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Necessary and sufficient conditions for a normalized weighted branching process in random environment to have a nondegenerate limit

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Abstract

In this paper we investigate the nonnegative martingale $W_n = Z_n/\mu_n(\mathbf{U})$, $n \geq 0$ and its a.s. limit W, when $(Z_n)_{n\geq 0}$ is a weighted branching process in random environment with stationary ergodic environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$ and $\mu_n(\mathbf{U})$ denotes the conditional expectation of Z_n given $\mathbf{U}$ for $n \geq 0$.
We find necessary and sufficient conditions for W to be nondegenerate, generalizing results by D. Tanny (see [19]) and R. Lyons, R. Pemantle and Y. Peres (see [10]) on ordinary branching processes in random environment and results by U. Rösler, V.A. Topchii, and V.A. Vatutin (see [14], [16]) on ordinary weighted branching processes.
In the important special case of i.i.d. random environment, a $Z \log Z$-condition turns out to be crucial.
Deterministic and nonvarying environments are treated as special cases.
Our arguments adapt the probabilistic proof of Biggins' theorem for branching random walks given by R. Lyons in [9] to our situation.

1 Introduction

We consider a weighted branching process $(Z_n)_{n\geq 0}$ in random environment with stationary ergodic environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$. One way to construct such a process is the following: Put $\Sigma : [0,\infty)^{\mathbb{N}} \to [0,\infty]$,

$$\Sigma((x_i)_{i\geq 1}) := \sum_{i\geq 1} x_i$$

and let $\mathbf{U} = (U_n)_{n\geq 0}$ be a stationary ergodic sequence of $(\mathbb{M}, \mathfrak{M})$-valued random variables (defined on some suitable probability space $(\Omega, \mathfrak{A}, P)$), where $\mathbb{M}$ denotes the collection of probability measures Q on $([0,\infty)^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}_{[0,\infty)})$ fulfilling

$$0 < \mu(Q) := \int_{[0,\infty)^{\mathbb{N}}} \Sigma \, dQ = \int_{[0,\infty]} y \; Q^{\Sigma}(dy) < \infty$$

and $\mathfrak{M}$ is the σ-algebra generated by the total variation norm.
Additionally, let $\mathcal{N} := \{\emptyset\} \cup \bigcup_{n\geq 1} \mathbb{N}^n$ and

$$T(v) = (T_i(v))_{i\geq 1}, \; v \in \mathcal{N},$$

be a family of $[0,\infty)^{\mathbb{N}}$-valued random variables that are conditionally independent given $\mathbf{U}$ with conditional distribution given by

$$P(T(v) \in \cdot | \mathbf{U}) = U_{|v|} \qquad (v \in \mathcal{N}),$$

$|v|$ denoting the length of v, in particular $|\emptyset| = 0$. Note that for fixed v, we allow arbitrary dependencies of the random variables $T_i(v), i \geq 1$.

Recursively, define the random weights $L(v)$, $v \in \mathcal{N}$, by $L(\emptyset) := 1$ and

$$L(v,i) := L(v) \cdot T_i(v) \qquad (v \in \mathcal{N}, i \geq 1).$$

Furthermore, let

$$Z_n := \sum_{|v|=n} L(v) \qquad (n \geq 0)$$

be the sum of weights of all individuals in generation n. Loosely speaking, $(Z_n)_{n\geq 0}$ forms a branching process in random environment with particles having random weights.

For simplicity, we abbreviate $T := (T_i)_{i\geq 1} := T(\emptyset)$ and for $\mathbf{Q} = (Q_n)_{n\geq 0} \in \mathbb{M}^{\mathbb{N}_0}$

$$\mu_0(\mathbf{Q}) := 1$$

and

$$\mu_n(\mathbf{Q}) := \prod_{i=0}^{n-1} \mu(Q_i) \in (0,\infty) \qquad (n \geq 1).$$

In addition, we assume that

$$\beta := E\left[\log \mu(U_0)\right] \in (-\infty,\infty)$$

and that

$$\gamma := E\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log \frac{T_i}{\mu(U_0)}\right] \text{ exists in } \overline{\mathbb{R}}.$$

In case $\gamma_+ := E\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{T_i}{\mu(U_0)}\right]$ or $\gamma_- := E\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^- \frac{T_i}{\mu(U_0)}\right]$ is finite, we have the decomposition $\gamma = \gamma_+ - \gamma_-$. Finally, note that the sequence $(\mu(U_n))_{n\geq 0}$ is stationary ergodic as well.

Our main object of interest is the sequence $(W_n)_{n\geq 0}$, defined by

$$W_n := Z_n / \mu_n(\mathbf{U}) \qquad (n \geq 0).$$

We will see in Lemma 9 that $(W_n)_{n\geq 0}$ forms a nonnegative martingale with respect to an appropriate filtration $(\mathcal{G}_n)_{n\geq 0}$ of $(\Omega, \mathfrak{A}, P)$ and therefore converges a.s. to a nonnegative random variable W. Our aim is to find necessary and sufficient conditions for W to be nondegenerate.

The following section contains our main results. In analogy to the investigations by R. Lyons in [9] for branching random walks and by R. Lyons, R. Pemantle and Y. Peres in [10] for ordinary Galton-Watson processes, their proofs to be given in Section 6 are based on viewing a weighted branching process as the sequence of generation sizes of an appropriate (weighted) random family tree. Therefore we examine the latter object in Section 3. In Section 4 we construct a size-biased version of the random tree mentioned above and find an important connection between the distributions of these two types of trees. Section 5 provides some lemmata required in the proofs of our main results that are given in Section 6. The final section contains some additional remarks.

2 Main results

Recall the definitions $\gamma_+ = E\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^+ \frac{T_k}{\mu(U_0)}\right]$, $\gamma_- = E\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^- \frac{T_k}{\mu(U_0)}\right]$ and the assumption that $\gamma = E\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log \frac{T_k}{\mu(U_0)}\right]$ exists in $\overline{\mathbb{R}}$.

Theorem 1. *Suppose that $\gamma_+ < \infty$, $-\infty \leq \gamma < 0$ and there is some $a > 1$ such that*

$$\begin{aligned} \mathbb{G}(\mathbf{U}, a) &:= \sum_{n\geq 0} \mu(U_n)^{-1} \int_{\{\Sigma > a^n\}} \Sigma \, dU_n \\ &= \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x \; U_n^{\Sigma}(dx) < \infty \quad \textit{with positive probability.} \end{aligned}$$

Then $W = \lim_{n\to\infty} W_n$ satisfies $EW = 1$.

Theorem 2. *Suppose that $\gamma_- < \infty$. If $0 < \gamma \leq \infty$, or $\mathbb{G}(\mathbf{U}, a)$ is infinite with positive probability for some $a > 1$, then $P(W = 0) = 1$.*

The proof of the following corollary shows that the series $\mathbb{G}(\mathbf{U}, a)$ introduced in Theorem 1 converges a.s. for all $a > 1$ if $Z_1 \log^+ Z_1/\mu(U_0)$ is integrable. However, we will see in Remark 14 that in general, the integrability of this random variable is not a necessary condition for W to be nondegenerate.

Corollary 3. *If $-\infty \leq \gamma < 0$ and $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$, then $EW = 1$.*

Corollary 4. *Suppose that there is some $\ell \geq 1$ such that*

$$P(T_i = 0 \textit{ for all } i > \ell) = 1. \tag{1}$$

Then $-\infty \leq \gamma < 0$ implies $EW = 1$.

The following theorem deals with the important special case where the environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$ consists of independent components. As in the case of ordinary Galton-Watson processes or branching processes in i.i.d. random environment, a $Z \log Z$-condition plays an essential role (see [10], [19]).

Theorem 5. *Suppose that the random variables $U_n, n \geq 0$ are i.i.d.*

(a) If $\gamma_- < \infty$ and $c := P\left(\bigcap_{i\geq 1}\{T_i = 0 \text{ or } T_i = \mu(U_0)\}\right) < 1$, the following conditions are equivalent:

  (i) $P(W = 0) < 1$,

  (ii) $EW = 1$,

  (iii) $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$ and $\gamma < 0$.

(b) If $c = 1$ and $P(Z_1 = \mu(U_0)) = 1$, then $W = EW = 1$ a.s.

(c) If $c = 1$ and $P(Z_1 = \mu(U_0)|\mathbf{U}) < 1$ a.s., then $W = 0$ a.s.

Corollary 6. *If $U_n, n \geq 0$ are i.i.d., $c < 1$ and (1) holds, then*

$$P(W = 0) < 1 \Longleftrightarrow EW = 1 \Longleftrightarrow \gamma < 0.$$

We now consider ordinary weighted branching processes (see for example [13], [14], [15], [16], [17]). In this special case, the environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$ is deterministic and nonvarying, i.e. $P(U_0 \in \cdot) = P(U_1 \in \cdot) = ... = \delta_\Gamma$ for some $\Gamma \in \mathbb{M}$ and

$$\mu(U_i) = \int_{[0,\infty]} y \; \Gamma^\Sigma(dy) =: \mu \in (0, \infty) \quad \text{a.s. for all } i \geq 0.$$

Consequently, $E(Z_n|\mathbf{U}) = EZ_n = \mu^n \in (0, \infty)$ a.s., $\gamma_\pm = E\left[\sum_{i\geq 1} \frac{T_i}{\mu} \log^\pm \frac{T_i}{\mu}\right]$ and $\gamma = E\left[\sum_{i\geq 1} \frac{T_i}{\mu} \log \frac{T_i}{\mu}\right] \in \overline{\mathbb{R}}$.

Theorem 7. *Suppose that $(Z_n)_{n\geq 0}$ is weighted branching process with $\mu = EZ_1 \in (0, \infty)$.*

(a) If $\gamma_- < \infty$ and $c = P(T \in \{0, \mu\}^{\mathbb{N}}) < 1$, then the following conditions are equivalent:

  (i) $P(W = 0) < 1$,

  (ii) $EW = 1$,

  (iii) $EZ_1 \log^+ Z_1 < \infty$ and $\gamma < 0$.

(b) If $c = 1$ and $P(Z_1 = \mu) = 1$, then $W = EW = 1$ a.s.

(c) If $c = 1$ and $P(Z_1 = \mu) < 1$, then $W = 0$ a.s.

(d) If $-\infty \leq \gamma < 0$ and $EZ_1 \log^+ Z_1 < \infty$, then $EW = 1$.

3 Weighted trees

We have already mentioned that our method of proof is based on analysing the weighted family tree associated with $(Z_n)_{n\geq 0}$. For this reason, we formally introduce the space of weighted trees in this section and endow it with an appropriate σ-field, following Chauvin's and Neveu's approaches (see [7], [11]).

Denote by $\mathbb{T}$ the set of all nonnegative mappings defined on $\mathcal{N}$, i.e.

$$\mathbb{T} := \{t \mid t : \mathcal{N} \to [0,\infty)\}.$$

Any $t \in \mathbb{T}$ is called a *weighted tree*. Additionally, given any $v \in \mathcal{N}$, define

$$\ell_v : \mathbb{T} \to [0,\infty),$$

$$\ell_v(t) := t(v).$$

The filtration $(\mathcal{F}_n)_{n\geq 0}$, given by

$$\mathcal{F}_n := \sigma(\ell_v, |v| \leq n) \qquad (n \geq 0),$$

will play an important role in our analysis. Finally, endow $\mathbb{T}$ with the σ-field

$$\mathcal{F} := \sigma(\cup_{n\geq 0}\mathcal{F}_n) = \sigma(\ell_v, v \in \mathcal{N}).$$

To get the connection to the weighted branching process $(Z_n)_{n\geq 0}$ introduced in Section 1, denote by $\mathcal{T}$ the (uniquely determined) random weighted tree fulfilling

$$\mathcal{T}(v) = \ell_v \circ \mathcal{T} = L(v) \qquad (v \in \mathcal{N}).$$

In fact, $\mathcal{T}$ is $\mathcal{F}$-measurable since for any $A \in \mathbb{B}^{\mathcal{N}}$,

$$\mathcal{T}^{-1}(\{(\ell_v)_{v\in\mathcal{N}} \in A\}) = \{(L(v))_{v\in\mathcal{N}} \in A\} \in \mathfrak{A}. \tag{2}$$

4 Size-biased weighted trees

The proofs of our main results exploit a fundamental relation between the random weighted tree $\mathcal{T}$ and the so-called *size-biased tree* $\widehat{\mathcal{T}}$ to be defined hereafter. Lemma 8 compares the distributions of $\mathcal{T}$ and $\widehat{\mathcal{T}}$.

Let $((\widehat{T}(n), C_n))_{n\geq 0} = (((\widehat{T}_i(n))_{i\geq 1}, C_n))_{n\geq 0}$ be a sequence of random variables (defined on $(\Omega, \mathfrak{A}, P)$) meeting the following conditions:

- Conditionally upon $\mathbf{U}$, $(T(v))_{v\in\mathcal{N}}$ and $((\widehat{T}(n), C_n))_{n\geq 0}$ are independent.

- Conditionally upon $\mathbf{U}$, the random variables $(\widehat{T}(n), C_n), n \geq 0$ are independent with conditional distributions determined by

$$\begin{aligned} P(\widehat{T}(n) \in A, C_n = i | \mathbf{U}) &= \mu(U_n)^{-1} E\left(\mathbb{1}_{\{T(v_n) \in A\}} T_i(v_n) | \mathbf{U}\right) \\ &= \mu(U_n)^{-1} \int_A \pi_i \, dU_n \quad \text{a.s.} \qquad (A \in \mathbb{B}^{\mathbb{N}}, i \geq 1), \end{aligned} \tag{3}$$

where v_n is an arbitrary element of $\mathbb{N}^n$ and $\pi_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ denotes the projection to the i-th coordinate. Note that this implies

$$\begin{aligned} P(\widehat{T}(n) \in A | \mathbf{U}) &= \mu(U_n)^{-1} E\left(\mathbb{1}_{\{T(v_n) \in A\}} \cdot \Sigma \circ T(v_n) | \mathbf{U}\right) \\ &= \mu(U_n)^{-1} \int_A \Sigma \, dU_n \quad \text{a.s.} \qquad (A \in \mathbb{B}^{\mathbb{N}}, |v_n| = n) \end{aligned} \tag{4}$$

and

$$P\left(0 < \sum_{i \geq 1} \widehat{T}_i(n) < \infty\right) = 1 \tag{5}$$

because

$$P(\widehat{T}(n) \in \Sigma^{-1}((0,\infty)) | \mathbf{U}) = \mu(U_n)^{-1} \int_{\Sigma^{-1}((0,\infty))} \Sigma \, dU_n = 1 \quad \text{a.s.}$$

Now define the random variables $V_0 := \emptyset$, $\widehat{L}(\emptyset) = 1$,

$$V_n := (C_0, ..., C_{n-1}) \in \mathbb{N}^n \qquad (n \geq 1)$$

and

$$\widehat{L}(w, i) := \begin{cases} \widehat{L}(V_n) \cdot \widehat{T}_i(n), & w = V_n \text{ for some } n \geq 0, \\ \widehat{L}(w) \cdot T_i(w), & \text{otherwise} \end{cases}$$

for all $w \in \mathcal{N}$ and $i \geq 1$. After these preliminaries, let $\widehat{\mathcal{T}}$ be the random tree with

$$\widehat{\mathcal{T}}(v) = \ell_v \circ \widehat{\mathcal{T}} = \widehat{L}(v) \qquad (v \in \mathcal{N}).$$

By the same argument as in (2) it can be seen that $\widehat{\mathcal{T}}$ is $\mathcal{F}$-measurable.

In the remainder of this paper we shall make frequent use of the relation between

$$\widehat{\mathcal{Q}} := P((\widehat{\mathcal{T}}, \mathbf{U}) \in \cdot)$$

and

$$\mathcal{Q} := P((\mathcal{T}, \mathbf{U}) \in \cdot)$$

described in Lemma 8 below. For this purpose, put

$$\begin{aligned} z_n(t) &:= \sum_{|v|=n} \ell_v(t) \qquad (t \in \mathbb{T}, n \geq 0), \\ w_n(t,u) &:= z_n(t)/\mu_n(u) \qquad (t \in \mathbb{T}, u \in \mathbb{M}^{\mathbb{N}_0}, n \geq 0), \\ w &:= \limsup_{n\to\infty} w_n, \\ \widehat{W}_n &:= w_n \circ (\widehat{\mathcal{T}}, \mathbf{U}), \\ \mathcal{F}_n' &:= \ \mathcal{F}_n \otimes \mathfrak{M}^{\mathbb{N}_0} \end{aligned}$$

and note that w_n is $\mathcal{F}_n'$-measurable ($n \geq 0$). In addition, we have the representations

$$W = w \circ (\mathcal{T}, \mathbf{U}) \text{ and } W_n = w_n \circ (\mathcal{T}, \mathbf{U}).$$

Lemma 8. *Let $n \geq 0$.*

(a) For any $A \in \mathcal{F}_n$, we have the identity

$$P(\widehat{\mathcal{T}} \in A|\mathbf{U}) = E(W_n \cdot \mathbb{1}_{\{\mathcal{T}\in A\}}|\mathbf{U}) \quad a.s.$$

(b) For any $B \in \mathcal{F}_n'$, we have

$$\widehat{\mathcal{Q}}(B) = E[W_n \cdot \mathbb{1}_{\{(\mathcal{T},\mathbf{U})\in B\}}] = \int_B w_n(t,u)\, \mathcal{Q}(dt,du),$$

i.e.

$$\widehat{\mathcal{Q}}_{|\mathcal{F}_n'} \ll \mathcal{Q}_{|\mathcal{F}_n'}$$

with

$$\frac{d\widehat{\mathcal{Q}}_{|\mathcal{F}_n'}}{d\mathcal{Q}_{|\mathcal{F}_n'}}(t,u) = w_n(t,u).$$

(c) We have the dichotomy

(i) $\widehat{\mathcal{Q}}(w < \infty) = 1 \Longleftrightarrow EW = 1$ *and*

(ii) $\widehat{\mathcal{Q}}(w = \infty) = 1 \Longleftrightarrow \mathcal{Q}(w = 0) = 1.$

Proof. (a) Obviously, it suffices to show that for all $n \geq 0$ and $A_v \in \mathbb{B}$ ($|v| \leq n$),

$$P(\widehat{L}(v) \in A_v,\ |v| \leq n|\mathbf{U}) = E\left(\mathbb{1}_{\{L(v)\in A_v,\,|v|\leq n\}} W_n|\mathbf{U}\right) \quad \text{a.s.}$$

For this purpose, we introduce the auxiliary random variables $\widetilde{T}(v)$, $v \in \mathcal{N}$, defined by

$$\widetilde{T}(v) := \begin{cases} \widehat{T}(n), & v = V_n \text{ for some } n \geq 0, \\ T(v), & \text{otherwise.} \end{cases}$$

Since $\widehat{L}(\emptyset) = L(\emptyset) = 1$, we may suppose $n \geq 1$. Considering the representations

$$(\widehat{L}(v))_{|v| \leq n} = \Psi \circ (\widetilde{T}(v))_{|v| \leq n-1}$$

and

$$(L(v))_{|v| \leq n} = \Psi \circ (T(v))_{|v| \leq n-1}$$

for some measurable mapping $\Psi : \times_{v \in \mathcal{N}, |v| \leq n-1} \mathbb{R}^{\mathbb{N}} \to \times_{v \in \mathcal{N}, |v| \leq n} \mathbb{R}$, it is enough to prove that for all $B_v \in \mathbb{B}^{\mathbb{N}}$ $(|v| \leq n-1)$,

$$P(\widetilde{T}(v) \in B_v,\, |v| \leq n-1 | \mathbf{U}) = E\left(\mathbb{1}_{\{T(v) \in B_v,\, |v| \leq n-1\}} W_n | \mathbf{U}\right) \quad \text{a.s.} \tag{6}$$

Choose $\sigma \in \mathbb{N}^n$. We claim (and prove by induction) that

$$P(\widetilde{T}(v) \in B_v,\, |v| \leq n-1,\, V_n = \sigma | \mathbf{U}) = \mu_n(\mathbf{U})^{-1} E\left(\mathbb{1}_{\{T(v) \in B_v,\, |v| \leq n-1\}} L(\sigma) | \mathbf{U}\right) \quad \text{a.s.} \tag{7}$$

Once this identity is verified, summation over all $\sigma \in \mathbb{N}^n$ yields (6).

First note that the case $n = 1$ in (7) is obviously true because

$$\begin{aligned} P(\widetilde{T}(\emptyset) \in B,\, V_1 = \sigma | \mathbf{U}) &= P(\widehat{T}(0) \in B, C_0 = \sigma | \mathbf{U}) \\ &= \mu(U_0)^{-1} E\left[T_\sigma \cdot \mathbb{1}_{\{T \in B\}} | \mathbf{U}\right] \end{aligned}$$

for any $B \in \mathbb{B}$ and $\sigma \geq 1$. Now fix n and suppose that (7) is proved for all $B_v \in \mathbb{B}^{\mathbb{N}}$ $(|v| \leq n-1)$ and all $\sigma \in \mathbb{N}^n$. Let $\tau = (i_0, ..., i_n) \in \mathbb{N}^{n+1}$ and $\sigma := (i_0, ..., i_{n-1})$. Then it follows by hypothesis and construction that

$$\begin{aligned} & P(\widetilde{T}(v) \in B_v,\, |v| \leq n,\, V_{n+1} = \tau | \mathbf{U}) \\ = \; & P(\widetilde{T}(v) \in B_v,\, |v| \leq n-1,\, V_n = \sigma | \mathbf{U}) \cdot P(T(v) \in B_v, |v| = n, v \neq \sigma | \mathbf{U}) \\ & \quad \cdot P(\widehat{T}(n) \in B_\sigma, C_n = i_n | \mathbf{U}) \\ = \; & \mu_n(\mathbf{U})^{-1} E\left[L(\sigma) \cdot \mathbb{1}_{\{T(v) \in B_v,\, |v| \leq n-1\}} | \mathbf{U}\right] \cdot E\left[\mathbb{1}_{\{T(v) \in B_v,\, |v| = n, v \neq \sigma\}} | \mathbf{U}\right] \\ & \quad \cdot \mu(U_n)^{-1} E\left[T_{i_n}(\sigma) \mathbb{1}_{\{T(\sigma) \in B_\sigma\}} | \mathbf{U}\right] \\ = \; & \mu_{n+1}(\mathbf{U})^{-1} E\left[\mathbb{1}_{\{T(v) \in B_v,\, |v| \leq n\}} \cdot L(\sigma) T_{i_n}(\sigma) | \mathbf{U}\right] \\ = \; & \mu_{n+1}(\mathbf{U})^{-1} E\left[\mathbb{1}_{\{T(v) \in B_v,\, |v| \leq n\}} \cdot L(\tau) | \mathbf{U}\right] \quad \text{a.s.} \end{aligned}$$

which finishes the proof of (a).

(b) Without loss of generality, we may suppose that B is of the form $B = C \times D$ for some $C \in \mathcal{F}_n$ and $D \in \mathfrak{M}^{\mathbb{N}_0}$. Now (a) gives

$$\begin{aligned}\widehat{\mathcal{Q}}(B) &= \int_{\{\mathbf{U}\in D\}} P(\widehat{\mathcal{T}} \in C|\mathbf{U})\, dP \\ &= \int_{\{\mathbf{U}\in D\}} E(W_n \cdot \mathbb{1}_{\{\mathcal{T}\in C\}}|\mathbf{U})\, dP \\ &= E[W_n \cdot \mathbb{1}_{\{(\mathcal{T},\mathbf{U})\in C\times D\}}] \\ &= \int_B w_n(t,u)\, \mathcal{Q}(dt,du)\end{aligned}$$

(c) Since $(\mathcal{F}'_n)_{n\geq 0}$ is a filtration of $\mathbb{T} \times \mathbb{M}^{\mathbb{N}_0}$ satisfying $\mathcal{F} \otimes \mathfrak{M}^{\mathbb{N}_0} = \cup_{n\geq 0}\mathcal{F}'_n$, part (b) and Theorem (4.3.4) in [8] imply that for all $C \in \mathcal{F} \otimes \mathfrak{M}^{\mathbb{N}_0}$,

$$\widehat{\mathcal{Q}}(C) = \int_C w\, d\mathcal{Q} + \widehat{\mathcal{Q}}(C \cap \{w = \infty\}),$$

in particular

$$EW = \int_{\mathbb{T}\times\mathbb{M}^{\mathbb{N}_0}} w\, d\mathcal{Q} = 1 - \widehat{\mathcal{Q}}(w = \infty),$$

ensuring (i) and (ii). □

5 Auxiliary lemmata

Introducing the filtration $(\mathcal{G}_n)_{n\geq 0}$, given by $\mathcal{G}_0 := \sigma(\mathbf{U})$ and $\mathcal{G}_n := \sigma(\mathbf{U}, T(v), |v| \leq n-1)$ for $n \geq 1$, we can easiliy establish the a.s. convergence of $(W_n)_{n\geq 0}$.

Lemma 9. *(a) Given any $w \in \mathcal{N}$, $E\left(\sum_{i\geq 1} T_i(w)|\mathbf{U}\right) = \mu(U_{|w|})$ a.s.*

(b) $E(Z_n|\mathbf{U}) = \mu_n(\mathbf{U})$ a.s. for all $n \geq 0$.

(c) The sequence $(W_n)_{n\geq 0}$ forms a nonnegative martingale with respect to $(\mathcal{G}_n)_{n\geq 0}$ with $EW_0 = 1$ and therefore converges a.s. to a nonnegative random variable W satisfying $EW \leq 1$.

Proof. (a) By construction, we have that

$$\begin{aligned}E\left(\sum_{i\geq 1} T_i(w)|\mathbf{U}\right) &= E(\Sigma \circ T(w)|\mathbf{U}) \\ &= \int_{[0,\infty)^{\mathbb{N}}} \Sigma\, dP^{T(w)|\mathbf{U}} \\ &= \int_{[0,\infty)^{\mathbb{N}}} \Sigma\, dU_{|w|} = \mu(U_{|w|}) \quad \text{a.s.}\end{aligned}$$

(b) We prove the claim by induction. As $Z_0 = \mu_0(\mathbf{U}) = 1$ a.s., we may suppose that $E(Z_n|\mathbf{U}) = \mu_n(\mathbf{U})$ a.s. for some $n \geq 0$. Then using the decomposoition

$$Z_{n+1} = \sum_{|v|=n+1} L(v) = \sum_{|w|=n} L(w) \sum_{i\geq 1} T_i(w) = \sum_{|w|=n} L(w) \cdot \Sigma \circ T(w),$$

the conditional independence of $L(w)$ and $\Sigma \circ T(w)$, the monotone convergence theorem and part (a), we infer that

$$\begin{aligned} E(Z_{n+1}|\mathbf{U}) &= \sum_{|w|=n} E(L(w) \cdot \Sigma \circ T(w)|\mathbf{U}) \\ &= \sum_{|w|=n} E(L(w)|\mathbf{U}) \cdot E(\Sigma \circ T(w)|\mathbf{U}) \\ &\overset{(a)}{=} \mu(U_n) \cdot \sum_{|w|=n} E(L(w)|\mathbf{U}) \\ &= \mu(U_n) \cdot E(Z_n|\mathbf{U}) \\ &= \mu(U_n) \cdot \mu_n(\mathbf{U}) = \mu_{n+1}(\mathbf{U}) \quad \text{a.s.}, \end{aligned}$$

as demanded.

(c) Fix $n \geq 0$. Obviously, W_n is measurable with respect to $\mathcal{G}_n$. Since $L(w)$ is $\mathcal{G}_n$-measurable if $|w| = n$, it follows that

$$\begin{aligned} E(W_{n+1}|\mathcal{G}_n) &= \mu_{n+1}(\mathbf{U})^{-1} E\left(\sum_{|v|=n+1} L(v)\middle|\mathcal{G}_n\right) \\ &= \mu_{n+1}(\mathbf{U})^{-1} \sum_{|w|=n} E\left(L(w) \cdot \Sigma \circ T(w)|\mathcal{G}_n\right) \\ &= \mu_{n+1}(\mathbf{U})^{-1} \sum_{|w|=n} L(w) E\left(\Sigma \circ T(w)|\mathcal{G}_n\right) \\ &\overset{(\star)}{=} \mu_{n+1}(\mathbf{U})^{-1} \mu(U_n) \sum_{|w|=n} L(w) \\ &= \mu_n(\mathbf{U})^{-1} Z_n = W_n \quad \text{a.s.} \end{aligned}$$

To justify $(\star)$, observe that for all $A \in \mathfrak{M}^{\mathbb{N}_0}$ and $B \in \bigotimes_{v \in \mathcal{N}, |v| \leq n-1} \mathbb{B}^{\mathbb{N}}$

$$\begin{aligned} \int_{\{\mathbf{U}\in A, (T(v))_{|v|\leq n-1} \in B\}} \Sigma \circ T(w)\, dP &= \int_{\{\mathbf{U}\in A\}} E\left(\mathbb{1}_{\{(T(v))_{|v|\leq n-1}\in B\}} \cdot \Sigma \circ T(w)\middle|\mathbf{U}\right) dP \\ &= \int_{\{\mathbf{U}\in A\}} \mu(U_n) \cdot P\left((T(v))_{|v|\leq n-1} \in B\middle|\mathbf{U}\right) dP \\ &= \int_{\{\mathbf{U}\in A\}} E\left(\mu(U_n)\mathbb{1}_{\{(T(v))_{|v|\leq n-1}\in B\}}\middle|\mathbf{U}\right) dP \\ &= \int_{\{\mathbf{U}\in A, (T(v))_{|v|\leq n-1}\in B\}} \mu(U_n)\, dP, \end{aligned}$$

for the random variables $\Sigma \circ T(w)$ and $(T(v))_{|v| \leq n-1}$ are conditionally independent with $E(\Sigma \circ T(w) | \mathbf{U}) = \mu(U_n)$ a.s. as seen in (a). Since $W_0 = 1$ we can finish the proof by applying the martingale convergence theorem and Fatou's lemma.

□

Denote by $\mathbb{M}'$ the set of all probability measures on $(\mathbb{R}, \mathbb{B})$ and endow $\mathbb{M}'$ with the σ-algebra $\mathfrak{M}'$ generated by the total variation norm. Besides, put $\Phi : \mathbb{R} \times \mathbb{M}' \to \mathbb{R}$,

$$\Phi(t, Q') := Q'(t) := Q'((-\infty, t]) \qquad (t \in \mathbb{R}, Q' \in \mathbb{M}').$$

Lemma 10. *(a) The mapping $\Phi^{-1} : (0,1) \times \mathbb{M}' \to \mathbb{R}$,*

$$\Phi^{-1}(t, Q') := \inf\{x \in \mathbb{R} : \Phi(x, Q') \geq t\} \qquad (0 < t < 1, Q' \in \mathbb{M}'),$$

is $\mathbb{B} \otimes \mathfrak{M}'$-$\mathbb{B}$-measurable.

(b) The sequence $(X_n)_{n \geq 0}$, defined by $X_n := \Sigma \circ \widehat{T}(n) = \sum_{i \geq 1} \widehat{T}_i(n)$ for $n \geq 0$, is stationary ergodic.

(c) The same holds true for the sequence $(\widetilde{X}_n)_{n \geq 0}$, where $\widetilde{X}_n := \frac{\widehat{T}_{C_n}(n)}{\mu(U_n)}$ $(n \geq 0)$.

(d) If $\mathbf{U} = (U_n)_{n \geq 0}$ consists of i.i.d. components, the same is true for the sequences $(X_n)_{n \geq 0}$ and $(\widetilde{X}_n)_{n \geq 0}$, respectively.

Proof. (a) Fix any $\alpha \in \mathbb{R}$. Then we have that

$$\begin{aligned} \{(t, Q') : \Phi^{-1}(t, Q') \leq \alpha\} &= \{(t, Q') : \inf\{x \in \mathbb{R} : Q'(x) \geq t\} \leq \alpha\} \\ &= \{(t, Q') : Q'(\alpha) \geq t\} \\ &= \{(t, Q') : t - Q'(\alpha) \leq 0\}. \end{aligned}$$

Since for fixed α, the mapping $Q' \mapsto Q'(\alpha)$ is continuous (and therefore measurable), the claim follows.

(b) Let $\mathbf{Y} = (Y_n)_{n \geq 0}$ be a sequence of i.i.d. random variables (defined on $(\Omega, \mathfrak{A}, P)$) with $P(Y_0 \in \cdot) = \mathcal{R}(0,1)$ such that $\mathbf{Y}$ and $\mathbf{U} = (U_n)_{n \geq 0}$ are independent. First note that by (5), $P(0 < X_n < \infty) = 1$. Now (4) says that for any $A \in \mathbb{B}$,

$$P(X_n \in A | \mathbf{U}) = \mu(U_n)^{-1} \int_{\{\Sigma \in A\}} \Sigma \, dU_n =: \widetilde{U}_n(A) \quad \text{a.s.} \tag{8}$$

Since $\mathbf{U}$ is stationary ergodic, the same is true for $\widetilde{\mathbf{U}} = (\widetilde{U}_n)_{n \geq 0}$. Now put $X'_n := \Phi^{-1}(Y_n, \widetilde{U}_n), n \geq 0$. Then $(X'_n)_{n \geq 0}$ is stationary ergodic as well (see Proposition I.4.1.6 in [6] and Proposition 6.31 in [5]), and it is readily checked that the random variables $X'_n, n \geq 0$ are conditionally independent given $\mathbf{U}$ with $P(X'_n \in \cdot | \mathbf{U}) = \widetilde{U}_n$ a.s., $n \geq 0$. Thus, $(X_n)_{n \geq 0} \stackrel{d}{=} (X'_n)_{n \geq 0}$ and $(X_n)_{n \geq 0}$ is stationary ergodic, as asserted.

(c) The conditional independence of the random variables $(\widehat{T}(n), C_n), n \geq 0$ entails that for any sequence $(A_n)_{n\geq 0}$ in $\mathbb{B}$,

$$P(\widehat{T}_{C_n}(n) \in A_n, n \geq 0|\mathbf{U}) = \prod_{n\geq 0} U_n^*(A_n) \quad \text{a.s.},$$

where for $n \geq 0$, (3) shows that

$$\begin{aligned} U_n^*(A_n) &:= P(\widehat{T}_{C_n}(n) \in A_n|\mathbf{U}) \\ &= \sum_{i\geq 1} P(\widehat{T}_i(n) \in A_n, C_n = i|\mathbf{U}) \\ &= \mu(U_n)^{-1} \sum_{i\geq 1} \int_{\{\pi_i \in A_n\}} \pi_i \, dU_n \quad \text{a.s.} \end{aligned}$$

Now define $X_n'' := \Phi^{-1}(Y_n, U_n^*)$ and $X_n''' := X_n''/\mu(U_n), n \geq 0$. Then it is again easy to see that the random variables $X_n'', n \geq 0$ are conditionally independent given $\mathbf{U}$ with conditional distribution given by $P(X_n'' \in \cdot|\mathbf{U}) = U_n^*$ a.s. for all $n \geq 0$. Thus, $(X_n''')_{n\geq 0}$ forms a copy of $(\widetilde{X}_n)_{n\geq 0}$, and since $(X_n''')_{n\geq 0}$ is stationary ergodic (see Proposition I.4.1.6 in [6] and Proposition 6.31 in [5]), the proof is complete.

(d) As the random variables $X_n, n \geq 0$ are conditionally independent given $\mathbf{U}$ with

$$P(X_n \in \cdot|\mathbf{U}) = \widetilde{U}_n \quad \text{a.s.},$$

the independence of $\widetilde{U}_n, n \geq 0$ ensures that for any sequence $(A_n)_{n\geq 0}$ in $\mathbb{B}$,

$$\begin{aligned} P(X_n \in A_n, n \geq 0) &= E\left[P(X_n \in A_n, n \geq 0|\mathbf{U})\right] \\ &= E\left[\prod_{n\geq 0} \widetilde{U}_n(A_n)\right] \\ &= \prod_{n\geq 0} E\widetilde{U}_n(A_n) \\ &= \prod_{n\geq 0} P(X_n \in A_n), \end{aligned}$$

i.e. the independence of $X_n, n \geq 0$. The assertion on $(\widetilde{X}_n)_{n\geq 0}$ follows by an analogous argument. □

Recall that $\gamma_+ = E\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^+ \frac{T_k}{\mu(U_0)}\right]$, $\gamma_- = E\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^- \frac{T_k}{\mu(U_0)}\right]$ and $c = P\left(\bigcap_{i\geq 1}\{T_i = 0 \text{ or } T_i = \mu(U_0)\}\right)$.

Lemma 11. *(a) Suppose that* γ_+ *or* γ_- *is finite. Then*

$$E\log\widetilde{X}_0 = E\left[\log\frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}\right] = \gamma_+ - \gamma_- = \gamma = E\left[\sum_{k\ge 1}\frac{T_k}{\mu(U_0)}\log\frac{T_k}{\mu(U_0)}\right] \in \overline{\mathbb{R}}.$$

(b) $E\log^+ X_0 = E\left[\log^+\sum_{i\ge 1}\widehat{T}_i(0)\right] = E\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right].$

(c) $P(\widetilde{X}_0 = 1|\mathbf{U}) = 1$ *a.s.* $\Longleftrightarrow c = 1.$

Proof. (a) By construction, it follows that for all $A \in \mathbb{B}^{\mathbb{N}}, i \ge 1$ and $B \in \mathfrak{M}^{\mathbb{N}_0}$,

$$\begin{aligned} P(\widehat{T}(0)\in A, C_0 = i, \mathbf{U}\in B) &= \int_{\{\mathbf{U}\in B\}} P(\widehat{T}(0)\in A, C_0 = i|\mathbf{U})\,dP \\ &= \int_{\{\mathbf{U}\in B\}} E\left[\mu(U_0)^{-1}\mathbb{1}_{\{T\in A\}}T_i|\mathbf{U}\right]\,dP \qquad (9)\\ &= E\left[\mathbb{1}_{\{T\in A,\mathbf{U}\in B\}}\frac{T_i}{\mu(U_0)}\right], \end{aligned}$$

proving that

$$P((\widehat{T}(0),\mathbf{U})\in C, C_n = i) = E\left[\frac{T_i}{\mu(U_0)}\mathbb{1}_{\{(T,\mathbf{U})\in C\}}\right]$$

for all $C \in \mathbb{B}^{\mathbb{N}}\otimes\mathfrak{M}^{\mathbb{N}_0}$ and $i \ge 1$. Consequently,

$$P\left(\frac{\widehat{T}_i(0)}{\mu(U_0)}\in D, C_n = i\right) = \int_{\left\{\frac{T_i}{\mu(U_0)}\in D\right\}}\frac{T_i}{\mu(U_0)}\,dP \qquad (D\in\mathbb{B}, i\ge 1). \qquad (10)$$

Now first turning to the positive part of $\log\frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}$, we obtain the decomposition

$$\log^+\frac{\widehat{T}_{C_0}(0)}{\mu(U_0)} = \sum_{k\ge 1}\Upsilon_k,$$

where

$$\Upsilon_k := \mathbb{1}_{\{C_0=k\}}\log^+\frac{\widehat{T}_k(0)}{\mu(U_0)} = \mathbb{1}_{\{C_0=k,\widehat{T}_k(0)\ge\mu(U_0)\}}\log\frac{\widehat{T}_k(0)}{\mu(U_0)}, \quad k\ge 1.$$

On the basis of (10), we infer that for arbitrary $t > 0$ and $k \geq 1$,

$$\begin{aligned} P(\Upsilon_k > t) &= P\left(C_0 = k, \log \frac{\widehat{T}_k(0)}{\mu(U_0)} > t\right) \\ &= P\left(C_0 = k, \frac{\widehat{T}_k(0)}{\mu(U_0)} > e^t\right) \\ &= \int_{\left\{\frac{T_k}{\mu(U_0)} > e^t\right\}} \frac{T_k}{\mu(U_0)}\, dP \\ &= \int_{\left\{\log \frac{T_k}{\mu(U_0)} > t\right\}} \frac{T_k}{\mu(U_0)}\, dP \end{aligned}$$

and therefore by Fubini's theorem

$$\begin{aligned} E\Upsilon_k &= \int_{(0,\infty)} P(\Upsilon_k > t)\, \boldsymbol{\lambda}(dt) \\ &= E\left[\frac{T_k}{\mu(U_0)} \int_{(0,\infty)} \mathbb{1}_{\left\{t < \log \frac{T_k}{\mu(U_0)}\right\}}\, \boldsymbol{\lambda}(dt)\right] \\ &= E\left[\frac{T_k}{\mu(U_0)} \log^+ \frac{T_k}{\mu(U_0)}\right]. \end{aligned}$$

Thus, by monotone convergence, we have

$$E\left[\log^+ \frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}\right] = \sum_{k \geq 1} E\Upsilon_k = \sum_{k \geq 0} E\left[\frac{T_k}{\mu(U_0)} \log^+ \frac{T_k}{\mu(U_0)}\right] = \gamma_+.$$

Since an analogous calculation gives

$$E\left[\log^- \frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}\right] = \gamma_-$$

and one of the terms γ_+, γ_- is finite by assumption, it follows that

$$E\left[\log \frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}\right] = \gamma_+ - \gamma_- = \gamma = E\left[\sum_{k \geq 1} \frac{T_k}{\mu(U_0)} \log \frac{T_k}{\mu(U_0)}\right] \in \overline{\mathbb{R}},$$

as demanded.

(b) As seen in the proof of Lemma 10(b),

$$\begin{aligned} P(\log^+ X_0 > t) &= P(X_0 > e^t) \\ &= E\widetilde{U}_0((e^t, \infty]) \\ &= E\left[\mu(U_0)^{-1} Z_1 \mathbb{1}_{\{Z_1 > e^t\}}\right] \\ &= E\left[\mu(U_0)^{-1} Z_1 \mathbb{1}_{\{\log^+ Z_1 > t\}}\right] \quad \text{for all } t > 0 \end{aligned}$$

because $Z_1 = \sum_{i\geq 1} T_i$. This implies by Fubini's theorem that

$$\begin{aligned} E \log^+ X_0 &= \int_{(0,\infty)} P(\log^+ X_0 > t)\, \boldsymbol{\lambda}(dt) \\ &= E\left[\mu(U_0)^{-1} Z_1 \int_{(0,\infty)} \mathbb{1}_{\{\log^+ Z_1 > t\}}\, \boldsymbol{\lambda}(dt)\right] \\ &= E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]. \end{aligned}$$

(c) In analogy to (9), we have that for all $A, B \in \mathbb{B}$,

$$\begin{aligned} P(\widehat{T}_{C_0}(0) \in A, \mu(U_0) \in B|\mathbf{U}) &= \mathbb{1}_{\{\mu(U_0)\in B\}} \sum_{i\geq 1} \mu(U_0)^{-1} E(\mathbb{1}_{\{T_i \in A\}} T_i|\mathbf{U}) \\ &= \sum_{i\geq 1} \mu(U_0)^{-1} E(\mathbb{1}_{\{T_i\in A, \mu(U_0)\in B\}} T_i|\mathbf{U}) \quad \text{a.s.}, \end{aligned}$$

and letting $D := \{(x,x) : x > 0\} \in \mathbb{B}^2$,

$$\begin{aligned} P(\widetilde{X}_0 = 1|\mathbf{U}) &= P((\widehat{T}_{C_0}(0), \mu(U_0)) \in D|\mathbf{U}) \\ &= \sum_{i\geq 1} E(\mu(U_0)^{-1} T_i \mathbb{1}_{\{T_i = \mu(U_0)\}}|\mathbf{U}) \\ &= \sum_{i\geq 1} P(T_i = \mu(U_0)|\mathbf{U}) \quad \text{a.s.} \end{aligned}$$

Now the claim follows from the decomposition

$$\begin{aligned} \mu(U_0) = E(Z_1|\mathbf{U}) &= \sum_{i\geq 1} E(T_i \mathbb{1}_{\{T_i=\mu(U_0)\}}|\mathbf{U}) + \sum_{i\geq 1} E(T_i \mathbb{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \\ &= \mu(U_0) \sum_{i\geq 1} P(T_i = \mu(U_0)|\mathbf{U}) + \sum_{i\geq 1} E(T_i \mathbb{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \\ &= \mu(U_0) P(\widetilde{X}_0 = 1|\mathbf{U}) + \sum_{i\geq 1} E(T_i \mathbb{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \quad \text{a.s.} \end{aligned}$$

because $P(0 < \mu(U_0) < \infty) = 1$. □

Lemma 12. *(a) For $a > 1$, $\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} P(X_n > a^n|\mathbf{U})$ a.s.*

(b) If $\mathbb{G}(\mathbf{U}, a) < \infty$ with positive probability for some $a > 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \log^+ X_n = 0 \quad \textit{a.s.}$$

(c) If $\mathbb{G}(\mathbf{U}, a) = \infty$ *with positive probability for some* $a > 1$, *then*

$$\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = \infty \quad a.s.$$

(d) $\mathbb{G}(\mathbf{U}, a)$ *is finite with positive probability for some* $a > 1$ *if and only if it is finite a.s. for all* $a > 1$.

Proof. (a) Fix $n \geq 0$ and $a > 1$. Then (5) and (8) yield

$$P(X_n > a^n | \mathbf{U}) = \widetilde{U}_n((a^n, \infty)) = \mu(U_n)^{-1} \int_{\{\Sigma > a^n\}} \Sigma \, dU_n \quad \text{a.s.}$$

and therefore

$$\sum_{n\geq 0} P(X_n > a^n | \mathbf{U}) = \mathbb{G}(\mathbf{U}, a) \quad \text{a.s.}$$

(b) By Theorem 1 in [18], Lemma 4 in [12] or Lemma 7.2 in [10], $\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = 0$ a.s. or $\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = \infty$ a.s., for the sequence $(X_n)_{n\geq 0}$ is stationary ergodic (Lemma 10). Now if $\mathbb{G}(\mathbf{U}, a)$ is finite with positive probability for some $a > 1$, the Borel-Cantelli Lemma implies that

$$P(X_n > a^n \text{ i.o.} | \mathbf{U}) = P\left(\frac{1}{n} \log X_n > \log a \text{ i.o.} \middle| \mathbf{U}\right) = 0 \quad \text{with positive probability,}$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n} \log X_n = 0 \quad \text{a.s.}$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \log^+ X_n = 0 \quad \text{a.s.}$$

as well.

(c) Considering the conditional independence of $X_n, n \geq 0$, a similar argument gives

$$P(X_n > a^n \text{ i.o.} | \mathbf{U}) = P\left(\frac{1}{n} \log X_n > \log a \text{ i.o.} \middle| \mathbf{U}\right) = 1 \quad \text{with positive probability,}$$

i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n \geq \limsup_{n\to\infty} \frac{1}{n} \log X_n = \infty \quad \text{a.s.}$$

(d) is an immediate consequence of (b) and (c). □

6 Proofs of the main results

We can now proceed to the proofs of our results stated in Section 2.

Proof of Theorem 1. Define $\mathcal{G} := \sigma((\widehat{T}_k, C_k)_{k\geq 0}, \mathbf{U})$ and fix $n \geq 0$. We start by calculating the conditional expectation of $\widehat{W}_n$ given $\mathcal{G}$. For this purpose, we decompose $\widehat{Z}_n = z_n \circ \widehat{\mathcal{T}}$, the sum of all weights in the n-th generation of $\widehat{\mathcal{T}}$ in the following way: Given any $k \in \{0, ..., n-1\}$, let $\widehat{R}_k$ be the sum of weights of all individuals in generation n stemming from V_k, but not from V_{k+1}. Then we have the representation

$$\widehat{Z}_n = \widehat{L}(V_n) + \sum_{k=0}^{n-1} \widehat{R}_k, \tag{11}$$

or more explicitly

$$\widehat{Z}_n = \widehat{L}(V_n) + \sum_{|\sigma|=n} \mathbb{1}_{\{V_n=\sigma\}} \sum_{k=0}^{n-1} \widehat{L}(V_k) D_{k,\sigma},$$

where

$$\begin{aligned} D_{k,\sigma} &:= \sum_{j\geq 1, j\neq \sigma_{k+1}} \sum_{|w|=n-k-1} \widehat{T}_j(k) \underbrace{T_{w_1}(\sigma_1, ..., \sigma_k, j) ... T_{w_{n-k-1}}(\sigma_1, ..., \sigma_k, j, w_1, ..., w_{n-k-2})}_{:=1 \text{ if } k=n-1} \\ &= \sum_{j\geq 1, j\neq \sigma_{k+1}} \widehat{T}_j(k) \cdot \Xi_{\sigma,k,j} \end{aligned}$$

and

$$\Xi_{\sigma,k,j} := \sum_{|w|=n-k-1} T_{w_1}(\sigma_1, ..., \sigma_k, j) \cdot ... \cdot T_{w_{n-k-1}}(\sigma_1, ..., \sigma_k, j, w_1, ..., w_{n-k-2}),$$

writing $\sigma = (\sigma_1, ..., \sigma_n)$ and $w = (w_1, ..., w_{n-k-1})$. We claim that

$$E(\Xi_{\sigma,k,j}|\mathcal{G}) = E(\Xi_{\sigma,k,j}|\mathbf{U}) = \prod_{l=k+1}^{n-1} \mu(U_l) = \frac{\mu_n(\mathbf{U})}{\mu_{k+1}(\mathbf{U})} \quad \text{a.s.} \tag{12}$$

To establish (12), let $B \in \bigotimes_{n\geq 0}(\mathbb{B}^{\mathbb{N}} \otimes \mathbb{B})$, $C \in \mathfrak{M}^{\mathbb{N}_0}$ and

$$A := \{(\widehat{T}(n), C_n)_{n\geq 0} \in B, \mathbf{U} \in C\} \in \mathcal{G}.$$

Then the conditional independence of $\Xi_{\sigma,k,j}$ and $(\widehat{T}(n), C_n)_{n\geq 0}$ (given $\mathbf{U}$) implies

$$\begin{aligned}
\int_A \Xi_{\sigma,k,j}\, dP &= \int_{\{\mathbf{U}\in C\}} E\left(\Xi_{\sigma,k,j}\cdot \mathbb{1}_{\{(\widehat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) dP \\
&= \int_{\{\mathbf{U}\in C\}} E\left(\Xi_{\sigma,k,j}|\mathbf{U}\right)\cdot E\left(\mathbb{1}_{\{(\widehat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) dP \\
&= \int_{\{\mathbf{U}\in C\}} E\left(E\left(\Xi_{\sigma,k,j}|\mathbf{U}\right)\cdot \mathbb{1}_{\{(\widehat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) dP \\
&= \int_A E\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) dP \quad \text{a.s.},
\end{aligned}$$

i.e.

$$E\left(\Xi_{\sigma,k,j}|\mathcal{G}\right) = E\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \quad \text{a.s.}$$

because $\sigma(\mathbf{U})\subset\mathcal{G}$. Besides, it follows by construction that for all $k\geq 0$,

$$\begin{aligned}
\prod_{l=k+1}^{n-1}\mu(U_l) &= \prod_{l=k+1}^{n-1} E\left[\Sigma\circ T(v_l)|\mathbf{U}\right] \\
&= E\left[\prod_{l=k+1}^{n-1}\Sigma\circ T(v_l)\middle|\mathbf{U}\right] \\
&= E\left[\prod_{l=k+1}^{n-1}\sum_{m_l\geq 1} T_{m_l}(v_l)\middle|\mathbf{U}\right] \\
&= E\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \quad \text{a.s.}
\end{aligned}$$

where for $l\geq 1$, v_l is an arbitrary element of $\mathbb{N}^l$. Thus, (12) is proved, and consequently,

$$E(D_{k,\sigma}|\mathcal{G}) = \sum_{j\geq 1, j\neq\sigma_{k+1}} \widehat{T}_j(k)\cdot\prod_{l=k+1}^{n-1}\mu(U_l) = \sum_{j\geq 1, j\neq\sigma_{k+1}} \widehat{T}_j(k)\cdot\frac{\mu_n(\mathbf{U})}{\mu_{k+1}(\mathbf{U})} \quad \text{a.s.}$$

Now invoking the $\mathcal{G}$-measurability of $V_0,\ldots,V_n$,

$$\begin{aligned}
E(\widehat{W}_n|\mathcal{G}) &= \frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{|\sigma|=n}\mathbb{1}_{\{V_n=\sigma\}}\sum_{k=0}^{n-1}\frac{\widehat{L}(V_k)}{\mu_n(\mathbf{U})}\cdot E(D_{k,\sigma}|\mathcal{G}) \\
&= \frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{|\sigma|=n}\mathbb{1}_{\{V_n=\sigma\}}\sum_{k=0}^{n-1}\frac{\widehat{L}(V_k)}{\mu_{k+1}(\mathbf{U})}\cdot\sum_{j\geq 1, j\neq\sigma_{k+1}}\widehat{T}_j(k) \\
&\leq \frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{k=0}^{n-1}\frac{1}{\mu(U_k)}\cdot\frac{\widehat{L}(V_k)}{\mu_k(\mathbf{U})}\cdot\exp\left(\log^+ X_k\right) \quad \text{a.s.}
\end{aligned}$$

Our aim is to see that the last expression converges a.s. as $n\to\infty$. Now using the obvious product representation

$$\widehat{L}(V_n) = \prod_{k=0}^{n-1}\widehat{T}_{C_k}(k),$$

the ergodic theorem (see e.g. Corollary 6.23 in [5]) and Lemma 11(a) show that

$$\left[\frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})}\right]^{1/n} = \left(\prod_{k=0}^{n-1} \frac{\widehat{T}_{C_k}(k)}{\mu(U_k)}\right)^{1/n} = \exp\left[\frac{1}{n}\sum_{k=0}^{n-1} \log \widetilde{X}_k\right] \underset{n\to\infty}{\longrightarrow} \exp(E \log \widetilde{X}_0) = e^\gamma < 1 \quad \text{a.s.},$$

i.e.

$$\lim_{n\to\infty} \frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})} = 0 \quad \text{a.s.},$$

and it remains to verify that

$$\sum_{k\geq 0} \frac{1}{\mu(U_k)} \cdot \frac{\widehat{L}(V_k)}{\mu_k(\mathbf{U})} \cdot \exp\left(\log^+ X_k\right) \tag{13}$$

converges a.s. For this purpose, note that the ergodic theorem gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log \mu(U_k) = \beta = E[\log \mu(U_0)] \in (-\infty, \infty) \quad \text{a.s.}$$

and therefore

$$\lim_{n\to\infty} (\mu(U_n))^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{1}{n}\log \mu(U_n)\right) = 1 \quad \text{a.s.}$$

Referring to Lemma 12, the assumption $\mathbb{G}(\mathbf{U}, a) < \infty$ w.p.p. for some $a > 1$ ensures

$$\lim_{n\to\infty} \frac{1}{n} \log^+ X_n = 0 \quad \text{a.s.}, \tag{14}$$

and this guarantees that there is some $\delta \in (e^\gamma, 1)$ such that for almost every $\omega \in \Omega$, fixed $\eta \in (1, \delta^{-1})$ and all sufficiently large k,

$$\frac{\widehat{L}(V_k)(\omega)}{\mu_k(\mathbf{U})(\omega) \cdot \mu(U_k)(\omega)} \leq \delta^k$$

as well as

$$\exp\left(\log^+ X_k\right)(\omega) \leq \eta^k.$$

Now putting these estimates together, the a.s. convergence of the series (13) is evident since $\delta\eta < 1$. Therefore, it follows by Fatou's lemma that

$$E(\liminf_{n\to\infty} \widehat{W}_n | \mathcal{G}) \leq \liminf_{n\to\infty} E(\widehat{W}_n | \mathcal{G}) < \infty \quad \text{a.s.},$$

i.e. $\liminf_{n\to\infty} \widehat{W}_n < \infty$ a.s., in other words

$$\widehat{\mathcal{Q}}(\liminf_{n\to\infty} w_n < \infty) = 1. \tag{15}$$

In addition, note that $\widehat{\mathcal{Q}}(w_n > 0) = 1$ for all n since by Lemma 8(b),

$$\widehat{\mathcal{Q}}(w_n = 0) = \int_{\{w_n = 0\}} w_n \, d\mathcal{Q} = 0.$$

Obviously, w_n^{-1} is $\mathcal{F}'_n$-measurable for all n. In the next step we show that the sequence $((w_n^{-1}, \mathcal{F}'_n))_{n\geq 0}$ is a nonnegative supermartingale with respect to $\widehat{\mathcal{Q}}$. In fact, letting $n \geq 0, A \in \mathcal{F}'_n \subset \mathcal{F}'_{n+1}$ and recalling Lemma 8(b), the computation

$$\begin{aligned}
\int_A w_{n+1}^{-1} \, d\widehat{\mathcal{Q}} &= \int_{A\cap\{w_{n+1}>0\}} w_{n+1}^{-1} \, d\widehat{\mathcal{Q}}_{|\mathcal{F}'_{n+1}} \\
&= \int_{A\cap\{w_{n+1}>0\}} w_{n+1}^{-1} \cdot w_{n+1} \, d\mathcal{Q}_{|\mathcal{F}'_{n+1}} \\
&= \mathcal{Q}(A) - \mathcal{Q}(A \cap \{w_{n+1} = 0\}) \\
&= \mathcal{Q}_{|\mathcal{F}'_n}(A \cap \{w_n > 0\}) + \mathcal{Q}(A \cap \{w_n = 0\}) - \mathcal{Q}_{|\mathcal{F}'_{n+1}}(A \cap \{w_{n+1} = 0\}) \\
&= \int_{A\cap\{w_n>0\}} w_n^{-1} \cdot w_n \, d\mathcal{Q}_{|\mathcal{F}'_n} + \underbrace{\mathcal{Q}(A \cap \{w_n = 0\}) - \mathcal{Q}(A \cap \{w_{n+1} = 0\})}_{\leq 0} \\
&\leq \int_A w_n^{-1} \, d\widehat{\mathcal{Q}}
\end{aligned}$$

proves that $E_{\widehat{\mathcal{Q}}}(w_{n+1}^{-1} | \mathcal{F}'_n) \leq w_n^{-1}$ $\widehat{\mathcal{Q}}$-a.s. as well as the integrability of w_n^{-1} with respect to $\widehat{\mathcal{Q}}$. To justify the last inequality, consider that $\{W_n = 0\} \subset \{W_{n+1} = 0\}$ for $n \geq 0$. Therefore, $(w_n^{-1})_{n\geq 0}$ converges $\widehat{\mathcal{Q}}$-a.s., in particular

$$\widehat{\mathcal{Q}}(\liminf_{n\to\infty} w_n = w) = 1.$$

Consequently, (15) gives

$$\widehat{\mathcal{Q}}(w < \infty) = 1,$$

which in turn implies

$$EW = \int_{\mathbb{T}} w \, d\mathcal{Q} = 1$$

by another appeal to Lemma 8. This is the claim. □

Proof of Theorem 2. First suppose that $\gamma = \gamma_+ - \gamma_- = E\left[\log \frac{\widehat{T}_{C_0}(0)}{\mu(U_0)}\right] \in (0, \infty]$. Then (11), the ergodic theorem and Lemma 11(a) show that

$$\widehat{W}_n^{1/n} = \left[\frac{\widehat{Z}_n}{\mu_n(\mathbf{U})}\right]^{1/n} \geq \left[\frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})}\right]^{1/n} = \exp\left[\frac{1}{n}\sum_{k=0}^{n-1} \log \widetilde{X}_k\right] \underset{n\to\infty}{\longrightarrow} \exp(E \log \widetilde{X}_0) = e^\gamma > 1 \tag{16}$$

a.s. Thus,

$$\widehat{\mathcal{Q}}(w=\infty) = P(\limsup_{n\to\infty} \widehat{W}_n = \infty) = 1$$

and

$$P(W=0) = \mathcal{Q}(w=0) = 1$$

by Lemma 8(c).

If on the other hand $\mathbb{G}(\mathbf{U}, a)$ is infinite with positive probability for some $a > 1$, then Lemma 12 shows that

$$\limsup_{n\to\infty} \frac{1}{n} \log X_n = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i\geq 1} \widehat{T}_i(n) = \infty \quad \text{a.s.} \tag{17}$$

Besides, we have the estimate

$$\widehat{Z}_{n+1} \geq \widehat{L}(V_n) \cdot X_n = \widehat{L}(V_n) \cdot \sum_{i\geq 1} \widehat{T}_i(n)$$

and therefore

$$\begin{aligned} \left(\widehat{W}_{n+1}\right)^{1/n} &\geq \left[\frac{1}{\mu(U_{n+1})} \cdot \frac{\widehat{L}(V_n)}{\mu_n(\mathbf{U})} \cdot X_n\right]^{1/n} \\ &= \exp\left(-\frac{1}{n} \log \mu(U_{n+1})\right) \cdot \exp\left(\frac{1}{n} \sum_{k=0}^{n-1} \log \widetilde{X}_k\right) \cdot \exp\left(\frac{1}{n} \log X_n\right) \end{aligned} \tag{18}$$

As seen in the proof of Theorem 1,

$$\lim_{n\to\infty} \exp\left(-\frac{1}{n} \log \mu(U_{n+1})\right) = 1 \quad \text{a.s.},$$

and considering the fact that $\gamma_- < \infty$ implies $\gamma > -\infty$ and hence

$$\lim_{n\to\infty} \exp\left(\frac{1}{n} \sum_{k=0}^{n-1} \log \widetilde{X}_k\right) = e^{\gamma} > 0 \quad \text{a.s.},$$

(17) and (18) yield

$$\limsup_{n\to\infty} \widehat{W}_n = \infty \quad \text{a.s.},$$

i.e. once more $W = 0$ a.s. by Lemma 8(c). □

Proof of Corollary 3. We have seen in Lemma 11(b) that $E\log^+ X_0 = E\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]$. Consequently, the stationarity of $(X_n)_{n\geq 0}$ implies that for arbitrary $a>1$,

$$\begin{aligned}\infty > E\left[\frac{\log^+ X_0}{\log a}\right] &\geq \sum_{n\geq 0} P(\log^+ X_0 > n\log a)\\ &= \sum_{n\geq 0} P(X_n > a^n)\\ &= \sum_{n\geq 0}\int P(X_n > a^n|\mathbf{U})\,dP\\ &= \int\left(\sum_{n\geq 0} P(X_n > a^n|\mathbf{U})\right)dP\\ &= E\mathbb{G}(\mathbf{U},a),\end{aligned}$$

i.e. $P(\mathbb{G}(\mathbf{U},a)<\infty)=1$. Besides, the integrability of $\frac{Z_1\log^+ Z_1}{\mu(U_0)}$, the inequality $T_i\leq Z_1$ for $i\geq 1$ and the fact that $x\mapsto\log^+ x$ is nondecreasing in $[0,\infty]$ ensure that

$$\begin{aligned}0\leq\gamma_+ &= E\left[\sum_{i\geq 1}\frac{T_i}{\mu(U_0)}\log^+\frac{T_i}{\mu(U_0)}\right]\\ &\leq E\left[\sum_{i\geq 1}\frac{T_i}{\mu(U_0)}\log^+\frac{Z_1}{\mu(U_0)}\right]\\ &= E\left[\frac{Z_1}{\mu(U_0)}\log^+\frac{Z_1}{\mu(U_0)}\right]\\ &= E\left[\frac{Z_1}{\mu(U_0)}(\log Z_1-\log\mu(U_0))\mathbb{1}_{\{Z_1\geq\mu(U_0)\}}\right]\\ &\leq E\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+E\left[\frac{Z_1|\log\mu(U_0)|}{\mu(U_0)}\right]\\ &= E\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+E\left[\frac{|\log\mu(U_0)|}{\mu(U_0)}\cdot E(Z_1|\mathbf{U})\right]\\ &= E\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+E|\log\mu(U_0)|<\infty.\end{aligned}$$

Hence, the assertion follows from Theorem 1. □

Proof of Corollary 4. Put $\psi(x) := x\log x$, $x\geq 0$ and note that ψ is convex with

$\sup_{0<x<1} |\psi(x)| = 1/e$. Consequently, $\gamma_- \leq \ell/e < \infty$, $-\infty < \gamma < 0$ and additionally

$$\begin{aligned} -\infty < E\psi\left(\frac{Z_1}{\mu(U_0)}\right) &= \log \ell \, E\left[\frac{Z_1}{\mu(U_0)}\right] + \ell \, E\psi\left(\frac{Z_1}{\ell\mu(U_0)}\right) \\ &= \log \ell + \ell \, E\psi\left(\frac{1}{\ell}\sum_{i=1}^{\ell} \frac{T_i}{\mu(U_0)}\right) \\ &\leq \log \ell + E\left[\sum_{i=1}^{\ell} \psi\left(\frac{T_i}{\mu(U_0)}\right)\right] \\ &= \log \ell + \gamma < \infty. \end{aligned}$$

This gives

$$\begin{aligned} E\left[\frac{Z_1 \log Z_1}{\mu(U_0)}\right] &= E\psi\left(\frac{Z_1}{\mu(U_0)}\right) + E\left[\frac{Z_1 \log \mu(U_0)}{\mu(U_0)}\right] \\ &= E\psi\left(\frac{Z_1}{\mu(U_0)}\right) + E \log \mu(U_0) \\ &= E\psi\left(\frac{Z_1}{\mu(U_0)}\right) + \beta \in (-\infty, \infty), \end{aligned}$$

in particular $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$. Now Corollary 3 gives the claim. □

Proof of Theorem 5. (a) The implication "$(ii) \Rightarrow (i)$" is trivial.

"$(i) \Rightarrow (iii)$": First suppose $0 < \gamma \leq \infty$. If $\gamma > 0$, we have already found the contradiction $P(W = 0) = 1$ in the proof of Theorem 2. In case $\gamma = 0$, the assumption $c < 1$ and Lemma 11(c) ensure that

$$P(\log \widetilde{X}_0 = 0) = P(\widetilde{X}_0 = 1) < 1.$$

Consequently, the Chung-Fuchs theorem, Lemma 10(d) and (16) give

$$\limsup_{n\to\infty} \widehat{W}_n \geq \limsup_{n\to\infty} \exp\left(\sum_{k=0}^{n-1} \log \widetilde{X}_k\right) \geq \limsup_{n\to\infty} \sum_{k=0}^{n-1} \log \widetilde{X}_k = \infty \quad \text{a.s.},$$

i.e. $W = 0$ a.s. in this case as well. Thus, we have $\gamma < 0$, and it remains to prove the finiteness of $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]$. For this purpose, assume $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] = \infty$, i.e.

$$E \log^+ X_0 = \infty \quad \text{(Lemma 11)}.$$

Since the random variables $X_n, n \geq 0$ are independent and identically distributed (Lemma 10(d)), this guarantees (see for example Lemma 1.1. in [10])

$$\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = \infty \quad \text{a.s.}$$

and therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log X_n = \infty \quad \text{a.s.}$$

as well. Now $W = 0$ a.s. is established in the same way as in the proof of Theorem 2. Summarizing, we have proved that (i) implies $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]$ and $\gamma < 0$.

"$(iii) \Rightarrow (ii)$": This implication is basically verified by copying the proof of Theorem 1. It is merely equation (14) that requires a new argument: As $E \log^+ X_0 = E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$, the independence of $X_n, n \geq 0$ (Lemma 10(d)) in combination with Lemma 1.1. in [10] yields

$$\limsup_{n\to\infty} \frac{1}{n} \log X_n = \limsup_{n\to\infty} \frac{1}{n} \log^+ \sum_{i\geq 1} \widehat{T}_i(n) = 0 \quad \text{a.s.},$$

and the proof of (a) is complete.

(b) As for all $A \in \mathbb{B}^{\mathbb{N}}, B \in \mathbb{B}$ and $v \in \mathcal{N}$

$$\begin{aligned} P\left(T(v) \in A, \mu(U_{|v|}) \in B\right) &= E\left[\mathbb{1}_{\{\mu(U_{|v|})\in B\}} U_{|v|}(A)\right] \\ &= E\left[\mathbb{1}_{\{\mu(U_0)\in B\}} U_0(A)\right] \\ &= P\left(T \in A, \mu(U_0) \in B\right), \end{aligned}$$

it follows that for all $v \in \mathcal{N}$, $P\left(\Sigma \circ T(v) = \mu(U_{|v|})\right) = 1$. Hence, $c = 1$ implies $Z_n = \mu_n(U)$ a.s. for all $n \geq 0$ and $W = 1$ a.s.

(c) Let $T^*(v) := (T_i^*(v))_{i\geq 1} := T(v)/\mu(U_{|v|})$ and $U_{|v|}^* := P(\Sigma \circ T^*(v) \in \cdot|\mathbf{U})$, $v \in \mathcal{N}$. Then $(W_n)_{n\geq 0}$ is the uniquely determined weighted branching process with the factors $T^*(v)$, $v \in \mathcal{N}$. Consequently, $P(\Sigma \circ T^*(v) \in \mathbb{N}_0) = 1$ for all $v \in \mathcal{N}$. More precisely, we will see that $(W_n)_{n\geq 0}$ may be viewed as an ordinary branching process in stationary ergodic random environment. To be rigorous, let $\mathbf{U}^* := (\mathbf{U}_n^*)_{n\geq 0}$ and observe that $\mathbf{U}^*$ is stationary ergodic, too. Additionally, let $Y(v)$, $v \in \mathcal{N}$ be a family of $\mathbb{N}_0$-valued random variables (defined on $(\Omega, \mathfrak{A}, P)$) that are conditionally independent given $\mathbf{U}^* := (\mathbf{U}_n^*)_{n\geq 0}$ with

$$P(Y(v) \in \cdot|\mathbf{U}^*) = \mathbf{U}_{|v|}^* \quad \text{a.s. for all } v \in \mathcal{N},$$

and recursively define $Z_0^* := 1$, $I_0^* := \{\emptyset\}$,

$$I_{n+1}^* := \{(w,i) \in \mathbb{N}^{n+1} : w \in I_n \text{ and } 1 \leq i \leq Y(w)\}$$

and

$$Z_{n+1}^* := |I_{n+1}^*| = \sum_{w\in I_n} Y(w)$$

for $n \geq 0$. Thus, $(Z_n^*)_{n\geq 0}$ forms a branching process with environmental sequence $\mathbf{U}^*$ fulfilling

$$\mu^*(U_0^*) := E(Z_1^*|\mathbf{U}^*) = \sum_{j\geq 0} jU_0^*(\{j\}) = E(\Sigma\circ T^*(\emptyset)|\mathbf{U}) = \mu(U_0)^{-1}E(Z_1|\mathbf{U}) = 1 \quad \text{a.s.}$$

To get the connection to the original process $(W_n)_{n\geq 0}$, note that for all finite $\mathcal{N}' \subset \mathcal{N}$ and all $A_v \subset \mathbb{N}_0$ $(v \in \mathcal{N}')$, the conditional independence of the random variables $(T(v), \mu(U_{|v|}))$, $v \in \mathcal{N}$ (given $\mathbf{U}$) on the one hand and of $Y(v)$, $v \in \mathcal{N}$ (given $\mathbf{U}^*$) on the other hand yields

$$\begin{aligned} P(\Sigma \circ T^*(v) \in A_v, v \in \mathcal{N}') &= EP(\Sigma \circ T^*(v) \in A_v, v \in \mathcal{N}'|\mathbf{U}) \\ &= E\left[\prod_{v\in\mathcal{N}'} P(\Sigma \circ T^*(v) \in A_v|\mathbf{U})\right] \\ &= E\left[\prod_{v\in\mathcal{N}'} U_{|v|}^*\right] \\ &= E\left[\prod_{v\in\mathcal{N}'} P(Y(v) \in A_v|\mathbf{U}^*)\right] \\ &= EP(Y(v) \in A_v, v \in \mathcal{N}'|\mathbf{U}^*) \\ &= P(Y(v) \in A_v, v \in \mathcal{N}'), \end{aligned}$$

showing that $(\Sigma \circ T^*(v))_{v\in\mathcal{N}}$ and $(Y(v))_{v\in\mathcal{N}}$ have the same distribution. From this it easily follows that $(W_n)_{n\geq 0}$ and $(Z_n^*)_{n\geq 0}$ are identically distributed, in particular

$$P(W = 0) = P(W_n \to 0) = P(Z_n^* \to 0). \tag{19}$$

In the next step we show that $A := \{U_0^*(\{0,1\}) = 1\}$ has $P(A) = 0$. To see this, observe that $P\Big(U_0^*(\{0,1\}) = 1, U_0^*(\{0\}) > 0\Big) = 0$ because $\sum_{j\geq 0} jU_0^*(\{j\}) = 1$ a.s. Therefore,

$$P(A) = P\Big(U_0^*(\{1\}) = 1\Big) = P\Big(P(Z_1 = \mu(U_0)|\mathbf{U}) = 1\Big) = 0.$$

Now we may apply (19) and Theorem 1 in [2] to infer

$$P(W = 0) = EP(Z_n^* \to 0|\mathbf{U}^*) = 1,$$

for $E \log \mu^*(U_0^*) = 0$. □

Proof of Corollary 6. In the proof of Corollary 4 we have seen that γ_- is finite and that $\gamma < 0$ implies $E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$. Now Theorem 5(a) completes the proof. □

Proof of Theorem 7. (a) and (b) are immediate consequences of Theorem 5 as $\mu(U_0) = \mu$ a.s.

(c) By replacing $T(v), v \in \mathcal{N}$ with $\check{T}(v) := \mu^{-1}T(v), v \in \mathcal{N}$, we may suppose $\mu = 1$. Thus, $(Z_n)_{n\geq 0}$ forms a critical Galton-Watson process with $p_1 := P(Z_1 = 1) < 1$. Consequently, Lemma I.3.1 in [3] gives

$$W = \lim_{n\to\infty} Z_n = 0 \quad \text{a.s.}$$

(d) has been proved in Corollary 3. □

7 Additional remarks

At the end of this paper we give some supplementary remarks.

Remark 13. (a) Suppose that $P(T \in \{0,1\}^{\mathbb{N}}) = 1$ and $\beta = E(\log \mu(U_0)) \in (0, \infty)$. Then $(Z_n)_{n\geq 0}$ may be viewed as an ordinary branching process in random environment with stationary ergodic environmental sequence $\mathbf{U}^{\Sigma} := (U_n^{\Sigma})_{n\geq 0}$, for if $|v_n| = n \geq 0$,

$$P\left(\sum_{i\geq 1} T_i(v_n) = k \,\middle|\, \mathbf{U}\right) = P(\Sigma \circ T(v_n) = k|\mathbf{U}) = U_n^{\Sigma}(\{k\}) \quad \text{a.s. for all } k \geq 0.$$

Besides,

$$\mu(U_n) = \sum_{k\geq 1} kU_n^{\Sigma}(\{k\}) \quad \text{a.s. for all } n \geq 0,$$

$$\begin{aligned}
\gamma_+ = E\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{T_i}{\mu(U_0)}\right] &= E\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{1}{\mu(U_0)} \mathbb{1}_{\{T_i=1\}}\right] \\
&= E\left[\log^+ \frac{1}{\mu(U_0)} \cdot \frac{Z_1}{\mu(U_0)}\right] \\
&= E\left[\mu(U_0)^{-1} \log^+ \frac{1}{\mu(U_0)} \cdot E(Z_1|\mathbf{U})\right] \\
&= E\left[\log^+ \frac{1}{\mu(U_0)}\right] \leq E|\log \mu(U_0)| < \infty,
\end{aligned}$$

$$\gamma_- = E\left[\log^- \frac{1}{\mu(U_0)}\right] < \infty$$

and

$$\gamma = \gamma_+ - \gamma_- = E\left[\log \frac{1}{\mu(U_0)}\right] = -\beta \in (-\infty, 0).$$

Now if

$$\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x\ U_n^{\Sigma}(dx) < \infty \quad \text{w.p.p. for some } a > 1,$$

Theorem 1 says that $W = \lim_{n\to\infty} Z_n/\mu_n(\mathbf{U})$ satisfies $EW = 1$, in accordance with Theorem 7.1. in [10]. As

$$\lim_{n\to\infty} \mu_n(\mathbf{U})^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{k=0}^{n-1} \log \mu(U_k)\right) = e^{\beta} > 1 \quad \text{a.s.}$$

by the ergodic theorem, we have that for almost every ω, there are constants $a_1, a_2 \in (1, \infty)$ with

$$a_1^n \leq \mu_{n+1}(\mathbf{U})(\omega) \leq a_2^n$$

for all sufficiently large n. Since $\mathbb{G}(\mathbf{U}, a)$ is finite w.p.p. for some $a > 1$ if and only if it is finite a.s. for all $a > 1$, this shows that

$$\mathbb{G}(\mathbf{U}, a) < \infty \text{ w.p.p. for some } a > 1 \Longleftrightarrow \sum_{n\geq 0} \frac{1}{\mu(U_n)} \int_{(\mu_{n+1}(\mathbf{U}),\infty)} x\ U_n^{\Sigma}(dx) < \infty \text{ a.s.},$$

the latter condition being part of Theorem 1 in [19]. Note that the additional technical assumption made in [19] is dispensable.

Summarizing, we can state that Theorem 1 forms a generalization of known results on branching processes in stationary ergodic random environment.

(b) If in the situation of (a) the random variables U_n, $n \geq 0$ are even i.i.d., then $\gamma = -\beta < 0$, and Theorem 5 implies that

$$EW = 1 \Longleftrightarrow E\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty,$$

in accordance with Theorem 2 in [19].

(c) If additionally, the sequence $(U_n)_{n\geq 0}$ is deterministic and nonvarying, then $\gamma = -\log \mu < 0 \Leftrightarrow \mu > 1$, and Theorem 7 says that for the classical Galton-Watson process $(Z_n)_{n\geq 0}$ with offspring distribution $(p_k)_{k\geq 0}$ satisfying $p_1 = P(Z_1 = 1) < 1$,

$$EW = 1 \quad \Longleftrightarrow \quad EZ_1 \log^+ Z_1 = \sum_{k\geq 2} p_k k \log k < \infty \text{ and } \mu > 1,$$

where $W := \lim_{n\to\infty} \frac{Z_n}{\mu^n}$. This gives a new proof of the classical result by Kesten and Stigum (see e.g. Theorem II.2.1 in [1] or Theorm I.10.1 in [3]).

Remark 14. The following example shows that in the general case of stationary ergodic random environment the finiteness of $E[Z_1 \log^+ Z_1/\mu(U_0)]$ is not a necessary condition for W to be nondegenerate (see Example 3.1 in [19] for a similar construction):
Let $(X_n)_{n\geq 0}$ be a stationary ergodic sequence of $\mathbb{N}$-valued random variables (defined on $(\Omega, \mathfrak{A}, P)$) satisfying

(i) $EX_0 = \infty$ and

(ii) $\lim_{n\to\infty} \frac{X_n}{n} = 0$ a.s.

(see Example (a) in [18] for the construction of such a sequence). Besides, let

$$\lambda_n := 2^{-n}\delta_{v^{(n)}} + (1-2^{-n})\delta_{(1,0,0,\ldots)},$$

where $v^{(n)} := (v_i^{(n)})_{i\geq 1}$ is defined by

$$v_i^{(n)} := \begin{cases} 1, & 1 \leq i \leq 2^{n+2}, \\ 0, & i > 2^{n+2} \end{cases} \qquad (n \geq 1).$$

Then $\mathbf{U} := (U_n)_{n\geq 0} := (\lambda_{X_n})_{n\geq 0}$ is stationary ergodic with

$$\mu(U_0) = \int_{[0,\infty]} x \, U_0^{\Sigma}(dx) = (1-2^{-X_n}) + 2^{X_n+2}\cdot 2^{-X_n} = 5 - 2^{-X_n} \in (4;5) \quad \text{a.s.} \tag{20}$$

In the next step we will show that the conditions of Theorem 1 are fulfilled: We have already seen in Remark 13 that $\gamma_+ < \infty$ and $\gamma = -E[\log \mu(U_0))]$, i.e. $-\infty < \gamma < 0$ by (20). Furthermore, it follows from (ii) that for almost every $\omega \in \Omega$ and all sufficiently large n that $X_n(\omega)/n < 1 - 2/n$, i.e. $2^{X_n(\omega)+2} < 2^n$ and therefore

$$\int_{(2^n,\infty)} x \, U_n^{\Sigma}(\omega)(dx) = 0.$$

Consequently, $\mathbb{G}(\mathbf{U}, 2)$ is a.s. finite, whence $EW = 1$ by Theorem 1.

It remains to show that $E[Z_1 \log^+ Z_1/\mu(U_0)]$ is infinite. But (20) and (i) ensure that

$$\begin{aligned} E[Z_1 \log^+ Z_1/\mu(U_0)] &= E\left[\mu(U_0)^{-1} E(Z_1 \log^+ Z_1 | \mathbf{U})\right] \\ &\geq \frac{1}{5} E\left[\int_{[0,\infty]} x \log^+ x \, U_0^{\Sigma}(dx)\right] \\ &\geq \frac{1}{5} E\left[2^{-X_0}\cdot 2^{X_0+2} \log 2^{X_0+2}\right] \\ &= \frac{4\log 2}{5} E\left[X_0 + 2\right] = \infty, \end{aligned}$$

as claimed.

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