ANGEWANDTE MATHEMATIK

UND

INFORMATIK

Markov Renewal Theory for Stationary (m+1)-Block Factors: First Passage Time and Overshoot

G. Alsmeyer and V. Hoefs

FB 10, Institut für Mathematische Statistik
Einsteinstraße 62, D-48149 Münster, Germany
e-mail: gerolda@math.uni-muenster.de (1st author)

UNIVERSITÄT MÜNSTER

Markov Renewal Theory for Stationary (m+1)-Block Factors: First Passage Time and Overshoot*

Gerold Alsmeyer and Volker Hoefs

Institut für Mathematische Statistik
Fachbereich Mathematik
Westfälische Wilhelms-Universität Münster
Einsteinstraße 62
D-48149 Münster, Germany

Given a sequence of i.i.d. random variables $Y_{-m}, Y_{-m+1}, ...$ with common distribution F and values in an arbitrary measurable space $(\mathcal{S}, \mathfrak{S})$, let $(S_n)_{n\geq 0}$ be a random walk whose increments X_n are nonnegative $(m+1)$-block factors of the form $\varphi(Y_{n-m}, ..., Y_n)$ for a measurable function $\varphi : \mathcal{S}^{m+1} \to [0,\infty)$. Defining $M_n = (Y_{n-m}, ..., Y_n)$ for $n \geq 0$, which is a Harris ergodic Markov chain, the sequence $(M_n, S_n)_{n\geq 0}$ constitutes a Markov renewal process with stationary drift $\mu = E\varphi(Y_{-m}, ..., Y_0)$. Suppose $\mu > 0$, and let $\tau(t) = \inf\{n : S_n > t\}$ be the first passage time of $(S_n)_{n\geq 0}$ beyond level $t \geq 0$. An important variable related to $\tau(t)$ is the $(m+1)$-step overshoot $R_{m,t} = S_{\tau(t)+m} - t$ which reduces to the familiar overshoot $R_t = S_{\tau(t)} - t$ if $m = 0$. The main results of this article are a second order approximation of the variance of $\tau(t)$ and bounds for $\sup_{t\geq 0} ER^p_{m,t}$ for $p \geq 1$ similar to those derived by Lorden in the i.i.d. case ($m = 0$).

AMS 1991 subject classifications. 60J05, 60J15, 60K05, 60K15.

Keywords and phrases. Random walks, m-dependence, $(m+1)$-block factors, first passage time, overshoot, Janson's equation, Markov renewal theory.

* This work was supported by the Deutsche Forschungsgemeinschaft (DFG) under grant Al 448/1-1

1. Introduction

Let $m \in \mathbb{N}$. A stochastic sequence $(X_n)_{n\geq 0}$ is called *m-dependent* if $X_0,...,X_n$ and $X_{n+m+1}, X_{n+m+2},...$ are independent for all $n \in \mathbb{N}$. Our concern here is a special class of such sequences, called *stationary* $(m+1)$*-block factors*, given by

$$X_n \;=\; \varphi(Y_{n-m},...,Y_n), \quad n \geq 0, \tag{1.1}$$

where $\varphi : \mathcal{S}^{m+1} \to \mathbb{R}$ is a measurable function and $Y_{-m}, Y_{-m+1},...$ are i.i.d. random variables on a probability space $(\Omega, \mathfrak{A}, P)$ with common distribution F and values in a measurable space $(\mathcal{S}, \mathfrak{S})$ with countably generated σ-field $\mathfrak{S}$. Let $S_n = \sum_{k=0}^n X_k$, $n \geq 0$, be the random walk associated with $(X_n)_{n\geq 0}$ and put $\mu \stackrel{\text{def}}{=} EX_1$. Our standing assumption throughout this article is that $X_0, X_1,...$ *are nonnegative (i.e.* $\varphi \geq 0$*) with finite positive mean* μ.

For $t \geq 0$, let

$$\tau(t) \;\stackrel{\text{def}}{=}\; \inf\{n \geq 0 : S_n > t\} \tag{1.2}$$

be the first passage time of $(S_n)_{n\geq 0}$ beyond level t. The associated overshoot (excess over the boundary) and $(l+1)$-step overshoot are defined as

$$R_t \;\stackrel{\text{def}}{=}\; S_{\tau(t)} - t \quad \text{and} \quad R_{l,t} \;\stackrel{\text{def}}{=}\; S_{\tau(t)+l} - t, \tag{1.3}$$

respectively. Note that $R_t = R_{0,t}$. As will be seen later, the variables $R_{m,t}$ and $R_{2m,t}$ will be of special interest in our analysis.

Janson [8] proved various results for $\tau(t)$ and R_t concerning moments and asymptotic behavior as $t \to \infty$. Some of them were improved or complemented in [2] by analyzing $(S_n)_{n\geq 0}$ within the framework of Markov renewal theory. This approach embarks on the observation that $(M_n, S_n)_{n\geq 0}$ constitutes a Markov renewal process when defining

$$M_n \;\stackrel{\text{def}}{=}\; (Y_{n-m},...,Y_n), \quad n \geq 0, \tag{1.4}$$

and that $(M_n)_{n\geq 0}$ is a positive Harris chain with stationary distribution F^{m+1}, the $(m+1)$-fold product of F. We call $(M_n, S_n)_{n\geq 0}$ hereafter a (φ, F)*-m-dependent Markov renewal process*, abbreviated as (φ, F, m)*-MRP*. For the definition of its lattice-span d, a notoriously important characteristic in renewal theory, see [2], Section 3. In this paper we always suppose that $(M_n, S_n)_{n\geq 0}$ *is nonarithmetic.*

Let us briefly summarize some notation from [2] which is kept throughout unless stated otherwise. Suppose a canonical model with probability measures $P_{x,y}$, $(x,y) \in \mathcal{S}^{m+1} \times \mathbb{R}^+$, such that $P_{x,y}(M_0 = x, S_0 = y) = 1$. Here $\mathbb{R}^+ \stackrel{\text{def}}{=} [0,\infty)$. For a distribution λ on $\mathcal{S}^{m+1} \times \mathbb{R}^+$ put $P_\lambda \stackrel{\text{def}}{=} \int_{\mathcal{S}^{m+1}\times\mathbb{R}^+} P_{x,y}\,\lambda(dx,dy)$, whereas $P_\lambda \stackrel{\text{def}}{=} P_{\lambda\otimes\delta_0}$ if λ is a distribution on $\mathcal{S}^{m+1}$. In the stationary case $\lambda = F^{m+1}$ we simply write P instead of $P_{F^{m+1}}$. As usual, the corresponding expectation operators are denoted $E_{x,y}, E_\lambda$ and E. Finally, let $\mathfrak{B}^+$ be the Borel σ-field over $\mathbb{R}^+$ and $\lambda\!\!\lambda$ be Lebesgue measure.

Given any (φ, F, m)-MRP $(M_n, S_n)_{n\geq 0}$, Janson [8] proved with the help of a simple martingale argument that

$$ES_{\tau+m} \;=\; \mu E(\tau+m) \tag{1.5}$$

holds true for any stopping time τ. In case of i.i.d. $X_1, X_2, ...$ $(m = 0)$ this is nothing but Wald's first identity. We showed in [2], that the very same martingale argument also gives

$$E_\lambda S_{\tau+m} \;=\; \mu E_\lambda \tau + E_\lambda S_m \;=\; \mu E_\lambda(\tau+m) + E_\lambda(S_m - m\mu) \tag{1.6}$$

for any initial distribution λ on $\mathcal{S}^{m+1} \times I\!\!R^+$. Let us note that (1.5) and (1.6) remain valid for real-valued $X_1, X_2, ...$ provided that $E\tau$ is finite in (1.5), respectively $E_\lambda S_m$ and $E_\lambda \tau$ are both finite in (1.6). We will refer to formulae (1.5) and (1.6) as *Janson's equation.*

As a direct application of the Markov renewal theorem, it was shown in [2] that, under each P_λ, $(M_{\tau(t)}, R_t)$ converges in distribution to a limiting variable (M_∞, R_∞) whose distribution does not depend on λ. This result is easily extended to the random vector $(M_{\tau(t)}, R_{0,t}, ..., R_{l,t})$, the distribution of the limiting variable $(M_\infty, R_{0,\infty}, ..., R_{l,\infty})$ being

$$\begin{aligned} P(M_\infty \in B, &R_{0,\infty} > r_1, ..., R_{l,\infty} > r_{l+1}) \\ &= \frac{1}{\mu}\int_{I\!\!R^+} P(M_1 \in B, S_1 > x + r_1, ..., S_{l+1} > x + r_{l+1}) \;\lambda\!\!\lambda(dx) \end{aligned} \tag{1.7}$$

for all $l \in I\!\!N_0$, $B \in \mathfrak{B}^+$ and $r_1, ..., r_{l+1} \in I\!\!R^+$, in particular

$$P(M_\infty \in B, R_{l,\infty} > r) \;=\; \frac{1}{\mu}\int_{I\!\!R^+} P(M_1 \in B, S_{l+1} > x + r) \;\lambda\!\!\lambda(dx) \tag{1.8}$$

for each $l \in I\!\!N$, $B \in \mathfrak{B}^+$ and $r \in I\!\!R^+$. Here it should be recalled that $(M_n, S_n)_{n\geq 0}$ is nonarithmetic with nonnegative increments. Integration of (1.8) with respect to r (and $B = \mathcal{S}^{m+1}$) immediately leads to

$$ER^p_{l,\infty} \;=\; \Delta_{l,p} \;\stackrel{\text{def}}{=}\; \frac{ES^{p+1}_{l+1} - ES^{p+1}_l}{(p+1)\mu} \tag{1.9}$$

for all $l \in I\!\!N_0$ and $p > 0$, and $\Delta_{l,p} < \infty$ if $\mu_{p+1} \stackrel{\text{def}}{=} ES^{p+1}_1 < \infty$. In case $l = 0$ we also write Δ_p instead of $\Delta_{0,p}$. Then (1.9) reads

$$ER^p_\infty \;=\; \Delta_p \;=\; \frac{\mu_{p+1}}{(p+1)\mu} \tag{1.10}$$

which is a well known formula in the i.i.d. case.

We will also need the random variable

$$\hat{R}_{m,t} \;\stackrel{\text{def}}{=}\; \sum_{k=1}^m X_{\tau(c)+k}(S_{\tau(c)+m+k} - S_{\tau(t)+m}) \;=\; \sum_{k=1}^m (R_{k,t} - R_{k,t-1})(R_{m+k,t} - R_{m,t}). \tag{1.11}$$

As seen from the above, $\hat{R}_{m,t}$ converges in distribution to a random variable $\hat{R}_{m,\infty}$, and a straightforward computation using (1.7) renders

$$E\hat{R}_{m,\infty} \;=\; \Lambda \;\stackrel{\text{def}}{=}\; \frac{1}{\mu}\sum_{j=1}^{m}\sum_{k=1}^{j} EX_0X_jX_{m+k}, \tag{1.12}$$

where $\Lambda < \infty$ if $\mu_3 < \infty$.

2. A short review of the i.i.d. case

Let us briefly recall some results on $\tau(t)$ and R_t in case where $(S_n)_{n\geq 0}$ is an ordinary nonarithmetic zero-delayed renewal process with drift μ and renewal function $U(t)$. The following bounds for the moments of R_t in terms of the respective moments of R_∞ were first established by Lorden [10].

$$\sup_{t\geq 0} ER_t \;\leq\; 2ER_\infty \;=\; 2\Delta_1 \tag{2.1}$$

and

$$\sup_{t\geq 0} ER_t^p \;\leq\; (p+2)ER_\infty^p \;=\; (p+2)\Delta_p \tag{2.2}$$

for general $p > 0$. Alternative proofs appeared in Carlsson and Nerman [6] (only (2.1)) and Chang [5]. Since $U(t) = E\tau(t)$ in the given situation and $t \leq \mu E\tau(t) = ES_{\tau(t)} = t + ER_t$, (2.1) further leads to the following inequality for the renewal function: For all $t \geq 0$,

$$\frac{t}{\mu} \;\leq\; U(t) \;\leq\; \frac{t}{\mu} + \frac{\mu_2}{\mu^2}. \tag{2.3}$$

Turning to the asymptotic behavior of expectation and variance of $\tau(t)$ we first note that $\mu_2 < \infty$ implies the uniform integrability of $\{R_t; t \geq 0\}$ and thus $ER_t = ER_\infty + o(1) = \Delta_1 + o(1)$ as $t \to \infty$, see e.g. [12]. Putting $\sigma^2 \stackrel{\text{def}}{=} \mathrm{Var}X_1 = \mu_2 - \mu^2$, the identity $\mu E\tau(t) = t + ER_t$ now leads to the well known expansion

$$E\tau(t) \;=\; \frac{t+\Delta_1}{\mu} + o(1) \;=\; \frac{t}{\mu} + \frac{\sigma^2}{2\mu^2} + \frac{1}{2} + o(1), \quad t \to \infty. \tag{2.4}$$

A corresponding result for $Var\tau(t)$ was derived by Smith [11], Lai and Siegmund [9], and Alsmeyer [1]. Suppose $\mu_3 < \infty$ and that X_1 is spread out. For the given situation of nonnegative $X_1, X_2, ...$ the result then reads

$$\mathrm{Var}\tau(t) \;=\; \frac{\sigma^2 t}{\mu^3} + \frac{5\mu_2^2}{4\mu^4} - \frac{2\mu_3}{3\mu^3} - \frac{\mu_2}{\mu^2} + o(1), \quad t \to \infty. \tag{2.5}$$

3. Results

The purpose of this article is to provide results corresponding to (2.1), (2.2) (in a weaker form), (2.3) and (2.5) for nonarithmetic (φ, F, m)-MRP. However, instead of the overshoot R_t we will rather focus on $R_{m,t}$ because in the m-dependent situation the latter variable is more relevant for renewal theoretic considerations. To see this note that Janson's equation with $\tau(t)$ gives

$$E_\lambda\tau(t) \;=\; \frac{1}{\mu}\Big(E_\lambda S_{\tau(t)+m} - E_\lambda S_m\Big) \;=\; \frac{1}{\mu}\Big(t + E_\lambda R_{m,t} - E_\lambda S_m\Big) \tag{3.1}$$

for any initial distribution λ on $\mathcal{S}^{m+1}\times I\!\!R^+$. Put

$$\mu_p(\lambda) \;\stackrel{\text{def}}{=}\; \sup_{n\ge 0} E_\lambda X_n^p$$

for $p>0$ and note that, by m-dependence, $\mu_p = E_\lambda X_{m+1}^p \le \mu_p(\lambda)$. We already proved in [2, Cor. 4.2] that $\mu_2(\lambda)<\infty$ implies $\lim_{t\to\infty} E_\lambda R_{m,t} = \Delta_{m,1}$ and then with (3.1)

$$E_\lambda\tau(t) \;=\; \frac{t+\Delta_{m,1}-E_\lambda S_m}{\mu} + o(1), \quad t\to\infty. \tag{3.2}$$

We put $\kappa^2 \stackrel{\text{def}}{=} \lim_{n\to\infty} n^{-1}\mathrm{Var}S_n$ (for existence of this limit see [8] or [2]) and note that κ^2 also equals $C\mathrm{ov}(X_n, S_{n+m})$ for $n\ge 1$. With this quantity, and after some elementary calculations, (3.2) may be rewritten as

$$E_\lambda\tau(t) \;=\; \frac{t}{\mu} + \frac{\kappa^2}{2\mu^2} + \frac{1}{2} - \frac{E_\lambda(S_m - m\mu)}{\mu} + o(1), \quad t\to\infty. \tag{3.3}$$

In the stationary case $\lambda = F^{m+1}$, where $E_\lambda(S_m - m\mu)=0$, (3.3) reduces exactly to the second formula for $E\tau(t)$ in (2.4) when replacing σ^2 with κ^2.

We continue with a presentation of our results to be proved in the subsequent sections.

Theorem 3.1. *If $(M_n,S_n)_{n\ge0}$ is a nonarithmetic (φ,F,m)-MRP with positive drift μ, then*

$$\sup_{t\ge0} ER_{m,t} \;\le\; 2\Delta_{m,1} \tag{3.4}$$

and

$$\sup_{t\ge0} E_\lambda R_{m,t} \;\le\; 2\Delta_{m,1} + E_\lambda S_m \tag{3.5}$$

for arbitrary initial distribution λ on $\mathcal{S}^{m+1}\times I\!\!R^+$.

We note that every (φ,F,m)-MRP is also a (φ,F,l)-MRP for $l>m$ so that the assertions of Theorem 3.1 remain valid with any such l instead of m.

If $U_\lambda(t) \stackrel{\text{def}}{=} \sum_{n\ge0} P_\lambda(S_n\le t)$ denotes the renewal function of $(S_n)_{n\ge0}$ under P_λ and if $U(t) \stackrel{\text{def}}{=} U_{F^{m+1}}(t)$, then $U_\lambda(t) = E_\lambda\tau(t)$ for all $t\ge0$ as in the i.i.d. case whence we have the following immediate corollary to Theorem 3.1.

Corollary 3.2. *In the situation of Theorem 3.1,*

$$\frac{t}{\mu} - m \;\le\; U(t) \;\le\; \frac{t+2\Delta_{m,1}}{\mu} - m \tag{3.6}$$

and

$$\frac{t - E_\lambda S_m}{\mu} \;\le\; U_\lambda(t) \;\le\; \frac{t + 2\Delta_{m,1} - E_\lambda S_m}{\mu} \tag{3.7}$$

for all $t \ge 0$ *and any initial distribution* λ *on* $\mathcal{S}^{m+1} \times I\!R^+$.

Our next result provides a bound for $\sup_{t\ge 0} E_\lambda R_{m,t}^p$ for $p > 1$ which however is weaker than the one in (2.2) for the i.i.d. case.

THEOREM 3.3. *Given the situation of Theorem 3.1, let* $p > 1$. *Then*

$$\sup_{t\ge 0} ER_{m,t}^p \;\le\; 2^{p+1}\Delta_{m,p} \tag{3.8}$$

and

$$\sup_{t\ge 0} E_\lambda R_{m,t}^p \;\le\; 2^p E_\lambda S_m + 2^{2p+1}\Delta_{m,p} \tag{3.9}$$

for arbitrary initial distribution λ *on* $\mathcal{S}^{m+1} \times I\!R^+$.

In order to finally present an expansion of $Var_\lambda \tau(t)$ we note that $(M_n, S_n)_{n\ge 0}$ is called *spread out* if, for some F^{m+1}-positive set A and some $n \in I\!N$, the conditional distribution of (M_n, S_n) given $M_0 = x$ is not singular with respect to $F^{m+1} \otimes \lambda\!\!\lambda$ for each $x \in A$.

THEOREM 3.4. *Given the situation of Theorem 3.1 suppose further that* $(M_n, S_n)_{n\ge 0}$ *is spread out and* $\mu_3 < \infty$. *For any initial distribution* λ *such that* $\mu_3(\lambda) < \infty$ *it then holds true that*

$$Var_\lambda \tau(t) \;=\; \frac{\kappa^2}{\mu^3} t + \frac{C}{\mu^2} + o(1), \quad t \to \infty, \tag{3.10}$$

where

$$\begin{aligned} C \;\stackrel{\text{def}}{=}\; & \frac{\kappa^2}{\mu}\Delta_{m,1} + 2\Delta_{m,1}^2 - \Delta_{m,2} + \Delta_{2m,1}^2 - \Delta_{2m,2} \\ & - 2\Lambda - \frac{\kappa^2}{\mu} E_\lambda S_m + E_\lambda (S_{2m} - 2m\mu)^2. \end{aligned} \tag{3.11}$$

The rest of the article is devoted to the proofs of the three theorems. We prove Theorem 3.1 and 3.3 in Section 4 and Theorem 3.4 in Section 5. An Appendix contains a Markov renewal theorem for spread out (φ, F, m)-MRP.

4. PROOFS OF THEOREM 3.1 AND 3.3

We start by giving straightforward extension of Janson's equations (1.5) and (1.6) which will be repeatedly used thereafter.

LEMMA 4.1. *Given any fixed* $r \ge 1$, *put* $W_n \stackrel{\text{def}}{=} (X_n, ..., X_{n+r})$ *for* $n \ge 0$. *Let* $f : \mathcal{S}^{r+1} \to I\!R^+$ *be a measurable function,* $V_n \stackrel{\text{def}}{=} \sum_{k=0}^n f(W_k)$ *for* $n \ge 0$ *and* $\nu \stackrel{\text{def}}{=} Ef(W_1)$. *Let further* τ *be a stopping time with respect to* $(M_n, S_n)_{n\ge 0}$. *Then*

$$E_\lambda V_{\tau+m} \;=\; \nu E_\lambda(\tau + m) + E_\lambda(V_{m+r} - (m+r)\nu) \tag{4.1}$$

for any initial distribution λ on $\mathcal{S}^{m+1} \times I\!R^+$, in particular

$$EV_{\tau+m} \;=\; \nu E(\tau+m). \tag{4.2}$$

PROOF. Note first that each stopping time with respect to $(M_n, S_n)_{n\geq 0}$ is also a stopping time with respect to $(M_n, W_n)_{n\geq 0}$. Since $(W_n)_{n\geq 1}$ is further $(m+r)$-dependent and stationary under $P = P_{F^{m+1}}$ a similar martingale argument as for (1.6) yields $E_\lambda V_{\tau+m+r} = \nu E_\lambda(\tau+m+r)$ $+E_\lambda(V_{m+r} - (m+r)\nu)$. The assertion now follows because $P_\lambda(f(W_{\tau+k}) \in \cdot) = P(f(W_1) \in \cdot)$ for $k = m+1, ..., m+r$ and any λ and thus $E\sum_{k=\tau+m+1}^{\tau+m+r} f(W_k) = \nu r$. ♢

The second lemma is basically the first part of the proof of Theorem 3.1, but it will also be needed in the proof of Theorem 3.4 in Section 5.

LEMMA 4.2. *Given the situation of Theorem 3.1,*

$$\int_0^c ER_{m,t}\;dt \;=\; \Delta_{m,1}(c + ER_{m,c}) - \frac{1}{2}(ER_{m,c})^2 - E\hat{R}_{m,c}. \tag{4.3}$$

for all $c > 0$.

PROOF. Fix any $c > 0$. The function $t \mapsto R_{m,t}$ is piecewise linear with slope -1 on each interval $[S_{n-1}, S_n)$ and $R_{m,S_n} = S_{n+m+1} - S_n$ for $n \geq 0$. Therefore

$$\int_{S_{n-1}}^{S_n} R_{m,t}\;dt \;=\; \frac{1}{2}(S_{n+m} - S_{n-1})^2 - \frac{1}{2}(S_{n+m} - S_n)^2 \tag{4.4}$$

for $n \geq 0$ and

$$\int_c^{S_{\tau(c)}} R_{m,t}\;dt \;=\; \frac{1}{2}R_{m,c}^2 - \frac{1}{2}(S_{\tau(c)+m} - S_{\tau(c)})^2. \tag{4.5}$$

With the help of (4.4) and after some algebra, we find

$$\begin{aligned}\int_{S_{\tau(c)+k-1}}^{S_{\tau(c)+k}} R_{m,t}\;dt \;&=\; \frac{1}{2}(S_{\tau(c)+m+k} - S_{\tau(c)+k-1})^2 - \frac{1}{2}(S_{\tau(c)+m+k} - S_{\tau(c)+k})^2 \\ &=\; \frac{1}{2}(S_{\tau(c)+m} - S_{\tau(c)+k-1})^2 - \frac{1}{2}(S_{\tau(c)+m} - S_{\tau(c)+k})^2 \\ &\qquad + X_{\tau(c)+k}(S_{\tau(c)+m+k} - S_{\tau(c)+m})\end{aligned}$$

for $1 \leq k \leq m$. Combining this with (4.5),

$$\begin{aligned}\int_c^{S_{\tau(c)+m}} R_{m,t}\;dt \;&=\; \int_c^{S_{\tau(c)}} R_{m,t}\;dt \;+\; \sum_{k=1}^m \int_{S_{\tau(c)+k-1}}^{S_{\tau(c)+k}} R_{m,t}\;dt \\ &=\; \frac{1}{2}R_{m,c}^2 - \frac{1}{2}(S_{\tau(c)+m} - S_{\tau(c)})^2 \\ &\qquad + \frac{1}{2}\sum_{k=1}^m \Big((S_{\tau(c)+m} - S_{\tau(c)+k-1})^2 - (S_{\tau(c)+m} - S_{\tau(c)+k})^2\Big)\end{aligned}$$

$$+ \sum_{k=1}^{m} X_{\tau(c)+k}(S_{\tau(c)+m+k} - S_{\tau(c)+m})$$
$$= \frac{1}{2}R_{m,c}^2 + \hat{R}_{m,c}$$

which in turn may be combined with (4.4) to show

$$\begin{aligned} \int_0^c R_{m,t}\,dt &= \sum_{n=1}^{\tau(c)+m} \int_{S_{n-1}}^{S_n} R_{m,t}\,dt - \int_c^{S_{\tau(c)+m}} R_{m,t}\,dt \\ &= \frac{1}{2}\sum_{n=1}^{\tau(c)+m} \Big((S_{n+m}-S_{n-1})^2 - (S_{n+m}-S_n)^2\Big) - \frac{1}{2}R_{m,c}^2 - \hat{R}_{m,c}. \end{aligned} \tag{4.6}$$

Putting $f(x_1,...,x_{m+1}) \stackrel{\text{def}}{=} \frac{1}{2}((\sum_{k=1}^{m+1} x_k)^2 - (\sum_{k=2}^{m+1} x_k)^2)$, we see that

$$f(X_n,...,X_{n+m}) = \frac{1}{2}\Big((S_{n+m}-S_{n-1})^2 - (S_{n+m}-S_n)^2\Big)$$

for $n \geq 1$. Hence an application of Lemma 4.1 and Janson's equation (1.5) yields

$$\begin{aligned} &\frac{1}{2}E\left(\sum_{n=1}^{\tau(c)+m} \Big((S_{n+m}-S_{n-1})^2 - (S_{n+m}-S_n)^2\Big)\right) \\ &= \frac{1}{2}(ES_{m+1}^2 - ES_m^2)E(\tau(c)+m) \\ &= \Delta_{m,1}\mu E(\tau(c)+m) \\ &= \Delta_{m,1}(c+ER_{m,c}). \end{aligned}$$

Taking expectations in (4.6) and using $ER_{m,c}^2 \geq (ER_{m,c})^2$, we are thus finally led to

$$\int_0^c ER_{m,t}\,dt = E\left(\int_0^c R_{m,t}\,dt\right) = \Delta_{m,1}(c+ER_{m,c}) - \frac{1}{2}(ER_{m,c})^2 - E\hat{R}_{m,c}$$

which is the asserted result. ◊

Proof of Theorem 3.1. Our proof is essentially an adaptation of the one given by Lorden for the i.i.d. case. We begin with the stationary situation where $\lambda = F^{m+1}$.

First use Lemma 4.2, $\hat{R}_{m,c} \geq 0$ and $ER_{m,c}^2 \geq (ER_{m,c})^2$ to infer

$$\int_0^c ER_{m,t}\,dt = E\left(\int_0^c R_{m,t}\,dt\right) \leq \Delta_{m,1}(c+ER_{m,c}) - \frac{1}{2}(ER_{m,c})^2 \tag{4.7}$$

for any $c > 0$.

The second step is to show that $t \mapsto ER_{m,t}$ is a subsadditive function. For $s \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$, put

$$S_{s,n} \stackrel{\text{def}}{=} S_{\tau(s)+m+n} - S_{\tau(s)+m} = \sum_{k=1}^{n} X_{\tau(s)+m+k}$$

and

$$M_{s,n} \stackrel{\text{def}}{=} M_{\tau(s)+m+n}.$$

Then $(M_{s,n}, S_{s,n})_{n\ge 0}$ forms a (φ, F, m)-MRP with initial distribution $P(M_{\tau(s)+m} \in \cdot) \otimes \delta_0$. Since $M_{\tau(s)+m+n}$ is independent of $M_{\tau(s)}$ with distribution F^{m+1} for $n \ge 1$ (this is not necessarily true for $n = 0$), we infer that $(M_{s,n})_{n\ge 1}$ forms a stationary (φ, F, m)-MRP independent of $(M_n, S_n)_{0\le n\le \tau(s)}$ and with the same distribution as $(M_n, S_n)_{n\ge 1}$ under $P = P_{F^{m+1}}$.

Now, defining

$$\tau_s(t) \stackrel{\text{def}}{=} \inf\{n \ge 0 : S_{s,n} > t\} \quad \text{and} \quad R^{(s)}_{m,t} \stackrel{\text{def}}{=} S_{s,\tau_s(t)+m} - t$$

for $t \in \mathbb{R}^+$, we have that $S_{\tau(s+t)+m} \le S_{\tau(S_{\tau(s)+m}+t)+m}$ and $R^{(s)}_{m,t} \stackrel{d}{=} R_{m,t}$ where $\stackrel{d}{=}$ means equality in distribution (under P). Hence

$$\begin{aligned} R_{m,s+t} &= S_{\tau(s+t)+m} - (s+t) \\ &\le S_{\tau(S_{\tau(s)+m}+t)+m} - (s+t) \\ &= (S_{s,\tau_s(t)+m} - t) + (S_{\tau(s)+m} - s) \\ &= R^{(s)}_{m,t} + R_{m,s} \end{aligned} \tag{4.8}$$

and thus

$$ER_{m,s+t} \le ER_{m,s} + ER_{m,t}$$

for all $s, t \in \mathbb{R}^+$.

Arguing as in Lorden [10], the subadditivity of $ER_{m,t}$ in t implies

$$\begin{aligned} \frac{c}{2} ER_{m,c} &\le \frac{c}{2} \inf_{0\le t\le c/2} (ER_{m,t} + ER_{m,c-t}) \\ &\le \int_0^{c/2} (ER_{m,t} + ER_{m,c-t})\, dt = \int_0^c R_{m,t}\, dt. \end{aligned} \tag{4.9}$$

which in combination with (4.7) leads to

$$(ER_{m,c})^2 + (c - 2\Delta_{m,1}) ER_{m,c} - 2c\Delta_{m,1} \le 0.$$

But the polynomial $x \mapsto x^2 + (c - 2\Delta_{m,1})x + 2c\Delta_{m,1}$ is nonpositive only between its roots $-c$ and $2\Delta_{m,1}$, thus showing the desired inequality $ER_{m,c} \le 2\Delta_{m,1}$ for any $c \in \mathbb{R}^+$.

We must finally derive inequality (3.5) for arbitrary initial distributions λ on $\mathcal{S}^{m+1} \times \mathbb{R}^+$ which turns out to be easy. By another appeal to (4.8),

$$E_\lambda R_{m,s+t} \le E_\lambda R_{m,s} + E_\lambda R^{(s)}_{m,t} = E_\lambda R_{m,s} + ER_{m,t}$$

for all $s, t \in \mathbb{R}^+$ because m-dependence implies $P_\lambda(R^{(s)}_{m,t} \in \cdot) = P(R_{m,t} \in \cdot)$ for all $s, t \in \mathbb{R}^+$. Choose $s = 0$ and note that $R_{m,0} = S_m$ to conclude

$$\sup_{t\ge 0} E_\lambda R_{m,t} \le E_\lambda R_{m,0} + \sup_{t\ge 0} ER_{m,t} \le E_\lambda S_m + 2\Delta_{m,1}. \qquad \diamondsuit$$

Proof of Theorem 3.3. We keep the notation from the previous proof. Along the same lines as there (see (4.2–5)) we obtain here for each $c > 0$

$$
\begin{aligned}
\int_0^c R_{m,t}^p\,dt &= \int_0^{S_{\tau(c)+m}} R_{m,t}^p\,dt \;-\; \int_c^{S_{\tau(c)+m}} R_{m,t}^p\,dt \\
&\le \sum_{k=1}^{\tau(c)+m} \int_{S_{k-1}}^{S_k} (S_{k+m}-t)^p\,dt \;-\; \int_c^{S_{\tau(c)+m}} (S_{\tau(c)+m}-t)^p\,dt \\
&= \frac{1}{p+1} \sum_{k=1}^{\tau(c)+m} \Big((S_{k+m}-S_{k-1})^{p+1} - (S_{k+m}-S_k)^{p+1} \Big) \;-\; \frac{1}{p+1} R_{m,c}^{p+1}
\end{aligned}
$$

and then by another use of Lemma 4.1 and Janson's equation

$$
\begin{aligned}
\int_0^c ER_{m,t}^p\,dt &\le \frac{ES_{m+1}^{p+1} - ES_m^{p+1}}{p+1} E(\tau(c)+m) \;-\; \frac{1}{p+1} ER_{m,c}^{p+1} \\
&= \Delta_{m,p}(c + ER_{m,c}) \;-\; \frac{1}{p+1} ER_{m,c}^{p+1}.
\end{aligned}
$$

Since $ER_{m,c} \le (ER_{m,c}^p)^{1/p}$ and $ER_{m,c}^{p+1} \ge (ER_{m,c}^p)^{(p+1)/p}$ by Jensen's inequality, we arrive at

$$
\int_0^c ER_{m,t}^p\,dt \;\le\; \Delta_{m,p}\Big(c + (ER_{m,c}^p)^{1/p}\Big) \;-\; \frac{1}{p+1}(ER_{m,c}^p)^{(p+1)/p}. \tag{4.10}
$$

With the help of (4.8) we next infer that

$$
ER_{m,s+t}^p \;\le\; E(R_{m,s} + R_{m,t}^{(s)})^p \;\le\; 2^p(ER_{m,s}^p + ER_{m,t}^p)
$$

for all $s, t \in I\!R^+$, where $R_{m,t}^{(s)} \stackrel{d}{=} R_{m,t}$ should be recalled. Consequently,

$$
cER_{m,c}^p \;\le\; c \inf_{0\le t\le c} 2^p(ER_{m,t}^p + ER_{m,c-t}^p) \;\le\; 2^{p+1} \int_0^c ER_{m,t}\,dt,
$$

which in combination with (4.10) leads to

$$
cER_{m,c}^p \;-\; 2^{p+1}\Delta_{m,p}\Big(c + (ER_{m,c}^p)^{1/p}\Big) \;-\; \frac{2^{p+1}}{p+1}(ER_{m,c}^p)^{(p+1)/p} \;\le\; 0. \tag{4.11}
$$

Inequality (3.8) follows because the function $x \mapsto cx - 2^{p+1}\Delta_{m,p}(c + x^{1/p}) + \frac{2^{p+1}}{p+1}x^{(p+1)/p}$ is positive for $x > 2^{p+1}\Delta_{m,p}$. As for (3.9), we finally note that

$$
\sup_{t\ge 0} E_\lambda R_{m,t}^p \;\le\; 2^p\Big(E_\lambda R_{m,0}^p + \sup_{t\ge 0} ER_{m,t}^p\Big) \;\le\; 2^p E_\lambda S_m + 2^{2p+1}\Delta_{m,p}. \qquad \diamondsuit
$$

Remark. In the i.i.d. case Lorden's bounds for higher order moments of R_t are based on the inequality

$$
R_{s+t} \;\le\; \max\{R_s, L_t^{(s)}\},
$$

where $L_t^{(s)} \stackrel{\text{def}}{=} X_{\tau_s(t)}$ and $\tau_s(t)$ is defined as in the proof of Theorem 3.1 (with $m = 0$). Then $ER_t^p \le ER_s^p + EX_{\tau(t)}^p$ which obviously provides a better bound than the use of subadditivity by which the unpleasant factor 2^p is incurred. Unfortunately, this inequality does not extend to the case $m \ge 1$ and we have been unable to come up with an inequality of similar type.

5. Proof of Theorem 3.4

Let us start by noting that Janson [8] proved the following identity as a substitute for Wald's second equation in the m-dependent case. If $\mu_2 < \infty$ then

$$E(S_{\tau+2m} - \mu(\tau + 2m))^2 \;=\; \kappa^2 E\tau + E(S_{2m} - 2m\mu)^2 \tag{5.1}$$

for any stopping time τ. His martingale argument also gives

$$E_\lambda(S_{\tau+2m} - \mu(\tau + 2m))^2 \;=\; \kappa^2 E_\lambda\tau + E_\lambda(S_{2m} - 2m\mu)^2 \tag{5.2}$$

for arbitrary initial distributions λ on $\mathcal{S}^{m+1} \times I\!R^+$ such that $E_\lambda(S_{2m} - 2m\mu)^2 < \infty$.

Similar to the i.i.d. case ($m = 0$) the crucial term in the expansion of $Var_\lambda\tau(t)$ turns out to be the covariance of $\tau(t)$ and $R_{2m,t}$. In order to compute this quantity up to vanishing terms as $t \to \infty$ we will need that the convergence of $ER_{2m,t}$ to $\Delta_{2m,1}$ is of the order $o(t^{-1})$. This is the result for which we require $(M_n, S_n)_{n\ge0}$ to be spread out. It is contained in the following proposition. Let us first give some necessary notation and facts from Markov renewal theory taken from [2] and [3]. For $C \in \mathfrak{S}^{m+1} \otimes \mathfrak{B}^+$ put

$$U_\lambda(C) \;\stackrel{\text{def}}{=}\; \sum_{n\ge0} P_\lambda((M_n, S_n) \in C) \;=\; E_\lambda\left(\sum_{n\ge0} \mathbf{1}_C(M_n, S_n)\right) \tag{5.3}$$

where $\mathbf{1}_C$ denotes the indicator function of C. U_λ is the Markov renewal measure associated with $(M_n, S_n)_{n\ge0}$ under P_λ. Note that the nonnegativity of $S_0, X_1, X_2, ...$ implies (in analogy to the i.i.d. case)

$$U_\lambda(\mathcal{S}^{m+1} \times [0, t]) \;=\; \sum_{n\ge0} P_\lambda(S_n \le t) \;=\; E_\lambda\tau(t) \tag{5.4}$$

for all $t \ge 0$. Defining ν^s through

$$\nu^s(A \times (r, \infty)) \;\stackrel{\text{def}}{=}\; \frac{1}{\mu}\int_{(r,\infty)} P(M_1 \in A, X_1 > t)\; \lambda\!\!\lambda(dt)$$

for $A \in \mathfrak{S}^{m+1}$ and $r \in I\!R^+$ one finds (see [2, eq. (5.1)]) that

$$U_{\nu^s} \;=\; \frac{1}{\mu} F^{m+1} \otimes \lambda\!\!\lambda^+ \tag{5.5}$$

where $\lambda\!\!\lambda^+ \stackrel{\text{def}}{=} \lambda\!\!\lambda(\cdot \cap I\!R^+)$. ν^s is the so-called *stationary Markov delay distribution*. One can easily check that

$$E_{\nu^s} S_m^\alpha < \infty \quad \text{iff} \quad \mu_{\alpha+1} < \infty \tag{5.6}$$

for any $\alpha > 0$. Moreover, $(R_{m,t})_{t\geq 0}$ is stationary under P_{ν^s}, in particular $ER_{m,t} = \Delta_{m,1}$ for all $t \geq 0$. Finally, let $\|\nu\|$ denote total variation of a bounded signed measure ν, i.e. $\|\nu\| \stackrel{\text{def}}{=} \sup_A |\nu(A)|$.

PROPOSITION 5.1. *Let $(M_n, S_n)_{n\geq 0}$ be a spread out (φ, F, m)-MRP with $\mu_\alpha < \infty$ for some $\alpha > 2$. If λ and λ' are two initial distributions on $\mathcal{S}^{m+1} \times I\!R^+$ such that $\mu_{\alpha-1}(\lambda) < \infty$ and $\mu_{\alpha-1}(\lambda') < \infty$, then*

$$|E_\lambda R_{m,t} - E_{\lambda'} R_{m,t}| \;=\; o(t^{-(\alpha-2)}), \quad t \to \infty, \tag{5.7}$$

and in case $\alpha = 3$ also

$$\int_{I\!R^+} |E_\lambda R_{m,t} - E_{\lambda'} R_{m,t}| \;\lambda\!\!\lambda(dt) \;<\; \infty. \tag{5.8}$$

In particular, $\mu_{\alpha-1}(\lambda) < \infty$ and $\mu_\alpha < \infty$ implies

$$|E_\lambda R_{m,t} - \Delta_{m,1}| \;=\; o(t^{-(\alpha-2)}), \quad t \to \infty, \tag{5.9}$$

and in case $\alpha = 3$ also

$$\int_{I\!R^+} |E_\lambda R_{m,t} - \Delta_{m,1}| \;\lambda\!\!\lambda(dt) \;<\; \infty. \tag{5.10}$$

PROOF. We showed in [3] the existence of two coupled (φ, F, m)-MRP $(M_n, S_n)_{n\geq 0}$ and $(M'_n, S'_n)_{n\geq 0}$ with initial distributions λ and λ', respectively, and the existence of two stopping times σ, σ' such that

$$(M_{\sigma+n}, S_{\sigma+n})_{n\geq 0} \;=\; (M'_{\sigma'+n}, S'_{\sigma'+n})_{n\geq 0}$$

and

$$E_{\lambda,\lambda'} S^{*\alpha-1} < \infty$$

for $S^* \stackrel{\text{def}}{=} S_\sigma = S'_{\sigma'}$. Here $E_{\lambda,\lambda'}$ clearly denotes expectation given initial distributions λ, λ' for the two processes. As a consequence of this coupling we derived that

$$\|U_\lambda(\cdot \cap [\mathcal{S}^{m+1} \times (t,\infty)]) - U_{\lambda'}(\cdot \cap [\mathcal{S}^{m+1} \times (t,\infty)])\| \;=\; o(t^{-(\alpha-2)}) \tag{5.11}$$

and

$$\|U_\lambda - U_{\lambda'}\| < \infty.$$

Put $\nu \stackrel{\text{def}}{=} U_\lambda - U_{\lambda'}$. ν is a bounded signed measure with (use (5.4))

$$\nu(\mathcal{S}^{m+1} \times [0,t]) \;=\; E_{\lambda,\lambda'}\tau(t) - E_{\lambda,\lambda'}\tau'(t) \;=\; E_{\lambda,\lambda'}\tau(t \wedge S^*) - E_{\lambda,\lambda'}\tau'(t \wedge S^*)$$

where $\tau'(t)$ has the obvious meaning. Hence, by letting $t \to \infty$,

$$\nu(\mathcal{S}^{m+1} \times I\!R^+) \;=\; E_{\lambda,\lambda'}\tau(S^*) - E_{\lambda,\lambda'}\tau'(S^*). \tag{5.12}$$

Now $E_{\lambda,\lambda'} S_{\tau(S^*)+m} = E_{\lambda,\lambda'} S'_{\tau'(S^*)+m}$. Using Janson's equation (which is possible because the coupling construction in [3] is Markov adapted), we obtain

$$E_\lambda S_m + \mu E_{\lambda,\lambda'} \tau(S^*) \;=\; E_{\lambda'} S_m + \mu \; E_{\lambda,\lambda'} \tau'(S^*)$$

and thus in (5.12)

$$\nu(\mathcal{S}^{m+1} \times I\!R^+) \;=\; \frac{E_\lambda S_m - E_{\lambda'} S_m}{\mu}.$$

This further yields

$$\begin{aligned} U_\lambda(t) - U_{\lambda'}(t) &= \nu(\mathcal{S}^{m+1} \times [0,t]) \\ &= \nu(\mathcal{S}^{m+1} \times I\!R^+) - \nu(\mathcal{S}^{m+1} \times (t,\infty)) \\ &= \frac{E_\lambda S_m - E_{\lambda'} S_m}{\mu} - \nu(\mathcal{S}^{m+1} \times (t,\infty)). \end{aligned}$$

where $U_\lambda(t) = U_\lambda(\mathcal{S}^{m+1} \times [0,t])$ as in Corollary 3.2. Now we conclude (5.7) with the help of (5.11) because

$$\begin{aligned} |E_\lambda R_{m,t} - E_{\lambda'} R_{m,t}| &= |E_\lambda S_m + \mu E_\lambda \tau(t) - E_{\lambda'} R_{m,t} - E_{\lambda'} \tau(t)| \\ &= \mu \left| U_\lambda(t) - U_{\lambda'}(t) + \frac{E_\lambda S_m - E_{\lambda'} S_m}{\mu} \right| \\ &= \mu |\nu(\mathcal{S}^{m+1} \times (t,\infty)) \\ &\le \mu \| U_\lambda(\cdot \cap [\mathcal{S}^{m+1} \times (t,\infty)]) - U_{\lambda'}(\cdot \cap [\mathcal{S}^{m+1} \times (t,\infty)]) \| \\ &= o(t^{-(\alpha-2)}). \end{aligned}$$

Choosing $\lambda' = \nu^s$ and using (5.6) we also get (5.9).

Left with the proof of (5.8) we first note that

$$|E_\lambda R_{m,t} - E_{\lambda'} R_{m,t}| \;=\; |E_{\lambda,\lambda'}(R_{m,t} - R'_{m,t})|$$

where $R'_{m,t} \stackrel{\text{def}}{=} S'_{\tau'(t)+m} - t$. Now $R_{m,t} = R'_{m,t}$ on $\{S^* \le t\}$ implies

$$|E_\lambda R_{m,t} - E_\lambda R_{m,t}| \;\le\; E_{\lambda,\lambda'}(R_{m,t} + R'_{m,t}) \mathbf{1}_{\{S^* > t\}}$$

for $t \ge 0$. As a consequence,

$$\int_{I\!R^+} |E_\lambda R_{m,t} - E_\lambda R_{m,t}| \; \lambda\!\!\lambda(dt) \;\le\; E_{\lambda,\lambda'} \left(\int_0^{S^*} (R_{m,t} + R'_{m,t}) \; dt \right).$$

Put $V_n \stackrel{\text{def}}{=} \sum_{k=1}^n ((S_{k+m} - S_{k-1})^2 - (S_{k+m} - S_k)^2)$ for $n \ge 0$ and $\nu \stackrel{\text{def}}{=} EV_1 = ES_{m+1}^2 - ES_m^2 = \mu \Delta_{m,1}$. Using (4.6) in Lemma 4.2 and then Lemma 4.1, we obtain

$$\begin{aligned} E_{\lambda,\lambda'} \left(\int_0^{S^*} R_{m,t} \; dt \right) &\le \frac{1}{2} E_{\lambda,\lambda'} V_{\tau(S^*)+m} \\ &= \Delta_{m,1} \mu E_{\lambda,\lambda'}(\tau(S^*) + m) + E_\lambda(V_{m+1} - (m+1)\nu) \end{aligned} \tag{5.13}$$

which is finite because $E_\lambda V_{m+1} \le K(E_\lambda S_m^2 + ES_m^2)$ for some constant K and $\tau(S^*) = \tau(S_\sigma) = \sigma$ has finite mean as may be deduced from the coupling construction in [3] and Lemma 6.2 there. The same arguments show that $E_{\lambda,\lambda'}(\int_0^{S^*} R'_{m,t}\, dt) < \infty$ which completes the proof of (5.8). $\diamondsuit$

Let us note that strictly speaking Lemma 4.2 does not apply directly in (5.13) because $\tau(S^*)$ is not a stopping time with respect to $(M_n, S_n)_{n\ge0}$ but only with respect to a suitable extended filtration under which the coupling process $(M_n, M'_n, S_n, S'_n)_{n\ge0}$ is Markov adapted. However, an extension of Lemma 4.2 of the required form is easily provided.

PROOF OF THEOREM 3.4. Put $\overline{S}_n = S_n - n\nu$ for $n \ge 0$. Using $E_\lambda S_{\tau(t)+2m} = \mu E_\lambda(\tau(t) + 2m) + E_\lambda \overline{S}_{2m}$ in (5.2), we find after some simple algebra

$$\mu^2 \mathrm{Var}_\lambda \tau(t) \;=\; \kappa^2 E_\lambda \tau(t) + 2\mu \mathrm{Cov}_\lambda(S_{\tau(t)+2m}, \tau(t)) - \mathrm{Var}_\lambda S_{\tau(t)+2m} + E_\lambda \overline{S}_{2m}^2. \tag{5.14}$$

An expansion of $E_\lambda \tau(t)$ up to vanishing terms is given by (3.3). Since μ_3 and $E_\lambda S_m^3$ are both finite, $(R_{2m,t}^2)_{t\ge0}$ is uniformly integrable by Theorem 4.1 in [2] which in combination with $R_{2m,t} \stackrel{d}{\to} R_{2m,\infty}$ under each P_λ further yields

$$\mathrm{Var}_\lambda S_{\tau(t)+2m} \;=\; \mathrm{Var}_\lambda R_{2m,t} \;=\; \Delta_{2m,2} - \Delta_{2m,1}^2 + o(1), \quad t \to \infty. \tag{5.15}$$

We are thus left with the computation of $\mathrm{Cov}_\lambda(S_{\tau(t)+2m}, \tau(t))$ up to vanishing terms as $t \to \infty$. Along similar lines as in [9] we obtain

$$\begin{aligned}
\mathrm{Cov}_\lambda(S_{\tau(t)+2m}, \tau(t)) \;&=\; \mathrm{Cov}_\lambda(S_{\tau(t)+m}, \tau(t)) \;=\; \mathrm{Cov}_\lambda(R_{m,t}, \tau(t)) \\
&= \sum_{n\ge1} n \int_{\{\tau(t)=n\}} (R_{m,t} - E_\lambda R_{m,t})\, dP_\lambda \\
&= \sum_{n\ge1} n \left(\int_{\{\tau(t)>n-1\}} - \int_{\{\tau(t)>n\}} \right) (R_{m,t} - E_\lambda R_{m,t})\, dP_\lambda \\
&= \sum_{n\ge0} \int_{\{\tau(t)>n\}} (R_{m,t} - E_\lambda R_{m,t})\, dP_\lambda \\
&= \sum_{n\ge0} \int_{\mathcal{S}^{m+1}\times[0,t]} \Big(E_\lambda(R_{m,t} | M_n = s, S_n = x) - E_\lambda R_{m,t} \Big) \\
&\qquad\qquad \times dP_\lambda(M_n \in ds, S_n \in dx) \\
&= \int_{\mathcal{S}^{m+1}\times[0,t]} \Big(E_{s,0} R_{m,t-x} - E_\lambda R_{m,t} \Big)\, U_\lambda(ds, dx) \\
&= g * U_\lambda(t) - (E_\lambda R_{m,t} - \Delta_{m,1}) E_\lambda \tau(t),
\end{aligned}$$

where

$$g(s,t) \;\stackrel{\mathrm{def}}{=}\; (E_{s,0} R_{m,t} - \Delta_{m,1}) \mathbf{1}_{\mathbb{R}^+}(t).$$

By Proposition 5.1 and (3.3),

$$(E_\lambda R_{m,t} - \Delta_{m,1}) E_\lambda \tau(t) \;=\; o(1), \quad t \to \infty.$$

Furthermore, Theorem A.1 in the Appendix implies

$$\begin{aligned} \lim_{t\to\infty} g * U_\lambda(t) \;&=\; \frac{1}{\mu}\int_{\mathbb{R}^+}\int_{\mathcal{S}^{m+1}} g(s,x)\; F^{m+1}(ds)\; \lambda\!\!\lambda(dx) \\ &=\; \frac{1}{\mu}\int_{\mathbb{R}^+} (ER_{m,x} - \Delta_{m,1})\; \lambda\!\!\lambda(dt) \end{aligned} \tag{5.16}$$

provided that g satisfies

$$\lim_{t\to\infty} g(s,t) \;=\; 0 \quad P_\lambda^{M_n}\text{-a.s. for all } n \ge 0; \tag{5.17}$$

$$\int_{\mathbb{R}^+}\int_{\mathcal{S}^{m+1}} |g(s,t)|\; F^{m+1}(ds)\; \lambda\!\!\lambda(dt) \;<\; \infty; \tag{5.18}$$

$$E_\lambda\left(\sum_{n=0}^{m+1} G(M_n)\right) \;<\; \infty, \tag{5.19}$$

where $G(s) \stackrel{\text{def}}{=} \sup_{t\ge 0} |g(s,t)|$. We verify these conditions below but will first finish the computation in (5.16). By Lemma 4.2 we have

$$\int_0^c ER_{m,t}\; dt \;=\; \Delta_{m,1}(c + ER_{m,c}) - \frac{1}{2}ER_{m,c}^2 - E\hat{R}_{m,c}.$$

Recall from the Introduction that $\hat{R}_{m,c}$ converges in distribution to a limiting variable $\hat{R}_{m,\infty}$ with mean Λ (see (1.12)). It is shown in [7, Lemma 6.6] that $\mu_3 < \infty$ implies $E\hat{R}_{m,c} = \Lambda + o(1)$ as $c \to \infty$. We are thus led to

$$\int_0^c ER_{m,t}\; dt \;=\; \Delta_{m,1}(c + \Delta_{m,1}) - \frac{1}{2}\Delta_{m,2} - \Lambda + o(1), \quad c \to \infty$$

and then in (5.16) to

$$\begin{aligned} g * U_\lambda(t) \;&=\; \frac{1}{\mu}\lim_{c\to\infty}\int_0^c (ER_{m,t} - \Delta_{m,1})\; dt \\ &=\; \frac{1}{\mu}\lim_{c\to\infty}\left(\int_0^c (ER_{m,t}\; dt \;-\; c\Delta_{m,1}\right) \\ &=\; \frac{1}{\mu}\Big(\Delta_{m,1}^2 - \frac{1}{2}\Delta_{m,2} - \Lambda\Big). \end{aligned} \tag{5.20}$$

A combination of (5.14), (3.3), (5.15) and (5.20) shows the asserted expansion for $Var_\lambda \tau(t)$.

We must finally check conditions (5.17–19). Note first that $\mu_2(\lambda) < \infty$ implies $\mu_2(\delta_{(s,0)} < \infty$ for $(\sum_{n\ge 0} P_\lambda^{M_n})$-almost all $s \in \mathcal{S}^{m+1}$. This in turn implies (5.17) by Corollary 4.2 in [2]. (5.18) is a direct consequence of (5.10) in Proposition 5.1 so that it remains to verify (5.19). But

$$G(s) \;=\; \sup_{t\ge 0} |g(s,t)| \;\le\; 2\Delta_{m,1} + E_{s,0}S_m$$

by (3.5) in Theorem 3.1, whence

$$\begin{aligned} E_\lambda\left(\sum_{n=0}^{m+1} G(M_n)\right) &\le 2(m+2)\Delta_{m,1} + \sum_{n=0}^{m+1} \int_{\mathcal{S}^{m+1}} E_{s,0}S_m \; P_\lambda(M_n \in ds) \\ &= 2(m+2)\Delta_{m,1} + \sum_{n=0}^{m+1} E_\lambda(S_{m+n} - S_n) \\ &\le 2(m+2)\Delta_{m,1} + (m+2)E_\lambda S_{2m+1} \;<\; \infty. \end{aligned}$$

This completes the proof of the theorem. ♢

Appendix

The following Markov renewal theorem for spread out (φ, F, m)-Markov random walks (MRW) $(M_n, S_n)_{n\ge 0}$ with positive drift is stated for completeness. We do not require here $X_0, X_1, ...$ to be nonnegative. The corresponding result for the larger class of nonarithmetic (φ, F, m)-MRW was given as Theorem 3.1 in [2] but requires stronger conditions. The result below may also be viewed as a specification of a Markov renewal theorem established by Arjas, Nummelin and Tweedie [4] for MRW possessing a recurrent atom.

Theorem A.1. *Let $(M_n, S_n)_{n\ge 0}$ be a spread out (φ, F, m)-Markov random walk with drift $\mu = EX_1 \in (0, \infty]$ and Markov renewal measure U_λ under P_λ (see (5.3)). Then $g*U_\lambda(t) = E_\lambda(\sum_{n\ge 0} g(M_n, t - S_n))$ satisfies*

$$\lim_{t\to\infty} g * U_\lambda(t) \;=\; \frac{1}{\mu} \int_{\mathcal{S}^{m+1}} \int_{I\!R} g(x,y) \;\lambda\!\!\lambda(dy)\; F^{m+1}(dy) \tag{A.1}$$

for every distribution λ on $\mathcal{S}^{m+1} \times I\!R$ and every measurable function $g : \mathcal{S}^{m+1} \times I\!R \to I\!R$ which satisfy the following conditions:

$$\lim_{t\to\infty} g(s,t) \;=\; 0 \quad P_\lambda^{M_n}\text{-a.s. for all } n \ge 0; \tag{A.2}$$

$$\int_{I\!R} \int_{\mathcal{S}^{m+1}} |g(s,t)| \; F^{m+1}(ds) \; \lambda\!\!\lambda(dt) \;<\; \infty; \tag{A.3}$$

$$E_\lambda\left(\sum_{n=0}^{m+1} G(M_n)\right) \;<\; \infty, \tag{A.4}$$

where $G(s) \stackrel{\text{def}}{=} \sup_{t\ge 0} |g(s,t)|$.

Proof. As shown in [3], a spread out (φ, F, m)-MRW can be constructed together with a sequence of stopping times $(T_n)_{n\ge 0}$ such that, for a F^{m+1}-positive set $\mathbb{B} \in \mathfrak{S}^{m+1}$

(i) $(S_{T_n})_{n\ge 0}$ constitutes an ordinary delayed random walk with continuous increment distribution;

(ii) S_{T_n} and M_{T_n} are independent for all $n \ge 0$;

(iii) $M_{T_0}, M_{T_1}, \ldots$ is a sequence of i.i.d. random variables with common distribution $F^{m+1}(\cdot|\mathbb{B}) = F^{m+1}(\cdot \cap \mathbb{B})/F^{m+1}(\mathbb{B})$;

(iv) $(T_n - T_{n-1})_{n\geq 0}$ is a sequence of i.i.d. random variables satisfying $T_n - T_{n-1} \geq m+1$, and $T_n - T_{n-1}$ is independent of $(M_{T_{n-1}}, S_{T_{n-1}})$ for all $n \geq 0$ ($T_{-1} \stackrel{\text{def}}{=} 0$).

Put

$$U_\lambda^{T_0}(C) \stackrel{\text{def}}{=} E_\lambda\left(\sum_{n=0}^{T_0-1} \mathbf{1}_C(M_n, S_n)\right) \quad \text{and} \quad \mathbb{U}_\lambda \stackrel{\text{def}}{=} \sum_{n\geq 0} P_\lambda(S_{T_n} \in \cdot).$$

As a consequence of (i)–(iii) (see Lemma 3.3 in [3]),

$$U_\lambda = U_\lambda^{T_0} + U_{F^{m+1}(\cdot|\mathbb{B})} * \mathbb{U}_\lambda$$

and thus

$$g * U_\lambda = g * U_\lambda^{T_0} + (g * U_{F^{m+1}(\cdot|\mathbb{B})}) * \mathbb{U}_\lambda. \tag{A.5}$$

Define $g_1 \stackrel{\text{def}}{=} g * U_\lambda^{T_0}$ and $g_2 \stackrel{\text{def}}{=} g * U_{F^{m+1}(\cdot|\mathbb{B})}^{T_0}$. Since $\mathbb{U}_\lambda$ is the renewal measure of an ordinary continuous (and thus spread out) random walk, we infer (A.1) from the key renewal theorem for such random walks (see [4, Theorem 1]) if we still verify that

$$\lim_{t\to\infty} g_1(t) = 0 \tag{A.6}$$

and that g_2 satisfies:

$$g_2 \text{ is bounded}; \tag{A.7}$$

$$\int_{I\!R} |g_2(t)| \, \lambda\!\!\lambda(dt < \infty; \tag{A.8}$$

$$\lim_{|t|\to\infty} g_2(t) = 0. \tag{A.9}$$

We start with the proof of (A.6), that is

$$\lim_{t\to\infty} \int_{\mathcal{S}^{m+1}\times I\!R} g(s, t-u) \, U_\lambda^{T_0}(ds, du) = 0.$$

Since $|g(s,t)| \leq G(s)$ for all $(s,t) \in \mathcal{S}^{m+1} \times I\!R$ the assertion follows from (A.2) in combination with the dominated convergence theorem if we still verify that

$$\int_{\mathcal{S}^{m+1}\times I\!R} G(s) \, U_\lambda^{T_0}(ds, du) = \int_{\mathcal{S}^{m+1}} G(s) \, U_\lambda^{T_0}(ds \times I\!R) < \infty. \tag{A.10}$$

As shown in [3], $ET_0 < \infty$ and

$$F^{m+1} = \frac{1}{ET_0} U_{F^{m+1}(\cdot|\mathbb{B})}^{T_0}(\cdot \times I\!R). \tag{A.11}$$

A combination with (iv) provides

$$U_{F^{m+1}(\cdot|\mathbb{B})}^{T_0}(\cdot \times I\!R) = ET_0 \, F^{m+1} + U_\lambda^{m+1}(\cdot \times I\!R) - U_{F^{m+1}(\cdot|\mathbb{B})}^{m+1}(\cdot \times I\!R),$$

where $U_\lambda^{m+1} \stackrel{\text{def}}{=} \sum_{n=0}^m P_\lambda((M_n, S_n) \in \cdot)$. Moreover,

$$U^{m+1}_{F^{m+1}(\cdot|\mathbb{B})}(\cdot \times I\!\!R) \;\le\; \frac{1}{F^{m+1}(\mathbb{B})} U^{m+1}_{F^{m+1}}(\cdot \times I\!\!R) \;=\; \frac{m+1}{F^{m+1}(\mathbb{B})} F^{m+1}$$

forms a bounded measure. (A.10) is now concluded from

$$\begin{aligned}
\int_{\mathcal{S}^{m+1}} G(s)\; U_\lambda^{T_0}(ds \times I\!\!R) = ET_0 \int_{\mathcal{S}^{m+1}} G(s)\; F^{m+1}(ds) \;&-\; \int_{\mathcal{S}^{m+1}} G(s)\; U_\lambda^{m+1}(ds \times I\!\!R) \\
+ \int_{\mathcal{S}^{m+1}} G(s)\; &U^{m+1}_{F^{m+1}(\cdot|\mathbb{B})}(ds \times I\!\!R) \\
\le ET_0\, EG(M_1) \;+\; &\sum_{n=0}^m E_\lambda G(M_n) \;-\; \frac{m+1}{F^{m+1}(\mathbb{B})} EG(M_1),
\end{aligned}$$

the last line being finite by (A.4).

By another use of (A.11), we obtain

$$\begin{aligned}
\sup_{t \in I\!\!R} |g_2(t)| \;&=\; \sup_{t \in I\!\!R} \int_{\mathcal{S}^{m+1} \times I\!\!R} g(s, t-u)\; U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(ds, du) \\
&\le\; \int_{\mathcal{S}^{m+1}} G(s)\; U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(ds \times I\!\!R) \qquad\qquad (A.12) \\
&=\; ET_0\, EG(M_1),
\end{aligned}$$

and furthermore

$$\begin{aligned}
\int_{I\!\!R} |g_2(t)|\; \lambda\!\!\lambda(dt) \;&\le\; \int_{I\!\!R} \int_{\mathcal{S}^{m+1} \times I\!\!R} |g(s, t-u)|\; U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(ds \times du)\; \lambda\!\!\lambda(dt) \\
&\le\; \int_{I\!\!R} \int_{\mathcal{S}^{m+1} \times I\!\!R} |g(s, t)|\; U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(ds \times du)\; \lambda\!\!\lambda(dt) \qquad (A.13) \\
&=\; \frac{1}{ET_0} \int_{\mathcal{S}^{m+1}} |g(s,t)|\; F^{m+1}(ds)\; \lambda\!\!\lambda(dt).
\end{aligned}$$

Since the last integral in (A.13) is finite by (A.3), we have shown (A.7), while (A.7) follows from (A.12) and the fact that $EG(M_1) = E_\lambda G(M_{m+1}) < \infty$ by (A.4). Finally, (A.9) may be proved by the same argument as (A.6). ◊

References

[1] ALSMEYER, G. (1988). Second-order approximations for certain stopped sums in extended renewal theory. *Adv. Appl. Probab.*, **20**, 391-410.

[2] ALSMEYER, G. and HOEFS, V. (2001). Markov renewal theory for stationary m-block factors. *Markov Proc. Rel. Fields* **7**, 325-348.

[3] ALSMEYER, G. and HOEFS, V. (2002). Markov Renewal theory for stationary $(m+1)$-block factors: Convergence rate results. *Stoch. Proc. Appl.* **98**, 77-112.

[4] ARJAS, E., NUMMELIN, E. and TWEEDIE, R. (1978). Uniform limit theorems for non-singular renewal and Markov renewal processes. *J. Appl. Probab.*, **15**, 112-125.

[5] CHANG, J.T. (1994). Inequalities for the overshoot. *Ann. Appl. Probab.* **4**, 1223-1233.

[6] CARLSSON, H. and NERMAN, O. (1986). An alternative proof of Lorden's renewal inequality. *Adv. Appl. Probab.* **16**, 1015-1016.

[7] HOEFS, V. (1999). *Markov-Erneuerungstheorie für stationäre* $(m+1)$*-Block-Faktoren.* Ph.D. Thesis (German), University of Münster.

[8] JANSON, S. (1983). Renewal theory for m-dependent variables *Ann. Probab.* **11** 558-568.

[9] LAI, T.L. and SIEGMUND, D. (1979). A nonlinear renewal theory with applications to sequential analysis II. *Ann. Statist.*, **7**, 60-76.

[10] LORDEN, G. (1970). On excess over the boundary. *Ann. Math. Statist.*, **41**, 520-527.

[11] SMITH, W.L. (1959). On the cumulants of renewal processes. *Biometrika*, **46**, 1-29.

[12] WOODROOFE, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis.* CBMS-NSF Regional Conf. Series in Appl. Math., **39**, SIAM, Philadelphia.