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On the Existence of ϕ-Moments of the Limit of a Normalized Supercritical Galton-Watson Process

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On the Existence of ϕ-Moments of the Limit of a Normalized Supercritical Galton-Watson Process

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Let $(Z_n)_{n\geq 0}$ be a supercritical Galton-Watson process with finite reproduction mean μ and normalized limit $W = \lim_{n\to\infty} \mu^{-n} Z_n$. Let further $\phi : [0,\infty) \to [0,\infty)$ be a convex differentiable function with $\phi(0) = \phi'(0) = 0$ and such that $\phi(x^{1/2^n})$ is convex with concave derivative for some $n \geq 0$. By using convex function inequalities due to Topchii and Vatutin, and Burkholder, Davis and Gundy, we prove that $0 < E\phi(W) < \infty$ if, and only if, $E\mathbb{L}\phi(Z_1) < \infty$, where

$$\mathbb{L}\phi(x) \stackrel{\text{def}}{=} \int_0^x \int_0^s \frac{\phi'(r)}{r}\, dr\, ds, \quad x \geq 0.$$

We further show that functions $\phi(x) = x^\alpha L(x)$ which are regularly varying of order $\alpha \geq 1$ at ∞ are covered by this result if $\alpha \notin \{2^n : n \geq 0\}$ and under an additonal condition also if $\alpha = 2^n$ for some $n \geq 0$. This was obtained in a slight weaker form and analytically by Bingham and Doney. If $\alpha > 1$, then $\mathbb{L}\phi(x)$ grows at the same order of magnitude as $\phi(x)$ so that $E\mathbb{L}\phi(Z_1) < \infty$ and $E\phi(Z_1) < \infty$ are equivalent. However, $\alpha = 1$ implies $\lim_{x\to\infty} \mathbb{L}\phi(x)/\phi(x) = \infty$ and hence that $E\mathbb{L}\phi(Z_1) < \infty$ is a strictly stronger condition than $E\phi(Z_1) < \infty$. If $\phi(x) = x\log^p x$ for some $p > 0$ it can be shown that $\mathbb{L}\phi(x)$ grows like $x\log^{p+1} x$, as $x \to \infty$. For this special case the result is due to Athreya. As a by-product we also provide a new proof of the Kesten-Stigum result that $Z_1 \log Z_1 < \infty$ and $EW > 0$ are equivalent.

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1. Introduction and Main Result

Let $(Z_n)_{n\geq 0}$ be a supercritical Galton-Watson process with offspring distribution $(p_j)_{j\geq 0}$ and finite mean offspring μ. Then the normalized process $W_n \stackrel{\text{def}}{=} \mu^{-n}Z_n$, $n \geq 0$, is a nonnegative and thus a.s. convergent martingale with limit W, say. The famous Kesten-Stigum Theorem (see e.g. [6, Thm. I.10.1]) states that in order for W to be positive on the set of non-extinction it is necessary and sufficient that $EZ_1 \log Z_1 = \sum_{j\geq 1} p_j \log j < \infty$. Athreya [5] showed for any $p \geq 0$ that $0 < EW|\log W|^p < \infty$ holds if, and only if, $EZ_1 \log^{p+1} Z_1 < \infty$. More precisely, he proved this equivalence be true for any integrable solution W of the stochastic fixed point equation

$$W \stackrel{d}{=} \frac{1}{\mu}\sum_{k=1}^{Z_1} W(k), \tag{1.1}$$

where "$\stackrel{d}{=}$" means equality in distribution and $Z_1, W(1), W(2), \ldots$ are mutually independent with $W(k) \stackrel{d}{=} W$ for $k \geq 1$. It is well-known that, even if $EZ_1 \log Z_1 = \infty$, a non-zero solution to (1.1) is unique up to a scaling factor, see [6, Thm. I.10.2]. Bingham and Doney [7] extended Athreya's result and considered the ϕ-moment of W when ϕ is a regularly varying function of order $\alpha \geq 1$; see also [8] for similar results in the case of more general branching processes. The present article will further extend their results by providing a necessary and sufficient moment condition on $(p_j)_{j\geq 0}$ for the existence of $E\phi(W)$ for an even larger class of functions ϕ to be described below.

However, rather than this improvement it is our approach we believe to be of main interest here because it differs completely from the analytic ones in [5], [7] and exploits more explicitly the inherent probabilistic nature of the branching model which expresses itself in a *double martingale structure*. To explain, a key observation on $(W_n)_{n\geq 0}$ is that besides forming a nonnegative martingale (the first one) its increments are also random sums of i.i.d. random variables and thus of a martingale after centering (the second one). Taking this as a starting point, a key step towards our results will be the repeated application of convex function inequalities for martingales. A somewhat similar approach was also used by the first author [1] in the quite different context of generalized renewal measures. As a by-product our approach will also produce a new proof of the famous Kesten-Stigum theorem [4, Thm. II.2.1]. Let us emphasize that it also differs from the recent probabilistic proof by Lyons, Pemantle and Peres [12] using spinal trees. The method developed here has also been employed in a recent paper by Kuhlbusch [11] for the more general class of weighted branching processes.

Let $\mathfrak{C}_0$ be the class of convex differentiable functions ϕ which are (strictly) increasing on $[0,\infty)$ with $\phi(0) = 0$ and concave derivative ϕ' on $(0,\infty)$ satisfying $\phi'(0+) = 0$. Observe that, by the last condition, the identity function $\phi(x) = x$ is not in $\mathfrak{C}_0$. We further note for each $\phi \in \mathfrak{C}_0$ that ϕ' is nondecreasing and positive on $(0,\infty)$ and that $\liminf_{x\to\infty} \frac{\phi(x)}{x} > 0$. For $n \geq 1$, we define recursively

$$\mathfrak{C}_n \stackrel{\text{def}}{=} \{\mathbb{S}\phi \in \mathcal{G} : \phi \in \mathfrak{C}_{n-1}\} = \mathbb{S}\mathfrak{C}_{n-1},$$

where the operator $\mathbb{S}$ is given by $\mathbb{S}\psi(x) \stackrel{\text{def}}{=} \psi(x^2)$. The functions ϕ to be considered throughout shall be elements from one of these classes, i.e. from $\mathfrak{C} \stackrel{\text{def}}{=} \cup_{n\ge 0}\mathfrak{C}_n$, and they are clearly always differentiable and convex, so $\mathbb{S} : \mathfrak{C} \to \mathfrak{C}$. As two further rather straightforward properties of functions in $\mathfrak{C}$ we mention

$$\phi(2x) \;\le\; C\,\phi(x), \quad x \ge 0, \tag{1.2}$$

for some $C = C_\phi \in (0,\infty)$, and

$$\phi \in \mathfrak{C}_n \quad \Rightarrow \quad \limsup_{x\to\infty} \frac{\phi(x)}{x^{2^{n+1}}} < \infty$$

for each $n \ge 0$ (see also Lemmata 3.3 and 3.4).

Let us stipulate hereafter that the usual primed notation for derivatives of convex or concave functions on $(0,\infty)$ is always to be understood in the right sense in cases where right and left derivatives are different. Whenever necessary and without further notice, a function $\phi \in \mathfrak{C}$ is extended to the real line by setting $\phi(-x) = \phi(x)$ for $x < 0$. This renders an even convex function on $I\!R$. We write $f \asymp g$ if $0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \le \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ holds true, while $f \sim g$ has the usual meaning $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Given any nondecreasing convex function $\phi : [0,\infty) \to [0,\infty)$, we next define the operator $\mathbb{L}$ through

$$\mathbb{L}\phi(x) \;\stackrel{\text{def}}{=}\; \int_0^x \int_0^s \frac{\phi'(r)}{r}\,dr\,ds, \quad x \ge 0. \tag{1.3}$$

It is crucial for the statement of our main result, Theorem 1.1 below. Plainly, $\mathbb{L}\phi$ is again nondecreasing with values in $[0,\infty]$ and convex on $\{x : \mathbb{L}\phi(x) < \infty\}$.

To each $\phi \in \mathfrak{C}$ there exists a function $\psi \in \mathfrak{C}$ satisfying $\psi \sim \phi$ and $\mathbb{L}\psi(x) < \infty$ for all $x \ge 0$. One may take for instance

$$\psi(x) \;=\; \int_0^x \int_0^y \Big(a\mathbf{1}_{[0,1]}(z) + \phi''(z)\mathbf{1}_{(1,\infty)}(z)\Big)\,dz\,dy, \quad x \ge 0,$$

for any $a > \phi''(1)$, in which case furthermore $\psi''(0) \in (0,\infty)$ and

$$\mathbb{S}^n\phi(x) \;=\; \phi(x^{2^n}) \;\sim\; \psi(x^{2^n}) \;=\; \mathbb{S}^n\psi(x)$$

for all $n \ge 0$. Therefore existence results for ϕ-moments with $\phi \in \mathfrak{C}$ can (and will) be confined without loss of generality to functions $\phi \in \mathfrak{C}^* \stackrel{\text{def}}{=} \cup_{n\ge 0}\mathfrak{C}^*_n$, where $\mathfrak{C}^*_n \stackrel{\text{def}}{=} \mathbb{S}^n\mathfrak{C}^*_0$ for $n \ge 1$, and

$$\mathfrak{C}^*_0 \;\stackrel{\text{def}}{=}\; \mathfrak{C}_0 \cap \{\phi : \mathbb{L}\phi < \infty\}.$$

Notice that, for $n \ge 1$ and $\phi \in \mathfrak{C}^*_n$, we have $\phi(0) = \phi'(0) = \phi''(0) = 0$ and thus integrability of $\frac{\phi'(x)}{x}$ at 0. This shows $\mathfrak{C}^* \subset \{\phi : \mathbb{L}\phi < \infty\}$. Notice also that $\{\phi \in \mathfrak{C} : \phi''(0) \in (0,\infty)\} \subset \mathfrak{C}^*$.

The functions $\phi_\alpha(x) \stackrel{\text{def}}{=} x^{\alpha+1}$, $\alpha > 0$, as well as $\phi_0(x) \stackrel{\text{def}}{=} x^2\mathbf{1}_{[0,1]}(x) + (2x-1)\mathbf{1}_{(1,\infty)}(x)$ are all elements of $\mathfrak{C}^*$ (as for ϕ_0, note that $\phi_0(x) \asymp x$, but that the identity function is neither in $\mathfrak{C}$ nor in $\{\phi : \mathbb{L}\phi < \infty\}$).

It is easily verified (see Lemma 4.4) that

$$\liminf_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} \;>\; 0$$

for $\phi \in \mathfrak{C}^*$ that is, $\mathbb{L}\phi$ grows at least at the same order of magnitude as ϕ. For the special functions ϕ_α, $\alpha \in [0,\infty)$, defined above we compute

$$\mathbb{L}\phi_\alpha(x) \;=\; \begin{cases} x^2\mathbf{1}_{[0,1]}(x) + (1+2x\log x)\mathbf{1}_{(1,\infty)}(x), & \text{if } \alpha = 0, \\ \dfrac{x^{\alpha+1}}{\alpha}, & \text{if } \alpha > 0, \end{cases} \tag{1.4}$$

and thus see that $\lim_{x\to\infty} \frac{\mathbb{L}\phi_0(x)}{\phi_0(x)} = \infty$, and $\mathbb{L}\phi_\alpha \asymp \phi_\alpha$ if $\alpha > 0$. If we consider functions $\phi \in \mathfrak{C}^*$ which are regularly varying at infinity with exponent $\alpha \geq 1$, then the discussion in Section 2 will confirm the very same result for this more general situation (see Lemma 2.2). It indicates that $\mathbb{L}\phi$ grows faster than ϕ only when ϕ is a function "close to the identity function".

Theorem 1.1. *Let $(Z_n)_{n\geq 0}$ be a supercritical Galton-Watson process with offspring distribution $(p_j)_{j\geq 0}$, finite mean offspring μ and normalized limit $W = \lim_{n\to\infty} \mu^{-n} Z_n$. Then for each $\phi \in \mathfrak{C}^*$ the equivalence*

$$0 < E\phi(W) < \infty \quad \textit{iff} \quad E\mathbb{L}\phi(Z_1) < \infty, \tag{1.5}$$

holds true with $\mathbb{L}\phi$ as in (1.3).

The convexity of $\phi \in \mathfrak{C}$ implies that $(\phi(W_n))_{n\geq 0}$ constitutes a nonnegative submartingale and thus $\lim_{n\to\infty} E\phi(W_n) = \sup_{n\geq 0} E\phi(W_n)$. Combining this fact with Theorem 5 in [13] (see also [2]) and the well-known tail estimate

$$P(\sup_{n\geq 0} W_n > bx) \;\leq\; c\,P(W > x), \quad x \geq 0 \tag{1.6}$$

for suitable $b, c > 0$ (see [4, Lemma II.2.6]), the next result is readily concluded and therefore stated without proof.

Theorem 1.2. *In the situation of Theorem 1.1 the following assertions are equivalent:*

$$0 < E\phi(W) < \infty; \tag{1.7}$$

$$\sup_{n\geq 0} E\phi(W_n) < \infty; \tag{1.8}$$

$$(\phi(W_n))_{n\geq 0} \textit{ is uniformly integrable}; \tag{1.9}$$

$$\lim_{n\to\infty} E|\phi(W_n) - \phi(W)| < \infty; \tag{1.10}$$

$$E\phi(\sup_{n\geq 0} W_n) < \infty. \tag{1.11}$$

The equivalence of (1.8)-(1.11) holds true for any ϕ-integrable submartingale $(W_n)_{n\geq 0}$, but the equivalence with (1.7) hinges on (1.6) which in turn follows from the special structure of the Galton-Watson process.

Given a supercritical Galton-Watson process $(Z_n)_{n\geq 0}$ with finite reproduction mean the crucial equivalence of the Kesten-Stigum theorem [4, Thm. II.2.1] states that

$$EW > 0 \quad \text{iff} \quad EZ_1 \log Z_1 < \infty. \tag{1.12}$$

Theorem 1.1 contains this result as a special case when choosing $\phi_0(x) \asymp x$ (in which case $\mathbb{L}\phi_0(x) \asymp x \log x$ by (1.4)). Our martingale proof for the more difficult "if"-conclusion of (1.12) is new and essentially furnished by Lemma 4.5. In contrast to the martingale proof in [4] it does not make use of truncation.

The further organization is as follows. Section 2 provides a discussion of our main result in the context of regularly varying functions and shows in particular that it implies the related ϕ-moment results of Bingham and Doney [7]. Some general facts on functions ϕ from the classes $\mathfrak{C}$ and $\mathfrak{C}^*$ and the associated $\mathbb{L}\phi$ are provided in Section 3, while Section 4 contains various inequalities for the ϕ-moments of W. They will furnish the proof of Theorem 1.1 presented in Section 5.

2. Functions of Regular Variation

It need not be further explained that functions of regular variation are of particular interest when dealing with moment results. This section is therefore devoted to a discussion of several aspects concerning these functions in the present context. For $\alpha \geq 0$, let $\mathfrak{R}_\alpha$ be the class of locally bounded functions from $[0,\infty)$ to $[0,\infty)$ which are regularly varying at infinity with exponent α (slowly varying in case $\alpha = 0$). Given $\phi(x) = x^\alpha L(x) \in \mathfrak{R}_\alpha$ for some $\alpha \geq 0$, the smooth variation theorem [9, Thm. 1.8.2] ensures the existence of a function $\psi \in \mathfrak{R}_\alpha$ which is smooth (infinitely often differentiable) on $(0,\infty)$ and satisfies $\phi \asymp \psi$. Let $\mathfrak{S}_\alpha$ denote the subclass of such functions. If $\alpha > 0$ and $\alpha \notin I\!N$ then ψ can also be chosen such that all its derivatives are monotone [9, Thm. 1.8.3] which implies that ϕ and all its derivatives are either convex or concave. Possibly after switching to $\psi(x+a) - \psi(a) - \psi'(a)x$ for some $a > 0$, we may assume $\psi(0) = \psi'(0) = 0$ and $\psi''(0) \in (0,\infty)$, hence $\psi \in \mathfrak{C}^*$. We note as a trivial observation that $\phi \in \mathfrak{R}_\alpha$ implies $\mathbb{S}^{-n}\phi \in \mathfrak{R}_{\alpha/2^n}$ for each $n \in I\!N_0$.

The following three questions will be addressed hereafter:

— How are the $\mathfrak{R}_\alpha$ related to the classes $\mathfrak{C}_n$?

— What can be said about the behavior of $\mathbb{L}\phi$ in (1.3) if $\phi \in \mathfrak{R}_\alpha$ for $\alpha \geq 1$?

— How can Theorem 1.1 be restated for regularly varying functions ϕ?

For any measurable $\phi : [0,\infty) \to [0,\infty)$, we put

$$\hat{\phi}(x) \stackrel{\text{def}}{=} \int_0^x \frac{\phi(y)}{y}\, dy \quad \text{and} \quad \tilde{\phi}(x) \stackrel{\text{def}}{=} \int_x^\infty \frac{\phi(y)}{y}\, dy$$

and stipulate the everywhere finiteness of such functions whereever they appear. Plainly, this is guaranteed for $\hat{\phi}$ if $\frac{\phi(x)}{x}$ is locally integrable on $[0,\infty)$, and for $\tilde{\phi}$ if $\frac{\phi(x)}{x}$ is integrable on $[0,\infty)$.

The following lemma addresses the first of the above questions.

Lemma 2.1. *Given $\phi(x) = x^\alpha L(x) \in \mathfrak{R}_\alpha$ for some $\alpha \geq 1$, the following assertions hold true:*

(a) If $2^n < \alpha < 2^{n+1}$ for $n \in I\!N_0$, then $\phi \asymp \varphi$ for some $\varphi \in \mathfrak{C}_n^* \cap \mathfrak{R}_\alpha$.

(b) If $\alpha = 2^n$ for $n \in I\!N_0$ and $L \asymp \hat{L}_0$ for some $L_0 \in \mathfrak{R}_0$, then $\phi \asymp \varphi$ for some $\varphi \in \mathfrak{C}_n^* \cap \mathfrak{R}_{2^n}$.

(c) If $\alpha = 2^n$ for $n \in I\!N$ and $L \asymp \tilde{L}_0$ for some $L_0 \in \mathfrak{R}_0$, then $\phi \asymp \varphi$ for some $\varphi \in \mathfrak{C}_{n-1}^* \cap \mathfrak{R}_{2^n}$.

Proof. (a) If $2^n < \alpha < 2^{n+1}$ for $n \in I\!N_0$ and $\phi \in \mathfrak{R}_\alpha$, then $\mathbb{S}^{-n}\phi \in \mathfrak{R}_\beta$ for $\beta \stackrel{\text{def}}{=} \alpha/2^n \in (1,2)$, thus $\mathbb{S}^{-n}\phi \asymp \psi \in \mathfrak{C}_0^* \cap \mathfrak{S}_\beta$ by what has been mentioned before the lemma.

(b) If $\phi(x) = x^{2^n} L(x)$ for some $n \geq 0$ and $L \asymp \hat{L}_0$ for some $L_0 \in \mathfrak{R}_0$, then

$$\mathbb{S}^{-n}\phi(x) \;=\; xL(x^{1/2^n}) \;\asymp\; \psi(x) \;\stackrel{\text{def}}{=}\; x\hat{L}_0(x^{1/2^n}).$$

Note that $\hat{L}_0 \in \mathfrak{R}_0$ with $\lim_{x\to\infty} \frac{\hat{L}_0(x)}{L_0(x)} = \infty$ [9, Prop. 1.5.9a], and that $\frac{L_0(x)}{x} \sim \overline{L}(x) \stackrel{\text{def}}{=} \sup_{y \geq x} \frac{L_0(y)}{y}$ [9, Thm. 1.5.3]. We infer

$$\psi'(x) \;=\; \hat{L}_0(x^{1/2^n}) + \frac{L_0(x^{1/2^n})}{2^n} \;\sim\; \hat{L}_0(x^{1/2^n}) \;\sim\; \int_0^{x^{1/2^n}} \overline{L}(y)\, dy \;\in\; \mathfrak{R}_0$$

and therefore, by an appeal to Karamata's theorem [9, Prop. 1.5.8],

$$\mathbb{S}^{-n}\phi(x) \;\asymp\; \psi(x) \;\asymp\; \overline{\psi}(x) \;\stackrel{\text{def}}{=}\; \int_0^x \int_0^{y^{1/2^n}} \overline{L}(z)\, dz\, dy \;\in\; \mathfrak{R}_1.$$

Since $\overline{L}$ is nonincreasing we see that $\overline{\psi}$ is also an element of $\mathfrak{C}_0^*$ so that $\phi(x) \asymp \mathbb{S}^n\psi(x) = \psi(x^{2^n}) \in \mathfrak{C}_n^* \cap \mathfrak{R}_{2^n}$.

(c) Note here that $\tilde{L}_0 \in \mathfrak{R}_0$ with $\lim_{x\to\infty} \frac{\tilde{L}_0(x)}{L_0(x)} = \infty$ [9, Prop. 1.5.9b]. Hence, having $\mathbb{S}^{-n+1}\phi(x) = x^2 L(x^{1/2^{n-1}}) \asymp \psi(x) \stackrel{\text{def}}{=} x^2 \tilde{L}_0(x^{1/2^{n-1}})$, we infer

$$\psi'(x) \;=\; 2x\tilde{L}_0(x^{1/2^{n-1}}) - \frac{xL_0(x^{1/2^{n-1}})}{2^{n-1}} \sim\; 2x\tilde{L}_0(x^{1/2^{n-1}})$$

as well as

$$\Big(x\tilde{L}_0(x^{1/2^{n-1}})\Big)' \;\asymp\; \tilde{L}_0(x^{1/2^{n-1}}) \;\in\; \mathfrak{R}_0.$$

Consequently, by another appeal to Karamata's theorem,

$$\mathbb{S}^{-n+1}\phi(x) \;\asymp\; \psi(x) \;\asymp\; \int_0^x \int_0^y \tilde{L}_0(z^{1/2^{n-1}})\, dz\, dy \;\in\; \mathfrak{R}_2$$

where the right-most function constitutes an element of $\mathfrak{C}_0^*$. The assertion now follows by a similar conclusion as in (b). ♢

The next lemma addresses the second question above.

LEMMA 2.2. *Let $\phi(x) = x^\alpha L(x) \in \mathfrak{C}^* \cap \mathfrak{R}_\alpha$ for some $\alpha \geq 1$. If $\alpha > 1$ then*

$$\mathbb{L}\phi(x) \sim \frac{\phi(x)}{\alpha - 1}, \tag{2.1}$$

while

$$\mathbb{L}\phi(x) \sim x\hat{L}(x) \tag{2.2}$$

and

$$\lim_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} = \infty \tag{2.3}$$

hold in case $\alpha = 1$.

PROOF. Given $\phi(x) = x^\alpha L(x) \in \mathfrak{C}^* \cap \mathfrak{R}_\alpha$ for some $\alpha \geq 1$ and $L \in \mathfrak{R}_0$, we have $\phi'(x) \sim \alpha x^{\alpha-1} L(x)$ by the Monotone Density Theorem [9, Thm. 1.7.2]. In case $\alpha > 1$, Karamata's theorem implies

$$\mathbb{L}\phi(x) \sim \int_0^x \int_0^s \alpha r^{\alpha-2} L(r)\, dr\, ds \sim \frac{x^\alpha L(x)}{\alpha - 1} = \frac{\phi(x)}{\alpha - 1}$$

that is (2.1).

If $\alpha = 1$, then $\phi' \sim L$, $\hat{L} \in \mathfrak{R}_0$ and once again Karamata's theorem give

$$\mathbb{L}\phi(x) \sim \int_0^x \hat{L}(y)\, dy \sim x\hat{L}(x),$$

i.e. (2.2). Moreover,

$$\lim_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} = \lim_{x\to\infty} \frac{\hat{L}(x)}{L(x)} = \infty$$

follows from [9, Prop. 1.5.9a]. ◊

Turning to the third question, a combination of Theorem 1.1 and the previous two lemmata leads directly to the following corollary which essentially contains the moment results first obtained by Bingham and Doney [7, Thm. 5–7].

COROLLARY 2.3. *Suppose the situation of Theorem 1.1 and let $L \in \mathfrak{R}_0$. Then*

$$0 < EW^\alpha L(W) < \infty \quad \textit{iff} \quad EZ_1^\alpha L(Z_1) < \infty \tag{2.4}$$

for any $\alpha > 1$ which is not a dyadic power. The same equivalence holds true if $\alpha = 2^n$ for some $n \geq 1$ and if either $L(x) \asymp \hat{L}_0(x)$ or $L(x) \asymp \tilde{L}_0(x)$ for some $L_0 \in \mathfrak{R}_0$. Finally,

$$0 < EW^\alpha L(W) < \infty \quad \textit{iff} \quad EZ_1^\alpha \hat{L}(Z_1) < \infty, \tag{2.5}$$

if $\alpha = 1$ and $L(x) \asymp \hat{L}_0(x)$ for some $L_0 \in \mathfrak{R}_0$.

The corollary is slightly more general than Bingham and Doney's result which needs an extra condition on L whenever α is an integer. In the special case $L(x) = \log^p x$ for some $p \geq 0$ one finds that $\hat{L}(x) \asymp \log^{p+1} x$ and thus that (2.5) reduces (as it must) to Athreya's result.

3. Some general facts on the classes $\mathfrak{C}$ and $\mathfrak{C}^*$

We proved in [2, Lemma 1] that each increasing convex function ϕ on $[0,\infty)$ with $\phi(0) = 0$ has a unique *Choquet representation* of the form

$$\phi \;=\; \int_{[0,\infty]} \varphi_t \; Q_\phi(dt) \tag{3.1}$$

where $\varphi_0(x) \stackrel{\text{def}}{=} x$, $\varphi_\infty(x) = x^2$, and

$$\varphi_t(x) \;\stackrel{\text{def}}{=}\; \begin{cases} x^2, & \text{if } x \leq t \\ 2xt - t^2, & \text{if } x > t \end{cases} \tag{3.2}$$

for $t \in (0,\infty)$. We note that $\varphi_t(x) = t^2 \varphi_1(x/t)$ for $t \in (0,\infty)$. The unique nonzero measure Q_ϕ is given by

$$Q_\phi \;\stackrel{\text{def}}{=}\; \phi'(0)\delta_0 + \Lambda_\phi \tag{3.3}$$

where $\Lambda_\phi((t,\infty]) \stackrel{\text{def}}{=} \phi''(t) - \phi''(\infty)$ for $t > 0$ and $\phi''(\infty) \stackrel{\text{def}}{=} \lim_{t\to\infty} \phi''(t)$. So we have that $Q_\phi((t,\infty]) = \frac{\phi''(t)}{2} < \infty$ for all $t > 0$ and

$$\int_{(0,c]} t \; Q_\phi(dt) \;=\; \int_0^c Q_\phi((t,c]) \; dt \;=\; \int_0^c (\phi''(t) - \phi''(c)) \; dt \;<\; \infty$$

for all $c > 0$. Q_ϕ is finite iff $\phi''(0) < \infty$.

When imposing the additonal restriction $\phi'(0) = 0$ we arrive at the class $\mathfrak{C}_0$. The following lemma is now easily established and thus stated without proof.

Lemma 3.1. *The mapping*

$$\phi \;\mapsto\; \int \varphi_t \; \mu(dt) \tag{3.3}$$

provides a bijection between functions $\phi \in \mathfrak{C}_0$ and nonzero measures μ on $(0,\infty]$ satisfying $\mu((x,\infty]) < \infty$ for all $x > 0$ and $\int_{(0,c]} t \, \mu(dt) < \infty$ for all $c > 0$. μ is finite iff $\phi''(0) < \infty$.

The natural question of a similar result for the subclass $\mathfrak{C}_0^* = \mathfrak{C}_0 \cap \{\phi : \mathbb{L}\phi < \infty\}$ is answered by the next lemma.

Lemma 3.2. *The mapping (3.3) provides a bijection between functions $\phi \in \mathfrak{C}_0$ and nonzero measures μ on $(0,\infty]$ satisfying $\mu((x,\infty]) < \infty$ for all $x > 0$ and $\int_{(0,c]} t|\log t| \, \mu(dt) < \infty$ for all $c > 0$.*

PROOF. It is obviously enough to show for a given $\phi=\int\varphi_t\mu(dt)\in\mathfrak{C}_0$, that $\int_0^1\frac{\phi'(s)}{s}\,ds<\infty$ holds iff $\int_{(0,1]}t|\log t|\,\mu(dt)<\infty$. By Fubini's theorem,

$$\int_0^1\frac{\phi'(s)}{s}\,ds\ =\ \int_{(0,\infty]}\int_0^1\frac{\varphi_t'(s)}{s}\,ds\ \mu(dt).$$

Since $\varphi_\infty(x)\ =\ x^2$ is clearly in $\mathfrak{C}_0^*$ suppose $\mu(\{\infty\})\ =\ 0$ without loss of generality. Using $\varphi_t(x)=t^2\varphi_1(x/t)$ for $0<t<\infty$ we then arrive at

$$\int_0^1\frac{\phi'(s)}{s}\,ds\ =\ \int_{(0,\infty)}\int_0^1\frac{\varphi_1'(s/t)}{s/t}\,ds\ \mu(dt)\ =\ \int_{(0,\infty)}t\int_0^{1/t}\frac{\varphi_1'(r)}{r}\,dr\ \mu(dt)$$

which after a simple integration yields

$$\int_0^1\frac{\phi'(s)}{s}\,ds\ =\ 2\mu((1,\infty))\ +\ \int_{(0,1]}2t\ \mu(dt)\ +\ \int_{(0,1]}2t|\log t|\ \mu(dt)$$

and thus provides the desired conclusion. ◊

The trivial observation that $\phi=\int_{(0,\infty]}\varphi_t\,\mu(dt)$ implies

$$\mathbb{S}^n\phi\ =\ \int_{(0,\infty]}\mathbb{S}^n\varphi_t\ \mu(dt) \tag{3.4}$$

for each $n\geq 1$ shows that the previous two lemmata carry over verbatim to the classes $\mathfrak{C}_n$ and $\mathfrak{C}_n^*$, respectively, when replacing the functions φ_t by $\mathbb{S}^n\varphi_t$. So $\mathfrak{C}_0$ and $\mathfrak{C}_n$ as well as $\mathfrak{C}_0^*$ and $\mathfrak{C}_n^*$ are isomorphic positive cones including the order.

The subsequent two lemmata, stated without proofs, collect some straightforward properties of functions in $\mathfrak{C}_0$, respectively $\mathfrak{C}_n$ for $n\geq 1$.

LEMMA 3.3. *Each element $\phi\in\mathfrak{C}_0$ has the following properties:*
(a) $2\phi(x)\leq\phi(2x)\leq 4\phi(x)$ *for all* $x\geq 0$.
(b) $\frac{\phi(x)}{x}$ *is nondecresing and* $\frac{\phi(x)}{x^2}$ *is nonincreasing in* $x\geq 0$.
(c) $\lim_{x\downarrow 0}\frac{\phi(x)}{x}=\phi'(0)=0$ *and* $\lim_{x\downarrow 0}\frac{\phi(x)}{x^2}=\phi''(0)\in[0,\infty]$.
(d) There exists $\psi\in\mathfrak{C}_0$ *with* $\psi\sim\phi$ *and* $\psi''(0)\in(0,\infty)$.

LEMMA 3.4. *For each $n\geq 1$ and $\phi=\mathbb{S}^n\varphi\in\mathfrak{C}_n$ the following assertions hold true:*
(a) $\phi(2x)\leq 2^{2^{n+1}}\phi(x)$ *for all* $x\geq 0$.
(b) $\frac{\phi(x)}{x^{2n}}$ *is nondecresing and* $\frac{\phi(x)}{x^{2n+1}}$ *is nonincreasing in* $x\geq 0$.
(c) $\lim_{x\downarrow 0}\frac{\phi(x)}{x^{2n}}=\varphi'(0)=0$ *and* $\lim_{x\downarrow 0}\frac{\phi(x)}{x^{2n+1}}=\varphi''(0)\in[0,\infty]$.
Morover, the classes $\mathfrak{C}_n$ are pairwise disjoint.

Our final lemma in this section collects a number of properties of the function $\mathbb{L}\phi$ associated with any $\phi\in\mathfrak{C}^*$. Let us note the general fact that $\phi(x)\leq x\phi'(x)\leq\phi(2x)$, $x\geq 0$, holds for any increasing convex function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. This further implies

$$\phi(x) \asymp x\phi'(x) \tag{3.5}$$

if ϕ satisfies (1.2) and will enter into our arguments in several places.

LEMMA 3.5. *For each $\phi \in \mathfrak{C}^*$ with associated function $\mathbb{L}\phi$ as in (1.3) the following assertions hold:*

$$\phi(x) \;=\; x(\mathbb{L}\phi)'(x) - \mathbb{L}\phi(x), \quad x \geq 0.$$

If $\phi \in \mathfrak{C}_n^*$ for $n \geq 1$, then

$$2\phi(x/2) \;\leq\; \mathbb{L}\phi(x) \;\leq\; \phi(x), \quad x \geq 0, \tag{3.6}$$

and $\mathbb{L}\phi \asymp \phi$. If $\phi \in \mathfrak{C}_0^*$, then $\mathbb{L}\phi \in \mathfrak{C}_0^*$ and

$$\mathbb{L}\phi \;\geq\; \phi. \tag{3.7}$$

Finally, for any $\phi \in \mathfrak{C}^*$,

$$\liminf_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} \;>\; 0 \tag{3.8}$$

as well as

$$\liminf_{x\to\infty} \frac{\mathbb{L}\phi(x)}{x\log x} \;>\; 0 \tag{3.9}$$

hold true.

REMARK. It is tempting to believe that $\phi \in \mathfrak{C}_n^*$ implies $\mathbb{L}\phi \in \mathfrak{C}_n^*$ for $n \geq 1$ (as it does for $n = 0$). However, one can check that this is not true in general. As an example one may take $\phi(x) \stackrel{\text{def}}{=} \phi_0(x^2) \in \mathfrak{C}_1^*$ where $\phi_0(x) = x^2\mathbf{1}_{[0,1]}(x) + (2x-1)\mathbf{1}_{(1,\infty)}(x) \in \mathfrak{C}_0^*$ is the function already defined in the Introduction. One obtains for this case that $\psi(x) \stackrel{\text{def}}{=} \mathbb{L}\phi(x^{1/2}) = \frac{1}{3}x^2\mathbf{1}_{[0,1]}(x) + (\frac{1}{3}+\frac{4}{3}(x^{1/2}-1)+2(x^{1/2}-1)^2)\mathbf{1}_{(1,\infty)}(x)$ has derivative $\psi'(x) = \frac{2}{3}x\mathbf{1}_{[0,1]}(x)+(2-\frac{4}{3}x^{1/2})\mathbf{1}_{(1,\infty)}(x)$ which is obviously not concave, and thus $\mathbb{L}\phi \notin \mathfrak{C}_1^*$.

PROOF. The differential equation follows easily when integrating $(\mathbb{L}\phi)''(x) = \frac{\phi'(x)}{x}$. Turning to (3.6), the following estimations utilize that $\frac{\phi'(x)}{x}$ is nondecreasing for $\phi \in \mathfrak{C}_n^*$, $n \geq 1$. We obtain

$$\mathbb{L}\phi(x) \;\leq\; \int_0^x \int_0^s \frac{\phi'(s)}{s}\, dr\; ds \;=\; \phi(x) \tag{3.10}$$

for $x \geq 0$, and

$$\mathbb{L}\phi(x) \;\geq\; \int_0^x \int_{s/2}^s \frac{\phi'(r)}{r}\, dr\; ds \;\geq\; \int_0^x \int_{s/2}^s \frac{\phi'(s/2)}{s/2}\, dr\; ds \;=\; 2\phi(x/2)$$

for $x \geq 0$. In particular, $\mathbb{L}\phi \asymp \phi$ because of (1.2). If $\phi \in \mathfrak{C}_0^*$, then $\frac{\phi'(x)}{x}$ is nonincreasing so that $\mathbb{L}\phi \in \mathfrak{C}_0^*$ is obvious and (3.10) holds with reversed inequality sign thus showing (3.7). (3.8) follows from (3.6), or (3.7), and another appeal to (1.2). The final assertion holds because $\phi'(1) > 0$ for each $\phi \in \mathfrak{C}^*$ implies

$$\mathbb{L}\phi(x) \ \geq \ \int_1^x \int_1^s \frac{\phi'(1)}{r} \, dr \, ds \ = \ \phi'(1)\Big(x \log x - x + 1\Big)$$

for all $x \geq 1$. ◇

4. Auxiliary Moment Results

We are now going to prove a number of lemmata which will furnish the proof of Theorem 1.1 provided in the next section. In order to state them a number of random variables must be introduced. First, let $(X_{n,k})_{k,n\geq 1}$ be a family of i.i.d. random variables with distribution $(p_j)_{j\geq 0}$ such that the Galton-Watson process $(Z_n)_{n\geq 0}$ is given as

$$Z_n \ = \ \sum_{k=1}^{Z_{n-1}} X_{n,k}, \quad n \geq 1,$$

where $Z_0 = W_0 = 1$. For $k, n \geq 1$ we further define $\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(W_0, ..., W_n)$,

$$W_n^* \stackrel{\text{def}}{=} \max_{0\leq k\leq n} W_n \quad \text{and} \quad W^* \stackrel{\text{def}}{=} \sup_{n\geq 0} W_n,$$

$Y_{n,k} \stackrel{\text{def}}{=} X_{n,k} - \mu$ with generic copy Y, and

$$D_n \ \stackrel{\text{def}}{=} \ W_n - W_{n-1} \ = \ \frac{1}{\mu^n} \sum_{k=1}^{Z_{n-1}} Y_{n,k}.$$

Put $D_0 \stackrel{\text{def}}{=} 1$. It is stipulated for the rest of this article that C always denotes a finite positive constant which may differ from line to line.

Lemma 4.1. *Suppose $\sigma^2 \stackrel{\text{def}}{=} VarZ_1 < \infty$. Let $(c_n)_{n\geq 0}$ be a bounded sequence of real numbers with $c \stackrel{\text{def}}{=} \sup_{n\geq 0} |c_n|$ and $\phi \in \mathfrak{C}$. Then*

$$E\phi\left(\sum_{n\geq 1} c_n D_n\right) \ \leq \ C\left(1 \ + \ E\mathbb{S}^{-1}\phi\left(\sum_{n\geq 1} \frac{c^2\sigma^2}{\mu^{n+1}(\mu-1)} \, D_n\right) \ + \ \sum_{n\geq 1} E\phi(c_n D_n)\right). \quad (4.1)$$

and this inequality further simplifies to

$$E\phi\left(\sum_{n\geq 1} c_n D_n\right) \ \leq \ C\left(1 \ + \ \phi\left(\frac{c\sigma}{\mu^{1/2}(\mu-1)}\right) \ + \ \sum_{n\geq 1} E\phi(D_n)\right) \quad (4.2)$$

if $\phi \in \mathfrak{C}_0$. It is furthermore always true that

$$E\phi\left(\sum_{n\geq 1} c_n D_n\right) \ \leq \ C\left(1 + \sum_{n\geq 1} E\phi(D_n)\right). \quad (4.3)$$

PROOF. Note first that a.s.

$$E(D_n^2|\mathcal{F}_{n-1}) \;=\; E\left(\frac{1}{\mu^{2n}}\left(\sum_{k=1}^{Z_{n-1}} Y_{n,k}\right)^2 \middle| Z_{n-1}\right) \;=\; \frac{\sigma^2}{\mu^{2n}} Z_{n-1} \;=\; \frac{\sigma^2}{\mu^{n+1}} W_{n-1} \tag{4.4}$$

for all $n \geq 1$. An application of the Burkholder-Davis-Gundy inequality [10, Theorem 11.3.2] yields

$$E\phi\left(\sum_{n\geq 1} c_n D_n\right) \;\leq\; C\left(E\mathbb{S}^{-1}\phi\left(\sum_{n\geq 1} c_n^2 E(D_n^2|\mathcal{F}_{n-1})\right) \;+\; E\sup_{n\geq 1}\phi(c_n D_n)\right).$$

For the last term on the right hand side, we further obtain

$$E\sup_{n\geq 1}\phi(c_n D_n) \;\leq\; \sum_{n\geq 1} E\phi(c_n D_n) \;\leq\; \sum_{n\geq 1} E\phi(cD_n).$$

As to the first term on the right hand side, (4.4) and partial summation leads to

$$\begin{aligned}
E\mathbb{S}^{-1}\phi\left(\sum_{n\geq 1} c_n^2 E(D_n^2|\mathcal{F}_{n-1})\right) \;&\leq\; E\mathbb{S}^{-1}\phi\left(\sum_{n\geq 1} \frac{c^2\sigma^2}{\mu^{n+1}} W_{n-1}\right) \\
&=\; E\mathbb{S}^{-1}\phi\left(\sum_{n\geq 0} \frac{c^2\sigma^2}{\mu^{n+2}} \sum_{k=0}^{n} D_k\right) \\
&=\; E\mathbb{S}^{-1}\phi\left(\sum_{k\geq 0} \frac{c^2\sigma^2}{\mu^{k+2}} D_k \sum_{n\geq k} \frac{1}{\mu^{n-k}}\right) \\
&=\; E\mathbb{S}^{-1}\phi\left(\sum_{k\geq 0} \frac{c^2\sigma^2}{\mu^{k+1}(\mu-1)} D_k\right) \\
&\leq\; C\left(1 \;+\; E\mathbb{S}^{-1}\phi\left(\sum_{k\geq 1} \frac{c^2\sigma^2}{\mu^{k+1}(\mu-1)} D_k\right)\right),
\end{aligned}$$

where $\mathbb{S}^{-1}\phi(x+y) \leq \mathbb{S}^{-1}(2x) + \mathbb{S}^{-1}\phi(2y) \leq C(\mathbb{S}^{-1}\phi(x) + \mathbb{S}^{-1}\phi(y)) \leq C(1 + \mathbb{S}^{-1}\phi(y))$ for all $x, y > 0$ was utilized for the final inequality. (4.1) now follows by combining the previous inequalities. If $\mathbb{S}^{-1}\phi$ is concave and hence subadditive on $[0,\infty)$, then (4.2) is a direct consequence of (4.1) when noting that $E|D_n| \leq EW_n = 1$.

In order to see (4.3) suppose $\phi \in \mathfrak{C}_n$ for some $n \geq 0$. Then $\mathbb{S}^{-n-1}\phi$ is concave which in combination with (1.2), $\lim_{x\to\infty} \frac{\mathbb{S}^{-k}\phi(x)}{\phi(x)} = 0$ for $k \geq 1$, and an n-fold iteration of (4.1) yields (4.3). ◇

LEMMA 4.2. *Suppose* $\sigma^2 \stackrel{\text{def}}{=} VarZ_1 < \infty$ *and* $\phi \in \mathfrak{C}$. *Then*

$$E\phi(D_n) \;\leq\; C\left(E\mathbb{S}^{-1}\phi\left(\frac{\sigma^2}{\mu^{n+1}} W_{n-1}\right) + \mu^{n-1} E\phi\left(\frac{Y}{\mu^n}\right)\right). \tag{4.5}$$

for all $n \geq 1$.

PROOF. Since, for each $n \geq 1$, D_n is the limit of the martingale transform

$$H_{n,k} \stackrel{\text{def}}{=} \mu^{-n} \sum_{j=1}^{k} Y_{n,j} \mathbf{1}_{\{Z_{n-1} \geq j\}}, \quad k \geq 0$$

we infer by another appeal to the Burkholder-Davis-Gundy inequality

$$\begin{aligned} E\phi(D_n) &\leq C \left(E\mathbb{S}^{-1}\phi\left(\frac{1}{\mu^{2n}} \sum_{k\geq 1} E(Y_{n,k}^2 \mathbf{1}_{\{Z_{n-1}\geq k\}} | \mathcal{F}_{n-1}) \right) + E \sup_{k\geq 1} \phi\left(\frac{Y_{n,k}}{\mu^n}\right) \mathbf{1}_{\{Z_{n-1}\geq k\}} \right) \\ &\leq C \left(E\mathbb{S}^{-1}\phi\left(\sum_{k\geq 1} \frac{\sigma^2}{\mu^{2n}} \mathbf{1}_{\{Z_{n-1}\geq k\}} \right) + E\left(\sum_{k=1}^{Z_{n-1}} \phi\left(\frac{Y_{n,k}}{\mu^n}\right)\right)\right) \\ &= C \left(E\mathbb{S}^{-1}\phi\left(\frac{\sigma^2}{\mu^{n+1}} W_{n-1} \right) + \mu^{n-1} E\phi\left(\frac{Y}{\mu^n}\right)\right). \end{aligned} \qquad \diamondsuit$$

LEMMA 4.3. *Given $\phi \in \mathfrak{C}^*$ with $\phi''(0) \in (0,\infty)$ and $\mu \in (1,\infty)$, let X be a random variable with $E\phi(X) < \infty$. Then*

$$\sum_{n\geq 1} \mu^n E\phi\left(\frac{X}{\mu^n}\right) \;\leq\; C\,(1 + E\mathbb{L}\phi(X)). \tag{4.6}$$

PROOF. Put $I_n \stackrel{\text{def}}{=} (n-1, n]$ for $n \geq 1$ and note that $\sum_{n\geq 0} \mu^n \phi(n\mu^{-n}) < \infty$ because $\phi(x) = O(x^2)$ as $x \to 0$. Then

$$\begin{aligned} \sum_{n\geq 1} \mu^n E\phi\left(\frac{X}{\mu^n}\right) &= \sum_{n\geq 1} \mu^n \sum_{k\geq 1} E\phi\left(\frac{X}{\mu^n}\right) \mathbf{1}_{I_k}(|X|) \\ &\leq \sum_{n\geq 1} \mu^n \left(\phi(n\mu^{-n}) + \sum_{k\geq n} \phi(k\mu^{-n}) P(|X| \in I_k) \right) \\ &\leq \frac{1}{\mu} \left(\sum_{n\geq 1} \mu^n \phi(n\mu^{-n}) + \sum_{k\geq 1} k\, P(|X| \in I_k) \sum_{n=1}^{k} \phi'(k\mu^{-n}) \right), \end{aligned} \tag{4.7}$$

where $\phi(x) \leq x\phi'(x)$ has been utilized for the last inequality. Since $\sum_{n\geq 1} \mu^n \phi(n\mu^{-n})$ is finite, it remains to further estimate the second expression in (4.7). We obtain for $k \geq 1$

$$\begin{aligned} \sum_{n=1}^{k} \phi'(k\mu^{-n}) &= \sum_{n=1}^{k} \left(\phi'(k\mu^{-k}) \;+\; \sum_{i=n+1}^{k} \int_{k\mu^{-i}}^{k\mu^{-i+1}} \phi''(z)\, dz \right) \\ &\leq k\phi'(k\mu^{-k}) + \sum_{i=1}^{k} \sum_{n=1}^{i} \int_{k\mu^{-i}}^{k\mu^{-i+1}} \phi''(z)\, dz \\ &\leq C\mu^k \phi(k\mu^{-k}) + \sum_{i=1}^{k} \int_{k\mu^{-i}}^{k\mu^{-i+1}} i\phi''(z)\, dz, \end{aligned} \tag{4.8}$$

the final bound on $k\phi'(k\mu^{-k})$ being a consequence of (3.5). Now $k\mu^{-i} \le z \le k\mu^{-i+1}$ is equivalent to $-\log_\mu(z/k) \le i \le 1 - \log_\mu(z/k)$, whence

$$\begin{aligned}\sum_{i=1}^{k} \int_{k\mu^{-i}}^{k\mu^{-i+1}} i\phi''(z)\,dz \;&\le\; \sum_{i=1}^{k} \int_{k\mu^{-i}}^{k\mu^{-i+1}} \left(1 - \log_\mu\left(\frac{z}{k}\right)\right)\phi''(z)\,dz \\ &=\; \int_{k\mu^{-k}}^{k} \left(1 - \log_\mu\left(\frac{z}{k}\right)\right)\phi''(z)\,dz.\end{aligned} \tag{4.9}$$

Partial integration leads to

$$\begin{aligned}&\int_{k\mu^{-k}}^{k} \left(1 - \log_\mu\left(\frac{z}{k}\right)\right)\phi''(z)\,dz \\ &\quad= \left[\left(1 - \log_\mu\left(\frac{z}{k}\right)\right)\phi'(z)\right]_{k\mu^{-k}}^{k} \;+\; \frac{1}{\log\mu}\int_{k\mu^{-k}}^{k} \frac{\phi'(z)}{z}\,dz \\ &\quad= \phi'(k) \;-\; (k+1)\phi'(k\mu^{-k}) \;+\; \frac{1}{\log\mu}\big((\mathbb{L}\phi)'(k) - (\mathbb{L}\phi)'(k\mu^{-k})\big) \\ &\quad\le \phi'(k) \;+\; \frac{1}{\log\mu}(\mathbb{L}\phi)'(k) \\ &\quad\le C(1 + (\mathbb{L}\phi)'(k))\end{aligned} \tag{4.10}$$

for all $k \ge 1$, where we have utilized that $(\mathbb{L}\phi)'(k) = \int_0^k \frac{\phi'(s)}{s}\,ds \ge \frac{1}{2}\phi'(k/2) \ge C\phi'(k)$ for all $k \ge 1$. Summarizing the results from (4.8-10) and recalling $\sum_{n\ge1} \mu^n \phi(n\mu^{-n}) < \infty$, we obtain

$$\begin{aligned}\sum_{n\ge1} n\,P(|X| \in I_n) \sum_{k=1}^{n} \phi'(n\mu^{-k}) \;&\le\; C\sum_{n\ge1} n(1 + (\mathbb{L}\phi)'(n))P(|X| \in I_n) \\ &\le\; C\,E|X|(1 + (\mathbb{L}\phi)'(|X|)) \;<\; \infty\end{aligned}$$

and thus the desired bound for the second expression in (4.7) because $\mathbb{L}\phi(x) \asymp x(\mathbb{L}\phi)'(x)$. $\diamondsuit$

LEMMA 4.4. *Let $\phi \in \mathfrak{C}^*$ and $X, X_1, X_2, \ldots$ be integrable i.i.d. random variables with partial sums $S_n = X_1 + \ldots + X_n$ for $n \ge 1$. Then $E\sup_{n\ge1} \phi(S_n/n) < \infty$ iff $E\mathbb{L}\phi(X) < \infty$.*

PROOF. Put $U_n \stackrel{\text{def}}{=} X_1^+ + \ldots + X_n^+$ and $V_n \stackrel{\text{def}}{=} X_1^- + \ldots + X_n^-$ for $n \ge 1$. Then $(n^{-1}U_n)_{n\ge1}$ and $(n^{-1}V_n)_{n\ge1}$ are both nonnegative reversed martingales. By Theorem 2.1 in [3],

$$E\sup_{n\ge1} \phi(U_n/n) \;\le\; C\,E\mathbb{L}\phi(X_1^+)$$

and similarly $E\sup_{n\ge1} \phi(V_n/n) \le C\,E\mathbb{L}\phi(X_1^-$. This proves the direct conclusion of the lemma.

The converse follows by the same proof as for the case $\phi(x) = x$ given by Chow and Teicher [10,Thm. 10.3.3]: It is easily seen that $E\sup_{n\ge1} \phi(S_n/n) < \infty$ implies $E\sup_{n\ge1} \phi(X_n/n) < \infty$. The integrability of X ensures

$$P\Big(\sup_{n\ge1} |X_n|/n > t\Big) \;\ge\; C\sum_{n\ge1} P(|X| \ge nt)$$

for all $t \geq T$, T sufficiently large. Consequently,

$$\begin{aligned}
\infty \;>\; E\sup_{n\geq 1}\phi(X_n/n) \;&\geq\; C\int_T^\infty \phi'(t)\sum_{n\geq 1}P(|X|\geq nt)\;dt \\
&\geq\; C\int_{\{|X|\geq T\}}\int_T^\infty \phi'(t)\sum_{n\geq 1}\mathbf{1}_{\{n\leq |X|/t\}}\;dt\;dP \\
&\geq\; C\int_{\{|X|\geq T\}}\int_T^{|X|} \phi'(t)\Big(\frac{|X|}{t}-1\Big)\;dt\;dP \\
&=\; C\int_{\{|X|\geq T\}} |X|((\mathbb{L}\phi)'(|X|)-(\mathbb{L}\phi)'(T))-(\phi(X)-\phi(T))\;dP \\
&\geq\; C\Big(E|X|(\mathbb{L}\phi)'(|X|)-1\Big),
\end{aligned}$$

which proves the lemma. ♢

Given a supercritical Galton-Watson process $(Z_n)_{n\geq 0}$ with finite mean offspring and normalized limit W, the Kesten-Stigum theorem provides the equivalence of the nondegeneracy of W with the so-called $(L\log L)$-condition $EZ_1\log Z_1<\infty$, which may also be stated as

$$EW>0 \quad \text{iff} \quad EZ_1\log Z_1<\infty. \tag{4.11}$$

Lemma 4.5 below will furnish a new and very short proof of the crucial "if"- part of (4.11) included in the proof of Theorem 1.1 in the next section.

LEMMA 4.5. *Let X be any nonnegative random variable with finite mean.*

(a) Then there exists a function $\phi\in\mathfrak{C}_0^*$ such that $\lim_{x\to\infty}\frac{\phi(x)}{x}=\infty$ and $E\phi(X)<\infty$.

(b) If $EX\log^+X<\infty$ then there exists a function $\phi\in\mathfrak{C}_0^*$ such that $\lim_{x\to\infty}\frac{\phi(x)}{x}=\infty$ and $E\mathbb{L}\phi(X)<\infty$.

The reader might expect at first glance that part (b) is a trivial consequence of (a). Namely, since $x\log x\sim\varphi_0(x)\stackrel{\text{def}}{=}(x+1)\log(x+1)-x\in\mathfrak{C}_0^*$, part (a) applied to $\varphi_0(X)$ implies the existence of a function $\psi\in\mathfrak{C}_0^*$ such that $\lim_{x\to\infty}\frac{\psi(x)}{x}=\infty$ and $E\psi\circ\varphi_0(X)<\infty$. However, to conclude assertion (b) we must have $\psi\circ\varphi_0\in\mathfrak{C}_0^*$ which may fail to hold.

PROOF. (a) Integrability of X implies the existence of $0\stackrel{\text{def}}{=}a_0<a_1<...\uparrow\infty$, such that

$$\int_{\{X>a_n\}}X\;dP\;\leq\;2^{-n}$$

for all $n\geq 1$. We may choose the a_n such that $a_n-a_{n-1}\uparrow\infty$. Defining the convex function $\psi:\mathbb{R}\to\mathbb{R}$, $x\mapsto\sum_{n\geq 0}(x-a_n)^+$, we obviously have

$$\frac{\psi(x)}{x}\;=\;\sum_{n\geq 1}\Big(1-\frac{a_n}{x}\Big)^+\;\uparrow\;\infty\quad(x\uparrow\infty)$$

and, by choice of the a_n,

$$\begin{aligned} E\psi(X) &= \sum_{n\geq 1} E(X-a_n)^+ \\ &\leq \sum_{n\geq 1}\int_{\{X>a_n\}} X\ dP \\ &\leq EX + \sum_{n\geq 1}\frac{1}{2^n} < \infty. \end{aligned}$$

We will now define a function $\phi \in \mathfrak{C}_0^*$ satisfying $\phi' \leq \psi'$, $\phi \leq \psi$ and $\lim_{x\to\infty}\frac{\phi(x)}{x} = \infty$. This clearly proves part (a) of the lemma.

Note that ψ is differentiable with derivative $\psi'(x) = \sum_{n\geq 1} n\,\mathbf{1}_{[a_{n-1},a_n)}(x)$ for all $x \notin \{a_n : n\geq 0\}$. Put

$$\phi'(x) \stackrel{\text{def}}{=} \sum_{n\geq 0}\left(n + \frac{x-a_n}{a_{n+1}-a_n}\right)\mathbf{1}_{[a_n,a_{n+1})}(x)$$

for $x \geq 0$. Then ϕ' is a continuous function dominated by ψ', starting at 0, and concave on $[0,\infty)$ because the differences $a_n - a_{n-1}$ are increasing. Consequently, its primitive $\phi(x) \stackrel{\text{def}}{=} \int_0^{|x|}\phi'(y)\,dy$ belongs to the class $\mathfrak{C}_0^*$. A comparison of the areas under the curves of ψ' and ϕ' also shows that $\phi \sim \psi$, hence $\lim_{x\to\infty}\frac{\phi(x)}{x} = \infty$.

(b) Consider the function $\varphi_0(x) \stackrel{\text{def}}{=} (x+1)\log(x+1) - x \sim x\log x$. Since $\varphi_0(0) = 0$, $\varphi_0'(x) = \log(x+1)$ and $\varphi_0''(x) = \frac{1}{x+1}$ we see that $\varphi_0 \in \mathfrak{C}_0^*$ and obtain by partial integration using Fubini's theorem

$$\begin{aligned} E\varphi_0(X) &= \int_0^\infty \log(x+1)P(X>x)\ dx \\ &= \int_0^\infty \frac{1}{y+1}\int_y^\infty P(X>x)\ dx\ dy = EX\,EY, \end{aligned}$$

where Y is a nonnegative random variable with survival function

$$P(Y>y) = \frac{1}{(y+1)EX}\int_y^\infty P(X>x)\ dx, \quad y\geq 0.$$

Part (a) ensures the existence of a function $\psi \in \mathfrak{C}_0^*$ such that $\psi(Y) < \infty$. Let Ψ be the associated function defined in (1.3). Using that $\Psi'(x) = \int_0^x \frac{\psi'(r)}{r}\,dr$ for $x\geq 0$ and $\psi'(0) = 0$ we then obtain

$$\begin{aligned} \infty > E\psi(Y) &= \int_0^\infty \psi'(y)P(Y>y)\ dy \\ &= \frac{1}{EX}\int_0^\infty \frac{\psi'(y)}{y+1}\int_y^\infty P(X>x)\ dx\ dy \\ &\leq \frac{1}{EX}\int_0^\infty\left(\int_0^x \frac{\psi'(y)}{y}\ dy\right)P(X>x)\ dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{EX}\int_0^\infty \Psi'(x)P(X>x)\,dx \\
&= E\Psi(X).
\end{aligned}
$$

Since $\Psi \in \mathfrak{C}_0^*$ (Lemma 4.4) and since $\lim_{x\to\infty}\psi'(x)=\infty$ implies $\lim_{x\to\infty}\frac{\Psi(x)}{x\log x}=\infty$ we have proved the asserted result. $\diamondsuit$

5. Proof of Theorem 1

Proof of Theorem 1 (direct part). We begin with the direct part and must therefore show that $E\mathbb{L}\phi(Z_1)<\infty$, or equivalently $EZ_1(\mathbb{L}\phi)'(Z_1)<\infty$, implies $0<E\phi(W)<\infty$. We first show that $E\phi(W)<\infty$ by distinguishing the cases $\phi\in\mathfrak{C}_n^*$ for $n\geq 0$ and using an induction over n.

Step 1. Let $\phi\in\mathfrak{C}_0^*$ in which case ϕ' is concave. Suppose $\phi''(0)\in(0,\infty)$ w.l.o.g. (by Lemma 3.3(d)). Since $W_0=1$, an application of the Topchii-Vatutin-inequality (see [2] or [14]) yields

$$
E\phi(W_n) \;\leq\; \phi(1)+C\sum_{k=1}^n E\phi(D_k) \tag{5.1}
$$

for all $n\geq 1$ (one may take $C=1$ as shown in [2]). We want to show

$$
\sup_{n\geq 0} E\phi(W_n)<\infty,
$$

which by the previous inequality follows if

$$
\sum_{k\geq 1} E\phi(D_k)<\infty \tag{5.2}
$$

The following estimation will use that the sequence $(\sum_{j=1}^n \frac{Y_{k,j}}{\mu^k}\mathbf{1}_{\{Z_{k-1}\geq j\}})_{n\geq 0}$ is a martingale and that Z_{k-1} is independent of $(Y_{k,j})_{j\geq 1}$. By another appeal to the Topchii-Vatutin-inequality,

$$
\begin{aligned}
E\phi(D_k) &= E\phi\left(\frac{1}{\mu^k}\sum_{j=1}^{Z_{k-1}} Y_{k,j}\right) \\
&= E\phi\left(\sum_{j\geq 1}\frac{Y_{k,j}}{\mu^k}\mathbf{1}_{\{Z_{k-1}\geq j\}}\right) \\
&\leq C\sum_{j\geq 1} E\phi\left(\frac{Y_{k,j}}{\mu^k}\right)\mathbf{1}_{\{Z_{k-1}\geq j\}} \\
&= C\,E\phi\left(\frac{Y}{\mu^k}\right)\sum_{j\geq 1}P(Z_{k-1}\geq j) \\
&= C\,E\phi\left(\frac{Y}{\mu^k}\right)EZ_{k-1} \\
&= C\mu^{k-1}E\phi\left(\frac{Y}{\mu^k}\right).
\end{aligned}
$$

So we obtain in combination with Lemma 4.3 (recall $\phi''(0) \in (0,\infty)$)

$$\sum_{k\geq 1} E\phi(D_k) \;\leq\; C\sum_{k\geq 1}\mu^{k-1}E\phi\left(\frac{Y}{\mu^k}\right) \;\leq\; C\,E\mathbb{L}\phi(Y) \;<\; \infty, \tag{5.3}$$

which is the desired conclusion because $Y \stackrel{d}{=} Z_1 - \mu$.

Step 2. Now let $\phi \in \mathfrak{C}_n^*$ for some $n \geq 1$ and suppose that $E\psi(W) < \infty$ for all $\psi \in \mathfrak{C}_k^*$ and $0 \leq k < n$. Note that $\mathbb{S}^{-1}\phi \in \mathfrak{C}_{n-1}^*$, $\mathbb{S}^{-1}\phi(x) \asymp x(\mathbb{S}^{-1}\phi)'(x)$ by (3.5), and thus (by the induction hypothesis)

$$\sup_{n\geq 0} EW_n(\mathbb{S}^{-1}\phi)'(W_n) \;\leq\; C\sup_{n\geq 0} E\mathbb{S}^{-1}\phi(W_n) \;\leq\; C\,E\mathbb{S}^{-1}\phi(W) \;<\; \infty.$$

Note also that $EZ_1^2 \leq C\,E\mathbb{L}\phi(Z_1) < \infty$ because $\liminf_{x\to\infty}\frac{\mathbb{L}\phi(x)}{x^2} > 0$ for $\phi \in \mathfrak{C}_n^*$ with $n \geq 1$. Combining these facts with Lemmata 3.5 and 4.1-3, we now infer

$$\begin{aligned} E\phi(W-1) \;&=\; E\phi\left(\sum_{n\geq 1} D_n\right) \;\leq\; C\left(1+\sum_{n\geq 1}E\phi(D_n)\right) \\ &\leq\; C\left(1+\sum_{n\geq 1}E\mathbb{S}^{-1}\phi\left(\frac{\sigma^2}{\mu^{n+1}}W_{n-1}\right)+\sum_{n\geq 1}\mu^{n-1}E\phi\left(\frac{Y}{\mu^n}\right)\right) \\ &\leq\; C\left(1+\sum_{n\geq 1}\frac{\sigma^2}{\mu^{n+1}}EW_{n-1}(\mathbb{S}^{-1}\phi)'\left(\frac{\sigma^2}{\mu^{n+1}}W_{n-1}\right)+E\mathbb{L}\phi(Y)\right) \\ &\leq\; C\left(E\mathbb{S}^{-1}\phi(W)\sum_{n\geq 1}\frac{\sigma^2}{\mu^{n+1}} \;+\; E\mathbb{L}\phi(Y)\right) \;<\; \infty. \end{aligned}$$

Step 3. The proof of the direct part of Theorem 1.1 is now completed by showing that $E\mathbb{L}\phi(Z_1) < \infty$ implies $E\phi(W) > 0$ for any $\phi \in \mathfrak{C}^*$. To that end note first that $E\mathbb{L}\phi(Z_1) < \infty$ implies $EZ_1 \log Z_1 < \infty$ by (3.9) in Lemma 3.5. If $EZ_1 \log Z_1 < \infty$ then Lemma 4.5 ensures $E\mathbb{L}\psi(Z_1) < \infty$ for some $\psi \in \mathfrak{C}_0^*$ satisfying $\lim_{x\to\infty}\frac{\phi(x)}{x} = \infty$. Consequently, by recalling (5.1-3) we infer

$$\sup_{n\geq 0} E\psi(W_n) \;\leq\; \psi(1)+\sum_{n\geq 1}E\phi(D_n) \;\leq\; C\,E\mathbb{L}\psi(Z_1) \;<\; \infty$$

and thus the uniform integrability of $(W_n)_{n\geq 0}$, in particular $EW = EW_0 = 1 > 0$ which completes the proof. ♢

Proof of Theorem 1 (converse). Since $P(W > 0) > 0$, there exist $0 < \eta < 1 < T$ such that $\gamma \stackrel{\text{def}}{=} \inf_{n\geq 0} P(\eta \leq W_n^* \leq t) > 0$. It follows that

$$P(W^* > t) \;=\; P(W_0 > t) + \sum_{n\geq 0} P(W_n^* \leq t, W_{n+1} > t)$$

$$
\begin{aligned}
&\geq \sum_{n\geq 0} P\left(\eta \leq W_n^* \leq t, \frac{1}{\mu^{n+1}} \sum_{j=1}^{\eta\mu^n} X_{n,j} > t\right) \\
&\geq \sum_{n\geq 0} P(\eta \leq W_n^* \leq t) P(\overline{S}_{\eta\mu^n} > \mu t/\eta) \\
&\geq \gamma \sum_{n\geq 1} P(\overline{S}_{\eta\mu^n} > \mu t/\eta)
\end{aligned}
$$

for $t \geq T$, where $S_k \stackrel{\text{def}}{=} X_{1,1} + ... + X_{1,k}$ for $k \geq 1$ and $\overline{S}_t \stackrel{\text{def}}{=} t^{-1} \sum_{k=1}^{\lceil t \rceil} X_{1,k}$ for $t \in (0, \infty)$. Since

$$
P\Big(\max_{\eta\mu^{n-1} < k \leq \eta\mu^n} \overline{S_k} > t/\eta\Big) \;\leq\; P\left(\frac{1}{\eta\mu^{n-1}} \sum_{k=1}^{\eta\mu^n} S_k > t/\eta\right) \;=\; P(\overline{S}_{\eta\mu^n} > \mu t/\eta)
$$

for all $t > 0$, we further obtain

$$
\begin{aligned}
P(W^* > t) &\geq \gamma \sum_{n\geq 1} P(\overline{S}_{\eta\mu^n} > \mu t/\eta) \\
&\geq \gamma \sum_{n\geq 1} P\Big(\max_{\eta\mu^{n-1} < k \leq \eta\mu^n} \overline{S_k} > t/\eta\Big) \\
&\geq \gamma P\Big(\sup_{k\geq 1} \overline{S}_k > t/\eta\Big)
\end{aligned}
$$

for all $t \geq T$. Consequently, given any $\phi \in \mathfrak{C}^*$, $E\phi(W^*)$ is finite if $E\phi(\sup_{k\geq 1} \overline{S}_k) < \infty$ which in turn holds iff $E\mathbb{L}\phi(Z_1) < \infty$ as we have proved in Lemma 4.4. ♢

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