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Renewal Theory for Stationary (m+1)-Block Factors: A Markov Renewal Approach*

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This article considers random walks $(S_n)_{n\geq 0}$ whose increments X_n are $(m+1)$-block factors of the form $\varphi(Y_{n-m},...,Y_n)$ for i.i.d. random variables $Y_{-m},Y_{-m+1},...$ taking values in an arbitrary measurable space $(\mathcal{S},\mathfrak{S})$. Providing $EX_1>0$ and by further introducing the Markov chain $M_n=(Y_{n-m},...,Y_n)$, $n\geq 0$, they can be perfectly analyzed within the framework of Markov renewal theory. Our approcach leads to new results for the associated sequence of ladder variables as well as for the first passage times $\tau(t)=\inf\{n\geq 1:S_n>t\}, t\geq 0$, and related quantities such as the excess over the boundary $R_t=S_{\tau(t)}-t$. In particular, we determine the asymptotic distribution of R_t, as $t\to\infty$, provide an asymptotic expansion of $E\tau(t)$ up to vanishing terms and are able to confirm a conjecture by Janson who earlier obtained related results.

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1. Introduction

Given any measurable space $(\mathcal{S},\mathfrak{S})$ with countably generated σ-field $\mathfrak{S}$ and $m \geq 1$, let $(Y_n)_{n\geq -m}$ a sequence of i.i.d. $\mathcal{S}$-valued random variables, defined on a common probability space $(\Omega,\mathfrak{A},P)$ and with common distribution F. Let further $\varphi_0,\varphi : \mathcal{S}^{m+1} \to I\!R$ be any measurable functions,

$$X_0 \;=\; \varphi_0(Y_{-m},...,Y_0) \quad \text{and} \quad X_n \;=\; \varphi(Y_{n-m},...,Y_n) \tag{1.1}$$

for $n \geq 1$. Then $(X_n)_{n\geq 0}$ is called a *sequence of* $(m+1)$*-block factors of* $(Y_n)_{n\geq -m}$. It is obviously m-dependent ($X_0,...,X_k$ and $(X_{k+j})_{j\geq m+1}$ are independent for every $k \geq 0$) and further stationary when leaving out X_0. It has recently been shown that there exist stationary m-dependent sequences that are not $(m+l)$-block factors for *any* $l \geq 1$, see [1], [7].

Providing the previous assumptions and further $EX_1 > 0$, we want to consider the random walk

$$S_n \;=\; X_0 + ... + X_n, \quad n \geq 0.$$

Due to the weak dependence structure of the increment sequence it is intuitively obvious that many renewal theoretic properties of $(S_n)_{n\geq 0}$ that are valid in the i.i.d. case ($m = 0$) should carry over in one way or another to the present m-dependent situation. Our purpose is to show that this can indeed be accomplished in an elegant way by employing Markov renewal theory, especially a general version of the Markov renewal theorem that has been recently established by the first author [2], [3]. In view of that general purpose the following questions will be addressed here:

1. *(Blackwell's and key renewal theorem)* If $V = \sum_{n\geq 0} P(S_n \in \cdot)$ denotes the renewal measure of $(S_n)_{n\geq 0}$, how does $V([t,t+h])$, and more generally $g * V(t)$ for suitable functions g behave as $t \to \infty$?
2. *(Ladder variables)* What kind of dependence structure do the given assumptions on $(X_n)_{n\geq 0}$ imply for the sequence of ladder epochs and ladder heights associated with $(S_n)_{n\geq 0}$?
3. *(First passage times)* If $\tau(t) = \inf\{n \geq 0 : S_n > t\}$ is the first passage time beyond $t \geq 0$, how do quantities like $E\tau(t)$ and the distribution of the excess (forward recurrence time) $S_{\tau(t)} - t$ behave, as $t \to \infty$?

The answers are provided in Sections 2–4 together with further ones and an account of their relation to earlier results in the literature, especially by Janson,[8] and [9], where totally different methods are employed.

Let us now describe the appropriate framework for our investigations which is nothing but a very particular and simple Markov renewal model. Defining the Markov chain

$$M_n \;=\; (Y_{n-m},...,Y_n), \quad n \geq 0, \tag{1.2}$$

we have $X_0 = \varphi_0(M_0)$, $X_n = \varphi(M_n)$ for $n \geq 1$ and see that the X_n's are conditionally deterministic and thus independent given $(M_n)_{n\geq 0}$. Consequently, $(M_n,X_n)_{n\geq 0}$ forms a bivariate Markov chain with transitions from $(x,y) \in \mathcal{S}^{m+1} \times I\!R$ into a set $A \times B \in \mathfrak{S}^{m+1} \otimes \mathfrak{B}$ depending on x only. Clearly, $\mathfrak{B}$ denotes here the Borel σ-field on $I\!R$. Given this special transition property the sequence $(M_n,S_n)_{n\ge 0}$ has been called a *Markov random walk (MRW)* in [2] and satisfies an extension of Blackwell's and the Key renewal theorem, the so-called Markov renewal theorem under a number of further conditions. Due to the particularly nice form of the M_n's, being random vectors of $m+1$ i.i.d. components and again a stationary m-dependent sequence, these conditions are easily verified except perhaps for the appearing lattice-condition which is more complicated than in classical renewal theory (see Lemma 3.1). Finally, we mention that the Markov renewal structure persists if, instead of $(M_n)_{n\ge 0}$, one takes

$$\hat{M}_n \stackrel{\text{def}}{=} (Y_{n-m+1},...,Y_n), \quad n\ge 0$$

as the driving chain for $(X_n)_{n\ge 1}$. We have chosen the former one only because then X_n can be written as a function of M_n alone instead of M_{n-1},M_n.

In the following we always assume $\varphi_0\equiv 0$, i.e. $S_0=X_0=0$ (the so-called zero-delayed case). This is only a minor loss of generality in the subsequent analysis because one can always prove results first for the zero-delayed MRW $(M_n,S_n-S_0)_{n\ge 0}$ and then check under what extra, typically moment conditions on the delay S_0 these results carry over to $(M_n,S_n)_{n\ge 0}$. This is a notorious procedure in classical renewal theory. On the other hand, this assumption simplifies the statement of a number of results.

Let us further introduce, for any $x\in\mathcal{S}^{m+1}$, the conditional probability measure $P_x\stackrel{\text{def}}{=}P(\cdot|M_0=x)$ on $(\Omega,\mathfrak{A})$ with expectation operator E_x. $(S_n)_{n\ge 0}$ then forms a zero-delayed random walk under P_x. Our results will be stated for $P_\lambda\stackrel{\text{def}}{=}\int P_x\,\lambda(dx)$ with λ being an arbitrary initial distribution on $\mathcal{S}^{m+1}$. If $\lambda=F^{m+1}$ we simply write P,E instead of $P_{F^{m+1}},E_{F^{m+1}}$, where F^{m+1} clearly denotes the $(m+1)$-fold product of F. Notice that F^{m+1} is the stationary distribution of $(M_n)_{n\ge 0}$ and that, by m-dependence $(M_n)_{n\ge 0}$ obviously forms an ergodic (aperiodic and positive recurrent) Harris chain, see at the beginning of Section 6 for a definition of this notion and a number of important related facts. The Harris recurrence constitutes an essential condition for the application of Markov renewal theory.

Given a distribution λ on $(\mathcal{S}^{m+1},\mathfrak{S}^{m+1})$, let

$$U_\lambda(A\times B) \;=\; \sum_{n\ge 0}P_\lambda(M_n\in A,S_n\in B), \quad A\in\mathfrak{S}^{m+1},B\in\mathfrak{B}, \tag{1.3}$$

be the associated *Markov renewal measure* of $(M_n,S_n)_{n\ge 0}$. Again, we write U instead of $U_{F^{m+1}}$ and U_x instead of U_{δ_x}. For a measurable function $g:\mathcal{S}^{m+1}\times I\!R\to I\!R$, we then define its convolution with U_λ by

$$g*U_\lambda(t) \;=\; E_\lambda\left(\sum_{n\ge 0}g(M_n,t-S_n)\right), \quad t\in I\!R, \tag{1.4}$$

providing the right-hand expression exists.

All previously introduced notation will be valid throughout the entire paper unless stated otherwise. The main results are stated in Sections 3–5, while proofs can be found in Sections 5–7.

2. Ladder variables

In view of an analysis of first passage times and related quantities like the excess over the boundary (question (3)) it is necessary to study the sequence of (strictly ascending) ladder epochs and ladder heights associated with $(S_n)_{n\geq 0}$. The former are recursively given by $\sigma_0 = 0$ and

$$\sigma_n = \inf\{j > \sigma_{n-1} : S_j > S_{\sigma_{n-1}}\}$$

for $n \geq 1$, the latter by $S_n^> = S_{\sigma_n}$ for $n \geq 0$. Let $\tau_n = \sigma_n - \sigma_{n-1}$ and $X_n^> = S_n^> - S_{n-1}^>$, $n \geq 1$, denote their respective increments and define $\tau_0 = X_0^> = 0$. Finally put

$$M_n^> \;=\; M_{\sigma_n} \;=\; (Y_{\sigma_n - m}, ..., Y_{\sigma_n}) \tag{2.1}$$

for $n \geq 0$ and note that this sequence is again a Markov chain.

Janson [8] pointed out by giving a counterexample (see Example 3.1 there) that one cannot expect $(\tau_n)_{n\geq 0}$ or $(X_n^>)_{n\geq 1}$ to be stationary nor k-dependent for some $k \geq m$. However, our next theorem shows that they can still be analyzed within the framework of Markov renewal theory. The notion *Markov renewal process (MRP)* is used there to denote a MRW whose second component has positive increments. The definition of its lattice-type and the associated shift function is given in the next section.

Theorem 2.1. *Given the previous notation and $\mu > 0$, the following assertions hold:*

(a) $(M_n^>)_{n\geq 0}$, $(M_n^>, \tau_n)_{n\geq 0}$, $(M_n^>, X_n^>)_{n\geq 0}$ *and* $(M_n^>, X_n^>, \tau_n)_{n\geq 0}$ *are each positive recurrent Harris chains.*

(b) $(M_n^>, \sigma_n)_{n\geq 0}$ *and* $(M_n^>, S_n^>)_{n\geq 0}$ *both constitute MRP, the former being 1-arithmetic with shift function 0, the latter of the same lattice-type as* $(M_n, S_n)_{n\geq 0}$ *(with identical shift function if arithmetic).*

(c) If ξ^* *denotes the stationary distribution of* $(M_n^>)_{n\geq 0}$*, then* $\theta \stackrel{\text{def}}{=} E_{\xi^*}\tau_1 < \infty$.

(d) $\mu^> \stackrel{\text{def}}{=} E_{\xi^*}X_1^> < \infty$ *iff* $EX_1^> < \infty$ *iff* $\mu < \infty$.

(e) $\mu^> = \mu\theta$.

Apart from assertion (e), obviously a Wald-type equation for the first ladder height, this theorem is again a specialization of a more general one in [4]. Its difficult part is to prove the Harris recurrence of $(M_n^>)_{n\geq 0}$. Somewhat surprisingly, even in the present situation there seems to be no simplification of the necessary arguments. As for (e), a simple proof is given right after (4.2).

Remark. It is important to point out that $(M_n^>)_{n\geq 0}$ and thus also the other three chains in part (a) of Theorem 2.1 need not be aperiodic (though $(M_n)_{n\geq 0}$ always is in the present setup). The following counterexample is from [9] where strictly ascending ladder epochs in case of indicator functions φ were investigated: Given i.i.d. Bernoulli variables $Y_{-1}, Y_0, ...$ with $P(Y_{-1} = 1) = 1 - P(Y_{-1} = 0) = p \in (0,1)$, let $X_n = \mathbf{1}_{\{Y_{n-1}\neq Y_n\}}$. $(X_n)_{n\geq 0}$ then forms a stationary 1-dependent sequence. Now observe that a ladder epoch for the associated random walk occurs at n iff $M_n = (Y_{n-1}, Y_n) = (0,1)$, or $= (1,0)$, and that these two states are attained in alternating order at consecutive ladder epochs. Hence $(M_n^>)_{n\geq 0}$ is the 2-periodic discrete

Markov chain with state space $\{(0,1),(1,0)\}$ and transition matrix $\binom{0\ 1}{1\ 0}$. As a consequence, unless $p = \frac{1}{2}$, the distribution of τ_n under $P_{(0,1)}$ as well as $P_{(1,0)}$ does not converge to a stationary limit, but is either geometric with parameter p or with parameter $1-p$, in alternating order for each $n \geq 1$. The ladder height increments $X_n^>$ are, of course, always equal to 1 here.

Janson showed further in [9] that, within the class of m-dependent and *m-separated* ($X_n X_{n+k} = 0$ for all $1 \leq k \leq m-1$) indicator sequences $X_n = \mathbf{1}_A(Y_{n-m}, ..., Y_n)$ based on real-valued i.i.d. $Y_0, Y_1, ...$, the previous example is the only one where the τ_n do not have a limit distribution.

3. Markov renewal theorems

Within the described Markov renewal framework we can now formulate the fundamental result, the Markov renewal theorem for $(m+1)$-block factors and its ladder variables. As is well-known most renewal theorems have an arithmetic and a nonarithmetic version. Unfortunately, in Markov renewal theory the distinction of the two cases requires greater care than in classical renewal theory. The right definition of an arithmetic MRW has been provided by Shurenkov [12] and looks as follows: $(M_n, S_n)_{n\geq 0}$ is called *d-arithmetic* if $d > 0$ denotes the maximal number for which there exists a measurable function $\gamma : \mathcal{S}^{m+1} \to [0, d)$, called *shift function*, such that

$$P(X_1 \in \gamma(M_0) - \gamma(M_1) + d\mathbb{Z}) = 1. \tag{3.1}$$

If no such d exists, $(M_n, S_n)_{n\geq 0}$ is called *nonarithmetic*. The following lemma gives the translation of that lattice condition in terms of φ where "$\equiv_{d\mathbb{Z}}$" means congruence modulo elements from $d\mathbb{Z}$.

Lemma 3.1. *$(M_n, S_n)_{n\geq 0}$ is d-arithmetic iff $d > 0$ denotes the maximal number for which there exists a function $\psi : \mathcal{S}^m \to [0, d)$ such that*

$$\varphi(x_0, ..., x_m) \ \equiv_{d\mathbb{Z}} \ \psi(x_0, ..., x_{m-1}) - \psi(x_1, ..., x_m) \quad F^{m+1}\text{-a.s.} \tag{3.2}$$

In this case the shift function γ is given by $\gamma(x_0, ..., x_m) = \psi(x_1, ..., x_m)$.

The proof is easy and can be omitted. Our Markov renewal theorems are stated for the cases where $(M_n, S_n)_{n\geq 0}$ is nonarithmetic or 1-arithmetic with shift function 0. Assuming $d = 1$ in the arithmetic case is clearly no loss of generality. As for the second restriction note that if $(M_n, S_n)_{n\geq 0}$ has shift function $\gamma \neq 0$ then $(M_n, S_n - \gamma(M_0) + \gamma(M_n))_{n\geq 0}$ has shift function 0 (and the same lattice-span). We denote by $\lambda\!\!\!\lambda_d$ Lebesgue measure on $I\!\!R$ if $d = 0$ and counting measure on $\mathbb{Z}$ if $d = 1$. We further define for convenience

$$d\text{-}\lim_{t\to\infty} f(t) \ = \ \begin{cases} \lim\limits_{t\to\infty} f(t), & \text{if } d = 0 \\ \lim\limits_{n\to\infty} f(nd), & \text{if } d > 0 \end{cases}.$$

Finally, throughout the whole paper we make the following

Standing assumption: Whenever in the 1-arithmetic case, initial distributions λ are such that $P_\lambda(X_n \in \mathbb{Z}) = 1$ for all $n \geq 1$.

Theorem 3.2. *Suppose* $0 < \mu = EX_1 \le \infty$. *If* $(M_n, S_n)_{n\ge 0}$ *is nonarithmetic* ($d = 0$) *or 1-arithmetic with shift function 0* ($d = 1$) *then*

$$\underset{t\to\infty}{d\text{-lim}}\ g * U_\lambda(t) \ = \ \frac{1}{\mu}\int_{\mathcal{S}^{m+1}}\int_{I\!R} g(x,y)\ \lambda\!\!\lambda_d(dy)\ F^{m+1}(dx), \tag{3.3}$$

$$\lim_{t\to-\infty} g * U_\lambda(t) \ = \ 0 \tag{3.4}$$

for each initial distribution λ *on* $\mathcal{S}^{m+1}$ *and each measurable function* $g : \mathcal{S}^{m+1} \times I\!R \to I\!R$ *satisfying*

$$E_\lambda\left(\sum_{n=0}^{m} G(M_n)\right) \ < \ \infty, \quad G(x) \ \stackrel{\text{def}}{=} \ \sup_{t\in I\!R} |g(x,t)|, \tag{3.5}$$

$$\lim_{|t|\to\infty} |g(x,t)| \ = \ 0 \quad P_\lambda^{M_n}\text{-a.s. for all } n \ge 0 \tag{3.6}$$

and further

$$g(x,\cdot) \text{ is } \lambda\!\!\lambda_0\text{-almost everywhere continuous for each } x \in \mathcal{S}^{m+1}; \tag{3.7}$$

$$\int_{\mathcal{S}^{m+1}} \sum_{n\in\mathbb{Z}} \sup_{n\rho\le y<(n+1)\rho} |g(x,y)|\ F^{m+1}(dx) \ < \ \infty \quad \text{for some } \rho > 0 \tag{3.8}$$

in the nonarithmetic case, and

$$\int_{\mathcal{S}^{m+1}} \sum_{n\in\mathbb{Z}} |g(x,n)|\ F^{m+1}(dx) \ < \ \infty \tag{3.9}$$

in the arithmetic case.

Theorem 3.2 is essentially a specification of the Markov renewal theorem in [2] and [3] to the present situation. It goes beyond the latter one only by allowing for more general distributions λ where condition (3.5) is a consequence of the particular dependence structure of $(M_n, X_n)_{n\ge 0}$. The necessary arguments for this generalization are given in Section 6. Since in the 1-arithmetic case $g * U_\lambda(n), n \in \mathbb{Z}$, only depends on the values of $g(x,n)$ we always assume in this case that $g(x,t) = g(x,n)$ for $t \in [n, n+1)$, $n \in \mathbb{Z}$. Note that, given a function g satisfying (3.7) and (3.8), resp. (3.9), condition (3.5) is always fulfilled with $\lambda = F^{m+1}$. Moreover, validity of (3.6) for any λ implies validity of that condition for $\lambda = F^{m+1}$ because $P_\lambda^{M_n} = F^{m+1}$ for all $n > m$.

The following Blackwell-type result yields as a direct consequence of Theorem 3.2 when specializing there to functions g of the form $g = 1_{A\times I}$ for $A \in \mathfrak{S}^{m+1}$ and finite intervals I.

Corollary 3.3. *In the situation of Theorem 3.2*

$$\underset{t\to\infty}{d\text{-lim}}\, U_\lambda(A \times t + I) \ = \ \mu^{-1} F^{m+1}(A)\lambda\!\!\lambda_d(I) \quad \text{and} \quad \lim_{t\to-\infty} U_\lambda(A \times t + I) \ = \ 0 \tag{3.10}$$

for all $A \in \mathfrak{S}^{m+1}$*, finite intervals* I *and initial distributions* λ.

Turning to the ladder variables associated with $(M_n, S_n)_{n\ge 0}$, more precisely to the MRPs $(M_n^>, \sigma_n)_{n\ge 0}$ and $(M_n^>, S_n^>)_{n\ge 0}$, we can also easily formulate a Markov renewal theorem for these sequences because their driving chain $(M_n^>)_{n\ge 0}$ is positive Harris recurrent by Theorem 2.1. Denote by

$$V_\lambda^> \;=\; \sum_{n\ge 0} P_\lambda^{(M_n^>,\sigma_n)} \quad \text{and} \quad U_\lambda^> \;=\; \sum_{n\ge 0} P_\lambda^{(M_n^>,S_n^>)}$$

their respective Markov renewal measures. Recall from Theorem 2.1 that $(M_n^>, S_n^>)_{n\ge 0}$ is of the same lattice-type as $(M_n, S_n)_{n\ge 0}$ while $(M_n^>, \sigma_n)_{n\ge 0}$ is 1-arithmetic with shift function 0.

THEOREM 3.4. *Given the situation of Theorem 3.2 and the notation of Theorem 2.1,*

$$\lim_{n\to\infty} g * V_\lambda^>(n) \;=\; \frac{1}{\theta}\int_{\mathcal{S}^{m+1}} \sum_{n\in\mathbb{Z}} g(x,n)\ \xi^*(dx) \tag{3.11}$$

for each initial distribution λ on $\mathcal{S}^{m+1}$ and each measurable function $g : \mathcal{S}^{m+1}\times\mathbb{R}\to\mathbb{R}$ (w.l.o.g. constant on each $[n, n+1)$, $n\in\mathbb{N}_0$) satisfying (3.5), (3.6) and (3.9).

THEOREM 3.5. *Given the situation of Theorem 3.2 and the notation of Theorem 2.1,*

$$d\text{-}\lim_{t\to\infty} g * U_\lambda^>(t) \;=\; \frac{1}{\mu^>}\int_{\mathcal{S}^{m+1}}\int_{\mathbb{R}} g(x,y)\ \lambda\!\!\lambda_d(dy)\ \xi^*(dx) \tag{3.12}$$

for each initial distribution λ on $\mathcal{S}^{m+1}$ and each measurable function $g : \mathcal{S}^{m+1}\times[0,\infty)\to\mathbb{R}$ (w.l.o.g. constant on each $[n, n+1)$, $n\in\mathbb{N}_0$, if $d=1$) satisfying (3.5), (3.6), and further (3.7), (3.8) in the nonarithmetic case and (3.9) in the arithmetic case.

It is easily seen that Blackwell-type versions of Theorem 3.4 and 3.5 hold again for all initial distributions λ. Their statement is omitted. The results are again specifications of the Markov renewal theorem in [2] or [3] apart from allowing a wider class of initial distributions λ. However, since $(\sigma_n)_{n\ge 0}$ and $(S_n^>)_{n\ge 0}$ in general do not possess m-dependent increments this extension is more difficult than in Theorem 3.2. Its proof, to be found in Section 6, will be based on certain uniform integrability results provided in Section 5. Note finally that validity of (3.6) for any λ implies validity of that condition for $\lambda = \xi^*$ as well because $\xi^* \le \theta F^{m+1}$ (use $F^{m+1}(A) = \theta^{-1}E_{\xi^*}(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_A(M_n))$ and recall the remark after Theorem 3.2).

4. FIRST PASSAGE TIMES

We keep all notation from the previous sections and put further $\mathcal{F}_n = \sigma(Y_{-m},...,Y_n)$ for $n\ge 0$. Before turning to question (3) posed in the Introduction we first note a useful substitute of Wald's identity in the present setup. Suppose μ is finite. By applying the optional sampling theorem to the P-martingale $W_n = E(S_{n+m}-(n+m)\mu|\mathcal{F}_n), n\ge 0$, Janson [8] obtained

$$ES_{\tau+m} \;=\; \mu E(\tau+m) \tag{4.1}$$

for every stopping time τ with finite mean. But $(W_n)_{n\ge 0}$ is also a martingale under every P_λ with $E_\lambda|S_m|<\infty$, whence by the same argument

$$E_\lambda W_{\tau+m} \;=\; E_\lambda W_0 \;=\; E_\lambda(S_m - m\mu),$$

that is

$$E_\lambda S_{\tau+m} \;=\; E_\lambda(S_m - m\mu) \;+\; \mu E_\lambda(\tau+m) \;=\; E_\lambda S_m \;+\; \mu E_\lambda \tau. \tag{4.2}$$

An alternative proof may be based on occupation measures, see [11]. Here is a quick argument for assertion (e) in Theorem 2.1: By stationarity we have $S_{\sigma_1+m} - S_{\sigma_1} \sim_{P_{\xi^*}} S_m$ which in combination with (4.2) gives

$$\mu^> \;=\; E_{\xi^*} S_{\sigma_1} \;=\; E_{\xi^*} S_{\sigma_1+m} - E_{\xi^*} S_m \;=\; \mu E_{\xi^*}\tau \;=\; \mu\theta.$$

Recall that

$$\tau(t) \;=\; \inf\{n \geq 1 : S_n > t\}$$

for $t \geq 0$. Assuming $\mu > 0$, Janson [8] showed that $E\tau(t) < \infty$ for all $t \geq 0$ and a number of moment and limit results for $\tau(t), S_{\tau(t)}$ and, most importantly, the *excess* $R_t = S_{\tau(t)} - t$, as $t \to \infty$ (see his Sections 2 and 3). Providing $\mu \in (0,\infty)$, $E_\lambda|X_n| < \infty$ for all $n \geq 1$ and $E_\lambda \tau(t) < \infty$ (which then already follows as we will see further below) and applying (4.2) to $\tau(t)$ yields

$$E_\lambda \tau(t) \;=\; \mu^{-1}\Big(t + E_\lambda R_t + E_\lambda(S_{\tau(t)+m} - S_{\tau(t)}) - E_\lambda S_m\Big) \tag{4.3}$$

By using this identity with $\lambda = F^{m+1}$ Janson could prove

$$E\tau(t) \;=\; \mu^{-1} t \;+\; O(1) \quad (t \to \infty) \tag{4.4}$$

in case of nonnegative $X_1, X_2, \ldots$ with finite second moment. He also gave weaker remainder term estimates under weaker moment conditions as well as for the case where $P(X_1 < 0) > 0$. A combination of his results with ours from the previous sections leads to number of extensions stated as Theorem 4.1 and Corollary 4.2 below.

Let us further define $\tau^>(t) = \inf\{n \geq 0 : S_n^> > t\}$ and notice that $R_t = S^>_{\tau^>(t)} - t$ for all $t \geq 0$ as in the i.i.d. case. Moreover, we have $M_{\tau(t)} = M^>_{\tau^>(t)}$ for all $t \geq 0$. In view of (4.3) above we will have to consider also the variables $S_{\tau(t)+m} - S_{\tau(t)}$ in order to obtain an expansion for $E_\lambda \tau(t)$ up to vanishing terms as $t \to \infty$. Put $S_{n,k} = S_{n+k} - S_n$.

THEOREM 4.1. *Given the previous notation let $(M_n, S_n)_{n\geq 0}$ be nonarithmetic ($d = 0$) or 1-arithmetic with shift function 0 ($d = 1$). Providing $0 < \mu < \infty$, the following assertions hold for every initial distribution λ on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$:*

(a) For all $C \in \mathfrak{S}^{m+1}$, $r \geq 0$ and $s \in I\!R$

$$d\text{-}\lim_{t\to\infty} P_\lambda(M_{\tau(t)} \in C, R_t > r) \;=\; \frac{1}{\mu^>}\int_{[r,\infty)} P_{\xi^*}(M_{\sigma_1} \in C, S_{\sigma_1} > u)\; \lambda\!\!\lambda_d(du); \tag{4.5}$$

$$d\text{-}\lim_{t\to\infty} P_\lambda(M_{\tau(t)} \in C, S_{\tau(t),m} > s) \;=\; \frac{1}{\mu^>}\int_{[0,\infty)} P_{\xi^*}(M_{\sigma_1} \in C, S_{\sigma_1} > u, S_{\sigma_1,m} > s)\; \lambda\!\!\lambda_d(du). \tag{4.6}$$

(b) For each $p > 0$, $E_\lambda X_n^- < \infty$ and $E_\lambda (X_n^+)^{p+1} < \infty$ for all $n \geq 1$ imply $E_\lambda S_{\sigma_n}^{p+1} < \infty$ for all $n \geq 1$ as well as uniform integrability of $(R_t^p)_{t\geq 0}$ and $(S^p_{\tau(t),m})_{t\geq 0}$ under P_λ.

(c) Providing $E_\lambda X_n^- < \infty$ and $E_\lambda (X_n^+)^2 < \infty$ for all $n \geq 1$,

$$\underset{t\to\infty}{d\text{-lim}}\, E_\lambda R_t \;=\; \frac{E_{\xi^*}S_{\sigma_1}^2}{2\mu^>}+\frac{d}{2}; \tag{4.7}$$

$$\underset{t\to\infty}{d\text{-lim}}\, E_\lambda S_{\tau(t),m} \;=\; \frac{E_{\xi^*}S_{\sigma_1}S_{\sigma_1,m}}{\mu^>}; \tag{4.8}$$

$$\underset{t\to\infty}{d\text{-lim}}\left(E_\lambda\tau(t)-\frac{t}{\mu}\right) \;=\; \frac{E_{\xi^*}S_{\sigma_1}^2}{2\mu\mu^>}+\frac{E_{\xi^*}S_{\sigma_1}S_{\sigma_1,m}}{\mu\mu^>}-\frac{E_\lambda S_m}{\mu}+\frac{d}{2\mu}. \tag{4.9}$$

Note that $E_\lambda(X_n^{(\pm)})^p = E(X_1^{(\pm)})^p$ for all $n>m$ follows from m-dependence and stationarity of the X_n.

If the X_n themselves are already nonnegative ($\varphi \ge 0$) with positive mean, the formulae above, notably (4.8), simplify because we do not need ladder variables and can replace ξ^* by the product distribution F^{m+1}, P_{ξ^*} by P and σ_1 by 1. In that case we can actually confirm a conjecture by Janson, (4.14) below, who proved the limit with $E\tau(t)-t/\mu$ replaced by the integrated average $\frac{1}{T}\int_0^T (E\tau(t)-t/\mu)\,dt$, see [8, Remark 3.1], and also Remark (a) following Corollary 4.2 below. Put $\kappa^2 \stackrel{\text{def}}{=} \mathrm{Cov}(X_n, S_{n+m})$ for $n>m$ and suppose $EX_1^2<\infty$. It follows by stationarity

$$\kappa^2 \;=\; \mathrm{Var}X_1 + 2\sum_{i=2}^{m+1}\mathrm{Cov}(X_1,X_i) \;=\; EX_1^2+2EX_1S_{1,m}-(2m+1)\mu^2,$$

and further $\kappa^2 = \lim_{n\to\infty} n^{-1}\mathrm{Var}S_n$.

Corollary 4.2. *If φ is nonnegative, then the assertions of Theorem 4.1 simplify to*

$$\underset{t\to\infty}{d\text{-lim}}\, P_\lambda(M_{\tau(t)}\in C, R_t>r) \;=\; \mu^{-1}\int_{[r,\infty)} P(M_1\in C, X_1>s)\;\lambda\!\!\lambda_d^+(ds); \tag{4.10}$$

$$\underset{t\to\infty}{d\text{-lim}}\, P_\lambda(M_{\tau(t)}\in C, S_{\tau(t),m}>r) \;=\; \mu^{-1}\int_0^\infty P(M_1\in C, X_1>s, S_{1,m}>r)\;ds, \tag{4.11}$$

$$\underset{t\to\infty}{d\text{-lim}}\, E_\lambda R_t \;=\; \frac{EX_1^2}{2\mu}+\frac{d}{2}; \tag{4.12}$$

$$\underset{t\to\infty}{d\text{-lim}}\, E_\lambda S_{\tau(t),m} \;=\; \frac{EX_1S_{1,m}}{\mu}; \tag{4.13}$$

$$\underset{t\to\infty}{d\text{-lim}}\left(E_\lambda\tau(t)-\frac{t}{\mu}\right) \;=\; \frac{\kappa^2}{2\mu^2}+\frac{1}{2}+\frac{d}{2\mu}-\frac{E_\lambda(S_m-m\mu)}{\mu}, \tag{4.14}$$

keeping all other assumptions there fixed.

Remarks. (a) Choosing $\lambda = F^{m+1}$ in (4.14) gives

$$\underset{t\to\infty}{d\text{-lim}}\left(E\tau(t)-\frac{t}{\mu}\right) \;=\; \frac{\kappa^2}{2\mu^2}+\frac{1}{2}+\frac{d}{2\mu}. \tag{4.15}$$

It is this formula with $d=0$ which was conjectured in [8].

(b) It is interesting to note that (4.15) formally coincides with the result in the i.i.d. case if one observes that the asymptotic average variance κ^2 then equals the variance of each increment.

(c) It is clear that part (a) and (b) of Theorem 4.1 do also lead to convergence results for higher order moments of R_t and $S_{\tau(t),m}$, as $t \to \infty$. The asymptotic moments can be computed with the help of (4.5) and (4.6), although nice formula do only yield in the nonarithmetic case, see the proof of Theorem 4.1(b) for further details

5. The marked point process view and uniform integrability

The main purpose of this section is to provide a number of uniform integrability results which will particularly furnish the proofs of Theorem 3.4 and 3.5 in Section 6. It is useful to adopt a different viewpoint by considering the marked point process (MPP)

$$N \stackrel{\text{def}}{=} \sum_{n\ge 0} \delta_{(M_n,S_n)}$$

associated with $(M_n,S_n)_{n\ge 0}$. Naturally, the M_n are the marks, the mark space thus being $(\mathcal{S}^{m+1},\mathfrak{S}^{m+1})$. N is a random measure in the space of counting measures on $(\mathcal{S}^{m+1}\times I\!R, \mathfrak{S}^{m+1}\otimes\mathfrak{B})$ endowed with the smallest σ-field such that all projections $N(C)$, $C\in\mathfrak{S}^{m+1}\otimes\mathfrak{B}$ are measurable. The Markov renewal measure U_λ is the intensity measure of N under P_λ, i.e.

$$U_\lambda(A\times B) \;=\; E_\lambda N(A\times B), \quad A\in\mathfrak{S}^{m+1}, B\in\mathfrak{B}.$$

For $t\in I\!R$ let

$$\theta_t N \;=\; \sum_{n\ge 0}\delta_{(M_n,S_n-t)}$$

be the translation (shift) of N by t so that

$$\theta_t N(A\times B) \;=\; N(A\times t+B)$$

for all $A\in\mathfrak{S}^{m+1}, B\in\mathfrak{B}$. Denote by N^+ the restriction of N to $\mathcal{S}^{m+1}\times[0,\infty)$. Similarly, given a measure ν on $(I\!R,\mathfrak{B})$, let ν^+ be its restriction to $[0,\infty)$.

Theorem 5.1 below shows weak convergence of $(\theta_t N)^+$, as $t\to\infty$, to a stationary limit providing $\mu\in(0,\infty)$. Although it could be stated for arbitrary MPPs associated with positive drift MRWs driven by an ergodic Harris chain we restrict ourselves to the setup of $(m+1)$-block factors. Berbee [6, Chapter 6] obtained very similar results within the wider class of weak Bernoulli processes, but instead of referring to his work we provide a proof based on a coupling argument that was already used in [2] and [3]. More important for our purposes than Theorem 5.1 is its extension Theorem 5.2 because it will lead to a number of uniform integrability conclusions when combined with Theorem 3.2 and Corollary 3.3.

We continue to assume that $(M_n,S_n)_{n\ge 0}$ is either nonarithmetic or 1-arithmetic with shift function 0. Moreover, suppose $\mu\in(0,\infty)$. In order to specify the stationary limit of $(\theta_t N)^+$, define the *stationary Markov delay distribution* ν^s on $(\mathcal{S}^{m+1}\times I\!R, \mathfrak{S}^{m+1}\otimes\mathfrak{B})$ through

$$\nu^s(A\times B) \;=\; \frac{1}{\mu^>}\int_B P_{\xi^*}(M_{\sigma_1}\in A, S_{\sigma_1}>s)\;\lambda\!\!\lambda_d^+(ds), \quad A\in\mathfrak{S}^{m+1}, B\in\mathfrak{B} \qquad (5.1)$$

where d is the lattice-span of $(M_n, S_n)_{n\geq 0}$. As will be seen in the following section (Theorem 5.1), ν^s is the limiting distribution of $(M_{\tau(t)}, S_{\tau(t)} - t)$, as $t \to \infty$ (through $\mathbb{Z}$ if $d = 1$). It was further shown in [3] in a more general framework that

$$U_{\nu^s}(A \times B) \;=\; E_{\xi^*}\left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_A(M_n)\lambda\!\!\lambda_d^+(B - S_n)\right), \quad A \in \mathfrak{S}^{m+1}, B \in \mathfrak{B}, \tag{5.2}$$

and in particular

$$U_{\nu^s}^+ \;=\; \mu^{-1} F^{m+1} \otimes \lambda\!\!\lambda_d^+. \tag{5.3}$$

This may easily be used to obtain that $(M_{\tau(t)+n}, S_{\tau(t)+n} - t)$, $t \in [0, \infty)$ resp. $\mathbb{N}_0$, is a stationary process under P_{ν^s} for every $n \geq 0$. Since

$$(\theta_t N)^+ \;=\; \left(\sum_{n\geq 0} \delta_{(M_{\tau(t)+n}, S_{\tau(t)+n} - t)}\right)^+$$

we also conclude stationarity of $(\theta_t N)^+$, $t \in [0, \infty)$ resp. $\mathbb{N}_0$, under P_{ν^s}. In other words, N is a (time-)stationary MPP under P_{ν^s} when restricted to $\mathcal{S}^{m+1} \times [0, \infty)$. We prove now

THEOREM 5.1. *For every initial distribution λ on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$*

$$\underset{t\to\infty}{d\text{-}\lim}\, P_\lambda(\theta_t N(A \times I) = n) \;=\; \begin{cases} P_{\nu^s}(N(A \times I) = n), & \text{if } \mu \in (0, \infty) \\ 0, & \text{if } \mu = \infty \end{cases} \tag{5.4}$$

for all $n \in \mathbb{N}_0$, $A \in \mathfrak{S}^{m+1}$ and bounded intervals $I \subset [0, \infty)$. Moreover, in either case

$$\lim_{t\to -\infty} P_\lambda(\theta_t N(A \times t + I) = n) \;=\; 0. \tag{5.5}$$

So we have weak convergence of $P_\lambda(\theta_t N(A \times I) \in \cdot)$ to $P_{\nu_s}(N(A \times I) \in \cdot)$, as $t \to \infty$, if $\mu \in (0, \infty)$, and convergence in probability to 0, if $\mu = \infty$. Moreover, the latter convergence holds in any case if $t \to -\infty$.

In order to motivate the following extension of Theorem 5.1 note first that

$$g * U_\lambda(t) \;=\; E_\lambda g * \theta_t N(0) \tag{5.6}$$

for every function g such that $g * U_\lambda(t)$ exists for all $t \in \mathbb{R}$. Theorem 5.2 provides a weak convergence result for $g * \theta_t N(0)$ as t tends to $\pm\infty$.

THEOREM 5.2. *For every initial distribution λ on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$ and every measurable function $g : \mathcal{S}^{m+1} \times \mathbb{R} \to \mathbb{R}$ satisfying the assumptions of Theorem 3.2*

$$\underset{t\to\infty}{d\text{-}\lim}\, P_\lambda(g * \theta_t N(0) \in \cdot) \;=\; \begin{cases} P_{\nu^s}(g * N(0) \in \cdot), & \text{if } \mu \in (0, \infty) \\ \delta_0, & \text{if } \mu = \infty \end{cases} \tag{5.7}$$

in the sense of weak convergence. Moreover, in either case

$$\lim_{t\to -\infty} P_\lambda(g * \theta_t N(0) \in \cdot) \;=\; \delta_0. \tag{5.8}$$

Theorem 5.1 is obviously a special case of Theorem 5.2. Hence it suffices to give a proof of the latter one which is postponed to the end of the section. Since weak convergence plus convergence of absolute moments implies uniform integrability, a combination of Theorem 5.2 and Theorem 3.2 now easily leads to a number of such conclusions.

Given a measurable $g : \mathcal{S}^{m+1} \times I\!R \to I\!R$, define

$$g_\varepsilon \;=\; \inf_{t\in[(n-1)\varepsilon,(n+2)\varepsilon]} g(x,t) \quad \text{and} \quad g^\varepsilon \;=\; \sup_{t\in[(n-1)\varepsilon,(n+2)\varepsilon]} g(x,t) \tag{5.9}$$

for $\varepsilon > 0$. Notice that, if the conditions of Theorem 3.2 hold for a pair (λ, g), then they hold also for $(\lambda, g^\pm)$, $(\lambda, |g|)$ and (λ, g_ε), (λ, g^ε) for sufficiently small $\varepsilon > 0$.

COROLLARY 5.3. *For every initial distribution λ on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$ and every measurable function $g : \mathcal{S}^{m+1} \times I\!R \to I\!R$ satisfying the assumptions of Theorem 3.2, the family $\{|g|*\theta_t N(0), t \in I\!R\}$ is uniformly integrable (u.i.) under P_λ. In particular, $\{N(A\times t+I), t \in I\!R\}$ is u.i. under each P_λ for all $A \in \mathfrak{S}^{m+1}$ and bounded intervals I.*

PROOF. In view of the previous remark it is no loss of generality to assume $g \geq 0$. Theorem 5.2 in combination with

$$d\text{-}\lim_{t\to\infty} E_\lambda g * \theta_t N(0) \;=\; E_{\nu_s} g * N(0), \text{ resp. } 0 \quad \text{and} \quad \lim_{t\to-\infty} E_\lambda g * \theta_t N(0) \;=\; 0$$

by Theorem 3.2 implies the uniform integrability of $\{g * \theta_{n\varepsilon} N(0), n \in \mathbb{Z}\}$, and the same conclusion can be drawn for $\{g^\varepsilon * \theta_{n\varepsilon} N(0), n \in \mathbb{Z}\}$ for all $\varepsilon > 0$. But $g * \theta_t N(0) \leq g^\varepsilon * \theta_n N(0)$ for all $t \in [n\varepsilon, (n+1)\varepsilon)$ then gives the asserted result.

Similar results are now more easily obtained for the MPPs

$$N^> \;\stackrel{\text{def}}{=}\; \sum_{n\geq 0} \delta_{(M_n^>, S_n^>)} \quad \text{and} \quad N^{(\sigma)} \;\stackrel{\text{def}}{=}\; \sum_{n\geq 0} \delta_{(M_n^>, \sigma_n)}$$

associated with $(M_n^>, S_n^>)_{n\geq 0}$ and $(M_n^>, \sigma_n)_{n\geq 0}$, respectively. Recall from Theorem 2.1(b) that the former MRP is of the same lattice-type as $(M_n, S_n)_{n\geq 0}$ (with identical shift function if arithmetic).

COROLLARY 5.4. *For every initial distribution λ on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$ and every measurable function $g : \mathcal{S}^{m+1} \times I\!R \to I\!R$ satisfying the assumptions of Theorem 3.2 the familiy $\{|g| * \theta_t N^>(0), t \in I\!R\}$ is u.i. In particular, $\{N^>(A \times t + I), t \in I\!R\}$ is u.i. under each P_λ for all $A \in \mathfrak{S}^{m+1}$ and bounded intervals I.*

PROOF. Again let $g \geq 0$. The assertions are trivial consequences of Corollary 5.3 because $(M_n^>, S_n^>)_{n\geq 0}$ is a subsequence of $(M_n, S_n)_{n\geq 0}$, thus implying $N^> \leq N$ and particularly $g * \theta_t N^>(0) \leq g * \theta_t N(0)$ for all $t \in I\!R$.

In order to obtain the same result with $N^>$ replaced by $N^{(\sigma)}$ notice that $(M_n^>, \sigma_n)_{n\geq 0}$ is a subsequence of $(M_n, n)_{n\geq 0}$, that both are 1-arithmetic MRPs with shift function 0 and that Theorems 3.2, 5.1 and 5.2 do also hold for $(M_n, n)_{n\geq 0}$ (choose $\varphi \equiv 1$ in the given setup).

COROLLARY 5.5. *For every initial distribution λ on $(\mathcal{S}^{m+1},\mathfrak{S}^{m+1})$ and every measurable function $g:\mathcal{S}^{m+1}\times I\!R\to I\!R$ satisfying the assumptions of Theorem 3.2 with $d=1$ the familiy $\{|g|*\theta_t N^{(\sigma)}(0),t\in I\!R\}$ is u.i. In particular, $\{N^{(\sigma)}(A\times t+I),t\in I\!R\}$ is u.i. under each P_λ for all $A\in\mathfrak{S}^{m+1}$ and bounded intervals I.*

Note finally that we even have uniform integrability of the families

$$\{N(A\times t+I),t\in I\!R,A\in\mathfrak{S}^{m+1}\},$$
$$\{N^{>}(A\times t+I),t\in I\!R,A\in\mathfrak{S}^{m+1}\},$$
$$\{N^{(\sigma)}(A\times t+I),t\in I\!R,A\in\mathfrak{S}^{m+1}\},$$

since they are bounded by the respective families in the above corollaries with $A=\mathcal{S}^{m+1}$.

PROOF OF THEOREM 5.2. W.l.o.g. suppose $g\ge 0$. Consider first the case $\mu\in(0,\infty)$ and let $(M_n^*,S_n^*)_{n\ge0}$ be a MRW with the same transition kernel as $(M_n,S_n)_{n\ge0}$ but with initial distribution ν^s under each P_λ. Given any $\varepsilon>0$, resp. $\varepsilon=0$ if $d=1$, by essentially proceeding as in [2, Section 4], one can construct the two MRWs in such a way that there exists a coupling time T_ε, a.s. finite under every P_λ, such that

$$M_n=M_n^* \quad\text{and}\quad |S_n-S_n^*|\le\varepsilon \quad \text{for all } n\ge T_\varepsilon. \tag{5.10}$$

Denoting N^* the MPP associated with $(M_n^*,S_n^*)_{n\ge0}$, it follows from (5.10)

$$g*\theta_tN(0)\ \le\ g^\varepsilon\theta_tN^*(0)\ +\ \sum_{n=0}^{T_\varepsilon-1}g(M_n,t-S_n)$$

and similarly

$$g*\theta_tN(0)\ \ge\ g_\varepsilon*\theta_tN^*(0)\ -\ \sum_{n=0}^{T_\varepsilon-1}g_\varepsilon(M_n^*,t-S_n^*)$$

for all $t\ge0$. Note that (3.6) does also hold with $\lambda=\nu^s$. Now use

$$\lim_{t\to\infty}\sum_{n=0}^{T_\varepsilon-1}g(M_n,t-S_n)\ =\ \lim_{t\to\infty}\sum_{n=0}^{T_\varepsilon-1}g_\varepsilon(M_n^*,t-S_n^*)\ =\ 0\quad P_\lambda\text{-a.s.},$$

following from the afore-mentioned condition, and the stationarity of $(\theta_tN^*)^+$ $(\sim P_{\nu^s}(N^+\in\cdot))$ to conclude

$$\begin{aligned}P_{\nu^s}(g^\varepsilon*N(0)\le x-\eta)\ &\le\ \liminf_{t\to\infty}P_\lambda(g*\theta_tN(0)\le x)\\ &\le\ \limsup_{t\to\infty}P_\lambda(g*\theta_tN(0)\le x)\ \le\ P_{\nu^s}(g_\varepsilon*N(0)\le x+\eta)\end{aligned}$$

for all $x\in I\!R$ and $\eta>0$. If $\varepsilon=0$ (1-arithmetic case) this immediatley gives the desired result (5.7), otherwise it follows by further letting ε tend to 0 and using

$$\lim_{\varepsilon\downarrow0}g_\varepsilon*N(0))\ =\ \lim_{\varepsilon\downarrow0}g^\varepsilon*N(0)\ =\ g*N(0)\quad P_{\nu^s}\text{-a.s.}$$

which holds by (3.7) and the $\lambda\!\!\lambda_0$-continuity of $\nu^s(\mathcal{S}^{m+1}\times\cdot)$.

If $\mu = \infty$ there is no stationary Markov delay distribution and we must employ instead an additional truncation argument. For $a > 0$ let $\mu_a^> = E(S_{\sigma_1} \wedge a)$ and ν_a^s be the restriction of ν^s to $\mathcal{S}^{m+1} \times [0, a]$, i.e.

$$\nu_a^s(A \times B) \;=\; \frac{1}{\mu_a^>} \int_{B \cap [0,a]} P_{\xi^*}(M_{\sigma_1} \in A, S_{\sigma_1} > s)\; \lambda_d^+(ds), \quad A \in \mathfrak{S}^{m+1}, B \in \mathfrak{B}. \tag{5.11}$$

We infer the following modification of (5.2) from the results in [4, Section 6]

$$U_{\nu_a^s}(A \times B) \;\leq\; \mu_a^{>-1} E_{\xi^*}\left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_A(M_n) \lambda_d^+(B - S_n) \right), \quad A \in \mathfrak{S}^{m+1}, B \in \mathfrak{B}, \tag{5.12}$$

in particular

$$U_{\nu_a^s} \;\leq\; \mu_a^{>-1} F^{m+1} \otimes \lambda_d. \tag{5.13}$$

Consequently, for all $\eta > 0$ and sufficiently small $\varepsilon > 0$ (use (3.7))

$$P_{\nu_a^s}(g^\varepsilon * N(0) > \eta) \;\leq\; \eta^{-1} g^\varepsilon * U_{\nu_a^s}(0) \;=\; \frac{1}{\eta \mu_a^>} \int_{\mathcal{S}^{m+1}} \int_{I\!R} g^\varepsilon(x, y)\; \lambda_d(dy)\; F^{m+1}(dx)$$

which tends to 0 as $a \to \infty$. Now one can prove (5.7) with $\mu = \infty$ by applying the same coupling argument as before but with $(M_n^*, S_n^*)_{n \geq 0}$ having initial distribution ν_a^s under each P_λ for arbitrary $a > 0$. Note once more, as being crucial, that (3.6) does also hold with $\lambda = \nu_a^s$. We thus obtain

$$\limsup_{t \to \infty} P_\lambda(g * \theta_t N(0) > \eta) \;\leq\; \lim_{a \to \infty} P_{\nu_a^s}(g^\varepsilon * N(0) > \eta) \;=\; 0$$

for all $\eta > 0$.

Finally, assertion (5.8) follows similarly when combining (3.7) with (5.2), resp. (5.12), and dominated convergence. Indeed, for all $\eta, \varepsilon > 0$

$$\begin{aligned}
\limsup_{t \to -\infty} P_\lambda(g * \theta_t N(0) > \eta) \;&\leq\; \lim_{t \to -\infty} P_\nu(g^\varepsilon * \theta_t N(0) > \eta) \;\leq\; \lim_{t \to -\infty} \eta^{-1} g^\varepsilon * U_\nu(t) \\
&=\; \frac{1}{\eta c} E_{\xi^*}\left(\sum_{n=0}^{\sigma_1 - 1} \int_{I\!R} g^\varepsilon(M_n, t - x - S_n)\; \lambda_d^+(dx) \right) \\
&=\; \frac{1}{\eta c} E_{\xi^*}\left(\sum_{n=0}^{\sigma_1 - 1} \sum_{k \in \mathbb{Z}} g^\varepsilon(M_n, k\varepsilon) \lambda_d^+(t - S_n - (k+1)\varepsilon, t - S_n - k\varepsilon] \right)
\end{aligned}$$

where $(\nu, c) = (\nu^s, \mu^>)$, resp. $= (\nu_a^s, \mu_a^>)$ for some $a > 0$. The final expectation is bounded by

$$E_{\xi^*}\left(\sum_{n=0}^{\sigma_1 - 1} \int_{I\!R} g^\varepsilon(x, y)\; \lambda_d(dx, dy) \right) \;=\; \theta \int_{\mathcal{S}^{m+1}} \int_{I\!R} g^\varepsilon(x, y)\; \lambda_d(dy)\; F^{m+1}(dx),$$

being finite for sufficiently small ε (by (3.7)), and its integrand converges P_{ξ^*}-a.s. to 0, as $t \to -\infty$.

6. Proof of the results in Section 3

Before turning to the proof of Theorem 3.2 we must briefly collect some basic facts on Harris recurrence and regeneration which are needed hereafter. For a moment we leave the given framework and suppose that $(M_n)_{n\geq 0}$ is an arbitrary Markov chain with state space $(\mathcal{S},\mathfrak{S})$ and r-step transition kernel $I\!P_r$ ($I\!P = I\!P_1$). Let a canonical model be given with probability measures P_s such that $P_s(M_0 = s) = 1$ and $P_\lambda = \int_{\mathcal{S}} P_s\,\lambda(ds)$ for any distribution λ on $\mathcal{S}$. $(M_n)_{n\geq 0}$ is called *Harris recurrent* or just *Harris chain*, if it possesses a *recurrence set* $\Re$, i.e. $P_s(M_n \in \Re \text{ i.o.})=1$ for all $s \in \mathcal{S}$, such that for some $\alpha \in (0,1]$, $r \geq 1$ and a distribution ϕ on $\mathcal{S}$ the minorization condition

$$I\!P_r(s,\cdot) \geq \alpha\phi \quad \text{for all } s \in \Re \tag{6.1}$$

holds. Given the latter condition, $\Re$ is called a *regeneration set* because it induces a regenerative structure for $(M_n)_{n\geq 0}$ that divides the chain into stationary (possibly except for the first one), 1-dependent cycles. This has been shown in the fundamental paper by Athreya and Ney [5] for $r = 1$ in which case the cycles are even independent, see also [10] for a similar technique. Indeed, (6.1) is equivalent to the existence of a *sequence* $(T_n)_{n\geq 0}$ *of regeneration epochs*, characterized through the following four conditions:

- (R.1) $0 = T_0 < T_1 < T_2 < ... < \infty$ a.s. under each P_λ.
- (R.2) There is a filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $(M_n)_{n\geq 0}$ is Markov-adapted and each T_k a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$.
- (R.3) $(T_{n+j} - T_n, M_{T_n+j})_{j\geq 0}$ is independent of $T_0, ..., T_n$.
- (R.4) For $n \geq 1$, the distribution of $(T_{n+j} - T_n, M_{T_n+j})_{j\geq 0}$ is the same under every $P_s, s \in \mathcal{S}$, and given by $P_\zeta((T_j, M_j)_{j\geq 0} \in \cdot)$ where $\zeta = P(M_{T_1} \in \cdot)$.

$(T_n)_{n\geq 0}$ may be chosen in such a way that

$$\xi(A) \stackrel{\text{def}}{=} E_\zeta\left(\sum_{j=0}^{T_1-1} \mathbf{1}_{\{M_j \in A\}}\right), \quad A \in \mathfrak{S} \tag{6.2}$$

forms the unique (up to multiplicative constants) σ-finite stationary measure of $(M_n)_{n\geq 0}$, and $\xi^* = \xi/E_\zeta T_1$ the unique stationary distribution, providing a finite stationary mean cycle length $E_\zeta T_1$. In the latter case the chain is called *positive recurrent*. It is called *aperiodic* if the distribution of T_1 under P_ζ is aperiodic (1-arithmetic), and *ergodic* if being aperiodic and positive recurrent in which case $P_\lambda(M_n \in \cdot)$ converges to ξ^* in total variation.

Returning to the situation of the previous sections, $M_n = (Y_{n-m}, ..., Y_n), n \geq 0$ obviously forms an ergodic Harris chain with stationary distribution F^{m+1}. Moreover, $T_n^{(k)} \equiv kn, n \geq 0$ for any fixed integer $k \geq m + 1$ is a sequence of regeneration epochs.

Proof of Theorem 3.2. In order to prove Theorem 3.2 for all initial distributions λ such that (3.5) holds we first recall a lemma from [2] (Lemma 3.1 there) adapted to the present, much simpler situation (put $r = m + 1$ and $\tau_1 \equiv 2m + 2$ in the notation from there).

Lemma 6.1. *There is some $c \in I\!R^{2m+2}$ such that for all $\varepsilon > 0$ there are $C_\varepsilon, D_\varepsilon \in \mathfrak{S}^{m+1}$ satisfying $F^{m+1}(C_\varepsilon) > 0$, $F^{m+1}(D_\varepsilon) > 0$ and*

$$\eta(\varepsilon) \stackrel{\text{def}}{=} \inf_{(x,y)\in C_\varepsilon \times D_\varepsilon} P(\|(X_1,...,X_{2m+2}) - c\|_\infty \le \tfrac{\varepsilon}{2m+2} | M_0 = x, M_{2m+2} = y) \ > \ 0, \quad (6.3)$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $I\!R^{2m+2}$. If $(M_n, S_n)_{n\ge0}$ is 1-arithmetic with shift function 0, then (6.3) remains true for $\varepsilon = 0$ if $c \in \mathbb{Z}^{2m+2}$ is suitably chosen.

The crucial consequence of this lemma is that we can construct a sequence of regeneration epochs $(T_n)_{n\ge0}$ in such a way that the increments $X_{T_n-2m-1}, ..., X_{T_n}$ and thus $S_{T_n} - S_{T_n-2m-2}$ are at least almost constant. For the rest of the proof we confine ourselves to the more difficult nonarithmetic case ($d = 0$). Here is a description of how to obtain the T_n: Fix $\varepsilon > 0, C = C_\varepsilon$ and $D = D_\varepsilon$ in the following. Let $\chi_0, \chi_1, ...$ be i.i.d. geometric(1/2) random variables under each P_λ and further independent of the "rest of the world". Define $T_0 = 0$ and then recursively

$$T_n \ = \ \inf\{k \ge T_{n-1} + \chi_{n-1} + 2m + 2 : (M_{k-2m-2}, M_k) \in C \times D\}$$

for $n \ge 1$. It is then readily verified that $(T_n)_{n\ge0}$ forms indeed a sequence of regeneration epochs with cycle debut distribution $\zeta = P.(M_{T_1} \in \cdot) = F^{m+1}(\cdot \cap D)/F^{m+1}(D)$ and $E_\lambda T_1 < \infty$ for every initial distribution λ. The insertion of geometrically distributed variables between successive regenerations guarantees

$$\inf_{n\ge1} E(e^{itS_{T_n}} | M_0, M_{T_n}) \ < \ 1 \quad P\text{-a.s.}$$

for all $0 < |t| < 2\pi/d$, a property needed for a successful coupling argument that leads to (3.3) and (3.4) with $\lambda = \zeta$ (see Lemma 3.3 and Section 5 in [2]), i.e.

$$\lim_{t\to\infty} g * U_\zeta(t) \ = \ \frac{1}{\mu} \int_{\mathcal{S}^{m+1}} \int_{I\!R} g(x,y) \ \lambda\!\!\lambda_0(dy) \ F^{m+1}(dx), \quad (6.4)$$

$$\lim_{t\to-\infty} g * U_\zeta(t) \ = \ 0. \quad (6.5)$$

Recall from (5.9) the definition of $g_\varepsilon, g^\varepsilon$ and further that these functions also satisfy (3.5-8) for all sufficiently small ε as well as

$$g_\varepsilon(x,\cdot) \uparrow g(x,\cdot) \quad \text{and} \quad g^\varepsilon(x,\cdot) \downarrow g(x,\cdot) \quad \lambda\!\!\lambda_0\text{-a.s.} \quad (\varepsilon \downarrow 0) \quad (6.6)$$

As is shown in [2], we then have

$$E_\lambda\left(\sum_{n=0}^{T_1-1} g_\varepsilon(M_n, t - S_n)\right) \ + \ \int_{I\!R} g_\varepsilon * U_\zeta(t-s) \ P_\lambda(S_{T_1} \in ds) \ \le \ g * U_\lambda(t)$$

$$\le \ E_\lambda\left(\sum_{n=0}^{T_1-1} g^\varepsilon(M_n, t - S_n)\right) \ + \ \int_{I\!R} g^\varepsilon * U_\zeta(t-s) \ P_\lambda(S_{T_1} \in ds).$$

$g_\varepsilon * U_\zeta(t)$ and $g^\varepsilon * U_\zeta(t)$ are bounded functions as following from (6.4) and (6.5) for $g_\varepsilon, g^\varepsilon$. Hence a combination with (6.6) and a successive application of the dominated and the monotone convergence theorem yields for $h \in \{g_\varepsilon, g^\varepsilon\}$

$$\lim_{\varepsilon\downarrow 0}\lim_{t\to\infty}\int_{\mathbb{R}} h * U_\zeta(t-s)\ P_\lambda(S_{T_1}\in ds) \ = \ \frac{1}{\mu}\int_{\mathcal{S}^{m+1}}\int_{\mathbb{R}} g(x,y)\ \lambda\!\!\lambda_0(dy)\ F^{m+1}(dx),$$

respectively 0 if $t\to-\infty$. The desired assertions follow if we finally prove that, providing (3.5) and (3.6),

$$\lim_{|t|\to\infty} E_\lambda\left(\sum_{n=0}^{T_1-1} h(M_n, t-S_n)\right) \ = \ 0 \tag{6.7}$$

for $\varepsilon > 0$ sufficiently small. This will follow from the subsequent lemma because assumptions (3.5-8) for g and λ do also hold for h and λ.

Lemma 6.2. *For every function g and initial distribution λ satisfying (3.5-8)*

$$\lim_{|t|\to\infty} E_\lambda\left(\sum_{n=0}^{T_1-1} g(M_n, t-S_n)\right) \ = \ 0. \tag{6.8}$$

Proof. Recall $G(x) = \sup_{t\in\mathbb{R}}|g(x,t)|$ and notice that

$$EG(M_0) \ \le \ \int_{\mathcal{S}^{m+1}}\sum_{n\in\mathbb{Z}}\sup_{n\rho\le y<(n+1)\rho}|g(x,y)|\ F^{m+1}(dx) \ < \ \infty$$

by (3.8). Since

$$\left|\sum_{n=0}^{T_1-1} g(M_n, t-S_n)\right| \ \le \ \sum_{n=0}^{T_1-1} G(M_n)$$

and $\lim_{|t|\to\infty}\sum_{n=0}^{T_1-1} g(M_n,t-S_n) = 0$ P_λ-a.s. by (3.6), we obtain (6.8) by the dominated convergence theorem if we still show

$$E_\lambda\left(\sum_{n=0}^{T_1-1} G(M_n)\right) \ < \ \infty.$$

To this end we use the inequality

$$E_\lambda\left(\sum_{n=0}^{T_1-1} G(M_n)\right) \ \le \ E_\lambda\left(\sum_{n=0}^{m} G(M_n)\right) \ + \ E_\lambda\left(\sum_{n=m+1}^{T_1+m} G(M_n)\right).$$

The first expectation on the right-hand side is finite by (3.5). The second equals $EG(M_0)E_\lambda T_1 < \infty$ because $P_\lambda^{M_{m+1}} = P_\lambda^{M_{T_1+m+1}} = F^{m+1}$ implies

$$E_\lambda\left(\sum_{n=m+1}^{T_1+m} \mathbf{1}_A(M_n)\right) \ = \ E_\lambda T_1 F^{m+1}(A),$$

which is a well-known fact about occupation measures for Markov chains.

Proof of Theorem 3.4. Here the arguments are precisely the same as in the previous proof except for the one that finally shows

$$\lim_{|t|\to\infty} E_\lambda\left(\sum_{n=0}^{T_1-1} g(M_n^>, t-\sigma_n)\right) \;=\; 0 \tag{6.9}$$

where $(T_n)_{n\geq 0}$ is now a sequence of regeneration epochs for $(M_n^>, \sigma_n)_{n\geq 0}$. We cannot proceed as in Lemma 6.2 because the σ_n do not in general possess m-dependent increments which was crucial there. Instead we will utilize Corollary 5.5 which gives uniform integrability of $\{|g| * \theta_t N^{(\sigma)}, t \in I\!R\}$ and thus also of $\{\sum_{n=0}^{T_1-1} g(M_n^>, t-\sigma_n), t \in I\!R\}$ under P_λ. By combining the latter with $\lim_{|t|\to\infty}\sum_{n=0}^{T_1-1} g(M_n^>, t-\sigma_n) = 0$ P_λ-a.s. (use (3.5)) we obtain (6.9).

Proof of Theorem 3.5. Identical with the previous one when exchanging Corollary 5.5 by Corollary 5.4.

7. Proof of the results in Section 4

We first note without proof a simple but useful lemma for checking conditions (3.7) and (3.8) for a given function $g \geq 0$ on $\mathcal{S}^{m+1} \times [0,\infty)$.

Lemma 7.1. *Suppose we are given a measurable function $g : \mathcal{S}^{m+1} \times [0,\infty) \to [0,\infty)$ which decreases in the second variable. Then (3.7) holds and*

$$\int_{\mathcal{S}^{m+1}} \int_{[0,\infty)} g(x,y)\; \lambda\!\!\lambda_0(dy)\; \nu(dx) \;<\; \infty$$

for a distribution ν on $(\mathcal{S}^{m+1}, \mathfrak{S}^{m+1})$ implies

$$\int_{\mathcal{S}^{m+1}} \sum_{n\in I\!N_0} \sup_{n\rho\leq y<(n+1)\rho} g(x,y)\; \nu(dx) \;<\; \infty \quad \text{for all } \rho > 0.$$

A typical example for a function g to which the lemma applies is

$$g(x,y) \;=\; P_x(M_{\sigma_1} \in C, S_{\sigma_1} > y)\mathbf{1}_{[0,\infty)}(y)$$

for arbitrary $C \in \mathfrak{S}^{m+1}$.

Proof of Theorem 4.1. We restrict ourselves to the little more complicated nonarithmetic case.

(a) For all $C \in \mathfrak{S}^{m+1}$, $r \geq 0$, $s \in I\!R$ and initial distributions λ we obtain

$$\begin{aligned} &P_\lambda(M_{\tau(t)} \in C, R_t > r, S_{\tau(t),m} > s)\\ &\quad = \sum_{n\geq 0} P_\lambda(M_{\sigma_{n+1}} \in C, S_{\sigma_n} \leq t, S_{\sigma_{n+1}} > t+r, S_{\sigma_{n+1},m} > s)\\ &\quad = g * U_\lambda^>(t), \quad g(x,y) \overset{\text{def}}{=} P_x(M_{\sigma_1} \in C, S_{\sigma_1} > y+r, S_{\sigma_1,m} > s)\mathbf{1}_{[0,\infty)}(y). \end{aligned}$$

As one can easily check with the help of Lemma 7.1, the conditions of Theorem 3.5 are fulfilled for this g and each λ whence

$$\begin{aligned}\lim_{t\to\infty} P_\lambda(M_{\tau(t)} \in C, R_t > r, S_{\tau(t),m} > s) \\ = \frac{1}{\mu^>}\int_{[r,\infty)} P_{\xi^*}(M_{\sigma_1} \in C, S_{\sigma_1} > y, S_{\sigma_1,m} > s)\ \lambda\!\!\lambda_d(dy).\end{aligned} \tag{7.1}$$

So we have shown convergence in distribution of $(M_{\tau(t)}, R_t, S_{\tau(t),m})$, as $t \to \infty$. By choosing $s = -\infty$, resp. $r = 0$ in (7.1), we get (4.5), resp. (4.6).

(b) Let $p > 0$, $E_\lambda X_n^- < \infty$ and $E_\lambda (X_n^+)^{p+1} < \infty$ for all $n \ge 1$. By considering the truncated sequence $(X_n \wedge a)_{n\ge1}$ for sufficiently large $a > 0$, which then satisfies $E_\lambda|X_n \wedge a| < \infty$ for all $n \ge 1$, and applying (4.2) we infer $E_\lambda\sigma_n < \infty$ for all $n \ge 1$ as well as $E_\lambda\tau(t) < \infty$ for all $t \ge 0$. Given this, we further obtain

$$\begin{aligned} n^{-(p+1)}E_\lambda S_{\sigma_n}^{p+1} &= E_\lambda\left(\frac{1}{n}\sum_{j=1}^n (S_{\sigma_j} - S_{\sigma_{j-1}})\right)^{p+1} \le E_\lambda\left(\frac{1}{n}\sum_{j=1}^n X_{\sigma_j}^+\right)^{p+1} \\ &\le E\Big(\max_{1\le j\le n} X_{\sigma_j}^+\Big)^{p+1} \le E_\lambda\left(\sum_{k=1}^{\sigma_n}(X_k^+)^{p+1}\right) \\ &= E_\lambda\left(\sum_{k=1}^m (X_k^+)^{p+1}\right) + E(X_1^+)^{p+1}E_\lambda\sigma_n < \infty \end{aligned} \tag{7.2}$$

for all $n \ge 1$, the final equality by another appeal to (4.2). As being needed below, note that (7.2) remains true with λ replaced by $\lambda_k = P_\lambda^{M_k}$ for each $k \ge 1$, i.e.

$$n^{-(p+1)}E_{\lambda_k} S_{\sigma_n}^{p+1} \le E_{\lambda_k}\left(\sum_{j=1}^m (X_j^+)^{p+1}\right) + E(X_1^+)^{p+1}E_{\lambda_k}\sigma_n < \infty \tag{7.3}$$

for all $k, n \ge 1$. One must only observe that $E_{\lambda_k}X_n^- = E_\lambda X_{k+n}^- < \infty$ and $E_{\lambda_k}(X_n^+)^{p+1} = E_\lambda(X_{k+n}^+)^{p+1} < \infty$ for all $k, n \ge 1$. As for R_t, a similar estimation gives

$$E_\lambda R_t^{p+1} \le E_\lambda\left(\sum_{k=1}^{\tau(t)}(X_k^+)^{p+1}\right) = E_\lambda\left(\sum_{j=1}^m (X_j^+)^{p+1}\right) + E(X_1^+)^{p+1}E_\lambda\tau(t) < \infty \tag{7.4}$$

for all $t \ge 0$.

Since, under each P_λ, R_t converges in distribution to a variable R_∞ with distribution $P_{\xi^*}(S_{\sigma_1} > u)\lambda\!\!\lambda_0(du)/\mu^>$ (by (4.5)), the uniform integrability of $\{R_t^p, t \ge 0\}$ under P_λ follows if we can show convergence of $E_\lambda R_t^p$ to $E_\lambda R_\infty^p = E_{\xi^*}S_{\sigma_1}^{p+1}/\mu^>$. To this end we note that

$$E_\lambda R_t^p = g * U_\lambda(t), \quad g(x,y) \stackrel{\text{def}}{=} E_x\big((S_{\sigma_1} - y)^+\big)^p \mathbf{1}_{[0,\infty)}(y).$$

Validity of (3.6), (3.7) and (3.8) are again easily checked (the latter two with the help of Lemma 7.1). As for (3.5), we obviously have $G(x) = E_x S_{\sigma_1}^p$ by monotonicity of $g(x,y)$ in y.

Consequently,

$$E_\lambda\left(\sum_{n=0}^m G(M_n)\right) \;=\; \sum_{n=0}^m E_{\lambda_n} S_{\sigma_1}^p \;<\; \infty$$

by (7.3). Now one may apply Theorem 3.5 and obtains the desired moment convergence after a simple computation. Further details can be omitted.

For the proof of uniform integrability of $\{S_{\tau(t),m}^p, t \geq 0\}$ under P_λ use

$$E_\lambda |S_{\tau(t),m}|^p \;=\; g * U_\lambda(t), \quad g(x,y) \;\stackrel{\text{def}}{=}\; E_x \mathbf{1}_{\{S_{\sigma_1}>y\}} |S_{\sigma_1,m}|^p \mathbf{1}_{[0,\infty)}(y)$$

and proceed as in the previous part.

(c) Here it suffices to note that all assertions are direct consequences of the previous parts in combination with (4.3).

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