A N G E W A N D T E   M A T H E M A T I K

U N D

I N F O R M A T I K

The Best Constant in the Topchii-Vatutin Inequality for Martingales

G. Alsmeyer und U. Rösler

Universität Münster und Universität Kiel
e-mail: gerolda@math.uni-muenster.de, roesler@math.uni-kiel.de

Bericht Nr. 11/02-S
(Last Update: 30. September 2003)

U N I V E R S I T Ä T   M Ü N S T E R

The Best Constant in the Topchii-Vatutin Inequality for Martingales

Gerold Alsmeyer

Institut für Mathematische Statistik
Fachbereich Mathematik
Westfälische Wilhelms-Universität Münster
Einsteinstraße 62
D-48149 Münster, Germany

Uwe Rösler

Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Straße 4
D-24098 Kiel, Germany

Consider the class of even convex functions $\phi : I\!R \to [0,\infty)$ with $\phi(0) = 0$ and concave derivative on $(0,\infty)$. Given any ϕ-integrable martingale $(M_n)_{n\geq 0}$ with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \geq 1$, the Topchii-Vatutin inequality [10] asserts that

$$E\phi(M_n) - E\phi(M_0) \;\leq\; C \sum_{k=1}^{n} E\phi(D_k)$$

with $C = 4$. It is proved here that the best constant in this inequality is $C = 2$ for general ϕ-integrable martingales $(M_n)_{n\geq 0}$, and $C = 1$ if $(M_n)_{n\geq 0}$ is further nonnegative or having symmetric conditional increment distributions.

AMS 2000 subject classifications. 60G42, 60E15.

Keywords and phrases. Martingale, Topchii-Vatutin inequality, convex function, Choquet representation.

1. Introduction and Result

Let $(M_n)_{n\ge 0}$ be a martingale with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \ge 1$, and associated absolute maxima $M_n^* \stackrel{\text{def}}{=} \max_{0\le k\le n} |M_k|$, $n \ge 0$. Let further $\mathcal{G}_0$ be the class of even convex functions $\phi : I\!\!R \to [0,\infty)$ with $\phi(0) = 0$ and $\mathcal{G}_1$ its subclass of $\phi \in \mathcal{G}_0$ with a concave derivative on $(0,\infty)$. Note that the latter class comprises the functions $\phi(x) = |x|^p$ for $p \in [1,2]$ as well as $\phi(x) = (|x| + a)^p \log^r(|x| + a) - a^p \log^r a$ for $p \in [1,2)$, $r > 0$ and $a > 0$ sufficiently large. The following convex function inequality is due to Topchii and Vatutin [10]: There exists a finite positive constant C such that for all $\phi \in \mathcal{G}_1$, all martingales $(M_n)_{n\ge 0}$ and all $n \ge 1$

$$E\phi(M_n) - E\phi(M_0) \;\le\; C\sum_{k=1}^{n} E\phi(D_k). \tag{1.1}$$

More precisely, they showed (1.1) be true with $C = 4$ and $M_0 = 0$. If $\phi(x) = |x|$ or $\phi(x) = x^2$, then it is well-known that (1.1) holds true with $C = 1$ and that this value cannot be improved. We shall prove in this note that the best constant for general $\phi \in \mathcal{G}_1$ and general ϕ-integrable martingales is $C = 2$, but that $C = 1$ is optimal when imposing certain additional restrictions on the class of considered martingales. The result is stated as the following theorem.

Theorem 1. *If $0 \not\equiv \phi \in \mathcal{G}_1$ and $M = (M_k)_{0\le k\le n}$ is a ϕ-integrable martingale, then*

$$E\phi(M_n) - E\phi(M_0) \;<\; 2\sum_{k=1}^{n} E\phi(D_k). \tag{1.2}$$

The constant 2 is sharp in the sense that, for each $\varepsilon \in (0,1)$, there exists a bounded martingale M and some $\phi \in \mathcal{G}_1$ such that

$$E\phi(M_n) - E\phi(M_0) \;\ge\; (2-\varepsilon)\sum_{k=1}^{n} E\phi(D_k). \tag{1.3}$$

If M is nonnegative or having symmetric conditional increment distributions, then inequality (1.1) holds true with $C = 1$.

An analogue of (1.1) for the maximum M_n^* can be quite easily inferred from the following Burkholder-Davis-Gundy inequality (see e.g. [2, Thm. 1 on p. 425]): Let $\nu > 0$ and $\mathcal{G}_0^{(\nu)}$ be the class of all $\phi \in \mathcal{G}_0$ satisfying $\phi(2x) \le \nu\phi(x)$ for all x. Then there exists a constant $C_\nu^* \in (0,\infty)$ such that for all $\phi \in \mathcal{G}_0^{(\nu)}$ and all martingales $(M_n)_{n\ge 0}$ having $M_0 = 0$

$$E\phi(M_n^*) \;\le\; C_\nu^* E\phi\left(\left(\sum_{k=1}^{n} D_k^2\right)^{1/2}\right). \tag{1.4}$$

This inequality applies to class $\mathcal{G}_1$ because $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$ as will be shown in Lemma 2 at the end of Section 2. Defining $\psi(t) \stackrel{\text{def}}{=} \phi(t^{1/2})$, the same lemma will further show that ψ is concave and subadditive on $[0,\infty)$, that is $\psi(\sum_{k=1}^n x_k) \le \sum_{k=1}^n \psi(x_k)$ for all $x_1,...,x_n \ge 0$ and $n \in I\!N$. Utilizing this last fact on the right hand side in (1.4), we obtain

$$E\phi(M_n^*) \ \le\ C_4^* \sum_{k=1}^n E\phi(D_n). \tag{1.5}$$

Let us finally mention that sharp inequalities similar to those considered here were derived in a recent paper by de la Peña, et al. [3] for infinite degree order statistics.

2. Proof of Theorem 1

The proof of Theorem 1 and in particular of the sharpness of the constant $C = 2$ in (1.1) are heavily based on several reductions, the main one being that it suffices to consider only certain extremal elements $\phi \in \mathcal{G}_1$. This was also used in [1] and [9] for the study of odd functional moments of positive random variables with a decreasing density. The general background is that the class of increasing convex (or concave) functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ as well as many important subclasses like $\mathcal{G}_1$ form a convex cone for which Choquet theory tells us that each element ϕ can be written as an integral of its extremal elements with respect to some measure on $[0,\infty]$ (depending on ϕ). For the given classes these integral representations are obtained by simple partial integration. The following lemma provides the result for the class $\mathcal{G}_1$ and exemplifies the general procedure.

Lemma 1. *For each $\phi \in \mathcal{G}_1$, there exists a unique finite measure Q_ϕ on $[0,\infty]$ such that*

$$\phi(x) \ =\ \int_{[0,\infty]} \phi_t(x)\ Q_\phi(dt), \quad x \ge 0, \tag{2.1}$$

where $\phi_0(x) = |x|$, $\phi_\infty(x) = x^2$, and

$$\phi_t(x) \ \stackrel{\text{def}}{=}\ \begin{cases} x^2, & \text{if } |x| \le t \\ 2xt - t^2, & \text{if } |x| > t \end{cases} \tag{2.2}$$

for $t \in (0,\infty)$.

Note that the functions ϕ_t also arise in problems of robust estimation and are known in statistics as Huber function or Huber's ρ-functions, see e.g. [5], [6].

Proof. Each $\phi \in \mathcal{G}_1$ has a concave derivative ϕ' with $\phi'_+(0) \stackrel{\text{def}}{=} \lim_{x\to+0}\phi'(x) \ge 0$ and thus also a nonincreasing second right derivative ϕ''_+ with asymptotic value $\phi''_+(\infty) \stackrel{\text{def}}{=} \lim_{x\to\infty}\phi''_+(x) \ge 0$. Therefore $\Lambda_{\phi'}((x,\infty)) \stackrel{\text{def}}{=} \phi''_+(x) - \phi''_+(\infty)$ for $x \ge 0$ defines a measure on $(0,\infty)$. Put

$$\mathcal{G}_1^* \stackrel{\text{def}}{=} \{\phi \in \mathcal{G}_1 : \phi'_+(0) = 0, \phi''_+(\infty) = 0\}.$$

and $\phi^*(x) \stackrel{\text{def}}{=} \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ which is an element of $\mathcal{G}_1^*$. Partial integration now gives for $x > 0$

$$\begin{aligned}\phi'(x) - \phi'_+(0) - \phi''_+(\infty)x &= \int_0^x \big(\phi''_+(y) - \phi''_+(\infty)\big)\, dy \\ &= \int_0^x \int_{(y,\infty)} \Lambda_{\phi'}(dt)\, dy \\ &= \int_{(0,\infty)} (x \wedge t)\, \Lambda_{\phi'}(dt)\end{aligned}$$

and then also

$$\begin{aligned}\phi^*(x) &= \int_0^x \big(\phi'(y) - \phi'_+(0) - \phi''_+(\infty)y\big)\, dy \\ &= \int_{(0,\infty)} \int_0^x (y \wedge t)\, dy\, \Lambda_{\phi'}(dt) \\ &= \int_{(0,\infty)} \phi_t(x)\, Q_{\phi^*}(dt),\end{aligned}$$

where $Q_{\phi^*} \stackrel{\text{def}}{=} \Lambda_{\phi'}/2$. We conclude (2.1) with $Q_\phi \stackrel{\text{def}}{=} \phi'_+(0)\delta_0 + \frac{1}{2}\phi''_+(\infty)\delta_\infty + Q_{\phi^*}$. $\diamondsuit$

Proof of Theorem 1. The following reduction arguments will show that it suffices to prove

$$E\phi_1(s + D) \;\leq\; \phi_1(s) + C\, E\phi_1(D) \tag{2.3}$$

for all $s \geq 0$ and all centered random variables D having a two point distribution, where $C = 2$ in the general case, while $C = 1$ if D is symmetric or $s + D \geq 0$. Of course, ϕ_1 is the function defined by (2.2). Note that in terms of the martingales under consideration the former means nothing but a reduction to martingales of the form $(M_0, M_1) = (s, s + D)$.

First reduction. As noted above, for each $\phi \in \mathcal{G}_1$ the even function $\phi^*(x) = \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ is an element of $\mathcal{G}_1^*$. Since

$$\begin{aligned}E\phi(M_n) &= E\phi^*(M_n) + \phi'_+(0)\, E|M_n| + \frac{\phi''_+(\infty)}{2}\, EM_n^2 \\ &\leq E\phi^*(M_n) + \phi'_+(0)\left(E|M_0| + \sum_{k=1}^n E|D_k|\right) + \frac{\phi''_+(\infty)}{2}\left(EM_0^2 + \sum_{k=1}^n ED_k^2\right)\end{aligned}$$

it suffices to prove Theorem 1 for functions $\phi \in \mathcal{G}_1^*$.

Second Reduction. Using (2.2), $\phi_t(x) = t^2\phi_1(x/t)$ for all $t \in (0,\infty)$ and $Q_\phi(\{0,\infty\}) = 0$ if $\phi'_+(0) = 0$ and $\phi''_+(\infty) = 0$ (see at the end of the proof of Lemma 1), we infer for each $\phi \in \mathcal{G}_1^*$

$$E\phi(M_n) \;=\; \int_{(0,\infty)} E\phi_t(M_n)\, Q_\phi(dt) \;=\; \int_{(0,\infty)} t^2 E\phi_1(M_n/t)\, Q_\phi(dt).$$

Since $(M_k/t)_{0\leq k\leq n}$ is still a martingale, it suffices to prove Theorem 1 with $\phi = \phi_1$.

Third Reduction. By conditioning

$$\begin{aligned} &E\phi_1(M_n) - E\phi_1(M_{n-1}) - C\,E\phi_1(D_n) \\ &\quad = \int \Big(E(\phi_1(s+D_n)|M_{n-1}=s) - \phi_1(s) - C\,E(\phi_1(D_n)|M_{n-1}=s) \Big)\; P(M_{n-1}\in ds) \end{aligned}$$

where, given $M_{n-1} = s$, D_n has conditonal mean 0. This reduces the proof to that of (2.3) for any centered random variable D and any $s \in I\!R$. We may further restrict to $s \geq 0$ because $E\phi_1(s+D) = E\phi_1(-s-D)$ and $-D$ is also centered.

Fourth Reduction. Finally, since every centered distribution is a mixture of centered two point distributions, we conclude that it is indeed enough to prove (2.3) for all $s \geq 0$ and all centered D taking only two values, see e.g. Hoeffding [4].

In the following we simply write f' and always mean f'_+ in those cases where left and right derivatives are different.

Proof of (2.3) with $C = 1$ for symmetric D. Suppose D has distribution $(\delta_{-a} + \delta_a)/2$ for some $a \geq 0$ and let

$$\Delta(s) \overset{\text{def}}{=} E\phi_1(s+D) - E\phi_1(D) - \phi_1(s), \quad s \geq 0.$$

Then

$$\begin{aligned} \Delta(s) &= \frac{\phi_1(s+a)+\phi_1(s-a)}{2} - \phi_1(a) - \phi_1(s), \\ \Delta'(s) &= \frac{\phi_1'(s+a)+\phi_1'(s-a)}{2} - \phi_1'(s), \\ \Delta''(s) &= \frac{\phi_1''(s+a)+\phi_1''(s-a)}{2} - \phi_1''(s) \end{aligned}$$

for $s \geq 0$. In particular $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) \leq 0$. Note that

$$\phi_1'(x) \overset{\text{def}}{=} \begin{cases} 2x, & \text{if } |x| \leq 1 \\ 2\,\text{sign}(x), & \text{if } |x| \geq 1 \end{cases} \quad \text{and} \quad \phi_1''(x) = 2\,\mathbf{1}_{[-1,1]}(x)\ \lambda\text{-a.e.}$$

where λ denotes Lebesgue measure on $I\!R$ and $\mathbf{1}_B$ the indicator function of a set B. Hence, if $a \in [0,1]$, then λ-a.e.

$$\Delta''(s) = \begin{cases} 0, & \text{if } 0 \leq s \leq 1-a \text{ or } s > a+1 \\ -1, & \text{if } 1-a < s \leq 1 \\ 1, & \text{if } 1 < s \leq a+1 \end{cases},$$

while in case $a \in (1,2]$

$$\Delta''(s) = \begin{cases} -2, & \text{if } 0 \leq s \leq a-1 \\ -1, & \text{if } a-1 < s \leq 1 \\ 1, & \text{if } 1 < s \leq a+1 \\ 0, & \text{if } s > a+1 \end{cases},$$

and in case $a > 2$

$$\Delta''(s) \;=\; \begin{cases} 0, & \text{if } 1 < s \le a-1 \text{ or } s > a+1 \\ -2, & \text{if } 0 \le s \le 1 \\ 1, & \text{if } a-1 < s \le a+1 \end{cases} \quad .$$

We also have that $\Delta(s)$ and $\Delta'(s)$ vanish at $s = 0$ and (by linearity of ϕ_1 on $(1,\infty)$) for sufficiently large s. From this we see that Δ' is everywhere nonpositive and unimodal which in turn yields $\Delta(s) \le 0$ for all $s \ge 0$ and thus (2.3) with $C = 1$.

Proof of (2.3) with $C = 1$ for nonnegative $s + D$. Let D be a centered random variable with distribution $p\delta_{-a} + q\delta_b$ for $a, b \ge 0$, hence $p + q = 1$ and $qb - pa = 0$. The function Δ now takes the form

$$\Delta(s) \;=\; p\phi_1(s-a) + q\phi_1(s+b) - p\phi_1(-a) - q\phi_1(b) - \phi_1(s)$$

and has derivative $\Delta'(s) = p\phi_1'(s-a) + q\phi_1'(s+b) - \phi_1'(s)$. By concavity of ϕ_1' on $[0,\infty)$,

$$\Delta'(s) \;\le\; \phi_1'(s - pa + qb) - \phi_1'(s) \;=\; 0$$

for all $s \ge a$. Consequently, $E\phi_1(s+D) \le E\phi_1(D) + \phi_1(s)$ follows for all $s > a$ if this is true for $s = a$.

If $s = a \le 1$, then $\phi_1(s) = s^2$ whence $\phi_1(s+x) - \phi_1(x) \le (s+x)^2 - x^2 = s(2x+s)$ for all $x \ge -s$ implies the asserted inequality, namely

$$E\phi_1(s+D) - E\phi_1(D) \;\le\; s\,E(2D+s) \;=\; s^2.$$

Now fix $s = a \ge 1$, note that $ED = 0$ implies $p = \frac{b}{s+b}$, and look at $\Delta(s)$ as a function $G(b)$, say, of b. We obtain

$$\begin{aligned} G(b) &= q\phi_1(s+b) - p\phi_1(-s) - q\phi_1(b) - \phi_1(s) \\ &= q\phi_1(s+b) - q\phi_1(b) - (1+p)\phi_1(s) \\ &= \frac{s\phi_1(s+b) - s\phi_1(b) - (s+2b)\phi_1(s)}{s+b}. \end{aligned}$$

This implies in case $b \ge 1$

$$G(b) \;=\; \frac{s(2(s+b)-1) - s(2b-1) - (s+2b)(2s-1)}{s+b} \;=\; \frac{s+2b-4sb}{s+b} \;\le\; 0,$$

and in case $0 < b < 1$

$$G(b) \;=\; \frac{s(2(s+b)-1) - sb^2 - (s+2b)(2s-1)}{s+b} \;=\; \frac{-2b(s-1) - sb^2}{s+b} \;\le\; 0.$$

So we have again shown that (2.3) holds with $C = 1$.

Proof of (2.3) with $C = 2$ for general D. The assertion to prove may be rephrased in terms of $\Delta(s)$ as

$$\Delta(s) \;\le\; E\phi_1(D) \;=\; p\phi_1(s-a) + q\phi_1(s+b)$$

for all $s \ge 0$. Since $s = 0$ is trivial, fix an arbitrary $s > 0$, let D have distribution $p\delta_{-a} + q\delta_b$ and suppose $\theta \overset{\text{def}}{=} a - s \ge 0$ (only this case needs to be considered after the previous part of the proof). Note that $ED = 0$ implies $b = (p/q)a$ and thus $D \overset{d}{=} p\delta_{-s-\theta} + q\delta_{(p/q)(s+\theta)}$. In order to prove (2.3) with $C = 2$, fix any $p \in (0,1)$ and consider

$$\begin{aligned} H(\theta) &\overset{\text{def}}{=} E\phi_1(s+D) - 2\,E\phi_1(D) - \phi_1(s) \\ &= p\Big(\phi_1(\theta) - 2\phi_1(s+\theta)\Big) + q\Big(\phi_1(s+(p/q)(s+\theta)) - 2\phi_1((p/q)(s+\theta))\Big) - \phi_1(s) \end{aligned}$$

for $\theta \ge 0$. Since $s + D \ge 0$ if $\theta = 0$, we infer $H(0) \le -E\phi_1(D) < 0$ from the previous part of the proof. Differentiation with respect to θ gives

$$\begin{aligned} H'(\theta) &= p\Big(\phi_1'(\theta) - 2\phi_1'(s+\theta)\Big) + p\Big(\phi_1'(s+(p/q)(s+\theta)) - 2\phi_1'((p/q)(s+\theta))\Big) \\ &= p\bigg(\Big(\phi_1'(s+(p/q)(s+\theta)) - \phi_1'((p/q)(s+\theta))\Big) - \Big(\phi_1'(s+\theta) - \phi_1'(\theta)\Big) \qquad (2.4) \\ &\qquad - \Big(\phi_1'(s+\theta) + \phi_1'((p/q)(s+\theta))\Big)\bigg). \end{aligned}$$

The function ϕ_1' is monotone and is subadditive as a nonnegative concave function on $[0,\infty)$. It follows that

$$\phi_1'(s+\theta) + \phi_1'((p/q)(s+\theta)) \;\ge\; \phi_1'(s+\theta+(p/q)(s+\theta)) \;\ge\; \phi_1'(s+(p/q)(s+\theta))$$

and thereby in (2.4)

$$H'(\theta) \;\le\; -p\Big(\phi_1'((p/q)(s+\theta)) + \big(\phi_1'(s+\theta) - \phi_1'(\theta)\big)\Big) \;\le\; 0.$$

Consequently, H is nonincreasing on $[0,\infty)$ with $H(0) < 0$ and therefore everywhere negative. This proves (2.3) with $C = 2$ and strict inequality.

Attaining the bound in (2.3) with $C = 2$. We finally have to provide examples showing that the bound $C = 2$ is sharp. Let $s \ge 1$ and D be distributed as $\frac{b}{a+b}\delta_{-a} + \frac{a}{a+b}\delta_b$ for some $a \ge 1 + s$ and $b \in [0,1]$. Then

$$\begin{aligned} &E\phi_1(s+D) - \phi_1(s) - (2-\varepsilon)E\phi_1(D) \\ &\quad= \frac{1}{a+b}\Big(b(2a-2s-1) + a(2s+2b-1) - (a+b)(2s-1) - (2-\varepsilon)\big(b(2a-1) + ab^2\big)\Big) \\ &\quad= \frac{b}{a+b}\Big(2 - 2ab - 4s + \varepsilon(2a-1+ab)\Big). \end{aligned}$$

Now it is easily seen that, for any $\varepsilon > 0$, a positive value is obtained when choosing $b = 1/a$ and a sufficiently large. The proof of Theorem 1 is herewith complete. ◇

Recall that $\mathcal{G}_0^{(\nu)}$ denotes the class of all $\phi \in \mathcal{G}_0$ satisfying $\phi(2x) \le \nu\phi(x)$ for all x. We claimed in the Introduction that $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$ as well as $\mathcal{G}_1 \subset \mathcal{G}_2$, where $\mathcal{G}_2$ denotes the subclass of $\mathcal{G}_0$ containing those ϕ for which $\psi(x) \stackrel{\text{def}}{=} \phi(x^{1/2})$ is concave on $[0,\infty)$. These claims are finally confirmed in the subsequent lemma.

LEMMA 2. *$\mathcal{G}_1 \subset \mathcal{G}_2$ and $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$.*

PROOF. Note that each nonnegative concave function f on $[0,\infty)$ is subadditive and that $f(x)/x$ is nonincreasing (because $-f$ is evidently star-shaped, see [8, p. 453]). Given any $\phi \in \mathcal{G}_1$, use this for $f = \phi'$ to see that the pertinent ψ is indeed concave because $\psi'(x) = \frac{\phi'(x^{1/2})}{2x^{1/2}}$. Moreover, the subadditivity of ϕ' on $[0,\infty)$ implies $\phi'(2x) \le 2\phi'(x)$ and thus

$$\phi(2x) \;=\; \int_0^{2x} \phi'(t)\, dt \;=\; \int_0^x 2\phi'(2t)\, dt \;\le\; \int_0^x 4\phi'(t)\, dt \;=\; 4\phi(x)$$

for all $x \ge 0$. ◇

ACKNOWLEDGEMENT

The authors are indebted to an anonymous referee for a very careful reading and for giving some additional related references.

NOTE ADDED IN PROOF

In a recent paper, Li [7, Thm. 2.1] proved the following large deviation inequality for a martingale $(M_n)_{n\ge 0}$ with $M_0 = 0$: If $1 < p \le 2$ and $K \stackrel{\text{def}}{=} \sup_{n\ge 1} E|M_n|^p < \infty$, then

$$P\left(\max_{1\le i\le n} |M_n| > nx\right) \;\le\; CKn^{1-p}x^{-p}$$

for all $x > 0$, $n \ge 1$ and $C = (18pq^{1/2})^p$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. A combination of Doob's maximal inequality (see [2, p. 255]) with our Theorem 1 for $\phi(x) = |x|^p$ immediately shows that Li's inequality actually holds true with the considerably smaller constant $C = 2$.

REFERENCES

[1] ALSMEYER, G. (1996). Nonnegativity of odd functional moments of positive random variables with decreasing density. *Statist. Probab. Letters* **26**, 75-82.

[2] CHOW, Y.S. and TEICHER, H. (1997). *Probability Theory: Independence, Interchangeability, Martingales (3rd Edition).* Springer, New York.

[3] DE LA PEÑA, V.H., IBRAGIMOV, R. and SHARAKHMETOV, S. (2002). On sharp Burkholder-Rosenthal type inequalities for infinite degree order statistics. *Ann. Inst. H. Poincaré* **38**, 973-990.

[4] HOEFFDING, W. (1955). The extrema of the expected value of a function of independent random variables. *Ann. Math. Statist.* **26**, 268-275.

[5] HUBER, P. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.* **35**, 73-101.

[6] HUBER, P. (1973). Robust regression: asymptotics, conjectures and Monte Carlo. *Ann. Statist.* **1**, 799-821.

[7] LI, Y. (2003). A martingale inequality and large deviations. *Statist. Probab. Letters* **62**, 317-321.

[8] MARSHALL, A.W. and OLKIN, I. (1979). *Inequalities: theory of majorization and its applications*. Mathematics in Science and Engineering, **143**. Academic Press, New York.

[9] RÖSLER, U. (1995). Distributions slanted to the right. *Statist. Neerlandica* **49**, 83-93.

[10] TOPCHII, V.A. and VATUTIN, V.A. (1997). Maximum of the critical Galton-Watson processes and left continuous random walks. *Theory Probab. Appl.* **42**, 17-27.