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A Useful Extension of Itô's Formula with Applications to Optimal Stopping

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A Useful Extension of Itô's Formula with Applications to Optimal Stopping

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Given a continuous semimartingale $M = (M_t)_{t \geq 0}$ and a d-dimensional continuous process of locally bounded variation $V = (V^1, ..., V^d)$, the multidimensional Itô Formula states that

$$
\begin{aligned}
f(M_t, V_t) - f(M_0, V_0) &= \int_{[0,t]} D_{x_0} f(M_s, V_s)\, dM_s \\
&+ \sum_{i=1}^{d} \int_{[0,t]} D_{x_i} f(M_s, V_s)\, dV_s^i \\
&+ \frac{1}{2} \int_{[0,t]} D_{x_0}^2 f(M_s, V_s)\, d\langle M \rangle_s
\end{aligned}
$$

if $f(x_0, ..., x_d)$ is of C^2-type with respect to x_0 and of C^1-type with respect to the other arguments. This formula is very useful when solving various optimal stopping problems based on Brownian motion, but typically with a function f whose first partial derivative with respect to x_0 is only absolutely continuous. We prove that the formula remains true for such functions and demonstrate its use with two examples from Mathematical Finance.

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1. Introduction

Various infinite horizon optimal stopping problems for processes of type $(e^{-rt}X_t)_{t\geq 0}$, where $X_t = f(B_t, V_t)$, $t \geq 0$, is a smooth function f of a standard Brownian motion $(B_t)_{t\geq 0}$ and a process $(V_t)_{t\geq 0}$ of locally bounded variation, can be explicitly solved by first guessing a parametric class of stopping times in which the optimal one is expected to lie and by then using the (heuristic) principle of smooth fit in order to find the parameters of the optimal rule τ^*, say, in that class. Roughly speaking, this means to identify τ^* as the unique element for which the associated expected reward function $\mathbb{V}(x) \stackrel{\text{def}}{=} \mathbb{E}(e^{-r\tau^*}X_{\tau^*}|X_0 = x)$ possesses certain smoothness properties. Global optimality of τ^* is then obtained by finally showing that $(e^{-rt}\mathbb{V}(X_t))_{t\geq 0}$ forms a supermartingale and a subsequent simple estimation using the otional sampling theorem. The multidimensional Itô Formula for sufficiently smooth functions of a continuous semimartingale and a process of locally bounded variation appears to be the natural tool. However, it cannot be applied without further ado because the value function $\mathbb{V}$ is typically continuously differentiable but has a second derivative which is only continuous outside a finite set of exceptional points, namely those where smooth fit was used for the first derivative. On the other hand, second derivatives appear only in the integral with respect to quadratic variation measure which for standard Brownian motion is just Lebesgue measure and thus ignores discontinuities of functions on finite sets. It is therefore to be expected and in fact confirmed in this note that for situations as just described Itô's Formula remains true for functions $f(x, y)$ which are continuously differentiable with respect to x and y but whose second partial derivative with respect to x exists merely outside a Lebesgue null set. While such an extension is known for the one-dimensional case (functions of a continuous semimartingale), see e.g. [6, Thm. 22.5], we have not been able to find it in the literature for the multivariate situation.

We continue with a simple illustrating example in support of this rather informal introduction. Given any $(\mu, \sigma^2, r) \in I\!R \times (0,\infty)^2$, let $W_t \stackrel{\text{def}}{=} \sigma B_t + \mu$ for $t \geq 0$. We consider optimal stopping of

$$e^{-rt}W_t \;=\; e^{-rt}(\sigma B_t + \mu), \quad t \geq 0,$$

which means to find a stopping time τ^* for $(W_t)_{t\geq 0}$ such that

$$\mathbb{V}(x) \;\stackrel{\text{def}}{=}\; \mathbb{E}_x e^{-r\tau^*}W_{\tau^*} \;=\; \max_{\tau} \mathbb{E}_x e^{-r\tau}W_\tau$$

for all $x \in I\!R$ where the maximum is taken over all stopping times for $(W_t)_{t\geq 0}$ and where $\mathbb{P}_x \stackrel{\text{def}}{=} \mathbb{P}(\cdot|W_0 = x)$ with associated expectation operator $\mathbb{E}_x$. In Mathematical Finance the model is known as the classical Bachelier model [1] in which a bond paying interest at rate r is to be compared to a stock whose price at time t is given by W_t. It appears very plausible here that the optimal stopping time should be an element of the class of threshold rules

$$\tau_a \;\stackrel{\text{def}}{=}\; \inf\{t \geq 0 : W_t \geq a\}, \quad a \geq 0.$$

So we start by computing $\mathbb{V}_a(x) \stackrel{\text{def}}{=} \mathbb{E}_x e^{-r\tau_a} W_{\tau_a}$ which is easily accomplished by using well-known formulae for Brownian motion. The result is

$$\mathbb{V}_a(x) \;=\; \begin{cases} a \exp\left(\dfrac{(x-a)\left(\sqrt{\mu^2+2r\sigma^2}-\mu\right)}{\sigma^2}\right), & \text{if } x \leq a \\ x, & \text{if } x \geq a \end{cases}$$

and constitutes, for every a, a continuous function which is twice continuously differentiable function on $I\!\!R - \{a\}$. The smooth fit principle now means to determine a^* such that $\mathbb{V}_{a^*}$ is also differentiable at $x = a^*$, i.e. to find a^* such that

$$\frac{d}{dx}\left[a \exp\left(\frac{(x-a)\left(\sqrt{\mu^2+2r\sigma^2}-\mu\right)}{\sigma^2}\right)\right]_{x=a^*} \;=\; 1.$$

The unique solution is easily calculated as

$$a^* \;=\; \frac{\sigma^2}{\sqrt{\mu^2+2r\sigma^2}-\mu},$$

and the associated value function $\mathbb{V}^* \stackrel{\text{def}}{=} \mathbb{V}_{a^*}$ takes the form

$$\mathbb{V}^*(x) \;=\; \begin{cases} a^* \exp\left(\dfrac{x}{a^*}-1\right), & \text{if } x \leq a^* \\ x, & \text{if } x \geq a^* \end{cases}.$$

It is continuously differentiable on $I\!\!R$ with a continuous second derivative at all $x \neq a^*$. In order to finally verify that τ_{a^*} is indeed the optimal stopping rule or, equivalently, that $\mathbb{V}^*(x) = V(x)$ it suffices to prove that $(e^{-rt}\mathbb{V}^*(W_t))_{t\geq 0}$ forms a supermartingale because then we infer upon using $\mathbb{V}^*(x) \geq x$ and the optional sampling theorem

$$\mathbb{E}_x e^{-r\tau} W_\tau \;\leq\; \mathbb{E}_x e^{-r\tau}\mathbb{V}^*(W_\tau) \;\leq\; \mathbb{E}_x \mathbb{V}^*(W_0) \;=\; \mathbb{V}^*(x)$$

for all $x \in I\!\!R$ and any stopping time τ and thus $\mathbb{V}(x) \leq \mathbb{V}^*(x)$. Since the reverse inequality holds trivially true we have arrived at the desired conclusion.

Left with the supermartingale property to be proved write $e^{-rt}\mathbb{V}^*(W_t) = f(W_t, t)$ with $f(x,y) \stackrel{\text{def}}{=} e^{-ry}\mathbb{V}^*(x)$. The function f is bivariate and inherits the smoothness properties of $\mathbb{V}^*$. It has continuous partial derivatives $D_x f, D_y f$ of first order and a second order partial derivative $D_x^2 f$ which is continuous at all (x,y) with $x \neq a^*$. Ignoring the latter fact an application of the two-dimensional Itô Formula (see (2.1) in the following section) with a continuous semimartingale in the first component and a continuous process of local finite variation in the second one does indeed confirm that $(e^{-rt}\mathbb{V}^*(W_t))_{t\geq 0}$ is a supermartingale and completes the solution of the above optimal stopping problem.

Our extension of the multidimensional Itô Formula (Theorem 2.1 below) is tailored to situations just as the previous one and turns the argument into a rigorous result.

The further organization is as follows. Section 2 provides the main result and its proof while Section 3 contains another application to an optimal stopping problem in Mathematical Finance, namely the perpetual American put option, and also some references for examples of similar type.

2. Main Result

For $i \in I\!N \cup \{\infty\}$, let $\mathcal{C}^{i,0}(I\!R^{d+1})$ be the class of functions $f : I\!R^{d+1} \to I\!R$ which have continuous partial derivatives of order i with respect to the first argument and are continuous in the other ones. Its subclass of functions which have continuous first order partial derivatives with respect to each variable is denoted as $\mathcal{C}^{i,1}(I\!R^{d+1})$. Let also $\mathcal{C}_0^{\infty}(I\!R)$ be the class of functions from $I\!R$ to $I\!R$ which are infinitely often differentiable with compact support.

A special variant of the multidimensional Itô Formula (cf. [2]) states that a.s.

$$\begin{aligned} f(M_t, V_t) - f(M_0, V_0) &= \int_{[0,t]} D_{x_0} f(M_s, V_s)\, dM_s + \sum_{i=1}^{d} \int_{[0,t]} D_{x_i} f(M_s, V_s)\, dV_s^i \\ &+ \frac{1}{2} \int_{[0,t]} D_{x_0}^2 f(M_s, V_s)\, d\langle M \rangle_s \end{aligned} \tag{2.1}$$

if $f(x_0, ..., x_d)$ is an element of $\mathcal{C}^{2,1}(I\!R^{d+1})$, M is a continuous semimartingale with quadratic variation process $\langle M \rangle$, and $V = (V^1, ..., V^d)$ is a d-dimensional continuous process of locally finite variation. The subsequent theorem weakens the conditions on f under which (2.1) holds true.

Theorem 2.1. *Let $f(x_0, ..., x_d)$ be a real-valued function on $I\!R^{d+1}$ satisfying the following three conditions:*

(F.1) $f \in \mathcal{C}^{1,1}(I\!R^{d+1})$.

(F.2) $D_{x_0} f$ *is absolutely continuous in the first argument when keeping the others fixed.*

Then formula (2.1) holds true for any continuous semimartingale $M = (M_t)_{t \geq 0}$ *and any continuous process in* $I\!R^d$ *of locally bounded variation* $V = (V^1, ..., V^d)$.

The proof presented below hinges on a suitable approximation of f by $\mathcal{C}^{\infty}$-functions in combination with a deep result on the connection between the quadratic variation $\langle M \rangle$ and the local time of M. The latter is in fact also the key for the extension of the univariate Itô Formula to differentiable functions with absolutely continuous derivative we mentioned in the Introduction.

Various applications in Mathematical finance use geometric Brownian motion to describe the evolution of stock prices. With view to such applications including a particular one discussed in Section 3 the following corollary provides us with formula (2.1) for the special case where the semimartingale is geometric Brownian motion.

COROLLARY 2.2. *Let $(X_t)_{t\ge 0}$ be geometric Brownian motion with drift parameter μ and volatility $\sigma > 0$, so*

$$X_t \;=\; X_0 \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right), \quad t \ge 0,$$

where $(B_t)_{t\ge 0}$ denotes standard Brownian motion starting at 0. Let further V and f be as assumed in Theorem 2.1. Then a.s.

$$\begin{aligned} f(X_t, V_t) - f(X_0, V_0) \;&=\; \sigma \int_{[0,t]} X_s D_{x_0} f(X_s, V_s)\, dB_s \;+\; \sum_{i=1}^{d} \int_{[0,t]} D_{x_i} f(X_s, V_s)\, dV_s^i \\ &+ \int_{[0,t]} \left(\mu X_s D_{x_0} f(X_s, V_s) + \frac{1}{2}\sigma^2 X_s^2 D_{x_0}^2 f(X_s, V_s)\right) ds. \end{aligned} \tag{2.2}$$

PROOF. Define $g(x_0, ..., x_d, s) \stackrel{\text{def}}{=} f(X_0 e^{\sigma x_0 + (\mu - (\sigma^2/2))s}, x_1, ..., x_d)$ and notice that g satisfies (F.1) and (F.2) if f does. An application of Theorem 2.1 to $f(X_t, V_t) = g(B_t, V_t, t)$ now yields (2.2). ◊

For any $\varphi \in \mathcal{C}_0^\infty(I\!R)$ and any continuous function $g : I\!R^{d+1} \to I\!R$, we define

$$\varphi * g(x_0, ..., x_d) \;\stackrel{\text{def}}{=}\; \int_{-\infty}^{\infty} \varphi(y) f(x_0 - y, x_1, ..., x_d) dy$$

which is ordinary convolution, but only in the first argument. We will need the following standard smoothing lemma a proof of which may be found in [3].

LEMMA 2.3. *Let $g(x_0, ..., x_d)$ be a continuous function on $I\!R^{d+1}$ which is even absolutely continuous in the first variable for all fixed $(x_1, ..., x_d) \in I\!R^d$. Then $\varphi * g \in \mathcal{C}^{\infty,0}(I\!R^{d+1})$ with*

$$D_{x_0}(\varphi * g)(x) \;=\; \varphi * D_{x_0} g(x)$$

for all $\varphi \in \mathcal{C}_0^\infty(I\!R^{d+1})$.

The important connection between the quadratic variation and the local time of the continuous semimartingale M we mentioned after Theorem 2.1 enters into our second lemma which forms the key to cope with the weakened assumption of f concerning $D_{x_0} f$. Let $(L_t^u)_{t\ge 0, u\in I\!R}$ be a version of the local time process of M which is continuous in t and cádlág in u (see [6, Thm. 22.4]). Note that

$$L_t^u \;=\; |M_t - u| \;-\; |M_0 - u| \;-\; \int_0^t \operatorname{sgn}(M_s - u-)\, dM_s$$

where $\operatorname{sgn}(u-) \stackrel{\text{def}}{=} \mathbf{1}_{(0,\infty)}(u) - 1_{(-\infty,0]}(u)$ is the left continuous version of the sign function.

LEMMA 2.4. *Given the situation of Theorem 2.1,*

$$\int_0^t h(M_s, V_s)\ d\langle M\rangle_s \;=\; \int_{-\infty}^{\infty} h(u, V_s) L_t^u\ du \quad a.s. \tag{2.3}$$

for all $t \geq 0$ and nonnegative functions h. The $\mathbb{P}$-null set may be chosen independent of h.

PROOF. The occupation density result due to Meyer and Wang (see [6, Thm. 22.5]) states that outside a $\mathbb{P}$-null set we have

$$\int_0^t g(M_s)\ d\langle M\rangle_s \;=\; \int_{-\infty}^{\infty} g(u) L_t^u\ du, \quad t \geq 0, \tag{2.4}$$

for any nonnegative function g on $\mathbb{R}$. By using that the null set does not depend on g, a pathwise application of (2.4) to the functions $g_\omega = h(\cdot, V_s(\omega))$ for any $h \geq 0$ and ω outside the null set gives (2.3). ◇

PROOF OF THEOREM 2.1. Let us first assume that the processes M and V are bounded. Concerning $M = N + A$, we more precisely assume that both, its local martingale part N and its locally bounded variation part A, are bounded. The general case will follow by localization.

So we assume the existence of a $K > 0$ such that $|(M_s, V_s)| < K$ where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{d+1}$. For $x = (x_0, ..., x_d)$ put $\tilde{x} \stackrel{\text{def}}{=} (x_1, ..., x_d)$, i.e. $x = (x_0, \tilde{x})$. Further define $\varphi_n \in \mathcal{C}^\infty(\mathbb{R})$ for $n \geq 1$ by

$$\varphi_n(u) \;\stackrel{\text{def}}{=}\; c_n \exp\left(-\frac{1}{n^{-2} - u^2} \right) \mathbf{1}_{(-1/n, 1/n)}(u)$$

where $c_n > 0$ is a normalizing constant which renders $\int \varphi_n(u)\, du = 1$. Finally, let f_n be defined through

$$f_n(x) \;\stackrel{\text{def}}{=}\; \varphi_n * f(x) \;=\; \int_{-\infty}^{\infty} \varphi_n(y) f(x_0 - u, \tilde{x})\ du.$$

We claim that

(1) $f_n \in \mathcal{C}^{\infty,1}(\mathbb{R}^{d+1})$ for all $n \geq 1$;

(2) $f_n \to f$ uniformly on $[-K, K]^{d+1}$;

(3) $D_{x_i} f_n \to D_{x_i} f$ uniformly on $[-K, K]^{d+1}$ for $0 \leq i \leq d$;

The first claim is a direct consequence of Theorem 3.7 on p. 193 in [3]. Fix any $\varepsilon > 0$. By the uniform continuity of f on $[-K, K]^{d+1}$ there exists $0 < \delta < 1$ such that $|f(x_0 + y, \tilde{x}) - f(x)| < \varepsilon$ for all $x \in [-K, K]^{d+1}$ and $y \in (-\delta, \delta)$. Note that $\text{supp}(\varphi_n) \subset (-\delta, \delta)$ for all $n > 1/\delta$. For these n we hence infer

$$\begin{aligned} |\varphi_n * f(x) - f(x)| &= \left| \int_{-\delta}^{\delta} \varphi_n(y) f(x_0 - y, \tilde{x}) \, dy \; - \; f(x) \right| \\ &\leq \int_{-\delta}^{\delta} \varphi_n(y) |f(x_0 - y, \tilde{x}) - f(x)| \, dy \\ &\leq \varepsilon \int_{-\delta}^{\delta} \varphi_n(y) \, dy \\ &= \varepsilon \end{aligned}$$

that is Claim (2). Since Lemma 2.3 implies $D_{x_i} f_n(x) = \varphi_n * D_{x_i} f(x)$ for $x \in [-K-1, K+1]^{d+1}$ and $i = 0, ..., d$, the third claim is obtained analogously to the second one.

Applying Itô's Formula (2.1) to the functions $f_n \in \mathcal{C}^{\infty,1}(I\!R^{d+1})$, $n \geq 1$, gives

$$\begin{aligned} f_n(M_t, V_t) - f_n(M_0, V_0) &= \int_{[0,t]} \mathbb{D}_{x_0} f_n(M_s, V_s) \, dM_s \; + \; \sum_{i=1}^{d} \int_{[0,t]} D_{x_i} f_n(M_s, V_s) \, dV_s^i \\ &+ \; \frac{1}{2} \int_{[0,t]} D_{x_0}^2 f_n(M_s, V_s) \, d\langle M \rangle_s \end{aligned}$$

By (2),

$$\lim_{n \to \infty} f_n(M_t, V_t) \; = \; f(M_t, V_t) \quad \text{a.s.}$$

for all $t \geq 0$. For each $i = 0, ..., d$, the partial derivative $D_{x_i} f$ is bounded on $[-K, K]^{d+1}$ and, by (3),

$$\lim_{n \to \infty} D_{x_i} f_n(M_t, V_t) \; = \; D_{x_i} f(M_t, V_t) \quad \text{a.s.}$$

for all $t \geq 0$, the convergence being uniform. Hence the dominated convergence theorem for stochastic integrals (cf. [7, Thm. 2.12 on p. 142]) implies

$$\int_{[0,t]} D_{x_0} f_n(M_s, V_s) \, dM_s \; \overset{P}{\to} \; \int_{[0,t]} D_{x_0} f(M_s, V_s) \, dM_s \quad (n \to \infty)$$

as well as

$$\lim_{n \to \infty} \int_{[0,t]} D_{x_i} f_n(M_s, V_s) \, dV_s^i \; = \; \int_{[0,t]} D_{x_0} f(M_s, V_s) \, dV_s^i \quad \text{a.s.}$$

where $\overset{P}{\to}$ denotes convergence in probability.

So we must finally examine the behavior of $\int_{[0,t]} D_{x_0}^2 f_n(M_s, V_s) \, d\langle M \rangle_s$ as $n \to \infty$. Since M is bounded by K we clearly have $L_t^u = 0$ for $|u| > K$ and all $t \geq 0$. Moreover, $(L_t^u)_{u \in I\!R}$ as a cádlág process a.s. has at most countably many discontinuities, and it is locally bounded. Put $\mu_f(du, x_1, ..., x_d) \overset{\text{def}}{=} D_{x_0}^2 f(u, x_1, ..., x_d) \lambda\!\!\lambda(du)$ which gives a locally finite signed measure. Now use Lemma 2.4 (for the third and fifth equality) to see that

$$\int_{[0,t]} D_{x_0}^2 f_n(M_s, V_s) \, d\langle M \rangle_s \; = \; \int_{[0,t]} \int_{-\infty}^{\infty} \varphi_n(M_s - u) D_{x_0}^2 f(u, V_s) \, du \, d\langle M \rangle_s$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} D^2_{x_0} f(u,V_s) \int_{[0,t]} \varphi_n(M_s-u)\; d\langle M\rangle_s \; du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_n(z-u)\, L^z_t \; dz\; \mu_f(du,V_s) \\
&= \int_{-\infty}^{\infty} L^u_t \; \mu_f(du,V_s) \\
&+ \int_{-K-1}^{K+1} \int_{-\infty}^{\infty} \varphi_n(z-u)(L^z_t - L^u_t)\; dz\; \mu_f(du,V_s) \quad \text{a.s.} \\
&= \int_{[0,t]} D^2_{x_0} f(M_s,V_s)\; d\langle M\rangle_s \\
&+ \int_{-K-1}^{K+1} \int_{-\infty}^{\infty} \varphi_n(z-u)(L^z_t - L^u_t)\; dz\; \mu_f(du,V_s) \quad \text{a.s.}
\end{aligned}
$$

for all $n \geq 1$. Since $(L^u_t)_{u\in I\!R}$ is $\lambda\!\!\lambda$-a.e. continuous and locally bounded we get that (observe $\varphi_n(z) = \varphi_n(-z)$)

$$
\int_{-\infty}^{\infty} \varphi_n(z-u)(L^z_t - L^u_t)\; dz \;=\; \int_{-1/n}^{1/n} \varphi_n(z)(L^{u-z}_t - L^u_t)\; dz
$$

a.s. converges to 0 for $\lambda\!\!\lambda$-almost all $u \in [-K,K]$ and is bounded (as a function of u). So we finally conclude

$$
\lim_{n\to\infty} \int_{-K}^{K} \int_{-\infty}^{\infty} \varphi_n(z-u)(L^z_t - L^u_t)\; dz\; \mu_f(du,V_s) \;=\; 0 \quad \text{a.s.}
$$

by recalling the local finiteness of $\mu_f(\cdot,V_s)$ and by an appeal to the dominated convergence theorem (when applied pathwise). This completes the proof of Theorem 2.1 for bounded processes M and V.

To go from the bounded case to the general one we use the standard technique of localization. If $M = N + A$, this means to choose a localizing sequence $(\tau_n)_{n\geq 1}$ of stopping times such that $\tau_n \to \infty$ a.s. and the stopped processes $N^n \stackrel{\text{def}}{=} (M_{t\wedge\tau_n})$, $A^n \stackrel{\text{def}}{=} (M_{t\wedge\tau_n})$ and $V^n \stackrel{\text{def}}{=} (V_{t\wedge\tau_n})_{t\geq 0}$ are bounded. Note that $\langle M^n\rangle_t = \langle M\rangle_{t\wedge\tau_n}$ for all $t \geq 0$. By the first part of the proof we hence infer validity of (2.1) with (M^n,V^n) instead of (M,V), and the proof is completed by showing that all terms in that formula converge a.s. or in probability to their respective counterparts with (M,V). The continuity of f ensures

$$
\lim_{n\to\infty} f(M^n_t, V^n_t) \;=\; f(M_t,V_t) \quad \text{a.s.}
$$

for all $t \geq 0$. Furthermore it is easily seen that

$$
\int_{[0,t\wedge\tau_n]} D_{x_i} f(M^n_s,V^n_s)\; dV^{i,n}_s \;=\; \int_{[0,t\wedge\tau_n]} D_{x_i} f(M_s,V_s)\; dV^i_s
$$

for all $n \geq 1$ and $i = 1, ..., d$ which in combination with $\tau_n \to \infty$ a.s. implies

$$\int_{[0,t\wedge\tau_n]} D_{x_i} f(M_s^n, V_s^n)\, dV_s^{i,n} \overset{P}{\to} \int_{[0,t]} D_{x_i} f(M_s, V_s)\, dV_s^i.$$

By a similar argument we obtain

$$\int_{[0,t\wedge\tau_n]} D_{x_0} f(M_s^n, V_s^n)\, dM_s^n \overset{P}{\to} \int_{[0,t\wedge\tau_n]} D_{x_0} f(M_s, V_s)\, dM_s$$

for $i = 1, ..., d$, and

$$\int_{[0,t\wedge\tau_n]} D_{x_0}^2 f(M_s^n, V_s^n)\, d\langle M^n\rangle_s \overset{P}{\to} \int_{[0,t]} D_{x_0}^2 f(M_s, V_s)\, d\langle M\rangle_s.$$

Further details can be omitted. The proof of the theorem is herewith complete. ◇

3. An application: The perpetual American put option

In the famous Black-Scholes model an investor may either invest in a riskless bond with deterministic reward function

$$R_t = e^{rt} R_0, \quad R_0 > 0,$$

or in a stock whose price fluctuations follow a geometric Brownian motion

$$X_t = X_0 \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right), \quad t \geq 0,$$

with drift parameter $\mu \in I\!R$ and volatility $\sigma > 0$, where as before $(B_t)_{t\geq 0}$ denotes standard Brownian motion starting at 0. The perpetual American put option is a contract signed at time 0 which entitles the buyer to sell one unit of stock at any time $t \geq 0$ at a price $K > 0$. So his discounted reward is $e^{-rt}(K - X_t)^+$ if exercising the option at epoch t. To determine the optimal such striking epoch we must solve the optimal stopping problem

$$\mathbb{V}(x) = \sup_\tau \mathbb{E}_x e^{-r\tau}(K - X_\tau)^+, \quad x > 0,$$

where the supremum is taken over all stopping times τ with respect to $(X_t)_{t\geq 0}$ and $\mathbb{P}_x \overset{\text{def}}{=} \mathbb{P}(\cdot|X_0 = x)$ with associated expectation $\mathbb{E}_x$.

We note that $(e^{-rt}X_t)_{t\geq 0}$ is a submartingale for $r < \mu$, a supermartingale for $r > \mu$, and a martingale for $r = \mu$. Here we confine ourselves to the martingale case which is also the relevant situation under the no-arbitrage principle (cf. [10, Section VIII.2]).

It appears natural to look for the optimal stopping time within the class of threshold rules

$$\tau_a \overset{\text{def}}{=} \inf\{t \geq 0 : X_t \leq a\}, \quad 0 < a \leq K,$$

which however may be infinite with positive probability under $\mathbb{P}_x$. On the other hand, we have $\lim_{t\to\infty} e^{-rt}X_t = \lim_{t\to\infty} e^{-rt}(K - X_t)^+ = 0$ $\mathbb{P}_x$-a.s. for each $x > 0$ so that we can put

$e^{-r\tau_a}(K - X_{\tau_a})^+ \stackrel{\text{def}}{=} 0$ on $\{\tau_a = \infty\}$ in the definition of $\mathbb{V}_a(x)$ below. Following the procedure described in our introducing example the first step is then to compute

$$\mathbb{V}_a(x) \stackrel{\text{def}}{=} \mathbb{E}_x e^{-r\tau_a}(K - X_{\tau_a})^+, \quad 0 < a \le K.$$

Notice that in terms of the basic Brownian motion $(B_t)_{t\ge 0}$ the stopping time τ_a denotes the first epoch t where B_t is on or below the line $\sigma^{-1}\big(\log(a/X_0) - (r - \sigma^2/2)t\big)$. The Lebesgue density of $\mathbb{P}_x(\tau_a \in \cdot, \tau_a < \infty)$ is explicitly known which, after some calculations, leads to

$$\mathbb{V}_a(x) = \begin{cases} (K-a)\left(\dfrac{x}{a}\right)^{-2r/\sigma^2}, & \text{if } x \ge a \\ K - x, & \text{if } x \le a \end{cases}$$

for each $a > 0$. For the optimal threshold a^* the smooth fit principle requires that $\mathbb{V}_{a^*}$ be differentiable at a^*, so

$$\frac{d}{dx}\left[(K-a)\left(\frac{x}{a}\right)^{-2r/\sigma^2}\right]_{x=a^*} = -1.$$

We obtain

$$a^* = \frac{2rK}{\sigma^2 + 2r}$$

and must finally verify for $\mathbb{V}^* \stackrel{\text{def}}{=} \mathbb{V}_{a^*}$ that $(e^{-rt}\mathbb{V}^*(X_t))_{t\ge 0}$ is a supermartingale because then $\mathbb{V}^*(x) \ge (K-x)^+$ implies $\mathbb{V}^* \ge \mathbb{V}$ via

$$\mathbb{E}_x e^{-r\tau}(K - X_\tau)^+ \le \mathbb{E}_x e^{-r\tau}\mathbb{V}^*(X_\tau) \le \mathbb{E}_x \mathbb{V}^*(X_0) = \mathbb{V}^*(x)$$

for all $x > 0$ and all stopping times τ, and thus $\mathbb{V}^* = \mathbb{V}$.

The function $f(x,t) \stackrel{\text{def}}{=} e^{-rt}\mathbb{V}^*(x)$ obviously satisfies the conditions of Theorem 2.1. Hence we obtain by an application of Corollary 2.2 (with $\mu = r$) that

$$\begin{aligned} e^{-rt}\mathbb{V}^*(X_t) - \mathbb{V}^*(x) &= \int_{[0,t]} \sigma e^{-rs} X_s \mathbb{V}^{*\prime}(X_s)\, dB_s \\ &+ \int_{[0,t]} e^{-rs}\Big(- r\mathbb{V}^*(X_s) + rX_s\mathbb{V}^{*\prime}(X_s) + \frac{1}{2}\sigma^2 X_s^2 \mathbb{V}^{*\prime\prime}(X_s)\Big)\, ds. \end{aligned}$$

The first integral on the right hand side constitutes a martingale while the second one is a decreasing process because some simple calculations reveal

$$e^{-rs}\Big(- r\mathbb{V}^*(X_s) + rX_s\mathbb{V}^{*\prime}(X_s) + \frac{1}{2}\sigma^2 X_s^2 \mathbb{V}^{*\prime\prime}(X_s)\Big) \le 0 \quad \text{a.s.}$$

for all $s \ge 0$. Hence $(e^{-rt}\mathbb{V}^*(X_t))_{t\ge 0}$ is indeed a supermartingale as required.

For other optimal stopping problems based on Brownian which can be solved explicitly along the lines of the previous example the reader is referred to [9],[8],[4], and [5].

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