x-y)\, \mathbb{U}(dy), \quad x\in\mathbb{R}$$

is still well defined if ψ is bounded (instead of merely locally bounded), for $\mathbb{U}$ is finite with total mass $(1-\|Q\|)^{-1}$. Indeed, under this stronger condition on ψ the following result looks almost the same as in the standard case.

Theorem 5.3.1. *Given a defective distribution Q on $\mathbb{R}$ and a bounded measurable function $\psi:\mathbb{R}\to\mathbb{R}$, the following assertions hold true:*

(a) *There exists a unique bounded solution Ψ to equation* (5.3)*, namely $\Psi := \psi*\mathbb{U}$.*

(b) *For each $\alpha\in\{-\infty,\infty\}$, the existence of $\psi(\alpha)=\lim_{x\to\alpha}\psi(x)$ entails the same for $\Psi(\alpha)=\lim_{x\to\alpha}\Psi(x)$, and*

$$\Psi(\alpha) \;=\; \frac{\psi(\alpha)}{1-\|Q\|}. \tag{5.16}$$

Proof. Essentially the same as for Thm. 5.2.4 in the standard case and therefore left to the reader. ☐

Now let Q be a proper distribution on $\mathbb{R}$ with positive mean and consider first (5.3) with $\psi\equiv 0$. The obvious fact that any constant function $\Psi_a\equiv a$ for $a\in\mathbb{R}$ then constitutes a bounded solution marks a drastic change to the previous case in that uniqueness is lost. Moreover, as $\mathbb{U}$ is now only locally finite the canonical solution candidate $\psi*\mathbb{U}$ is not necessarily well defined even for bounded functions ψ and thus calls for a stronger condition to be imposed on these functions. In view of the key renewal theorem such a condition appears to be the direct Riemann integrability.

Theorem 5.3.2. *Let Q be a distribution on $\mathbb{R}$ with positive mean $\mu=\mu(Q)$ and lattice-span d and let $\psi:\mathbb{R}\to\mathbb{R}$ be a dRi function. Then any $\Psi_a := a+\psi*\mathbb{U}$, $a\in\mathbb{R}$, forms a bounded solution to equation* (5.3)*. Furthermore,*

$$d\text{-}\lim_{x\to\infty} \Psi_0(x+a) = \frac{1}{\mu}\int_{\mathbb{R}} \psi(x+a)\,\lambda\!\!\lambda_d(dx) \quad \textit{for all } a\in[0,d), \tag{5.17}$$

$$\lim_{x\to-\infty} \Psi_0(x) = 0, \tag{5.18}$$

and Ψ_0 *is the unique solution to* (5.3) *that vanishes at* $-\infty$. *If* Q *is spread out, then all previous assertions remain valid for any bounded,* $\lambda\!\!\lambda_0$*-integrable* ψ *with* $\lim_{|x|\to\infty}\psi(x)=0$.

Proof. It suffices to prove the uniqueness assertion for Ψ_0 or, equivalently, that $\Psi\equiv 0$ is the only bounded solution of the Choquet-Deny equation on the line (equation (5.3) with $\psi\equiv 0$) that vanishes at $-\infty$. To this end, let $(S_n)_{n\ge 0}$ be a SRW with increment distribution Q. Then $\Psi=\Psi*Q$ may be restated as $\Psi(x)=\mathbb{E}\Psi(x-S_1)$ for all $x\in\mathbb{R}$ which upon iteration yields

$$\Psi(x) = \mathbb{E}\Psi(x-S_n) \quad \text{for all } x\in\mathbb{R} \text{ and } n\in\mathbb{N}.$$

Since Q has positive mean, we have $S_n\to\infty$ a.s. which in combination with $\Psi(-\infty)=0$ and the boundedness of Ψ implies by an appeal to the dominated convergence theorem that

$$\Psi(x) = \mathbb{E}\left(\lim_{n\to\infty}\Psi(x-S_n)\right) = \Psi(-\infty) = 0 \quad \text{for all } x\in\mathbb{R}$$

as required. □

By what has just been verified we can finally state the following version of the Choquet-Deny lemma [☞ also Lemma 3.4.1 and Cor. 5.2.3].

Corollary 5.3.3. *If* Q *is a distribution on* $\mathbb{R}$ *with positive mean* $\mu=\mu(Q)$, *then any bounded solution* Ψ *to the Choquet-Deny equation* $\Psi=\Psi*Q$ *for which* $\Psi(-\infty)$ *exists must be constant.*

The renewal equation on the whole line has attracted much less interest than its one-sided counterpart. KARLIN [25] in an old paper studied the general case under the assumption that Q is an arithmetic or spread out distribution on $\mathbb{R}$ with $\mu(Q)\neq 0$. He also gave a discussion of an abstract generalization. In his textbook [21], FELLER proves a result for the case when ψ is continuous with compact support [☞ his Thm. VI.10.2] but only for the purpose of giving a proof of Blackwell's theorem in the general case.

5.4 Approximating solutions by iteration schemes

Returning to the standard case, more precisely equation (5.4) for a locally bounded function ψ and a distribution $Q \neq \delta_0$ on $\mathbb{R}_\geq$, this section wants to provide a brief discussion of how the unique locally bounded solution $\Psi = \psi * \mathbb{U}$ may be obtained via appropriate approximation sequences $(\Psi_n)_{n\geq 0}$. A natural approach, also taken by XIE in [48], is to consider iteration schemes of the form

$$\Psi_{n+1}(x) \;=\; \psi(x) \;+\; \int_{[0,x]} \Psi_n(x-y)\,Q(dy), \quad x \in \mathbb{R}_\geq \tag{5.19}$$

for suitable choices of the initialization Ψ_0. He embarks on the following simple observation.

Lemma 5.4.1. *Let $t \in \mathbb{R}_>$, $\Psi = \psi * \mathbb{U}$ and Ψ_0 a locally bounded function satisfying*

$$\Psi_0(x) \;\leq [\geq]\; \Psi(x) \quad \textit{for } x \in [0,t].$$

Then $\Psi_n(x) \leq [\geq] \Psi(x)$ for all $x \in [0,t]$ and $n \in \mathbb{N}$.

Proof. This is shown by induction over n. By assumption the assertion holds true for $n = 0$, and if it is assumed for general n, then it follows for $n+1$ from

$$\Psi_{n+1}(x) \;=\; \int_{[0,x]} \Psi_n(x-y)\,Q(dy) \;\leq [\geq]\; \int_{[0,x]} \Psi(x-y)\,Q(dy) \;=\; \Psi(x)$$

for $x \in [0,t]$ as Ψ forms a solution of (5.4). □

As shown by the next theorem, any iteration scheme (5.19) with locally bounded initialization Ψ_0 exhibits geometric convergence towards the unique locally bounded solution $\Psi = \psi * \mathbb{U}$ on any bounded interval $[0,t]$.

Theorem 5.4.2. *Let $t \in \mathbb{R}_>$, $\Psi = \psi * \mathbb{U}$ and $(\Psi_n)_{n\geq 0}$ be an iteration scheme as in (5.19) with bounded initialization Ψ_0 on $[0,t]$. Then*

$$\|\Psi_n - \Psi\|_{t,\infty} \;\leq\; \|\Psi_0 - \Psi\|_{t,\infty}\, Q^{*k}(t)^m$$

for all $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $mk \leq n < (m+1)k$, where $\|\cdot\|_{t,\infty}$ denotes the supremum norm on $[0,t]$.

Proof. Put $\Delta_n := \Psi_n - \Psi$ for $n \in \mathbb{N}_0$. Note that (5.19) implies

$$\Delta_n(x) \;=\; \int_{[0,x]} \Delta_{n-1}(x-y)\,Q(dy), \quad x \in \mathbb{R}_\geq$$

and thus $\Delta_n = \Delta_k * Q^{*(n-k)}$ for all $0 \le k < n$ as well as $\|\Delta_n\|_{t,\infty} \le \|\Delta_0\|_{t,\infty}$ for all $n \in \mathbb{N}$. Moreover, $Q^{*nk}(t) \le Q^{*k}(t)^n$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. By combining these facts, we obtain

$$\Delta_{mk+r}(x) \;\le\; \int_{[0,x]} |\Delta_r(x-y)|\, Q^{*mk}(dy) \;\le\; \|\Delta_0\|_{t,\infty} Q^{*k}(t)^m$$

for all $x \in [0,t]$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $r \in \{0,...,k-1\}$ which is the asserted result. □

Of course, if $Q(t) < 1$, then the theorem provides us with the geometrically decreasing estimate

$$\|\Psi_n - \Psi\|_{t,\infty} \;\le\; \|\Psi_0 - \Psi\|_{t,\infty} Q(t)^n \quad \text{for all } n \in \mathbb{N}_0 \tag{5.20}$$

which is the result given in [48].

5.5 Déjà vu: two applications revisited

5.6 The renewal density

5.7 A second order approximation for the renewal function

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Chapter 6
Stochastic fixed point equations and implicit renewal theory

Appendix A
Chapter Heading

A.1 Conditional expectations: some useful rules

In this section we collect some useful results on conditional expectations used in this text, in particluar those that might not immediately be seen.

Lemma A.1.1. *Let X,Y be nonnegative random variables and E be an event such that $\mathbb{P}(E\cap\{X>Y\})>0$. Let further F denote the cdf of X. Then*

$$\mathbb{P}(X>Y+x|Y,E\cap\{X>Y\}) \;=\; \frac{\mathbb{P}(X>Y+x|Y,E)}{\mathbb{P}(X>Y|Y,E)}\,\mathbf{1}_{E\cap\{X>Y\}} \quad a.s.$$

for all $x\ge 0$, and the right-hand ratio simplifies to $\frac{1-F(Y+x)}{1-F(Y)}$ if X is independent of Y and E.

Proof. The result follows from

$$\begin{aligned}
\int_{\mathbb{E}\cap\{Y\in B,X>Y\}} &\frac{\mathbb{P}(X>Y+x|Y,E)}{\mathbb{P}(X>Y|Y,E)}\,d\mathbb{P}\\
&= \int_{E\cap\{Y\in B\}} \frac{\mathbf{1}_{\{X>Y\}}\mathbb{P}(X>Y+x|Y,E)}{\mathbb{P}(X>Y|Y,E)}\,d\mathbb{P}\\
&= \int_{E\cap\{Y\in B\}} \frac{\mathbb{P}(X>Y|Y,E)\,\mathbb{P}(X>Y+x|Y,E)}{\mathbb{P}(X>Y|Y,E)}\,d\mathbb{P}\\
&= \int_{E\cap\{Y\in B\}} \mathbb{P}(X>Y+x|Y,E)\,d\mathbb{P}\\
&= \int_{E\cap\{Y\in B,X>Y\}} \mathbf{1}_{\{X>Y+x\}}\,d\mathbb{P}
\end{aligned}$$

for all measurable $B\subset\mathbb{R}_{\ge}$. If X and Y,E are independent, then the asserted simplification follows from

$$\mathbb{P}(X > Y + x|Y = y, E) \;=\; \mathbb{P}(X > x + y|Y = y, E) \;=\; 1 - F(x + y)$$

for all $x, y \geq 0$. □

Lemma A.1.2. *Let $(\mathscr{F}_n)_{n\geq 0}$ be a filtration, τ a $(\mathscr{F}_n)$-time and X a real-valued random variable such that $\mathbb{E}X$ exists. Then*

$$\mathbb{E}(X|\mathscr{F}_\tau) \;=\; \sum_{n\in\mathbb{N}_0\cup\{\infty\}} \mathbf{1}_{\{\tau=n\}}\, \mathbb{E}(X|\mathscr{F}_n) \quad a.s. \tag{A.1}$$

If τ is a.s. finite and $(X_n)_{n\geq 0}$ a sequence of random variables adapted to $(\mathscr{F}_n)_{n\geq 0}$, then particularly (choose $X = g(\mathbf{X}^\tau)$)

$$\mathbb{E}(g(\mathbf{X}^\tau)|\mathscr{F}_\tau) \;=\; \sum_{n\geq 0} \mathbf{1}_{\{\tau=n\}}\, \mathbb{E}(g(\mathbf{X}^n)|\mathscr{F}_n) \quad a.s. \tag{A.2}$$

for any measurable real function g such that $\mathbb{E}g(\mathbf{X}^\tau)$ exists.

As a consequence of the previous lemma we note the following result:

Lemma A.1.3. *Let $(\mathscr{F}_n)_{n\geq 0}$ be a filtration, τ an a.s. finite $(\mathscr{F}_n)$-time and $(S_n)_{n\geq 0}$ an adapted sequence of real valued integrable random variables with $S_0 = 0$ and increments $X_1, X_2, \ldots$ Suppose further that S_τ is integrable. Then*

$$\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) \;=\; \sum_{k>n} \mathbb{E}(X_k \mathbf{1}_{\{\tau\geq k\}}|\mathscr{F}_n) \quad a.s.$$

for all $n \in \mathbb{N}$. In the special case where $\mathbb{E}\tau < \infty$ and $(S_n)_{n\geq 0}$ is a RW with drift zero, this gives $\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) = 0$ a.s. for all $n \in \mathbb{N}$.

Proof. The assumptions ensure that $S_\tau - S_{\tau\wedge n} = \sum_{k>n} X_k \mathbf{1}_{\{\tau\geq k\}}$ is integrable for each $n \in \mathbb{N}$. Now use (A.1) to infer for any $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) \;&=\; \mathbb{E}\left(\sum_{k>n} X_k \mathbf{1}_{\{\tau\geq k\}} \middle| \mathscr{F}_{\tau\wedge n}\right) \\
&=\; \sum_{j=1}^{n} \mathbf{1}_{\{\tau\wedge n=j\}} \sum_{k>n} \mathbb{E}(X_k \mathbf{1}_{\{\tau\geq k\}}|\mathscr{F}_j) \\
&=\; \mathbf{1}_{\{\tau\geq n\}} \sum_{k>n} \mathbb{E}(X_k \mathbf{1}_{\{\tau\geq k\}}|\mathscr{F}_n) \\
&=\; \sum_{k>n} \mathbb{E}(X_k \mathbf{1}_{\{\tau\geq k\}}|\mathscr{F}_n) \quad \text{a.s.}
\end{aligned}$$

as claimed. The result in the special case follows easily from $\{\tau \ge k\} \in \mathscr{F}_{k-1}$ and

$$\mathbb{E}(X_k \mathbf{1}_{\{\tau\ge k\}}|\mathscr{F}_n) \;=\; \mathbb{E}(\mathbb{E}(X_k|\mathscr{F}_{k-1})\mathbf{1}_{\{\tau\ge k\}}|\mathscr{F}_n)) \;=\; 0 \quad \text{a.s.}$$

for all $k > n$. □

A.2 Uniform integrability: a list of criteria

A.3 Convergence of measures

A.3.1 Vague and weak convergence

A sequence $(Q_n)_{n\ge1}$ of locally finite measures on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ *converges vaguely* to a locally finite measure Q ($Q_n \stackrel{v}{\to} Q$) if

$$\lim_{n\to\infty}\int g\, dQ_n \;=\; \int g\, dQ \quad \text{for all } g\in\mathscr{C}_0(\mathbb{R}). \tag{A.3}$$

In the case where $Q, Q_1, ...$ are finite and (A.3) extends to all $g\in\mathscr{C}_b(\mathbb{R})$, the sequence is said to be *weakly convergent* to Q ($Q_n \stackrel{w}{\to} Q$). Clearly, weak convergence implies vague convergence. More precise information is stated in the following proposition.

Proposition A.3.1. *Let $Q, Q_1, ...$ be finite measures on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ such that $Q_n \stackrel{v}{\to} Q$. Then the following assertions are equivalent:*

(a) $Q_n \stackrel{w}{\to} Q$.
(b) $(Q_n)_{n\ge1}$ *is tight, i.e.* $\lim_{a\to\infty}\sup_{n\ge1} Q_n([-a,a]^c) = 0$.
(c) $\lim_{n\to\infty}\|Q_n\| = \|Q\|$.

The next result is tailored to an application in the Fourier analytic proof of Blackwell's renewal theorem in Section 3.5.

Proposition A.3.2. *Let $(Q_n)_{n\ge1}$ be a sequence of locally finite measures on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. Then the following assertions are equivalent.*

(a) $Q_n \stackrel{v}{\to} Q$ *for a locally finite measure* Q.
(b) *There exists a positive continuous function* h ($h\in\mathscr{C}(\mathbb{R})$) *such that* $hQ_n \stackrel{v}{\to} Q'$ *for a locally finite measure* Q'.

Proof. Since $h\mathscr{C}_0(\mathbb{R}) := \{hg : g \in \mathscr{C}_0(\mathbb{R})\} = \mathscr{C}_0(\mathbb{R})$, we have

$$\lim_{n\to\infty} \int g\, dQ_n \;=\; \int g\, dQ \quad \text{for all } g \in \mathscr{C}_0(\mathbb{R})$$

if, and only if,

$$\lim_{n\to\infty} \int gh\, dQ_n \;=\; \int gh\, dQ \quad \text{for all } g \in \mathscr{C}_0(\mathbb{R}).$$

Hence $Q_n \stackrel{v}{\to} Q$ holds iff $hQ_n \stackrel{v}{\to} hQ$ for some $h \in \mathscr{C}(\mathbb{R})$. But $hQ_n \stackrel{v}{\to} Q'$ may always be restated as $hQ_n \stackrel{v}{\to} hQ$ with $Q := (1/h)Q'$, for h is everywhere positive. The continuity of h and thus $1/h$ ensures that Q remains locally finite if this is the case for Q'. □

A.3.2 Total variation distance and exact coupling

Let $\mathscr{M}_\pm = \mathscr{M}_\pm(\mathfrak{X}, \mathscr{A})$ denote the space of finite signed measures on a given a measurable space $(\mathfrak{X}, \mathscr{A})$, i.e. the vector space of all differences $\lambda - \mu$ of finite measures λ, μ on this space. Endowed with the supremum norm $\|\cdot\|$, that is

$$\|\lambda\| \;:=\; \sup_{A\in\mathscr{A}} |\lambda(A)|,$$

$(\mathscr{M}_\pm, \|\cdot\|)$ becomes a complete normed space (Banach space). The induced metric is called *total variation distance*. Clearly, *convergence in total variation* of λ_n to λ means uniform convergence, that is

$$\lim_{n\to\infty} \sup_{A\in\mathscr{A}} |\lambda_n(A) - \lambda(A)| \;=\; 0.$$

By a standard extension argument the latter condition is further equivalent to

$$\lim_{n\to\infty} \sup_{g\in b\mathscr{A}:\|g\|_\infty\le 1} \left| \int g\, d\lambda_n - \int g\, d\lambda \right|, \tag{A.4}$$

where $b\mathscr{A}$ denotes the space of bounded $\mathscr{A}$-measurable functions $g : \mathfrak{X} \to \mathbb{R}$.

Confining to probability distributions λ, μ, we have

$$\|\lambda - \mu\| \;=\; \int_{\{f>g\}} (f-g)\, d\nu \;=\; \int_{\{f<g\}} (g-f)\, d\nu \;=\; \frac{1}{2}\int |f-g|\, d\nu, \tag{A.5}$$

where f, g are the densities of λ, μ with respect to an arbitrary dominating measure ν (e.g. $\lambda + \mu$). In the discrete case, where $\mathfrak{X}$ is countable, this implies

$$\|\lambda - \mu\| \;=\; \frac{1}{2}\sum_{x\in\mathfrak{X}} |\lambda_x - \mu_x\|$$

when choosing f,g as counting densities and setting $\lambda_x = \lambda(\{x\})$. Note further that λ,μ are mutually singular if, and only if, $\|\lambda - \mu\| = 1$ or, equivalently, $\lambda \wedge \mu = 0$, where

$$\lambda \wedge \mu(dx) \;:=\; (f \wedge g)(x)\,\nu(dx).$$

In general, we have by (A.5) that

$$\|\lambda - \mu\| \;=\; \int_{\{f>g\}} (f - f\wedge g)\,d\nu \;=\; \int (f - f\wedge g)\,d\nu \;=\; 1 - \|\lambda \wedge \mu\|. \quad \text{(A.6)}$$

A pair (X,Y) of random variables, defined on a probability space $(\Omega,\mathfrak{A},\mathbb{P})$, is called a *coupling for* λ *and* μ if $X \stackrel{d}{=} \lambda$ and $Y \stackrel{d}{=} \mu$. It satisfies the *coupling inequality*

$$\|\lambda - \mu\| \;\le\; \mathbb{P}(X \ne Y), \quad \text{(A.7)}$$

because, for all $A \in \mathscr{A}$,

$$\begin{aligned} |\lambda(A) - \mu(A)| \;&=\; |\mathbb{P}(X\in A) - \mathbb{P}(Y\in A)| \\ &=\; |\mathbb{P}(X\in A, X\ne Y) - \mathbb{P}(Y\in A, X\ne Y)| \;\le\; \mathbb{P}(X\ne Y). \end{aligned}$$

The following lemma shows that there is always a coupling that provides equality in (A.7).

Lemma A.3.3. (Maximal coupling lemma) *Given two distributions λ,μ on a measurable space $(\mathfrak{X},\mathscr{A})$ there exist random variables X,Y on a common probability space $(\Omega,\mathfrak{A},\mathbb{P})$ such that $X \stackrel{d}{=} \lambda$, $Y \stackrel{d}{=} \mu$ and*

$$\|\lambda - \mu\| \;=\; \mathbb{P}(X \ne Y).$$

The pair (X,Y) is called a ***maximal coupling of*** λ ***and*** μ.

Proof. Put $\alpha := \|\lambda \wedge \mu\|$, $\varphi := \alpha^{-1}(\lambda\wedge\mu) \in \mathscr{P}(\mathfrak{X})$ and observe that

$$\lambda \;=\; \alpha\varphi + (1-\alpha)\lambda' \quad \text{and} \quad \mu \;=\; \alpha\varphi + (1-\alpha)\mu'$$

with obviously defined $\lambda',\mu' \in \mathscr{P}(\mathfrak{X})$. Now let η, X', Y' and Z be independent random variables on a common probability space $(\Omega,\mathfrak{A},\mathbb{P})$ such that $X' \stackrel{d}{=} \lambda'$, $Y' \stackrel{d}{=} \mu'$, $Z \stackrel{d}{=} \varphi$ and $\mathbb{P}(\eta = 1) = 1 - \mathbb{P}(\eta = 0) = \alpha$. Defining

$$X \;:=\; \begin{cases} Z, & \text{if } \eta = 1, \\ X', & \text{if } \eta = 0 \end{cases} \quad \text{and} \quad Y \;:=\; \begin{cases} Z, & \text{if } \eta = 1, \\ Y', & \text{if } \eta = 0 \end{cases}$$

it is easily seen that $X \stackrel{d}{=} \lambda$, $Y \stackrel{d}{=} \mu$, and

$$\mathbb{P}(X \neq Y) \;=\; \mathbb{P}(\eta = 0) \;=\; 1 - \alpha \;=\; 1 - \|\lambda \wedge \mu\|$$

which, by (A.6), proves the assertion. □

In connection with the convolution of distributions the following result is an immediate consequence of the maximal coupling lemma.

Lemma A.3.4. *For any* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathscr{P}(\mathbb{R})$,

$$(\lambda_1 \otimes \lambda_2) \wedge (\mu_1 \otimes \mu_2) \;=\; (\lambda_1 \wedge \lambda_2) \otimes (\mu_1 \wedge \mu_2) \tag{A.8}$$

and therefore

$$\|(\lambda_1 \otimes \lambda_2) \wedge (\mu_1 \otimes \mu_2)\| \;=\; \|\lambda_1 \wedge \lambda_2\| \, \|\mu_1 \wedge \mu_2\|. \tag{A.9}$$

Furthermore,

$$\|(\lambda_1 * \lambda_2) \wedge (\mu_1 * \mu_2)\| \;=\; \|\lambda_1 \wedge \mu_1\| \, \|\lambda_2 \wedge \mu_2\| \tag{A.10}$$

or, equivalently,

$$\|\lambda_1 * \lambda_2 - \mu_1 * \mu_2\| \;=\; 1 - (1 - \|\lambda_1 - \mu_1\|)(1 - \|\lambda_2 - \mu_2\|). \tag{A.11}$$

Proof. Let f_1, f_2, g_1, g_2 be the densities of $\lambda_1, \lambda_2, \mu_1$ and μ_2, respectively, with respect to some $\nu \in \mathscr{P}(\mathbb{R})$. Then (A.8) follows from

$$\begin{aligned}(\lambda_1 \otimes \lambda_2) \wedge (\mu_1 \otimes \mu_2)(dx, dy) \;&=\; (f_1(x) g_1(y)) \wedge (f_2(x) g_2(y)) \; \nu^2(dx, dy) \\ &=\; (f_1(x) \wedge f_2(x)) \cdot (g_1(y) \wedge g_2(y)) \; \nu^2(dx, dy) \\ &=\; (\lambda_1 \wedge \lambda_2) \otimes (\mu_1 \wedge \mu_2)(dx, dy).\end{aligned}$$

while all subsequent assertions are straightforward consequences. We therefore omit further details. □

Remark A.3.5. The maximal coupling lemma shows that we can define independent pairs (X_1, Y_1) and (X_2, Y_2) on a common probability space which are maximal couplings of (λ_1, μ_1) and (λ_2, μ_2), respectively. Then (A.8) together with $(X_1, X_2) \stackrel{d}{=} \lambda_1 \otimes \lambda_2$ and $(Y_1, Y_2) \stackrel{d}{=} \mu_1 \otimes \mu_2$ confirms the intuitively obvious fact that the coupling $((X_1, X_2), (Y_1, Y_2))$ of $(\lambda_1 \otimes \lambda_2, \mu_1 \otimes \mu_2)$ is maximal.

As a another useful property of the total variation distance we point put its invariance under bimeasurable transformations, a case of particular interest for us being the reflection $x \mapsto -x$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Lemma A.3.6. *Let λ, μ be two probability measures on a measurable space $(\mathfrak{X}, \mathscr{A})$ and $T : \mathfrak{X} \to \mathfrak{X}$ a bijective map such that T and T^{-1} are both measurable. Then*

$$\|\lambda(T \in \cdot) - \mu(T \in \cdot)\| \;=\; \|\lambda - \mu\|.$$

In particular, if $\lambda, \mu \in \mathscr{P}(\mathbb{R})$ and $\lambda^-(B) := \lambda(-B)$ for $B \in \mathscr{B}(\mathbb{R})$, then

$$\|\lambda^- - \mu^-\| \;=\; \|\lambda - \mu\|.$$

An important fact about the total variation distance in connection with Markov processes is the following contraction property, formulated in terms of a Markov pair (X,Y) in a standard model $(\Omega, \mathfrak{A}, (\mathbb{P}_\lambda)_{\lambda \in \mathscr{P}(\mathfrak{X})}, (X,Y))$. This means that X, Y are defined on $(\Omega, \mathfrak{A})$, take values in $(\mathfrak{X}, \mathscr{A})$ and, for a transition kernel P, satisfy

$$\mathbb{P}_\lambda(X \in A, Y \in B) = \int_A P(x,B)\,\lambda(dx) \quad \text{for all } A, B \in \mathscr{A} \text{ and } \lambda \in \mathscr{P}(\mathfrak{X}).$$

Of course, if $(X_t)_{t \ge 0}$ is a Markov process (in discrete or continuous time), then any (X_s, X_t) with $s < t$ forms a Markov pair.

Lemma A.3.7. (Contraction lemma) *Let (X,Y) be a Markov pair in a standard model with transition kernel P. Then*

$$\|\mathbb{P}_\lambda(Y \in \cdot) - \mathbb{P}_\mu(Y \in \cdot)\| \;\le\; \|\mathbb{P}_\lambda(X \in \cdot) - \mathbb{P}_\mu(X \in \cdot)\| \;=\; \|\lambda - \mu\|$$

for all $\lambda, \mu \in \mathscr{P}(\mathfrak{X})$.

Proof. Since, for any $g \in b\mathscr{A}$, we have that $Pg(x) := \int g(y)P(x,dy) \in b\mathscr{A}$ and $Pg\|_\infty \le \|g\|_\infty$, we infer

$$\begin{aligned}
\|\mathbb{P}_\lambda(Y \in \cdot) - \mathbb{P}_\mu(Y \in \cdot)\| \;&=\; \sup_{g \in b\mathscr{A}:\|g\|_\infty \le 1} |\mathbb{E}_\lambda g(Y) - \mathbb{E}_\mu g(Y)| \qquad \text{(by (A.4))}\\
&=\; \sup_{g \in b\mathscr{A}:\|g\|_\infty \le 1} |\mathbb{E}_\lambda Pg(X) - \mathbb{E}_\mu Pg(X)|\\
&\le\; \sup_{h \in b\mathscr{A}:\|h\|_\infty \le 1} |\mathbb{E}_\lambda h(X) - \mathbb{E}_\mu h(X)|\\
&=\; \|\mathbb{P}_\lambda(X \in \cdot) - \mathbb{P}_\mu(X \in \cdot)\|
\end{aligned}$$

and thus the asserted inequality. □

The following coupling inequality is tailored to the study of couplings of temporally homogeneous Markov chains and thus particularly RW's with the same transition kernel but arbitrary initial distributions. Given a transition kernel P on a measurable space $(\mathfrak{X}, \mathscr{A})$, a bivariate sequence $(X_n, Y_n)_{n \ge 0}$ with natural filtration $(\mathscr{F}_n)_{n \ge 0}$

is called a *Markov coupling (for P)* if both, $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$, are Markov chains w.r.t. $(\mathscr{F}_n)_{n\geq 0}$ with transition kernel P.

Lemma A.3.8. (Coupling inequality) *Let $(X_n, Y_n)_{n\geq 0}$ be a Markov coupling with associated coupling time $T := \inf\{n \geq 0 : X_n = Y_n\}$. For $n \in \mathbb{N}_0$, put $\lambda_n := \mathbb{P}(X_n \in \cdot)$ and $\mu_n := \mathbb{P}(Y_n \in \cdot)$. Then*

$$\|\lambda_n - \mu_n\| \;\leq\; \mathbb{P}(T > n) \quad \text{for all } n \in \mathbb{N}_0.$$

Proof. Defining the coupling process

$$Z_n \;:=\; X_n \mathbf{1}_{\{T>n\}} + Y_n \mathbf{1}_{\{T\leq n\}} \quad \text{for } n \in \mathbb{N}_0,$$

it follows with the help of the strong Markov property that $(Z_n)_{n\geq 0}$ is a copy of $(X_n)_{n\geq 0}$, and it coincides with the Y-process after time T. Now

$$|\mathbb{P}(Z_n \in A) - \mathbb{P}(Y_n \in A)| \;=\; |\mathbb{P}(X_n \in A, T > n) - \mathbb{P}(Y_n \in A, T > n)| \;\leq\; \mathbb{P}(T > n)$$

for each measurable A and $n \in \mathbb{N}_0$ yields the assertion. □

A.3.3 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.1.

For multiline equations we recommend to use the `eqnarray` environment.

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$
$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \tag{A.12}$$

A.3.3.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.3.3.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

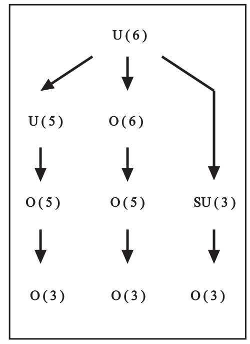

Fig. A.1 Please write your figure caption here

Table A.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

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Solutions

Problems of Chapter 1

?? The solution is revealed here.

?? Problem Heading
(a) The solution of first part is revealed here.
(b) The solution of second part is revealed here.

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