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The Minimal Subgroup of a Random Walk

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The Minimal Subgroup of a Random Walk

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It is proved that for each random walk $(S_n)_{n\geq 0}$ on $\mathbb{R}^d$ there exists a smallest measurable subgroup $\mathbb{G}$ of $\mathbb{R}^d$, called minimal subgroup of $(S_n)_{n\geq 0}$, such that $P(S_n \in \mathbb{G}) = 1$ for all $n \geq 1$. $\mathbb{G}$ can be defined as the set of all $x \in \mathbb{R}^d$ for which the difference of the time averages $n^{-1}\sum_{k=1}^n P(S_k \in \cdot)$ and $n^{-1}\sum_{k=1}^n P(S_k + x \in \cdot)$ converges to 0 in total variation norm as $n \to \infty$. The related subgroup $\mathbb{G}^*$ consisting of all $x \in \mathbb{R}^d$ for which $\lim_{n\to\infty} \|P(S_n \in \cdot) - P(S_n + x \in \cdot)\| = 0$ is also considered and shown to be the minimal subgroup of the symmetrization of $(S_n)_{n\geq 0}$. In the final section we consider quasi-invariance and admissible shifts of probability measures on $\mathbb{R}^d$. The main result shows that, up to regular linear transformations, the only subgroups of $\mathbb{R}^d$ admitting a quasi-invariant measure are those of the form $\mathbb{G}_1' \times ... \times \mathbb{G}_k' \times \mathbb{R}^{l-k} \times \{0\}^{d-l}$, $0 \leq k \leq l \leq d$, with $\mathbb{G}_1', ..., \mathbb{G}_k'$ being countable subgroups of $\mathbb{R}$. The proof is based on a result recently proved by Kharazishvili [3] which states no uncountable proper subgroup of $\mathbb{R}$ admits a quasi-invariant measure.

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1. Introduction

Given i.i.d. random variables $X_1, X_2, ...$ taking values in $\mathbb{R}^d$ for some $d \geq 1$, let $(S_n)_{n\geq 0}$ be the associated zero-delayed random walk, i.e. $S_0 = 0$ and $S_n = X_1 + ... + X_n$ for $n \geq 1$. If the X_n are a.s. concentrated on a measurable subgroup $\mathbb{G}$ of $\mathbb{R}^d$ then the same holds true for $(S_n)_{n\geq 0}$. Although a trivial fact this is of great importance when dealing with renewal theorems or local limit theorems for random walks on $\mathbb{R}$ (see e.g. [1]) and leads to a distinction of random walks with respect to their lattice-type, defined through the smallest *closed* subgroup of $\mathbb{R}$ on which all partial sums are concentrated. However, this definition appears to be inappropriate when asking for the *minimal (measurable) subgroup* on which a random walk is concentrated. Take, for example, a sequence $X_1, X_2, ...$ of i.i.d. Lapacians on $\{1, \pi\}$, i.e. $P(X_1 = 1) = P(X_1 = \pi) = \frac{1}{2}$. Then the minimal subgroup on which the associated random walk "lives" is evidently $\mathbb{G} = \mathbb{Z} + \pi\mathbb{Z} \stackrel{\text{def}}{=} \{m + n\pi : m, n \in \mathbb{Z}\}$ whereas the smallest closed subgroup is $\mathbb{R}$ itself. More generally, whenever the X_n take values only in a countable subset $\mathfrak{X} = \{x_1, x_2, ...\}$ of $\mathbb{R}^d$, the minimal subgroup is given by the countable set of all finite linear combinations of elements of $\mathfrak{X}$ with integral coefficients. The minimality of $\mathbb{G}$ may be expressed here by the following two properties:

(G.1) $P(S_n \in \mathbb{G}$ for all $n \geq 0) = 1$;

(G.2) If $\mathbb{G}'$ is any other measurable subgroup of $\mathbb{R}^d$ such that $P(S_n \in \mathbb{G}'$ for all $n \geq 0) = 1$, then $\mathbb{G} \subset \mathbb{G}'$. In other words,

$$\mathbb{G} \;=\; \bigcap_{\mathbb{G}' \text{ measurable subgroup of } \mathbb{R}^d \text{ satisfying (G.1)}} \mathbb{G}' \tag{1.1}$$

On the other hand, for random walks with a non-discrete increment distibution there seems to be no such obvious way to determine a measurable group $\mathbb{G}$ meeting conditions (G.1) and (G.2). To see why note that the set of all measurable subgroups of $\mathbb{R}$ has the cardinality of the continuum (cf. Theorem 17 of [7], p. 149). There even exists a continuum of uncountable, measurable subgroups $\mathbb{G}_i, i \in I$, of $\mathbb{R}$ for which $\mathbb{G}_i \cap \mathbb{G}_j = \{0\}$ for all $i \neq j$ (cf. Theorem 20 of [7], p. 150). It is hence conceivable that the intersection in (1.1), as taken over an uncountable number of subgroups $\mathbb{G}'$, may give a nonmeasurable group $\mathbb{G}$. The reader should now be convinced that the question of the existence of a minimal subgroup for a general random walk on $\mathbb{R}^d$ is a nontrivial problem which will be positively answered by Theorem 1 in the next section. A number of fairly straightforward implications can be found at the end of that section.

In Section 3, we will discuss an interesting connection between the minimal subgroup $\mathbb{G}$ of $(S_n)_{n\geq 0}$ and the following zero-one law. Theorem 2 will show that

$$\bar{\Delta}_n(x) \;\stackrel{\text{def}}{=}\; \frac{1}{n}\left\|\sum_{k=1}^{n}(P(S_k \in \cdot) - P(S_k + x \in \cdot))\right\| \tag{1.2}$$

either equals one for all $n \geq 1$ or converges to 0, as $n \to \infty$, and that

$$\mathbb{G} \;=\; \{x \in \mathbb{R}^d : \lim_{n\to\infty} \bar{\Delta}(x) = 0\}. \tag{1.3}$$

Here $\|\cdot\|$ denotes total variation norm, i.e. $\|\Psi\| = \sup_{B\in\mathfrak{B}^d}|\Psi(B)|$ for a finite signed measure Ψ on $(\mathbb{R}^d, \mathfrak{B}^d)$. The result will follow by a very similar coupling argument as the one given by Lindvall and Rogers [6] for the related stronger zero-one law[1)] that

$$\Delta_n(x) \stackrel{\text{def}}{=} \|P(S_n \in \cdot) - P(S_n + x \in \cdot)\| \tag{1.4}$$

either equals one for all $n \geq 1$ or converges to 0, as $n \to \infty$. Theorem 3 will add to that latter result the conclusion that

$$\mathbb{G}^* \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \lim_{n\to\infty} \Delta_n(x) = 0\} \tag{1.5}$$

is a measurable subgroup of $\mathbb{G}$ which either equals $\mathbb{G}$ or is a null event for S_1. Moreover, $\mathbb{G}^*$ will turn out the minimal subgroup of the symmetrization of $(S_n)_{n\geq 0}$. As described in [11] in a general framework, the convergence of the Césaro total variation $\bar{\Delta}_n(x)$ links to shift coupling theory and the invariant σ-field of $(S_n)_{n\geq 0}$, while the convergence of $\Delta_n(x)$ links to exact coupling theory and the tail σ-field of $(S_n)_{n\geq 0}$. This will also be briefly discussed.

Given a σ-finite measure Ψ on $(\mathbb{R}^d, \mathfrak{B}^d)$, an element $x \in \mathbb{R}^d$ is called an *admissible shift for* Ψ if $\Psi * \delta_x$ is absolutely continuous with respect to Ψ ($\Psi * \delta_x \ll \Psi$), see [10, p. 102] for this definition in the more general context of σ-finite measures on Hilbert spaces. If Ψ is concentrated on a measurable subgroup $\mathbb{G}$ of $\mathbb{R}^d$ and satisfies $\Psi * \delta_x \ll \Psi$ for all $x \in \mathbb{G}$ then Ψ is called *quasi-invariant on* $\mathbb{G}$ (see [8, p. 297] or [3, p. 18]). Let $\mathbb{H}_\Psi$ be the set of all admissible shifts for Ψ. It is known (see [10, p. 103]) and will be reproved in Section 4 that $\mathbb{H}_\Psi$ forms a measurable subsemigroup of $\mathbb{R}^d$ which is even a group if Ψ is symmetric. Since the previously defined properties remain unaffected when Ψ is replaced with an equivalent measure and since every σ-finite measure is equivalent to a probability measure it suffices to consider probability measures.

Now let $(S_n)_{n\geq 0}$ be any random walk with increment distribution Q. Since $Q * \delta_x \ll Q$ clearly implies $\Delta_1(x) = \|Q - Q * \delta_x\| < 1$ we infer $\mathbb{H}_Q \subset \mathbb{G}^*$. The final Section 4 provides a number of results concerning $\mathbb{H}_Q$. Its main result, Theorem 4, gives a description of all measurable subgroups $\mathbb{G}$ of $\mathbb{R}^d$ admitting a quasi-invariant σ-finite measure Ψ and states in particular that each such $\mathbb{G}$ is locally compact (not necessarily in the topology induced by $\mathbb{R}^d$) with a Haar measure $\lambda_{\mathbb{G}}$, say, equivalent to Ψ. The proof is basically a reduction to the one-dimensional case ($d = 1$) and a subsequent use of the result that no uncountable measurable proper subgroup $\mathbb{G}$ of $\mathbb{R}$ admits a quasi-invariant σ-finite measure. This was recently shown by Kharazishvili [3, Theorem 3 on p. 216] by making use of rather "heavy machinery", notably the Mackey-Weil theorem and deep results concerning the structure of locally compact Abelian groups. He also asked for a more elementary proof only based upon real analysis and classical measure theory. We have tried but failed to find such an alternative.

Despite the measure theoretic rather than probabilistic nature of the problems discussed in this article our principal technique in many of the proofs is a suitable *coupling* and thus based on probabilistic reasoning. For a good introduction of the coupling technique and its applications the reader may consult the monographs by Lindvall [5] and Thorisson [11].

[1)] a zero-*two* law when defining total variation as in [6] by $\|\Psi\| \stackrel{\text{def}}{=} \Psi^+(\mathbb{R}^d) + \Psi^-(\mathbb{R}^d)$ where $\Psi = \Psi^+ - \Psi^-$ is the Jordan decomposition of Ψ.

2. The minimal subgroup of a random walk

The following notation will be used hereafter. The Borel-σ-field on $\mathbb{R}^d$ is denoted $\mathfrak{B}^d$. In case $d=1$ we write $\mathfrak{B}$ instead of $\mathfrak{B}^1$. Let $\mathfrak{B}_A$ be the restriction of $\mathfrak{B}$ to a subset A. Let Q denote the distribution of X_1 and, for $n\ge 0$, Q_n the distribution of S_n, i.e. $Q=Q_1$ and $Q_n=Q^{*(n)}$. If $\lambda=(\lambda_n)_{n\ge 0}$ is a distribution on $\mathbb{N}_0$, the set of nonnegative integers, then $Q_\lambda \stackrel{\text{def}}{=} \sum_{n\ge 0}\lambda_n Q_n = \sum_{n\ge 0}\lambda_n P(S_n\in\cdot) = P(S_T\in\cdot)$, where T is a random variable with distribution λ and independent of $(S_n)_{n\ge 0}$. Finally, put $Q_{n,x}\stackrel{\text{def}}{=} Q_n * \delta_x = P(S_n+x\in\cdot)$ and $Q_{\lambda,x}\stackrel{\text{def}}{=} Q_\lambda * \delta_x = P(S_T+x\in\cdot)$ for $x\in\mathbb{R}^d$. Notice that $Q_{\lambda*\mu}(\cdot)=\int Q_{\lambda,x}(\cdot)\,Q_\mu(dx)$.

As usual, let $a\wedge b\stackrel{\text{def}}{=}\min(a,b)$ and $a\vee b\stackrel{\text{def}}{=}\max(a,b)$ for real numbers a,b. For two probability measures P_1,P_2 on the same measurable space $(\Omega,\mathfrak{A})$, their maximal common component (infimum) is defined as

$$P_1\wedge P_2(dx) \stackrel{\text{def}}{=} \left(\frac{dP_1}{d(P_1+P_2)}(x)\wedge\frac{dP_2}{d(P_1+P_2)}(x)\right)(P_1+P_2)(dx).$$

If $P_1\wedge P_2=0$ then P_1 and P_2 are mutually singular ($P_1\perp P_2$). We note that each pair (P_1,P_2) with $P_1\wedge P_2\ne 0$ possesses a *maximal coupling*, given by any pair of random variables (Z_1,Z_2) on the same probability space such that

$$\begin{aligned}\mathcal{L}(Z_i|Z_1=Z_2) &= \|P_1\wedge P_2\|^{-1}P_1\wedge P_2,\\ \mathcal{L}(Z_i) &= P_i\end{aligned}$$

for $i=1,2$, where $\mathcal{L}(X)$ stands for "distribution of X", see [5] for further details. Conversely, if there is a coupling (Z_1,Z_2) of (P_1,P_2), i.e. $\mathcal{L}(Z_i)=P_i$ for $i=1,2$, with $P(Z_1=Z_2)>0$, then $P_1\wedge P_2\ne 0$.

Finally, ν shall hereafter always be the geometric(1/2) distribution on the positive integers, that is $\nu_0=0$ and $\nu_n=2^{-n}$ for $n\in\mathbb{N}$. Hence Q_ν is the distribution of the first partial sum obtained by geometrically sampling (with parameter 1/2) $(S_n)_{n\ge 0}$.

Theorem 1. *Given the previous notation and assumptions, the set*

$$\mathbb{G} \stackrel{\text{def}}{=} \{x\in\mathbb{R}^d : Q_\nu\wedge Q_{\nu,x}\ne 0\} \tag{2.1}$$

defines a measurable subgroup of $\mathbb{R}^d$ which satisfies (G.1) and (G.2) and is thus the minimal subgroup of $(S_n)_{n\ge 0}$.

The proof of the theorem divides into two parts, given as Lemma 2 and 4, the first of which shows that $\mathbb{G}$ as defined above is indeed a measurable subset of $\mathbb{R}^d$. This is furnished by the following standard lemma based upon the separability of the Borel-σ-field $\mathfrak{B}^d$. In Lemma 4 we then show that $\mathbb{G}$ is also a group meeting the conditions (G.1) and (G.2).

Lemma 1. *There is a measurable function $f:(\mathbb{R}^d\times\mathbb{R}^d,\mathfrak{B}^d\otimes\mathfrak{B}^d)\to([0,1],\mathfrak{B}_{[0,1]})$ such that, for every $x\in\mathbb{R}^d$, $f(x,\cdot)$ is a density of Q_ν with respect to $Q_\nu+Q_{\nu,x}$.*

PROOF. For notational ease we only consider the case $d = 1$. The modifications for $d \geq 2$ are easily provided.

Put $B_{n,k} \stackrel{\text{def}}{=} (k/2^n, (k+1)/2^n]$ for $n \geq 0$ and $k \in \mathbb{Z}$ and $\mathfrak{B}_n \stackrel{\text{def}}{=} \sigma(B_{n,k}; k \in \mathbb{Z})$ for $n \geq 0$. Then $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots$ and $\mathfrak{B} = \sigma(\mathfrak{B}_n; n \geq 0)$. Let $Q_\nu | \mathfrak{B}_n$ be the restriction of Q_ν to the σ-field $\mathfrak{B}_n$. For each $x \in \mathbb{R}$, define $f_n : (\mathbb{R}^2, \mathfrak{B}^2) \to ([0,1], \mathfrak{B}_{|[0,1]})$ as

$$
\begin{aligned}
f_n(x,y) &\stackrel{\text{def}}{=} \frac{dQ_\nu|\mathfrak{B}_n}{d(Q_\nu + Q_{\nu,x})|\mathfrak{B}_n}(y) \\
&= \sum_{k \in \mathbb{Z}} \mathbf{1}_{B_{n,k}}(y) \left(\frac{Q_\nu(B_{n,k})}{Q_\nu(B_{n,k}) + Q_{\nu,x}(B_{n,k})} \right)
\end{aligned}
$$

with the convention $\frac{0}{0+0} \stackrel{\text{def}}{=} 0$. Notice that $f_n \leq 1$. Then

$$
f(x,y) \stackrel{\text{def}}{=} \liminf_{n \to \infty} f_n(x,y)
$$

is also jointly measurable and, for each $x \in \mathbb{R}$, it defines a density of Q_ν with respect to $Q_\nu + Q_{\nu,x}$ on $\mathfrak{B}$. Indeed, for each fixed x, $f_n(x,\cdot)$ equals the conditional expectation of $\frac{dQ_\nu}{d(Q_\nu + Q_{\nu,x})}$ given $\mathfrak{B}_n$ and with respect to $\frac{Q_\nu + Q_{\nu,x}}{2}$. Hence $(f_n(x,\cdot))_{n \geq 1}$ is a bounded martingale under $\frac{Q_\nu + Q_{\nu,x}}{2}$ and thus converges $(Q_\nu + Q_{\nu,x})$-a.e. to $f(x,y)$ by the martingale convergence theorem (see e.g. [1, p. 89]), i.e.

$$
N_x \stackrel{\text{def}}{=} \{y : \liminf_{n \to \infty} f_n(x,y) \neq \limsup_{n \to \infty} f_n(x,y)\}
$$

is a null set under $Q_\nu + Q_{\nu,x}$. Moreover, by the dominated convergence theorem (recall $f_n \leq 1$),

$$
\begin{aligned}
Q_\nu(B_{n,k}) &= \lim_{m \to \infty} \int_{B_{n,k} \cap N_x^c} f_{n+m}(x,y)\ (Q_\nu + Q_{\nu,x})(dy) \\
&= \int_{B_{n,k} \cap N_x^c} f(x,y)\ (Q_\nu + Q_{\nu,x})(dy) \\
&= \int_{B_{n,k}} f(x,y)\ (Q_\nu + Q_{\nu,x})(dy)
\end{aligned}
$$

for all $n \geq 0, k \in \mathbb{Z}$ and $x \in \mathbb{R}$, and this proves $f(x,\cdot) = \frac{dQ_\nu}{d(Q_\nu + Q_{\nu,x})}$ for *every* $x \in \mathbb{R}$ because the $B_{n,k}$ generate $\mathfrak{B}$ and form a system which is stable under intersections. $\diamondsuit$

LEMMA 2. *The set $\mathbb{G}$ defined in (2.1) is also given by*

$$
\mathbb{G} = \{x \in \mathbb{R}^d : Q_\nu(0 < f(x,\cdot) < 1) > 0\} \tag{2.2}
$$

and is an element of $\mathfrak{B}^d$.

PROOF. (2.2) is obvious in view of the definition of f. The joint measurability of f ensures the measurability of

$$
\mathbb{R}^d \ni x \mapsto Q_\nu(0 < f(x,\cdot) < 1)
$$

and thus also of $\mathbb{G}$. $\diamondsuit$

The following lemma collects some rather straightforward but useful characterizations for x being an element of $\mathbb{G}$. It is stated without proof.

LEMMA 3. *Let ν be as stated above. The following statements are equivalent:*
(i) $x \in \mathbb{G}$;
(ii) $Q_\nu \wedge Q_{\lambda,x} \neq 0$ *for each distribution* $\lambda = (\lambda_n)_{n\geq 0}$ *satisfying* $\lambda_n > 0$ *for all* $n \in I\!N$;
(iii) $Q_m \wedge Q_{n,x} \neq 0$ *for some* $m, n \in I\!N$.

We are now ready to prove the second part of Theorem 1.

LEMMA 4. $\mathbb{G}$ *is a measurable additive subgroup of* $I\!R^d$ *satisfying (G.1) and (G.2).*

PROOF. For $\mathbb{G}$ being a group we note first that $\mathbb{G}$ is not empty because it contains 0 ($f(0,\cdot) \equiv 1/2$). Hence it remains to prove that $x, y \in \mathbb{G}$ implies $x - y \in \mathbb{G}$. By Lemma 3(iii), we can choose $k, l, m, n \in I\!N$ such that $Q_k \wedge Q_{l,x} \neq 0$ and $Q_m \wedge Q_{n,y} \neq 0$. Hence we can define, on some common probability space, maximal couplings (Y_1, Y_2) and (Z_1, Z_2) for $(Q_k, Q_{l,x})$ and $(Q_m, Q_{n,y})$, respectively, which are further independent. Since

$$\mathcal{L}(Y_1 + Z_2 - y) = Q_k * Q_n = Q_{k+n}, \quad \mathcal{L}(Y_2 + Z_1 - y) = Q_l * Q_{m,x-y}$$

and

$$\begin{aligned} P(Y_1 + Z_2 - y = Y_2 + Z_1 - y) &= P(Y_1 + Z_2 = Y_2 + Z_1) \\ &\geq P(Y_1 = Y_2)P(Z_1 = Z_2) \ > \ 0, \end{aligned}$$

the pair $(Y_1 + Z_2 - y, Y_2 + Z_1 - y)$ provides a successful coupling for $(Q_{k+n}, Q_{l+m,x-y})$. It follows $Q_{k+n} \wedge Q_{l+m,x-y} \neq 0$ and thus $x - y \in \mathbb{G}$ by another appeal to Lemma 3.

In order to prove (G.1), which can be stated as $Q_\nu(\mathbb{G}) = \sum_{n\geq 1} 2^{-n} Q_n(\mathbb{G}) = 1$, we first prove $Q_\nu(\mathbb{G}) > 0$. For each $x \in \mathbb{G}^c$, we have

$$Q_{\nu,x}(0 < f(x,\cdot) < 1) \ = \ 0$$

because otherwise

$$Q_\nu(0 < f(x,\cdot) < 1) \ \geq \ \int_{\{0<f(x,\cdot)<1\}} f(x,y)\ Q_{\nu,x}(dy) \ > \ 0$$

would yield the contradiction $x \in \mathbb{G}$. Hence $f(x,\cdot)(1 - f(x,\cdot)) = 0$ $(Q_\nu + Q_{\nu,x})$-a.s. for every $x \in \mathbb{G}^c$. Now we conclude for such x and all $B \in \mathfrak{B}^d$

$$\begin{aligned} Q_\nu(B) &= \int_B f(x,y)\ (Q_\nu + Q_{\nu,x})(dy) \\ &= \int_B f(x,y)\ Q_\nu(dy) \ + \ \int_B f(x,y)\ Q_{\nu,x}(dy) \\ &= \int_B f(x,y)\ Q_\nu(dy) \ + \ \int_B f(x,y)(1 - f(x,y))\ (Q_\nu + Q_{\nu,x})(dy) \\ &= \int_B f(x,y)\ Q_\nu(dy) \end{aligned}$$

and thereby $f(x,\cdot)=1$ Q_ν-a.s.

Let T be a random variable with distribution ν and independent of $(S_n)_{n\ge 0}$. The simple computation

$$\int_B E\left(\frac{1}{f(S_T,y)}\right)\,Q_\nu(dy) \;=\; \iint_B \frac{1}{f(x,y)}\,Q_\nu(dy)\,Q_\nu(dx)$$
$$=\; \int (Q_\nu(B)+Q_{\nu,x}(B))\,Q_\nu(dx) \;=\; Q_\nu(B)+Q_{\nu*\nu}(B)$$

for $B\in\mathfrak{B}^d$ shows that

$$\frac{d(Q_\nu+Q_{\nu*\nu})}{dQ_\nu}(y) \;=\; E\left(\frac{1}{f(S_T,y)}\right) \quad Q_\nu\text{-a.s.}$$

Suppose $Q_\nu(\mathbb{G}^c)=1$. Then $f(S_T,y)=1$ Q_ν-a.s. and therefore $\frac{d(Q_\nu+Q_{\nu*\nu})}{dQ_\nu}(y)=1$ Q_ν-a.s. The latter implies

$$Q_\nu(\mathbb{G}^c)+Q_{\nu*\nu}(\mathbb{G}^c) \;=\; Q_\nu(\mathbb{G}^c) \;=\; 1,$$

hence $Q_{\nu*\nu}(\mathbb{G})=1$. But Q_ν dominates $Q_{\nu*\nu}$ (since $\nu*\nu\ll\nu$) whence $Q_\nu(\mathbb{G})$ must also be positive, a contradiction to $Q_\nu(\mathbb{G}^c)=1$. We have thus verified that $Q_\nu(\mathbb{G})>0$.

In order to complete the proof of (G.1) let us assume $Q_\nu(\mathbb{G})<1$ and produce a further contradiction. Notice that $Q_\nu(\mathbb{G}^c)>0$ implies $\alpha\overset{\text{def}}{=}Q(\mathbb{G}^c)>0$. Put $Q'=\alpha^{-1}Q(\cdot\cap\mathbb{G}^c)$ and let $\mathbb{G}'$ be the same group as $\mathbb{G}$ for a random walk with increment distribution Q'. We infer from the previous step of the proof that $Q'_\nu(\mathbb{G}')>0$ and thus $Q'_\nu(\mathbb{G}'\cap\mathbb{G}^c)>0$. Consequently, we can choose an element $x\in\mathbb{G}'\cap\mathbb{G}^c$. By Lemma 3, $Q'_\mu\wedge Q'_{\mu,x}\neq 0$ holds for each distribution $\mu=(\mu_n)_{n\ge0}$ on $\mathbb{N}_0$ with $\mu_n>0$ for all $n\in\mathbb{N}$. Take $\mu_0=0$ and $\mu_n=\frac{\alpha}{2-\alpha}(\alpha/2)^n$ for $n\ge 1$. Using $Q_n\ge\alpha^nQ'_n$ for all $n\ge1$, we obtain

$$Q_\nu \;=\; \sum_{n\ge1}2^{-n}Q_n \;\ge\; \sum_{n\ge1}\left(\frac{\alpha}{2}\right)^nQ'_n \;=\; \frac{\alpha}{2-\alpha}Q'_\mu$$

and thus also $Q_{\nu,x}=Q_\nu*\delta_x\ge\frac{\alpha}{2-\alpha}Q'_\mu*\delta_x=\frac{\alpha}{2-\alpha}Q'_{\mu,x}$. Consequently,

$$Q_\nu\wedge Q_{\nu,x} \;\ge\; \left(\frac{\alpha}{2-\alpha}\right)^2 Q'_\mu\wedge Q'_{\mu,x} \;\neq\; 0,$$

which is a contradiction to $x\in\mathbb{G}^c$.

The proof of (G.2) is easy. If $Q_\nu(\mathbb{G}')=1$ for a measurable subgroup $\mathbb{G}'$ of $\mathbb{R}$ we must conclude $\mathbb{G}'^c\subset\mathbb{G}^c$. But $x\in\mathbb{G}'^c$ implies $\mathbb{G}'-x\subset\mathbb{G}'^c$ and therefore

$$Q_{\nu,x}(\mathbb{G}') \;=\; P(S_\nu\in\mathbb{G}'-x) \;\le\; Q_\nu(\mathbb{G}'^c) \;=\; 0.$$

Hence $Q_\nu\wedge Q_{\nu,x}=0$, i.e. $x\in\mathbb{G}^c$. The proof of Lemma 4 and thus of Theorem 1 is herewith complete. ◊

We close this section with some rather straightforward implications of Theorem 1. Recall that every distribution Q on $(\mathbb{R}^d,\mathfrak{B}^d)$ can be uniquely decomposed as $Q=Q^{(d)}+Q^{(s)}+Q^{(ac)}$ where $Q^{(d)}$ denotes the *discrete* part (countable support), $Q^{(s)}$ the *singular* part (no atoms and orthogonal to d-dimensional Lebesgue measure $\lambda\!\!\lambda^d$ on $I\!R^d$) and $Q^{(ac)}$ the *absolutely continuous* part of Q.

COROLLARY 1. *Let $(S_n)_{n\ge 0}$ be a random walk on $I\!R$ with minimal subgroup $\mathbb{G}$. If $(S_n)_{n\ge 0}$ is*

(i) c-arithmetic for some $c > 0$, then $\mathbb{G} = c\mathbb{Z}$;

(ii) spread out, i.e. $Q_\nu^{(ac)} \ne 0$, then $\mathbb{G} = I\!R$;

(iii) nonarithmetic and discrete, i.e. $Q = Q^{(d)}$, then $\mathbb{G}$ is a countable dense subgroup of $I\!R$;

(iv) nonarithmetic but neither spread out nor discrete, then $\mathbb{G}$ is an uncountable subgroup of $I\!R$ with $\lambda\!\!\lambda(\mathbb{G}) = 0$.

PROOF. We only prove (iv) because (i)-(iii) are easily verified. Since $Q_\nu(\mathbb{G}) = 1$ and Q_ν is not discrete, $\mathbb{G}$ must be uncountable. For $\lambda\!\!\lambda$-positive sets possess the Steinhaus property (see e.g. [3, p. 75]), either $\lambda\!\!\lambda(\mathbb{G}) = 0$ or $\mathbb{G} = I\!R$ holds. Assuming the latter we will now produce the contradiction that $(S_n)_{n\ge 0}$ is spread out. Note first that $P_1 \wedge P_2$ is absolutely continuous (possibly $\equiv 0$) if P_1 or P_2 has this property. $\mathbb{G} = I\!R$ gives $Q_\nu \wedge Q_{\nu,x} \ne 0$, i.e. $\|Q_\nu - Q_{\nu,x}\| < 1$ for all $x \in I\!R$. Let φ be any absolutely continuous distribution on $I\!R$ so that $Q_\nu * \varphi$ is also absolutely continuous. The inequality

$$\|Q_\nu - Q_\nu * \varphi\| \;\le\; \int \|Q_\nu - Q_{\nu,x}\|\; \varphi(dx) \;<\; 1$$

implies $Q_\nu \wedge Q_\nu * \varphi \ne 0$ and thus the contradiction $Q_\nu^{(ac)} \ne 0$. ◇

An extension of Corollary 1 to higher dimensions ($d \ge 2$) could also be given but would be more difficult because there is no straightforward definition of lattice-type for multidimensional random walks. We confine ourselves to a consideration of the spread out case for which the following statement can easily be obtained by adapting the final argument in the proof of Corollary 1.

COROLLARY 2. *Let $(S_n)_{n\ge 0}$ be a random walk on $I\!R^d$ with minimal subgroup $\mathbb{G}$. Then $\mathbb{G} = I\!R^d$ iff $(S_n)_{n\ge 0}$ is spread out.*

Returning to one-dimensional random walks, our next result considers the minimal subgroups of associated ladder height sequences (again random walks) providing a.s. finite pertinent ladder epochs.

COROLLARY 3. *Let $(S_n)_{n\ge 0}$ be a random walk on $I\!R$ with minimal subgroup $\mathbb{G}$ and let $(\sigma_n)_{n\ge 0}$ be any pertinent sequence of a.s. finite ladder epochs. Then $\mathbb{G}$ is also the minimal subgroup of the associated ladder height process $(S_{\sigma_n})_{n\ge 0}$.*

PROOF. Let $\hat{\mathbb{G}}$ be the minimal subgroup of $(S_{\sigma_n})_{n\ge 0}$. Since $\hat{\mathbb{G}}$ is clearly a subgroup of $\mathbb{G}$ we must only prove $\mathbb{G} \subset \hat{\mathbb{G}}$.

Without loss of generality let the σ_n be the weakly ascending ladder epochs, in particular $\sigma_1 = \inf\{n \ge 1 : S_n \ge 0\}$. Given any $x \in \mathbb{G}$, we can choose, again appealing to Lemma 3, two copies $(S_j')_{j\ge 0}$ and $(S_j'')_{j\ge 0}$ of $(S_j)_{j\ge 0}$ such that, for suitable $k, l \in I\!N$, $P(S_k' = S_l' + x) > 0$

and $X'_{k+j} = X''_{l+j}$ for all $j \geq 1$. With $(\sigma'_j)_{j\geq 0}$ and $(\sigma''_j)_{j\geq 0}$ having the obvious meaning, put $M \stackrel{\text{def}}{=} S'_1 \vee ... \vee S'_k \vee S''_1 \vee ... \vee S''_l$, $\tau' \stackrel{\text{def}}{=} \inf\{j \geq 1 : S'_j \geq M\}$ and $\tau'' \stackrel{\text{def}}{=} \inf\{j \geq 1 : S''_j \geq M\}$. Then τ', τ'' are obviously ladder epochs for $(S'_j)_{j\geq 0}$ and $(S''_j)_{j\geq 0}$, respectively, which further satisfy $S'_{\tau'} = S''_{\tau''} + x$ on the coupling event $\{S'_k = S'_l + x\}$. Consequently, there must be $m, n, r \geq 1$ such that $P(\sigma'_m = k + r, \sigma''_n = l + r) > 0$ and therefore

$$P(S'_{\sigma'_m} = S''_{\sigma''_n} + x) \;\geq\; P(S'_k = S'_l + x, \sigma'_m = k + r, \sigma''_n = l + r) \;>\; 0.$$

This proves $x \in \hat{\mathbb{G}}$ and thus $\mathbb{G} \subset \hat{\mathbb{G}}$. ◇

Let us finally consider the following three symmetric random walks related to $(S_n)_{n\geq 0}$, namely

— its *symmetrization* $(S^s_n)_{n\geq 0}$ with increment distribution $Q^s \stackrel{\text{def}}{=} Q*Q^-$, $Q^-(B) \stackrel{\text{def}}{=} Q(-B)$;

— the random walk $(\hat{S}_n)_{n\geq 0}$ with increment distribution $(Q + Q^-)/2$;

— the random walk $(W_n)_{n\geq 0}$ with increment distribution $Q_\nu * (Q_\nu)^-$ which is a symmetrization of a geometric sample of $(S_n)_{n\geq 0}$.

Clearly, each of these random walks has a minimal subgroup contained in that of $(S_n)_{n\geq 0}$, that is $\mathbb{G}$. The inclusion is proper in general for the symmetrization of $(S_n)_{n\geq 0}$. Take, for example, $X_1, X_2, ...$ be i.i.d. Laplacians on $\{1, \pi\}$. Then $(\pi - 1)\mathbb{Z}$ is the minimal subgroup of $(S^s_n)_{n\geq 0}$ while $\mathbb{G} = \mathbb{Z} + \pi\mathbb{Z}$. However, for the other two random walks above the subsequent corollary shows that their minimal subgroup always equals $\mathbb{G}$. Its simple proof will be omitted.

COROLLARY 4. *Given a random walk $(S_n)_{n\geq 0}$ on $\mathbb{R}^d$ with minimal subgroup $\mathbb{G}$, the random walks $(\hat{S}_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ as defined above always have the same minimal subgroup.*

3. The minimal subgroup and a zero-one law

In this section, we regard $(S_n)_{n\geq 0}$ as a temporally homogeneous Markov chain on $\mathbb{R}^d$ with initial state $S_0 = 0$. Note that

$$\mathcal{L}((S_n)_{n\geq 0} | S_0 = x) \;=\; \mathcal{L}((x + S_n)_{n\geq 0})$$

for all $x \in \mathbb{R}^d$. Theorem 2 below provides another interesting characterization of the minimal subgroup $\mathbb{G}$ as the set of all initial states x for which, in a certain sense, the random walk forgets about its initial state. Recall from (1.2) that $\bar{\Delta}_n(x) = \frac{1}{n}\|\sum_{j=1}^n (P(S_j \in \cdot) - P(S_j + x \in \cdot))\| = \frac{1}{n}\|\sum_{j=1}^n (Q_j - Q_{j,x})\|$ for all $x \in \mathbb{R}^d$ and $n \geq 1$.

THEOREM 2. *Given a random walk $(S_n)_{n\geq 0}$ on $\mathbb{R}^d$ with minimal subgroup $\mathbb{G}$, for each $x \in \mathbb{G}$*

$$\lim_{n\to\infty} \bar{\Delta}_n(x) = 0,$$

while for $x \in \mathbb{G}^c$

$$\bar{\Delta}_n(x) = 1 \quad \textit{for all } n \geq 1.$$

Theorem 2 is a weaker version of another zero-one law which states that, for each $x \in I\!\!R^d$, $\Delta_n(x) = \|Q_n - Q_{n,x}\|$ either equals 1 for all $n \geq 1$ or converges to 0 as $n \to \infty$, see [6]. In view of the previous result it is natural to ask whether

$$\mathbb{G}^* \stackrel{\text{def}}{=} \{x \in I\!\!R^d : \lim_{n\to\infty} \Delta_n(x) = 0\} \;=\; \{x \in I\!\!R^d : \inf_{n\geq 1} \Delta_n(x) < 1\} \tag{3.1}$$

also defines a measurable subgroup of $I\!\!R^d$ (in fact of $\mathbb{G}$). Let $\mathbb{G}^s$ be the minimal subgroup of the symmetrization $(S_n^s)_{n\geq 0}$ and $\mathbb{G}^{s*}$ be the counterpart of $\mathbb{G}^*$ for this latter random walk.

THEOREM 3. *Given a random walk $(S_n)_{n\geq 0}$ on $I\!\!R^d$ with minimal subgroup $\mathbb{G}$, the following assertions hold:*
(i) The set $\mathbb{G}^*$ defines a measurable subgroup of $\mathbb{G}$.
(ii) Either $Q(\mathbb{G}^*) = 0$, or $Q(\mathbb{G}^*) = 1$ in which case $\mathbb{G}^* = \mathbb{G}$.
(iii) $\mathbb{G}^* = \mathbb{G}^{s*} = \mathbb{G}^s \subset \mathbb{G}$.

The proofs of Theorem 2 and 3 will be given below after Corollary 5 and the subsequent two examples. In view of Theorem 3(iii), $\mathbb{G}^*$ is nothing but the minimal subgroup of the symmetrization $(S_n^s)_{n\geq 0}$. A combination of this fact with Corollary 1 in the previous section immediately leads to the following corollary which is therefore stated without proof.

COROLLARY 5. *Let $(S_n)_{n\geq 0}$ be a random walk on $I\!\!R$ with minimal subgroup $\mathbb{G}$. If $(S_n)_{n\geq 0}$ is*
(i) d-arithmetic for some $d > 0$, then $\mathbb{G}^* = md\mathbb{Z}$ for some $m \in I\!\!N$ where md is the lattice span of Q^s;
(ii) spread out, i.e. $Q_\nu^{(ac)} \neq 0$, then $\mathbb{G}^* = I\!\!R$.

Given a random walk $(S_n)_{n\geq 0}$ on $I\!\!R$ with $\mathbb{G}$ and $\mathbb{G}^*$ as before, let $(\sigma_n)_{n\geq 0}$ be any pertinent sequence of a.s. finite ladder epochs. Denote by $\hat{\mathbb{G}}$ and $\hat{\mathbb{G}}^*$ the counterparts of $\mathbb{G}$ and $\mathbb{G}^*$, respectively, for the associated ladder height process $(S_{\sigma_n})_{n\geq 0}$ with increment distribution $\hat{Q}$. The following two counterexamples shall demonstrate that there is no general relation between these four groups:

EXAMPLES. (1) If $Q = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, then $Q^s = \frac{1}{4}\delta_{-2} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2$, $\hat{Q} = \delta_1$ and $\hat{Q}^s = \delta_0$. Hence we easily obtain $\hat{\mathbb{G}}^* = \{0\}$, $\mathbb{G}^* = 2\mathbb{Z}$ and $\hat{\mathbb{G}} = \mathbb{G} = \mathbb{Z}$, hence $\hat{\mathbb{G}}^* \subset \mathbb{G}^* \subset \hat{\mathbb{G}} = \mathbb{G}$.

(2) If $Q = \sum_{n\geq 0} 2^{-n-1}\delta_{2n-1}$, then Q^s is a distribution concentrated on $2\mathbb{Z}$, while $\hat{Q}$ has obviously positive mass at all positive integers and thus $\hat{Q}^s$ positive mass at all elements of $\mathbb{Z}$. In this case we obtain $\hat{\mathbb{G}}^* = \hat{\mathbb{G}} = \mathbb{G} = \mathbb{Z}$ and $\mathbb{G}^* = 2\mathbb{Z}$, hence $\mathbb{G}^* \subset \hat{\mathbb{G}}^* = \hat{\mathbb{G}} = \mathbb{G}$.

REMARK. (Connections to coupling theory) Given a random walk $S = (S_n)_{n\geq 0}$ on $I\!\!R^d$ with pertinent groups $\mathbb{G}$ and $\mathbb{G}^*$, let $\mathcal{I}$ be the *invariant σ-field* on the path space $((I\!\!R^d)^\infty, (\mathfrak{B}^d)^\infty)$ of S, i.e.

$$\mathcal{I} \stackrel{\text{def}}{=} \{A \in (\mathfrak{B}^d)^\infty) : \mathbf{1}_A = \mathbf{1}_A \circ \theta\}$$

where $\theta(x_0, x_1, ...) \stackrel{\text{def}}{=} (x_1, x_2, ...)$ denotes the shift operator. Let further $\mathcal{T}$ be the *tail σ-field*, defined as

$$\mathcal{T} \stackrel{\text{def}}{=} \{A \in (\mathfrak{B}^d)^\infty : \forall n \geq 1 : \exists\, A_n \in (\mathfrak{B}^d)^\infty : \mathbf{1}_A = \mathbf{1}_{A_n} \circ \theta^n\}.$$

Note that $\mathcal{I} \subset \mathcal{T}$ and that in our situation $\mathcal{T}$ and $\mathcal{I}$ are both trivial under $P(S \in \cdot)$ by Kolmogorov's zero-one law. It is shown in [11, Chapters 4 and 5] in a more general framework, that $\lim_{n\to\infty} \bar{\Delta}_n(x) = 0$ holds iff there exists a successful *shift-coupling* for S and $x + S$ and that this is further equivalent to $P(S \in \cdot)_{\mathcal{I}} = P(x + S \in \cdot)_{\mathcal{I}}$, where $P(S \in \cdot)_{\mathcal{I}}$ denotes the restriction of $P(S \in \cdot)$ to $\mathcal{I}$. Similarly, $\lim_{n\to\infty} \Delta_n(x) = 0$ holds iff there exists a successful *exact coupling* for S and $x + S$, and this is further equivalent to $P(S \in \cdot)_{\mathcal{T}} = P(x + S \in \cdot)_{\mathcal{T}}$. Hence our results show that $\mathbb{G}$ consists exactly of those x for which S and its translation $x + S$ have the same distribution on the invariant σ-field, while $\mathbb{G}^*$ contains those x for which $P(S \in \cdot)$ and $P(x + S \in \cdot)$ coincide on the larger tail σ-field. Moreover, since $\mathcal{I}$ and $\mathcal{T}$ are trivial under $P(S \in \cdot)$, we see that $P(S \in \cdot)_{\mathfrak{C}}$ and $P(x + S \in \cdot)_{\mathfrak{C}}$ are mutually singular for $\mathfrak{C} = \mathcal{I}$ in case $\bar{\Delta}_n(x) = 1$ for all $n \geq 1$, respectively for $\mathfrak{C} = \mathcal{T}$ in case $\Delta_n(x) = 1$ for all $n \geq 1$.

Proof of Theorem 2. By definition of $\mathbb{G}$, $x \in \mathbb{G}^c$ iff $Q_\nu \wedge Q_{\nu,x} = 0$, i.e. $\|Q_\nu - Q_{\nu,x}\| = 1$. But the latter is equivalent to $\|Q_m - Q_{n,x}\| = 1$ for all $m, n \in I\!N$ which in turn holds iff $\bar{\Delta}_n(x) = 1$ for all $n \in I\!N$ as one can easily check.

Suppose now $x \in \mathbb{G}$ so that $\|Q_\nu - Q_{\nu,x}\| < 1$. Let $(\tau(n))_{n\geq 0}$ be a zero-delayed renewal process independent of $(S_n)_{n\geq 0}$ and with $\mathcal{L}(\tau(1)) = \nu$. By using the coupling construction in [6] (a Mineka coupling), we can construct a sequence $(W'_n, W''_n)_{n\geq 0}$ such that $\mathcal{L}((W'_n)_{n\geq 0}) = \mathcal{L}((W''_n)_{n\geq 0}) = \mathcal{L}((S_{\tau(n)})_{n\geq 0})$ and $\alpha \stackrel{\text{def}}{=} \inf\{n \geq 1 : W'_n = W''_n + x\} < \infty$ a.s. This sequence can now easily be extended to a sequence $(\tau'(n), \tau''(n), S'_n, S''_n)_{n\geq 0}$ such that $(S'_{\tau'(n)}, S''_{\tau''(n)}) = (W'_n, W''_n)$ for all $n \geq 0$, $\mathcal{L}((S'_n)_{n\geq 0}) = \mathcal{L}((S''_n)_{n\geq 0}) = \mathcal{L}((S_n)_{n\geq 0})$ and $S'_{T'+n} = x + S''_{T''+n}$ for all $n \geq 0$ where $T' \stackrel{\text{def}}{=} \tau'(\alpha)$ and $T'' \stackrel{\text{def}}{=} \tau''(\alpha)$ are the a.s. finite coupling times. Now

$$
\begin{aligned}
\bar{\Delta}_n(x) &= \frac{1}{n} \sup_{B\in\mathfrak{B}^d} \left| \sum_{j=1}^n (Q_j - Q_{j,x})(B) \right| \\
&= \frac{1}{n} \sup_{B\in\mathfrak{B}^d} \left| \sum_{j=1}^n (P(S'_j \in B) - P(x + S''_j \in B)) \right| \\
&= \frac{1}{n} \sup_{B\in\mathfrak{B}^d} \left| E\left(\sum_{j=1}^n \Big(\mathbf{1}_B(S'_j) - \mathbf{1}_B(x + S''_j) \Big) \right) \right| \\
&\leq P(T' \vee T'' > n) \\
&\quad + \frac{1}{n} \sup_{B\in\mathfrak{B}^d} \left| E\left(\mathbf{1}_{\{T'\vee T''\leq n\}} \sum_{j=1}^n \Big(\mathbf{1}_B(S'_j) - \mathbf{1}_B(x + S''_j) \Big) \right) \right| \\
&\leq P(T' \vee T'' > n) \\
&\quad + \frac{1}{n} \sup_{B\in\mathfrak{B}^d} E\left(\mathbf{1}_{\{T'\vee T''\leq n\}} \left(\sum_{j=1}^{T'-1} \mathbf{1}_B(S'_j) + \sum_{j=n+1-|T'-T''|}^{n} \mathbf{1}_B(S'_j) \right) \right) \\
&\quad + \frac{1}{n} \sup_{B\in\mathfrak{B}^d} E\left(\mathbf{1}_{\{T'\vee T''\leq n\}} \left(\sum_{j=1}^{T''-1} \mathbf{1}_B(x + S''_j) + \sum_{j=n+1-|T'-T''|}^{n} \mathbf{1}_B(x + S''_j) \right) \right) \\
&\leq P(T' \vee T'' > n) \; + \; \frac{1}{n} \Big(E(T' \wedge n) + E(T'' \wedge n) + 2E\big(|T' - T''| \wedge n\big) \Big)
\end{aligned}
$$

and the latter line converges to 0 as $n \to \infty$. ◊

Proof of Theorem 3. (i) The following argument shows the measurability of $\mathbb{G}^*$ and provides also an alternative proof for the measurability of $\mathbb{G}$ although hinging on the same fact used in the previous section, namely the separability of $\mathfrak{B}^d$.

Let $\mathcal{E}$ be the countable field of all finite unions of dyadic intervals $(\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The tensor product $\mathcal{E}^d$ is a countable field which generates $\mathfrak{B}^d$ for every $d \geq 1$. Moreover, every finite signed measure Ψ on $(\mathbb{R}^d, \mathfrak{B}^d)$ is uniquely determined by its values on $\mathcal{E}^d$ in the sense that for every $B \in \mathfrak{B}^d$ and every $\varepsilon > 0$ there exists a set $C \in \mathcal{E}$ such that $\Psi(B\Delta C) < \varepsilon$. Consequently,

$$\|\Psi\| \;=\; \sup_{C\in\mathcal{E}^d} |\Psi(C)|.$$

Since $x \mapsto |Q_n(B) - Q_{n,x}(B)| = |Q_n(B) - Q_n(B-x)|$ is clearly measurable for all $n \in \mathbb{N}$ and $B \in \mathfrak{B}^d$, we now infer the measurability of

$$x \;\mapsto\; \inf_{n\geq 1} \|Q_n - Q_{n,x}\| \;=\; \inf_{n\geq 1} \sup_{C\in\mathcal{E}^d} |Q_n(C) - Q_n(C-x)|$$

and thus of $\mathbb{G}^*$ (see (3.1)).

To prove that $\mathbb{G}^*$ is a group, note first that it is not empty because $0 \in \mathbb{G}^*$. Choose $x, y \in \mathbb{G}^*$ so that $\|Q_n - Q_{n,x}\| \to 0$ as well as $\|Q_n - Q_{n,y}\| \to 0$ as $n \to \infty$. We must show that $x - y \in \mathbb{G}^*$. Recall the contraction property $\|(P_1 - P_2) * P_3\| \leq \|P_1 - P_2\|$ of the total variation norm for arbitrary probability distributions P_1, P_2, P_3 on $(\mathbb{R}^d, \mathfrak{B}^d)$. Even equality holds in case $P_3 = \delta_x$ for some $x \in \mathbb{R}^d$. Using this property and the triangular inequality, we conclude

$$\begin{aligned}\|Q_n - Q_{n,x-y}\| &= \|(Q_{n,y} - Q_{n,x}) * \delta_{-y}\| \\ &= \|Q_{n,y} - Q_{n,x}\| \\ &\leq \|Q_{n,y} - Q_n\| + \|Q_n - Q_{n,x}\|\end{aligned}$$

and thus $x - y \in \mathbb{G}^*$ because the final two expressions converge to 0 as $n \to \infty$.

(ii) Suppose $Q(\mathbb{G}^*) > 0$ and thus $Q_n(\mathbb{G}^*) > 0$ for all $n \geq 1$. Pick an arbitrary $x \in \mathbb{G}$. Then there exist $k, l \geq 1$, w.l.o.g. $l > k$, such that $\|Q_k - Q_{l,x}\| < 1$. Moreover,

$$\begin{aligned}\|Q_k - Q_l\| &\leq \int \|Q_k - Q_{k,y}\|\; Q_{l-k}(dy) \\ &\leq \int_{\mathbb{G}^*} \|Q_k - Q_{k,y}\|\; Q_{l-k}(dy) \;+\; Q_{l-k}(\mathbb{G}^{*c}) \;<\; 1.\end{aligned}$$

Hence we can construct two copies $(S_n')_{n\geq 0}$ and $(S_n'')_{n\geq 0}$ of $(S_n)_{n\geq 0}$ such that $P(S_k' = S_l''+x) > 0$ and $P(S_{k+l}' - S_k' = S_{k+l}'' - S_l'') > 0$ which together yield

$$P(S_{k+l}' = S_{k+l}'' + x) \;\geq\; P(S_k' = S_l'' + x)P(S_{k+l}' - S_k' = S_{k+l}'' - S_l'') \;>\; 0$$

and thus $x \in \mathbb{G}^*$. Consequently, $\mathbb{G}^* = \mathbb{G}$ and $Q(\mathbb{G}^*) = 1$. ◇

We postpone the proof of (the hardest) part (iii) of Theorem 3 after the following lemma which may also be of interest in its own right.

Lemma 5. *Given a distribution Q on $(\mathbb{R}^d, \mathfrak{B}^d)$, define*

$$\mathbb{S}_x \;\stackrel{\text{def}}{=}\; \{r \in \mathbb{Z} : \|Q_{k+r} - Q_{k,x}\| < 1 \textit{ for some } k \in \mathbb{N}\}.$$

for $x \in \mathbb{G}$ and similarly $\mathbb{S}_x^s$ for the symmetrization Q^s. Then the following assertions hold:

(i) $\mathbb{S}_0$ *is a subgroup of* $\mathbb{Z}$*, i.e.* $\mathbb{S}_0 = s_0\mathbb{Z}$ *for some* $s_0 \in \mathbb{N}_0$*.*

(ii) $r + \mathbb{S}_0 \subset \mathbb{S}_x$ *for all* $r \in \mathbb{S}_x$ *and all* $x \in \mathbb{G}$*.*

(iii) $Q(\mathbb{G}^*) = 1$*, and hence* $\mathbb{G}^* = \mathbb{G}$*, holds iff* $\mathbb{S}_0 = \mathbb{Z}$ *(and thus* $\mathbb{S}_x = \mathbb{Z}$ *for all* $x \in \mathbb{G}$ *by (ii)).*

(iv) There exists $x \in \mathbb{G}$ *such that* $1 \in \mathbb{S}_x$*.*

(v) If Q *is symmetric, i.e.* $Q = Q^-$*, then* $\mathbb{S}_0 = \mathbb{Z}$ *if* $Q(\mathbb{G}^*) = 1$*, and* $\mathbb{S}_0 = 2\mathbb{Z}$ *otherwise.*

(vi) $\mathbb{S}_x^s = \mathbb{Z}$ *for all* $x \in \mathbb{G}^s$ *and hence (by (iii))* $\mathbb{G}^{s*} = \mathbb{G}^s$*.*

PROOF. Note that, by the contraction property, $(\|Q_{k+r} - Q_{k,x}\|)_{k\geq 1}$ is always a decreasing sequence so that

$$\mathbb{S}_x \stackrel{\text{def}}{=} \{r \in \mathbb{Z} : \lim_{k\to\infty} \|Q_{k+r} - Q_{k,x}\| < 1\}.$$

Note further that for symmetric Q

$$\begin{aligned} \|Q_k - Q_{l,x}\| &= \|(Q_k)^- - (Q_{l,x})^-\| \\ &= \|(Q_k)^- - (Q_l)^- * \delta_{-x}\| \\ &= \|(Q_k)^- * \delta_x - (Q_l)^-\| \\ &= \|Q_k * \delta_x - Q_l\| \\ &= \|Q_{k,x} - Q_l\| \end{aligned} \tag{3.1}$$

for all $k, l \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Consequently, $r \in \mathbb{S}_x$ always gives $-r \in \mathbb{S}_x$ in the symmetric case, hence $\mathbb{S}_x = -\mathbb{S}_x = \mathbb{S}_{-x}$.

(i) We must show that $\mathbb{S}_0$ is a subgroup of $\mathbb{Z}$. Plainly, $0 \in \mathbb{S}_0$. Given $r, s \in \mathbb{S}_0$, there are $k, l \in \mathbb{N}$ such that $\|Q_{k+r} - Q_k\| < 1$ and $\|Q_{l+s} - Q_l\| < 1$. We can therefore construct two copies $(S'_n)_{n\geq 0}$ and $(S''_n)_{n\geq 0}$ of $(S_n)_{n\geq 0}$ such that $P(S'_{k+r} = S''_k) > 0$ and $P(S'_{k+l+r} - S'_{k+r} = S''_{k+l+s} - S''_k) > 0$. This implies

$$P(S'_{k+l+r} = S''_{k+l+s}) \;\geq\; P(S'_{k+r} = S''_k)P(S'_{k+l+r} - S'_{k+r} = S''_{k+l+s} - S''_k) \;>\; 0$$

and thus $\|Q_{k+l+r} - Q_{k+l+s}\| < 1$, i.e. $r - s \in \mathbb{S}_0$.

(ii) The same coupling argument under the assumption $r \in \mathbb{S}_x$ for any $x \in \mathbb{G} - \{0\}$ instead of $r \in \mathbb{S}_0$ (hence $P(S'_{k+r} + x = S''_k) > 0$) leads to the conclusion $\|Q_{k+l+r,x} - Q_{k+l+s}\| < 1$, i.e. $r - s \in \mathbb{S}_x$. Hence $r + \mathbb{S}_0 \subset \mathbb{S}_x$ for each $x \in \mathbb{S}_x$.

(iii) If $Q(\mathbb{G}^*) = 1$, then $1 \in \mathbb{S}_0$ and thus $\mathbb{S}_x = \mathbb{S}_0 = \mathbb{Z}$ for all $x \in \mathbb{G}$ (by (ii)) follows from

$$\lim_{k\to\infty} \|Q_{k+1} - Q_k\| \;\leq\; \int_{\mathbb{G}^*} \lim_{k\to\infty} \|Q_{k,x} - Q_k\|\; Q(dx) \;=\; 0.$$

The reverse conclusion is trivial.

(iv) By (iii), there is nothing to prove here if $Q(\mathbb{G}^*) = 1$. Hence suppose $Q(\mathbb{G}^*) = 0$. Then there exist $k \in \mathbb{N}$ and $r \in \mathbb{Z} - \{0\}$ such that $Q(\mathbb{G}_{k,r}) > 0$ where

$$\mathbb{G}_{k,r} \stackrel{\text{def}}{=} \{x \in \mathbb{G} : \|Q_{k+r,x} - Q_k\| < 1\}. \tag{3.2}$$

Notice that $-r \in \mathbb{S}_{-x}$ for each $x \in \mathbb{G}_{k,r}$ because $\|Q_{k+r,x} - Q_k\| = \|Q_{k,-x} - Q_{k+r}\|$.

The inequality

$$\begin{aligned} \|Q_{k+r+1}-Q_k\| &\le \int \|Q_{k+r,x}-Q_k\|\; Q(dx) \\ &\le Q(\mathbb{G}_{k,r}^c) \;+\; \int_{\mathbb{G}_{k,r}} \|Q_{k+r,x}-Q_k\|\; Q(dx) \;<\; 1 \end{aligned} \tag{3.3}$$

implies $r+1\in\mathbb{S}_0$ and hence in combination with $-r\in\mathbb{S}_{-x}$ and (ii) further $1\in\mathbb{S}_{-x}$ for all $x\in\mathbb{G}_{k,r}$.

(v) In view of (iii) it suffices to consider the case when Q is symmetric and $Q(\mathbb{G}^*)=0$. Let $\mathbb{G}_{k,r}$ be as defined in (3.2). The symmetry ensures $\mathbb{S}_x=-\mathbb{S}_x=\mathbb{S}_{-x}$ for all $x\in I\!R^d$ (see after (3.1)) as well as $\mathbb{G}_{k,r}=\mathbb{G}_{k+r,-r}=-\mathbb{G}_{k,r}$ because (3.1) further gives

$$\|Q_{k+r,x}-Q_k\| \;=\; \|Q_{k+r}-Q_{k,x}\| \;=\; \|Q_{k+r,-x}-Q_k\|.$$

So we may assume w.l.o.g. $r\ge 1$. By (3.3), $r+1\in\mathbb{S}_0$ which in combination with $r\in\mathbb{S}_x$ implies $1\in\mathbb{S}_x$ for all $x\in\mathbb{G}_{k,r}$. Another application of (3.3), now with $r=1$, shows $2\in\mathbb{S}_0$ and thus $\mathbb{S}_0=2\mathbb{Z}$ ($\mathbb{S}_0\neq\mathbb{Z}$ by (iii)).

(vi) By (iv), there exists $x\in\mathbb{G}$ such that $\|Q_{k+1,x}-Q_k\|<1$ for some $k\ge 1$. We can therefore construct two copies $(S_n')_{n\ge 0}$ and $(S_n'')_{n\ge 0}$ of $(S_n)_{n\ge 0}$ such that $P(S_{k+1}'+x=S_k'')>0$ and $P(S_{2k+2}'-S_{k+1}'+x=S_{2k}''-S_k'')>0$. We then obtain

$$P(S_{2k+2}'-2S_{k+1}'=S_{2k}''-2S_k'') \;\ge\; P(S_{k+1}'+x=S_k'')P(S_{2k+2}'-S_{k+1}'+x=S_{2k}''-S_k'') \;>\; 0. \tag{3.4}$$

But $S_{2k+2}'-2S_{k+1}'=\sum_{j=1}^{k+1}(X_{k+j+1}'-X_j')\sim Q_{k+1}^s$ and, similarly, $S_{2k}''-S_k''\sim Q_k^s$ whence (3.4) proves $1\in\mathbb{S}_0^s$, that is $\mathbb{S}_0^s=\mathbb{Z}$, and this in turn $Q^s(\mathbb{G}^{s*})=1$, by (iii). ♢

Proof of Theorem 3(iii). Obviously, $\mathbb{G}^{s*}\subset\mathbb{G}^s\subset\mathbb{G}$. We first show $\mathbb{G}^*\subset\mathbb{G}^{s*}$. Write Q_n^s for $Q_n*Q_n^-$ and notice that $Q_{n,x}^s=Q_{n,x}*Q_n^-$. Choose any $x\in\mathbb{G}^*$ and $n\ge 1$ such that $\|Q_n-Q_{n,x}\|<1$. By the contraction property,

$$\|Q_n^s-Q_{n,x}^s\| \;=\; \|(Q_n-Q_{n,x})*Q_n^-\| \;\le\; \|Q_n-Q_{n,x}\| \;<\; 1$$

whence $x\in\mathbb{G}^{s*}$.

The equality of $\mathbb{G}^{s*}$ and $\mathbb{G}^s$ follows directly from part (vi) of Lemma 5 but the following arguments will even prove $\mathbb{G}^*=\mathbb{G}^s$ and thus complete the proof of Theorem 3.

There is nothing to prove if $Q(\mathbb{G}^*)=1$ whence we assume $Q(\mathbb{G}^*)=0$. Since $Q^s(\mathbb{G})=1$, for each Q^s-positive event C there exist $k\in I\!N$ and $r\in\mathbb{Z}$ such that $Q^s(C\cap\mathbb{G}_{k,r})>0$ where $\mathbb{G}_{k,r}$ is as defined in (3.2). Let us fix any Q^s-positive and symmetric C (so $C=-C$). The symmetry of Q^s implies

$$\begin{aligned} Q^s(C\cap\mathbb{G}_{k,r}) &= Q^s(C\cap\{x\in\mathbb{G}:\|Q_{k+r,x}-Q_k\|<1\}) \\ &= Q^s(C\cap\{x\in\mathbb{G}:\|Q_{k+r,-x}-Q_k\|<1\}) \\ &= Q^s(C\cap\{x\in\mathbb{G}:\|Q_{k+r}-Q_{k,x}\|<1\}) \\ &= Q^s(C\cap\mathbb{G}_{k+r,-r}) \end{aligned}$$

whence we may assume w.l.o.g. that $r \geq 0$. We claim that there is an $l \in I\!N$ which may depend on C such that $Q^s(C \cap \mathbb{G}_{l,0}) > 0$. Since $\mathbb{G}_{l,0} \subset \mathbb{G}^*$ the latter implies $Q^s(C \cap \mathbb{G}^*) > 0$. For C was chosen arbitrarily among all symmetric Q^s-positive sets we then conclude $Q^s(\mathbb{G}^{*c}) = 0$ and thus the desired result $\mathbb{G}^* = \mathbb{G}^s$ because $Q^s(\mathbb{G}^{*c}) > 0$ would imply the impossible result $Q^s(\mathbb{G}^{*c} \cap \mathbb{G}^*) > 0$ ($\mathbb{G}^{*c}$ is symmetric).

To prove our claim we assume $r \geq 1$ (there is nothing to show otherwise) and employ a further coupling argument. Obviously, we can construct copies $(S'_n)_{n\geq 0}$ and $(S''_n)_{n\geq 0}$ of $(S_n)_{n\geq 0}$ such that

(1) $\mathcal{L}(S'_1, S''_1) = Q \otimes Q$, hence $\mathcal{L}(S'_1 - S''_1) = Q^s$;

(2) $\mathcal{L}(S'_{k+r+1} - S'_1 + y - z, S''_{k+1} - S''_1 | S'_1 = y, S''_1 = z)$ is a maximal coupling of $(Q_{k+r,y-z}, Q_k)$ and therefore successful if $y - z \in C \cap \mathbb{G}_{k,r}$.

Defining

$$p(y,z) \stackrel{\text{def}}{=} P(S'_{k+r+1} - S'_1 + y - z = S''_{k+1} - S''_1 | S'_1 = y, S''_1 = z),$$

we obtain from our assumption that

$$\int_{\{(u,v):p(u,v)>0\}} \mathbf{1}_C(y-z)\; Q \otimes Q(dy,dz) \;=\; Q^s(C \cap \mathbb{G}_{k,r}) \;>\; 0$$

and thereby further

$$\begin{aligned} P(S'_{k+r+1} &= S''_{k+1}) \\ &= \int P(y + S'_{k+r+1} - S'_1 = z + S''_{k+1} - S''_1 | S'_1 = y, S''_1 = z)\; Q \otimes Q(dy,dz) \\ &\geq \int_{\{(u,v):p(u,v)>0\}} \mathbf{1}_C(y-z) p(y-z)\; Q \otimes Q(dy,dz) \;>\; 0. \end{aligned}$$

Hence $\|Q_{k+r+1} - Q_{k+1}\| < 1$, i.e. $r \in \mathbb{S}_0$ (recall $r \geq 1$). We can therefore extend our coupling model above by further taking

(3) $\mathcal{L}(S'_{2k+r+2} - S'_{k+r+1}, S''_{2k+r+2} - S''_{k+1})$ to be any successful coupling of (Q_{k+1}, Q_{k+r+1}).

Now observe that

$$\begin{aligned} q(y,z) &\stackrel{\text{def}}{=} P(S'_{2k+r+2} - S'_1 + y - z = S''_{2k+r+2} - S''_1 | S'_1 = y, S''_1 = z) \\ &\geq p(y,z) P(S'_{2k+r+2} - S'_{k+r+1} = S''_{2k+r+2} - S''_{k+1}) \;>\; 0 \end{aligned}$$

and thus $\|Q_{2k+r+1,y-z} - Q_{2k+r+1}\| < 1$ for all (y,z) satisfying $y - z \in \mathbb{G}_{k,r}$. Consequently,

$$\begin{aligned} Q^s(C \cap \mathbb{G}_{2k+r+1,0}) &\geq \int_{\{(u,v):q(u,v)>0\}} \mathbf{1}_C(y-z)\; Q \otimes Q(dy,dz) \\ &\geq \int \mathbf{1}_{C \cap \mathbb{G}_{k,r}}(y-z)\; Q \otimes Q(dy,dz) \\ &= Q^s(C \cap \mathbb{G}_{k,r}) \;>\; 0 \end{aligned}$$

which proves our claim with $l = 2k + r + 1$. $\diamondsuit$

4. Admissible shifts and quasi-invariance

As already defined in the Introduction, let

$$\mathbb{H}_Q = \{x \in I\!R^d : Q_{1,x} \ll Q\}$$

be the set of admissible shifts for a given probability measure $Q = Q_1$ on $(I\!R^d, \mathfrak{B}^d)$. By Lemma 1 in Section 2, replacing Q_ν with Q, there is a measurable function $f : (I\!R^d \times I\!R^d, \mathfrak{B}^d \otimes \mathfrak{B}^d) \to ([0,1], \mathfrak{B}_{[0,1]})$ such that $f(x,\cdot) = \frac{dQ_{1,x}}{d(Q+Q_{1,x})}$ for all $x \in I\!R^d$. Consequently,

$$\mathbb{H}_Q = \{x \in I\!R^d : Q(f(x,\cdot) < 1) = 1\}. \tag{4.1}$$

We can now easily prove:

Lemma 6. *$\mathbb{H}_Q$ is a measurable subsemigroup of $I\!R^d$. It is a group if Q is symmetric.*

Proof. The measurability of $\mathbb{H}_Q$ follows directly from (4.1). Given $x, y \in \mathbb{H}_Q$, we have $Q_{1,x} \ll Q$, hence $Q_{1,x+y} = \delta_y * Q_{1,x} \ll \delta_y * Q = Q_{1,y}$, which together with $Q_{1,y} \ll Q$ implies $Q_{1,x+y} \ll Q$, i.e. $x + y \in \mathbb{H}_Q$. If Q is symmetric we also have that $x \in \mathbb{H}_Q$ implies $-x \in \mathbb{H}_Q$ because $Q_{1,-x} = (Q_{1,x})^- \ll Q^- = Q$.

Let $(S_n)_{n\geq 0}$ be a random walk with increment distribution Q and, as before, $\mathbb{G}^* = \mathbb{G}^s$ the minimal subgroup of its symmetrization. In the following, we will write $\mathbb{H}$ for $\mathbb{H}_Q$, $\mathbb{H}^s$ for $\mathbb{H}_{Q^s}$, $\mathbb{H}_n$ for $\mathbb{H}_{Q_n}$ and $\mathbb{H}_n^s$ for $\mathbb{H}_{Q_n^s}$.

Lemma 7. *Given the previous notation, the following assertions hold:*
(i) *For all $n \in I\!N$, $\mathbb{H}_n \subset \mathbb{H}_n^s$;*
(ii) *$\mathbb{H} = \mathbb{H}_1 \subset \mathbb{H}_2 \subset ... \subset \mathbb{G}^*$;*
(iii) *$\mathbb{H}^s = \mathbb{H}_1^s \subset \mathbb{H}_2^s \subset ... \subset \mathbb{G}^*$;*
(iv) *$\mathbb{H}_\infty \stackrel{\text{def}}{=} \cup_{n\geq 1}\mathbb{H}_n$ is a subsemigroup of $\mathbb{G}^*$ and a group if Q is symmetric;*
(v) *if Q has compact support then $\mathbb{H}_\infty = \mathbb{H}_\infty^s = \{0\}$.*

Proof. (i) follows because $\frac{dQ_{n,x}^s}{dQ_n^s} = \frac{dQ_{n,x}}{dQ_n} * (Q_n)^-$ for each $x \in \mathbb{H}_n$ and $n \in I\!N$. Similarly, $\frac{dQ_{n+1,x}}{dQ_{n+1}} = \frac{dQ_{n,x}}{dQ_n} * Q$ gives $\mathbb{H}_n \subset \mathbb{H}_{n+1}$ and $\mathbb{H}_n^s \subset \mathbb{H}_{n+1}^s$ for all $n \in I\!N$ in (ii) and (iii), respectively. Moreover, $Q_{n,x} \ll Q_n$ implies $\|Q_n - Q_{n,x}\| < 1$ whence $\mathbb{H}_n \subset \mathbb{H}_n^s \subset \mathbb{G}^*$ for all $n \in I\!N$. (iv) is an immediate consequence of (ii) in combination with Lemma 6. Finally, if Q has compact support, then all Q_n and Q_n^s also have compact support. Therefore it suffices to show $\mathbb{H} = \{0\}$. To this end let K be the support of Q and note that, for every $x \in I\!R^d - \{0\}$, $K \cap (x + K)$ is a proper compact subset of $x + K$ (the support of $Q_{1,x}$) whence $Q_{1,x}(K \cap (x + K)) < 1$. Consequently, $Q_{1,x}(K^c \cap (x + K)) > 0$ while $Q(K^c \cap (x + K)) = 0$. This clearly shows that $x \notin \mathbb{H}$ and completes the proof of (v).

We now turn to a discussion of *quasi-invariant* probability measures Q on measurable subgroups $\mathbb{G}$ of $I\!R^d$, defined through $Q(\mathbb{G}) = 1$ and $Q_{1,x} \ll Q$ for all $x \in \mathbb{G}$. Notice that a probability distribution Q whose set of admissible shifts $\mathbb{H}$ forms a group needs not be quasi-invariant because $Q(\mathbb{H})$ can be less than 1 and even 0, e.g. for a continuous symmetric Q having compact support. However, providing $Q(\mathbb{H}) > 0$, the restriction of Q to $\mathbb{H}$, i.e. $Q(\cdot \cap \mathbb{H})/Q(\mathbb{H})$, is quasi-invariant. As an immediate consequence of our previous results, we obtain:

LEMMA 8. *A probability measure Q is quasi-invariant on a measurable subgroup $\mathbb{G}$ of $I\!R^d$ iff $\mathbb{G}$ is the minimal subgroup of a random walk with increment distribution Q and $\mathbb{H} = \mathbb{G}^* = \mathbb{G}$.*

PROOF. The definition of quasi-invariance together with Lemma 7 gives $\mathbb{G} \subset \mathbb{H} \subset \mathbb{G}^*$. On the other hand, $Q(\mathbb{G}) = 1$ implies $Q(\mathbb{G}^*) = 1$ so that indeed $\mathbb{H} = \mathbb{G}^* = \mathbb{G}$ holds with $\mathbb{G}$ being the minimal subgroup of any random walk with increment distribution Q.

We are now ready to present the main result of this section:

THEOREM 4. *If $\mathbb{G}$ is a measurable subgroup of $I\!R^d$ admitting a quasi-invariant measure then there are $0 \le k \le l \le d$, countable infinite subgroups $\mathbb{G}_1', ..., \mathbb{G}_l'$ of $I\!R$ and a regular $d \times d$-matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^{-1}\mathbb{G} = \mathbb{G}_1' \times ... \times \mathbb{G}_k' \times I\!R^{l-k} \times \{0\}^{d-l}$. Moreover, Q is equivalent to the Haar measure $\lambda\!\!\lambda_{\mathbb{G}}$ on $\mathbb{G}$ defined through*

$$\lambda\!\!\lambda_{\mathbb{G}}(B) \stackrel{\text{def}}{=} \lambda\!\!\lambda_{\mathbb{G}_1'} \otimes ... \otimes \lambda\!\!\lambda_{\mathbb{G}_k'} \otimes \lambda\!\!\lambda^{l-k} \otimes \delta_0^{d-l}(\boldsymbol{C}^{-1}B), \quad B \in \mathfrak{B}^d,$$

where $\lambda\!\!\lambda_{\mathbb{G}_j'}$ denotes Haar measure (i.e. counting measure) on $\mathbb{G}_j'$ for $1 \le j \le k$.

The proof of Theorem 4 essentially consists of a reduction to the one-dimensional case for which the assertion comes down to the following result due to Kharazishvili:

THEOREM 5. [3, Theorem 3 on p. 216] *There is no uncountable measurable proper subgroup of $I\!R$ admitting a quasi-invariant measure.*

Let us also state two rather straightforward corollaries the proofs of which can be found after that of Theorem 4. For the one-dimensional case, they were proved by different methods in [9].

COROLLARY 6. *Given a probability measure Q on $I\!R^d$, the following statements are equivalent:*

(i) The family $(Q_{1,x})_{x \in I\!R^d}$ is dominated by some σ-finite measure Ψ;
(ii) $Q \ll \lambda\!\!\lambda^d$, where $\lambda\!\!\lambda^d$ denotes d-dimensional Lebesgue measure on $I\!R^d$;
(iii) for all $B \in \mathfrak{B}^d$, the mapping $x \mapsto Q_{1,x}(B)$, $x \in I\!R^d$, is continuous;
(iv) for all $\lambda\!\!\lambda^d$-null sets $N \in \mathfrak{B}^d$, the mapping $x \mapsto Q_{1,x}(N)$, $x \in I\!R^d$, is continuous.

COROLLARY 7. *Let Q be a probability measure on $I\!R^d$ whose set of admissible shifts $\mathbb{H}$ forms a group with $Q(\mathbb{H}) > 0$. If $x \mapsto Q_{1,x}(B), x \in \mathbb{H}$, is continuous for all $B \in \mathfrak{B}^d$, then there are $0 \le k \le l \le d$, $d_1, ..., d_k \in (0, \infty)$ and a regular $d \times d$-matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^{-1}\mathbb{H} = d_1\mathbb{Z} \times ... \times d_k\mathbb{Z} \times I\!R^{l-k} \times \{0\}^{d-l}$.*

PROOF OF THEOREM 4. Excluding the trivial case $\mathbb{G} = \{0\}^d$, there exists a unique maximal $1 \le l \le d$ for which we can find linearly independent $g_1, ..., g_l \in \mathbb{G}$. Then

$$\mathbb{G}_j' \stackrel{\text{def}}{=} \{t \in I\!R : tg_j \in \mathbb{G}\}, \quad j = 1, ..., l,$$

define measurable subgroups of $I\!R$ and $\mathbb{G} = g_1\mathbb{G}'_1 \oplus ... \oplus g_l\mathbb{G}'_l$. W.l.o.g. let $g_1, ..., g_l$ be labelled in such a way that $\mathbb{G}'_1, ..., \mathbb{G}'_k$ are countable and $\mathbb{G}'_{k+1}, ..., \mathbb{G}'_l$ are uncountable for some $k \geq 0$. Let $\boldsymbol{C}$ be any regular $d \times d$ matrix whose first l column vectors are $g_1, ..., g_l$. It is then obvious that $\boldsymbol{C}^{-1}\mathbb{G} = \mathbb{G}'_1 \times ... \times \mathbb{G}'_l \times \{0\}^{d-l}$ and that $\tilde{Q}(B) \stackrel{\text{def}}{=} Q(\boldsymbol{C}^{-1}B)$, $B \in \mathfrak{B}^d$, defines a quasi-invariant probability measure on $\boldsymbol{C}^{-1}\mathbb{G}$. We have thus reduced the remaining work, namely to verify $\mathbb{G}'_{k+1} = ... = \mathbb{G}'_l = I\!R$ (if at all $l > k$), to the case where $\mathbb{G}$ has the form $\mathbb{G}'_1 \times ... \times \mathbb{G}'_l \times \{0\}^{d-l}$. But since the quasi-invariance of Q on $\mathbb{G}$ entails the quasi-invariance of its j-th marginal $Q^{(j)}(B) \stackrel{\text{def}}{=} Q(I\!R^{j-1} \times B \times I\!R^{d-j}), B \in \mathfrak{B}$, on $\mathbb{G}'_j$ for each $k < j \leq l$, it is no loss of generality to further restrict to the case where $d = 1$ (hence $k = 0, l = 1$ and $\mathbb{G} = \mathbb{G}'_1$). After these simplifications the first statement of the theorem reduces to the statement of Theorem 5 that, unless $\mathbb{G}$ is countable, $\mathbb{G} = I\!R$ is the only measurable subgroup of $I\!R$ which supports a quasi-invariant measure.

In order to finally see that Q and $\lambda\!\!\lambda_{\mathbb{G}}$ are equivalent, suppose $\lambda\!\!\lambda_{\mathbb{G}}(N) = 0$ for some $N \in \mathfrak{B}^d$. Then, by the invariance of $\lambda\!\!\lambda_{\mathbb{G}}$,

$$0 \;=\; \int_{\mathbb{G}} \lambda\!\!\lambda_{\mathbb{G}}(N-x)\; Q(dx) \;=\; \int_{\mathbb{G}} Q(N-x)\; \lambda\!\!\lambda_{\mathbb{G}}(dx) \tag{4.2}$$

whence $Q(N-x) = 0$ for $\lambda\!\!\lambda_{\mathbb{G}}$-almost all $x \in \mathbb{G}$ which together with the quasi-invariance of Q implies $Q(N) = 0$. Hence $Q \ll \lambda\!\!\lambda_{\mathbb{G}}$. The reverse conclusion follows by interchanging the roles of Q and $\lambda\!\!\lambda_{\mathbb{G}}$ in the previous argument.

Proof of Corollary 6. "(i)⇒(ii)" By a well-known result of Halmos and Savage (see e.g. [4, p.575]), (ii) implies the existence of a sequence $(x_n)_{n\geq 1}$ such that

$$Q_{1,x} \;\ll\; \hat{Q} \;\stackrel{\text{def}}{=}\; \sum_{n\geq 1} 2^{-n} Q_{1,x_n}$$

for all $x \in I\!R^d$. This further implies $\hat{Q}_{1,x} = \sum_{n\geq 1} 2^{-n} Q_{1,x+x_n} \ll \hat{Q}$ for all $x \in I\!R^d$ and thus the quasi-invariance of $\hat{Q}$ on $I\!R^d$. Hence, by Theorem 4, $Q \ll \hat{Q} \ll \lambda\!\!\lambda^d$.

"(ii)⇒(iii)" By quasi-invariance $Q_{1,x} \ll Q \ll \lambda\!\!\lambda^d$ for all $x \in I\!R^d$, and if g denotes a $\lambda\!\!\lambda^d$-density of Q, then $g_x \stackrel{\text{def}}{=} g(\cdot - x)$ is a $\lambda\!\!\lambda^d$-density of $Q_{1,x}$. Now

$$\lim_{y\to x} \|Q_{1,x} - Q_{1,y}\| \;=\; \lim_{y\to x} I_{g_x}(y-x)/2 \;=\; 0 \tag{4.3}$$

for each $x \in I\!R^d$ where

$$I_h(y) \;\stackrel{\text{def}}{=}\; \int |h(z) - h(z-y)|\; \lambda\!\!\lambda^d(dz)$$

for $h \in \mathfrak{L}^1$, the space of $\lambda\!\!\lambda^d$-integrable functions on $I\!R^d$. In fact, I_h is continuous at $y = 0$ for every $h \in \mathfrak{L}^1$. This follows immediately for $h \in \mathfrak{C}_0$, the vector space of continuous functions on $I\!R^d$ with compact support, and then for general h because $\mathfrak{C}_0$ forms a dense subset of $\mathfrak{L}^1$ endowed with the usual norm and $\lambda\!\!\lambda^d$ is shift-invariant. Plainly, (4.3) implies (iii).

The implication "(iii)⇒(iv)" is trivial.

For the proof of "(iv)⇒(i)" let $N \in \mathfrak{B}^d$ be such that $\lambda\!\!\lambda^d(N) = 0$. Use (4.2) with $\lambda\!\!\lambda_{\mathbb{G}}$ replaced with $\lambda\!\!\lambda^d$ and integration over whole $I\!R^d$ to conclude $Q_{1,-x} = Q(N-x) = 0$ for all x outside a $\lambda\!\!\lambda^d$-null set N'. Since N'^c is dense in $I\!R^d$ the continuity of $x \mapsto Q_{1,x}(N)$ implies

$Q_{1,x}(N) = 0$ for all $x \in I\!R^d$, in particular $Q(N) = 0$ whence $Q \ll \lambda\!\!\lambda^d$ and then further $Q_{1,x} \ll \lambda\!\!\lambda^d$ for all $x \in I\!R^d$ showing (i) with $\Psi = \lambda\!\!\lambda^d$. $\diamondsuit$

Proof of Corollary 7. Since all assumptions on Q carry over to $Q(\cdot \cap \mathbb{H})/Q(\mathbb{H})$ which is quasi-invariant on $\mathbb{H}$ as mentioned before Lemma 8, it is no loss of generality to assume Q itself be quasi-invariant on $\mathbb{H}$. By Theorem 4, there are $0 \le k \le l \le d$, countable subgroups $\mathbb{G}'_1, ..., \mathbb{G}'_l$ of $I\!R$ and a regular $d \times d$-matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^{-1}\mathbb{H} = \mathbb{H}'_1 \times ... \times \mathbb{H}'_k \times I\!R^{l-k} \times \{0\}^{d-l}$. Hence it suffices to prove $\mathbb{H}'_i = d_i\mathbb{Z}$ for some $d_i \in (0, \infty)$ and all $1 \le i \le k$. Now consider the i-th marginal $Q^{(i)}$ of Q which is quasi-invariant on the countable subgroup $\mathbb{H}'_i$ of $I\!R$ and for which, by assumption, $x \mapsto Q^{(i)}(B)$, $x \in \mathbb{H}'_i$, is continuous for all $B \in \mathfrak{B}$. By quasi-invariance $Q(\{x\}) = Q_{1,-x}(\{0\}) > 0$ for all $x \in \mathbb{H}'_i$.

Suppose $\mathbb{H}'_i \neq d\mathbb{Z}$ for all $d > 0$. Then $\mathbb{H}'_i$ is dense in $I\!R$ and $\mathbb{H}'_i \cap [0,1] = \{x_n; n \ge 1\}$ an infinite set. In particular, there exists a sequence $(y_n)_{n \ge 1}$ in this set convergent to 0. But

$$\sum_{n \ge 1} Q(\{y_n\}) \;=\; \sum_{n \ge 1} Q_{1,-y_n}(\{0\}) \;\le\; Q([0,1]) \;<\; \infty$$

implies

$$\liminf_{\mathbb{H}'_i \ni y \to 0} Q_{1,y}(\{0\}) \;=\; \lim_{n \to \infty} Q_{1,-y_n}(\{0\}) \;=\; 0 \;\neq\; Q(\{0\}),$$

a contradiction to the continuity assumption. $\diamondsuit$

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