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Asexual Versus Promiscuous Bisexual Galton-Watson Processes: The Extinction Probability Ratio

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We consider the supercritical bisexual Galton-Watson process (BGWP) with promiscuous mating, that is a branching process which behaves like an ordinary supercritical Galton-Watson process (GWP) as long as at least one male is born in each generation. For a certain example, it was pointed out by Daley et al. [7] that the extinction probability of such a BGWP apparently behaves like a constant times the respective probability of its asexual counterpart (where males do not matter) if the number of ancestors grows to infinity. In an earlier paper [1] we provided general upper and lower bounds for the ratio between both extinction probabilities and also numerical results that seemed to confirm the convergence of that ratio. However, theoretical considerations rather led us to the conjecture that this does not generally hold. The present article turns this conjecture into a rigorous result. The key step in our analysis is to identify the extinction probability ratio as a certain functional of a subcritical ordinary GWP and to prove its continuity as a function of the number of ancestors in a suitable topology associated with the Martin entrance boundary of that GWP.

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1. Introduction and main results

The *bisexual Galton-Watson process with promiscuous mating* $(\mathcal{Z}_n)_{n\geq 0}$, shortly called promiscuous BGWP, is defined as follows: Consider a two sex population process $(\mathcal{Z}_n^F, \mathcal{Z}_n^M)_{n\geq 0}$ whose n-th generation consists of $\mathcal{Z}_n^F$ females and $\mathcal{Z}_n^M$ males. Females within one generation reproduce according to an ordinary two-type Galton-Watson process (GWP) with product reproduction law $\boldsymbol{p}^F \otimes \boldsymbol{p}^M$ as long as at least one male is alive. Plainly, $\boldsymbol{p}^F = (p_j^F)_{j\geq 0}$ and $\boldsymbol{p}^M = (p_j^M)_{j\geq 0}$ describe the number of female, respectively male offspring. We are therefore given

$$\mathcal{Z}_n \stackrel{\text{def}}{=} \mathcal{Z}_n^F \mathbf{1}_{(0,\infty)}(\mathcal{Z}_n^M)$$

mating units in the n-th generation, the pertinent mating function being $\zeta(x,\cdot) = x\mathbf{1}_{(0,\infty)}$. The formal definition of $(\mathcal{Z}_n^F, \mathcal{Z}_n^M)_{n\geq 0}$ thus takes the form

$$(\mathcal{Z}_{n+1}^F, \mathcal{Z}_{n+1}^M) = \sum_{j=1}^{\mathcal{Z}_n} (\xi_{n,j}, \eta_{n,j}) \tag{1.1}$$

with i.i.d. random vectors $(\xi_{n,j}, \eta_{n,j}), n \geq 0, j \geq 1$ with common distribution $\boldsymbol{p}^F \otimes \boldsymbol{p}^M$.

Bisexual GWPs with various mating functions were introduced by Daley [6] and further investigated in a series of papers [5],[7],[8],[9]. The present article is a continuation of [1] where we compared in some detail the extinctive behavior of a promiscuous BGWP $(Z_n)_{n\geq 0}$ with that of its asexual counterpart, henceforth denoted by $(F_n)_{n\geq 0}$. Let P_j be such that $P_j(\mathcal{Z}_0 = F_0 = j) = 1$ for each $j \geq 1$ and define the extinction probability function

$$\boldsymbol{q}(j) \stackrel{\text{def}}{=} P_j(\mathcal{Z}_n = 0 \text{ for some } n \geq 0), \quad j \in I\!N_0,$$

pertaining to $(\mathcal{Z}_n)_{n\geq 0}$. Plainly, the reproduction law of the ordinary GWP $(F_n)_{n\geq 0}$ is $\boldsymbol{p}^F$, its extinction probability function q^j for some $q \in [0,1]$. We are interested in the *supercritical case* when $\boldsymbol{q}(j) < 1$ for all $j \geq 1$, a standing assumption hereafter. For the promiscuous BGWP this is easily seen to be equivalent to $p_0^M < 1$ and $\mu \stackrel{\text{def}}{=} \sum_{j\geq 1} j p_j^F > 1$. Hence $(F_n)_{n\geq 0}$ is also supercritical and its extinction probability q less than 1. A numerical study in [7] showed for the case where $\boldsymbol{p}^F$ and $\boldsymbol{p}^M$ are Poisson with mean 1.2 that the extinction probability ratio

$$\boldsymbol{r}(k) \stackrel{\text{def}}{=} \frac{\boldsymbol{q}(k)}{q^k}, \quad k \in I\!N,$$

apparently converges very rapidly to approximately 1.33. On the other hand, they had no theoretical justification for this phenomenon and our analysis in [1] indeed showed that this can neither be given shortly nor by easy arguments. Let $\hat{P}_k = P_k(\cdot|F_n \to 0)$ with expectation operator $\hat{E}_k$ and put $\kappa \stackrel{\text{def}}{=} p_0^M$. By exploiting a functional equation for $\boldsymbol{r}(k)$, namely

$$\boldsymbol{r}(k) = \left(\frac{\kappa}{q}\right)^k + (1-\kappa^k)\,\hat{E}_k \boldsymbol{r}(F_1) \tag{1.2}$$

for each $k \geq 0$, we were led in [1] to lower and upper bounds for $\boldsymbol{r}(k)$ depending on the model parameters. Numerical studies for various sets of parameters further confirmed the observation of Daley et al. that $\boldsymbol{r}(k)$ rapidly stabilizes for increasing k if $\kappa < q$. However, based upon arguments beyond the scope of that article, we conjectured that $\boldsymbol{r}(k)$ may actually not always converge but oscillate very slowly, a "near-constancy" phenomenon also encountered for the so-called Harris function of certain supercritical ordinary GWP, see e.g. [4]. The main result of the present article, Theorem 2.1, shows that this conjecture is correct. The proof is based on potential theory for subcritical GWPs which is therefore shortly reviewed from [3] in the Section 3.

Iterating equation (1.2) leads to the fundamental identity (see (3.12) in [1])

$$\boldsymbol{r}(k) \;=\; \left(\frac{\kappa}{q}\right)^{k} \;+\; \hat{E}_k\left(\sum_{j=1}^{\tau}\left(\frac{\kappa}{q}\right)^{F_j}\prod_{i=0}^{j-1}\left(1-\kappa^{F_i}\right)\right) \tag{1.3}$$

where τ denotes the extinction time of $(F_n)_{n\geq 0}$. Note that, under $\hat{P}_k$, $(F_n)_{n\geq 0}$ forms an ordinary *subcritical* GWP with k ancestors, reproduction mean $\hat{\mu} = f'(q)$, offspring distribution $\hat{p}^F = (q^{j-1}p_j^F)_{j\geq 0}$ and offspring generating function $\hat{f}(s) = q^{-1}f(sq)$, f the generating function of $\boldsymbol{p}^F$, see [3, p. 37f]. Note that

$$\hat{P}_1(F_1 > 1) \;=\; \sum_{j\geq 2} q^{j-1}p_j^F \;>\; 0 \tag{1.4}$$

and that $q < 1$ clearly implies the $(X \log X)$-condition for $(F_n)_{n\geq 0}$ under the $\hat{P}_k$, i.e.

$$\hat{E}_k F_1 \log F_1 \;<\; \infty. \tag{1.5}$$

Our main concern hereafter will be the case $0 < \kappa < q$ where the near-constancy phenomenon turns up, but we will also provide a result for the case $\kappa = q$ (Theorem 2.2 below). If $\kappa > q$, we already gave a satisfactory answer in [1], Corollary 3.2 which states that $\kappa^{-k}\boldsymbol{q}(k)$ converges to 1 at an exponential rate.

2. Main results

A look at identity (1.3) shows that its further investigation does no longer require to deal with the original model of a promiscuous BGWP from which it came out. We may rather adopt the viewpoint of dealing with a certain functional in two arguments, κ and q, of an ordinary subcritical GWP. We will therefore simplify our previous notation and use the one for Galton-Watson branching processes in [3] to which we will frequently refer. So from now on let $(Z_n)_{n\geq 0}$ be a subcritical GWP with offspring distribution $(p_j)_{j\geq 0}$, offspring generating function $f(s) = \sum_{j\geq 0} p_j s^j$, reproduction mean $\mu = f'(1) < 1$ and extinction time τ. Notice that now $f(q) \neq q$. For each $k \geq 1$, P_k shall denote the probability measure under which

$Z_0 = k$. If $k = 1$, we also write P instead of P_1. We further assume (see also (1.4) and (1.5))

$$p_1 > 0, \quad p_0 + p_1 < 1 \tag{2.1}$$

and the $(X \log X)$-condition

$$EZ_1 \log Z_1 \ < \ \infty. \tag{2.2}$$

These conditions will in fact be needed in the course of our subsequent analysis. The first condition together with $p_0 > 0$ ensures that all states $i \geq 1$ are communicating for $(Z_n)_{n\geq 0}$ and, as a consequence, that all quasi invariant measures (see Section 3) have positive mass at each $i \geq 1$.

The function $\boldsymbol{r}(k) = \boldsymbol{r}(\kappa, q, k)$ now clearly takes the form

$$\boldsymbol{r}(k) \ = \ \left(\frac{\kappa}{q}\right)^k \ + \ E_k\left(\sum_{j=1}^{\tau}\left(\frac{\kappa}{q}\right)^{Z_j}\prod_{i=0}^{j-1}\left(1-\kappa^{Z_i}\right)\right) \tag{2.3}$$

for $k \in I\!N$ and $0 < \kappa \leq q < 1$. Since $\boldsymbol{r}(k)$ is also a functional of $(Z_{\tau-n})_{0\leq n\leq\tau}$ under P_k, its asymptotic behavior, as $k \to \infty$, should be linked to the limit behavior of $(Z_{\tau-n})_{0\leq n\leq\tau}$ under P_k. Unfortunately, there is not just one limiting distribution but infinitely many, essentially the Martin entrance boundary of $(Z_n)_{n\geq 0}$. This comes out from potential theoretic considerations for subcritical GWPs as described e.g. in [3]. A short review of the most important facts from there will be given in the following section. Here we confine ourselves to a sketchy description in order to formulate our results.

Let Q_k be the distribution of the time reversal $(Z_{\tau-n})_{0\leq n\leq\tau}$ under P_k. Any probability measure Q in the closure of $\{Q_k, k \geq 1\}$ with respect to weak convergence defines a Markov chain $(W_n)_{n\geq 0}$ on $I\!N_0$ with transition matrix $(q_{ij})_{i,j\geq 0}$, say, and corresponds uniquely to a quasi invariant measure $\eta = (\eta_i)_{i\geq 1}$ for $(Z_n)_{n\geq 0}$ (see Section 3) via the relation

$$q_{ij} \ = \ \begin{cases} 0, & \text{if } i = j = 0 \\ \eta_j p_0^j, & \text{if } i = 0, j \geq 1, \\ \dfrac{\eta_i p_{ij}}{\eta_j}, & \text{if } i, j \geq 1 \end{cases}, \tag{2.4}$$

where η is normalized such that $\sum_{j\geq 1} \eta_j p_0^j = 1$. In our setting, we are interested in sequences $k_n, n \geq 1$, approaching ∞ in such a way that $\boldsymbol{r}(k_n)$ converges, as $n \to \infty$. It suffices to consider sequences $k_n, n \geq 1$, such that Q_{k_n} converges weakly to some probability measure Q. We may identify Q with a quasi invariant measure η via (2.4). As shown in [2], these are exactly the extremal elements in the convex set of all quasi invariant measures (normalized as above), for which the circle forms a natural parametrization. We thus identify the closure of $\{Q_k, k \geq 1\}$ with the set $\mathfrak{N} \stackrel{\text{def}}{=} I\!N \cup (-1, 0]$. The Martin topology on $\mathfrak{N}$, rendering weak continuity of $x \mapsto Q_x$, is isomorphic to the topology generated by the metric ρ defined in Section 3. Taking these facts for granted, assertion (2.5) of Theorem 2.1 below should no longer be too surprising.

THEOREM 2.1. *Suppose (2.1), (2.2) and* $0 < \kappa < q$*. Then, for all* $x \in (-1, 0]$,

$$\lim_{k \overset{\rho}{\to} x} \boldsymbol{r}(k) \;=\; \boldsymbol{r}(x) \;\stackrel{\text{def}}{=}\; E\left(\sum_{n \geq 0} \left(\frac{\kappa}{q}\right)^{W_n(x)} \prod_{i > n} \left(1 - \kappa^{W_i(x)}\right) \middle| W_0(x) = 0 \right). \tag{2.5}$$

where $(W_n(x))_{n \geq 0}$ *is a Markov chain on* $I\!N_0$ *with distribution* Q_x*. Moreover, for each* $q \in (0, 1)$*, there exist infinitely many* $\kappa \in (0, q)$ *such that* $\boldsymbol{r}$ *is not a constant.*

THEOREM 2.2. *Suppose (2.1), (2.2),* $\kappa = q$*, and put* $a_k \stackrel{\text{def}}{=} E_k \tau$*. Then*

$$\lim_{k \to \infty} \frac{\boldsymbol{q}(k)}{a_k q^k} \;=\; 1. \tag{2.6}$$

3. QUASI INVARIANT MEASURES AND TIME REVERSAL

We begin with a review of some basic facts from potential theory for subcritical GWPs as described in Athreya and Ney [3, Chapter II]. The notation is kept from there as far as possible. So let $(Z_n)_{n \geq 0}$ be an ordinary subcritical GWP with reproduction distribution $(p_j)_{j \geq 0}$ and reproduction mean $\mu = \sum_{j \geq 0} j p_j < 1$. Let f be the generating function of $(p_j)_{j \geq 0}$, i.e. $f(s) = \sum_{j \geq 0} p_j s^j$ and f_n its n-fold iterate.

Denote by p_{ij} the transition probabilities of $(Z_n)_{n \geq 0}$. A σ-finite measure $\eta = (\eta_j)_{j \geq 1}$ on $I\!N$ is called *quasi invariant* or *quasi stationary* for $(Z_n)_{n \geq 0}$ if

$$\eta_j \;=\; \sum_{i \geq 1} \eta_i p_{ij}$$

for all $j \in I\!N$. Notice that we exclude the absorbing state 0 in the summation. The generating function $U(s) = \sum_{j \geq 1} \eta_j s^j$ of any such η is analytic for $|s| < 1$ and, if normalized so that $U(p_0) = 1$, satisfies the functional relation

$$1 + U(s) \;=\; U(f(s)).$$

Conversely, this relation characterizes quasi invariant measures [3, Theorem II.2]. Since all states $i \geq 1$ communicate and $\eta_j = \sum_{i \geq 1} \eta_i p_{ij}^{(n)}$ for all $n \geq 1$, we infer $\eta_i > 0$ for all $i \geq 1$, as already mentioned in Section 2.

In order to describe all quasi invariant measures for $(Z_n)_{n \geq 0}$, let [3, II.2, eq. (3)]

$$U(s, t) \;\stackrel{\text{def}}{=}\; \sum_{n \in \mathbb{Z}} \left(\exp(Q(s)\mu^{n-t}) - \exp(Q(0)\mu^{n-t}) \right), \quad |s| < 1, t \in (-1, 0].$$

Here

$$Q(s) \;\stackrel{\text{def}}{=}\; \lim_{n \to \infty} \mu^{-n}(f_n(s) - 1), \quad s \in [0, 1].$$

Note that $Q(f(s)) = \mu Q(s)$ [3], eq. (10) on p. 40] and $Q(s) = Q(0)(1 - B(s))$ [3, eq. (30) on p. 47], where B is the generating function

$$B(s) \stackrel{\text{def}}{=} \sum_{j\geq 1} b_j s^j, \quad b_j \;=\; \lim_{n\to\infty} P(Z_n = j | Z_n > 0).$$

The following result is shown in [2].

THEOREM 3.1. *If $EZ_1 \log Z_1 < \infty$, then the space of quasi invariant measures (up to positive scalars) is isomorphic to the set of probability measures on the circle. The bijection $\eta \leftrightarrow \nu$ can be stated as*

$$U_\eta \;=\; \int_{(-1,0]} U(\cdot, t)\; \nu(dt) \tag{3.1}$$

where U_η is the generating function of η.

Recall that $\mathfrak{N} = I\!N \cup (-1, 0]$. Define the map $\varphi : \mathfrak{N} \to \mathbb{C}$ by

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{x}{1+x} e^{2\pi i \log_\mu x}, & \text{if } x \in I\!N \\ e^{2\pi i x}, & \text{if } x \in (-1, 0] \end{cases}$$

and then the metric ρ on $\mathfrak{N}$ by

$$\rho(x, y) \;=\; |\varphi(x) - \varphi(y)|.$$

Notice that under this metric the closure of $I\!N$ is $(-1, 0]$ and that $(-1, 0]$ is endowed with the spherical topology. The latter is not true for the metric given in [3, p. 69]. An integer sequence $(k_n)_{n\geq 1}$ converges in the ρ-metric iff $(\mathbb{K}(k_n, \cdot))_{n\geq 1}$ converges pointwise on $I\!N_0$, where

$$\mathbb{K}(i, j) \stackrel{\text{def}}{=} \frac{\mathbb{G}(i, j)}{\sum_{k\geq 1} \mathbb{G}(i, k) p_0^k}$$

is the Martin kernel and

$$\mathbb{G}(i, j) \stackrel{\text{def}}{=} \sum_{n\geq 0} p_{ij}^{(n)}$$

is the Green kernel. Every such sequence $(k_n)_{n\geq 0}$ with a ρ-limit $t \in (-1, 0]$ will be called a Martin sequence hereafter, and we write $k_n \stackrel{\rho}{\to} t$ (equivalent is $\varphi(k_n) \to \varphi(t)$). The closure of $\{\mathbb{K}(\cdot, j), j \in I\!N\}$ is isomorphic to $(\mathfrak{N}, \rho)$. For such a Martin sequence we further have

$$\lim_{n\to\infty} \mathbb{K}(k_n, j) \;=\; \lim_{n\to\infty} \mathbb{G}(k_n, j) \;=\; \eta_j(t),$$

where $\eta(t) = (\eta_j(t))_{j\geq 1}$ is the quasi invariant measure with generating function $U(\cdot, t)$, $t \in (-1, 0]$ as defined above. For the first equality it should be noticed that

$$\sum_{l\geq 1} \mathbb{G}(k_n, l) p_0^l \;=\; \sum_{m\geq 1} (f_m^{k_n}(p_0) - f_m^{k_n}(0)) \;=\; \sum_{m\geq 1} (f_{m+1}^{k_n}(0) - f_m^{k_n}(0)) \;=\; 1 - f^{k_n}(0)$$

which converges to 1 as $n \to \infty$. Note also that $\eta(t)$ is continuous in t, see [3, p. 69].

The time reversal $(W_n(t))_{n\geq 0}$, say, of $(Z_n)_{n\geq 0}$ with respect to any quasi invariant measure $\eta(t)$ is a Markov chain with n-step transition matrix $\boldsymbol{Q}^n(t) = (q_{ij}^{(n)}(t))_{i,j\geq 0},\ n \geq 1$, where

$$q_{ij}^{(n)}(t) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } i = j = 0 \\ \eta_j(t) P_j(\tau = n), & \text{if } i = 0, j \geq 1 \\ \dfrac{\eta_j(t) p_{ji}^{(n)}}{\eta_i(t)}, & \text{if } i, j \geq 1 \end{cases}. \tag{3.2}$$

The associated Green function is denoted $H(i,j,t) = \sum_{n\geq 0} q_{ij}^{(n)}(t)$ and satisfies $H(0,0,t) = 1$, $H(0,j,t) = \eta_j(t)$ for $j \geq 1$ and $H(i,j,t) = \eta_j(t)\mathbb{G}(j,i)/\eta_i(t)$, otherwise.

LEMMA 3.2. *For any $i_1, ..., i_m \in I\!N$ und $m \in I\!N$, the function $f_{i_1,\dots,i_m} : I\!N \cup (-1,0] \to [0,1]$,*

$$f_{i_1,\dots,i_m}(t) \stackrel{\text{def}}{=} \begin{cases} P_t(Z_{\tau-m} = i_m, ..., Z_{\tau-1} = i_1, Z_\tau = 0), & \text{if } t \in I\!N \\ P(W_1(t) = i_1, ..., W_m(t) = i_m | W_0(t) = 0), & \text{if } t \in (-1,0] \end{cases}$$

is continuous in the ρ-metric.

PROOF. Let first $I\!N \ni k_n \xrightarrow{\rho} t \in (-1,0]$ be a Martin sequence. Then, as $n \to \infty$,

$$\begin{aligned} f_{i_1,\dots,i_m}(k_n) &= P_{k_n}(Z_{\tau-m} = i_m, ..., Z_{\tau-1} = i_1, Z_\tau = 0) \\ &= \sum_{l\geq 0} P_{k_n}(Z_l = i_m) p_{i_m i_{m-1}} \cdot ... \cdot p_{i_1 0} \\ &= \mathbb{G}(k_n, i_m) \frac{1}{\eta_{i_m}(t)} q_{i_{m-1} i_m}(t) \cdot ... \cdot q_{0 i_1}(t) \\ &\to q_{i_{m-1} i_m}(t) \cdot ... \cdot q_{0 i_1}(t) \\ &= f_{i_1,\dots,i_m}(t). \end{aligned}$$

For a sequence $(-1,0] \ni t_n \xrightarrow{\rho} t \in (-1,0]$, the assertion follows from the continuity of the $\eta_j(t)$ in t.

Notice that Lemma 3.2 states in particular that for every Martin sequence $k_n \xrightarrow{\rho} t$,

$$\lim_{n\to\infty} P_{k_n}(Z_{\tau-m} = i_m, ..., Z_{\tau-1} = i_1, Z_\tau = 0) = P(W_1(t) = i_1, ..., W_m(t) = i_m | W_0(t) = 0)$$

for all $i_1, ..., i_m \in I\!N$ und $m \in I\!N$ which explains the meaning of $(W_n(t))_{n\geq 0}$ as a time reversal of $(Z_n)_{n\geq 0}$.

4. Proof of Theorem 2.1

Let $R : \mathfrak{N} \times [0,1) \times [0,1) \to [0,\infty]$ be the function defined through

$$R(x,u,v) \;=\; \begin{cases} E_x\left(\sum_{n=0}^{\tau} u^{Z_n} \prod_{i=0}^{n-1}(1-v^{Z_i})\right), & \text{if } x \in I\!N \\ E\left(\sum_{n\geq 0} u^{W_n(x)} \prod_{i>n}(1-v^{W_i(x)}) \middle| W_0(x)=0\right), & \text{if } x \in (-1,0] \end{cases}.$$

The important fact we will prove in this section is

Proposition 4.1. *The function R is finite and continuous in the product topology induced by* $(\mathfrak{N},\rho) \otimes ([0,1)^2, \text{Euclidean})$.

The proof of this result is provided through a series of lemmata. Fix $N \in I\!N$ and define

$$R_N(x,u,v) \;=\; \begin{cases} E_x\left(\sum_{n=\tau-N}^{\tau} u^{Z_n} \prod_{i=\tau-N}^{n-1}(1-v^{Z_i})\right), & \text{if } x \in I\!N \\ E\left(\sum_{n=0}^{N} u^{W_n(x)} \prod_{i=n+1}^{N}(1-v^{W_i(x)}) \middle| W_0(x)=0\right), & \text{if } x \in (-1,0] \end{cases}.$$

Our program is to show first that R_N is continuous for each N (Lemma 4.2) and then in several steps that $R - R_N$ converges to 0 uniformly on compact sets (Lemma 4.3–5). This clearly implies the asserted continuity of R.

Lemma 4.2. *For each $N \in I\!N$, the function R_N is continuous in the product topology induced by* $(\mathfrak{N},\rho) \otimes ([0,1)^2, \text{Euclidean})$.

Proof. Fix $N \in I\!N$, take a sequence (x_n,u_n,v_n) convergent to (x,u,v) and write

$$|R_N(x_n,u_n,v_n) - R_N(x,u,v)| \;\leq\; |R_N(x_n,u_n,v_n) - R_N(x_n,u,v)| + |R_N(x_n,u,v) - R_N(x,u,v)|.$$

The second expression on the right-hand side tends to 0 by an application of Lemma 3.1 because $R_N(\cdot,u,v)$ is the expectation of a bounded function w.r.t. the weakly convergent discrete probability distributions $P_x((Z_{\tau-N},...,Z_\tau) \in \cdot)$. As for the first, it is easy to show uniform convergence in x_n. Indeed, if $|u'-u| < \varepsilon$ and $|v-v'| < \varepsilon$, then

$$\begin{aligned} &|R_N(x,u',v') - R_N(x,u,v)| \\ &\quad\leq\; E_x\left(\sum_{n=\tau-N}^{\tau} (u+\varepsilon)^{Z_n} \prod_{i=\tau-N}^{n-1}(1-(v-\varepsilon)^{Z_i}) - (u-\varepsilon)^{Z_n} \prod_{i=\tau-N}^{n-1}(1-(v+\varepsilon)^{Z_i})\right) \\ &\quad=\; E_x \Delta_{\varepsilon,u,v}(Z_{\tau-N},...,Z_\tau) \end{aligned}$$

where $\Delta_{\varepsilon,u,v}$ is a bounded function defined in the obvious manner. Notice that $\Delta_{\varepsilon,u,v} \downarrow 0$ as $\varepsilon \downarrow 0$. Hence, by another appeal to Lemma 3.1 and the monotone convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} |R_N(x_n,u_n,v_n) - R_N(x_n,u,v)| \\
&\quad \leq \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} E_{x_n}\Delta_{\varepsilon,u,v}(Z_{\tau-N},\ldots,Z_\tau) \\
&\quad = \lim_{\varepsilon\downarrow 0} E(\Delta_{\varepsilon,u,v}(W_N(x),\ldots,W_0(x))|W_0(x)=0) \;=\; 0.
\end{aligned}$$

In order to show uniform compact convergence of R_N to R, as $N\to\infty$, we first observe that for $k \in I\!N$

$$\begin{aligned}
|R(k,u,v) - R_N(k,u,v)| \;&\leq\; E_k\left(\sum_{n=0}^{\tau-N-1} u^{Z_n}\prod_{i=0}^{n-1}(1-v^{Z_i})\right) \\
&\quad +\; E_k\left(\sum_{n=\tau-N}^{\tau} u^{Z_n}\prod_{i=\tau-N}^{n-1}(1-v^{Z_i})\left(1-\prod_{i=0}^{\tau-N-1}(1-v^{Z_i})\right)\right) \\
&\leq\; E_k\left(\sum_{n=0}^{\tau-N-1} u^{Z_n}\right) \;+\; N\,E_k\left(1-\prod_{i=0}^{\tau-N-1}(1-v^{Z_i})\right) \qquad (4.1)\\
&\leq\; N\,E_k\left(\sum_{n=0}^{\tau-N-1}(u^{Z_n}+v^{Z_n})\right) \\
&\leq\; 2N\,E_k\left(\sum_{n=0}^{\tau-N-1}(u\vee v)^{Z_n}\right)
\end{aligned}$$

where

$$1-\prod_{i=0}^{n}(1-c_i) \;\leq\; \sum_{i=0}^{n} c_i \qquad (4.2)$$

for $c_0,\ldots,c_n \in [0,1]$ has been used for the penultimate inequality. In view of the subsequent estimations we note that (4.2) remains true if $n=\infty$. For $x\in(-1,0]$, we further have

$$\begin{aligned}
&|R(x,u,v) - R_N(x,u,v)| \\
&\quad \leq \sum_{n>N} E\left(u^{W_n(x)}\prod_{i>n}(1-v^{W_i(x)})\,\middle|\, W_0(x)=0\right) \\
&\quad + \sum_{n=0}^{N} E\left(u^{W_n(x)}\prod_{i=n+1}^{N}(1-v^{W_i(x)})\left(1-\prod_{j>N}(1-v^{W_j(x)})\right)\,\middle|\, W_0(x)=0\right) \qquad (4.3)\\
&\quad \leq \sum_{n>N} E\Big(u^{W_n(x)}|W_0(x)=0\Big) \;+\; N\,E\left(1-\prod_{j>N}(1-v^{W_j(x)})\,\middle|\, W_0(x)=0\right) \\
&\quad \leq 2N\sum_{n>N} E\Big((u\vee v)^{W_n(x)}\Big| W_0(x)=0\Big).
\end{aligned}$$

Since that latter inequality is easier to handle we first show

LEMMA 4.3. *For all $y, w < 1$,*

$$\lim_{N\to\infty} \sup_{x\le y; u,v\le w} |R(x,u,v) - R_N(x,u,v)| \;=\; 0.$$

PROOF. Recalling from (3.2) the definition of $q_{0i}^{(n)}$, we obtain

$$\begin{aligned}
N \sum_{n>N} & E\Big((u\vee v)^{W_n(x)}\Big| W_0(x)=0\Big) \\
&\le N \sum_{n>N} E(w^{W_n(x)}|W_0(x)=0) \\
&= N \sum_{n>N}\sum_{i\ge 1} w^i q_{0i}^{(n)} \\
&= N \sum_{n>N}\sum_{i\ge 1} w^i \eta_i(x) P_i(\tau=n) \\
&= N \sum_{i\ge 1} w^i \eta_i(x) P_i(\tau>N) \\
&= N \sum_{i\ge 1} w^i \eta_i(x) P_i(Z_N>0) \\
&= N \sum_{i\ge 1} w^i \eta_i(x)(1-f_N^i(0)) \\
&= N\big(U(w,x) - U(wf_N(0),x)\big) \\
&\le C(w,y)Nw(1-f_N(0)),
\end{aligned}$$

where $C(w,y) \stackrel{\text{def}}{=} \max_{x\le y; u\le w} D_u U(u,x) < \infty$ as one can easily check. The assertion now follows because $N(1-f_N(0)) \to 0$ as $N\to\infty$, see [3, Section I.11].

In order to further exploit (4.1) for our purposes, we have to consider the functions

$$\begin{aligned}
g_N(k,u) &\stackrel{\text{def}}{=} E_k\left(\sum_{n=0}^{\tau-N-1} u^{Z_n}\right), \\
h(k,u) &\stackrel{\text{def}}{=} \sum_{n\ge 0} E_k u^{Z_n} \mathbf{1}_{\{Z_n>0\}} \;=\; \sum_{n\ge 0}\Big(f_n^k(u) - f_n^k(0)\Big),
\end{aligned}$$

$k, N \in I\!N$, $u\in[0,1)$, which are related as follows:

$$\begin{aligned}
g_N(k,u) &= E_k\Big(\sum_{n\ge 0} u^{Z_n}\mathbf{1}_{\{\tau>n+N\}}\Big) \\
&= \sum_{n\ge 0} E_k u^{Z_n} \mathbf{1}_{\{Z_{n+N}>0\}} \\
&= \sum_{n\ge 0} E_k u^{Z_n} P_{Z_n}(Z_N>0)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0} E_k u^{Z_n}(1 - f_N^{Z_n}(0)) \\
&= \sum_{n\geq 0} \Big(f_n^k(u) - f_n^k(uf_N(0)) \Big) \\
&= h(k,u) - h(k, uf_N(0)).
\end{aligned} \tag{4.4}
$$

Lemma 4.4. *The function h satisfies*

$$
\sup_{k\in I\!N} h(k,u) \;\leq\; m(u) \tag{4.5}
$$

for each $u < 1$, where $m(u) \stackrel{\text{def}}{=} \inf\{n \geq 1 : f_n(0) \geq u\}$. Furthermore,

$$
\sup_{k\geq 1; u,v\leq w} |h(k,u) - h(k,v)| \;<\; m(w+\varepsilon)\left(2\left(\frac{w}{w+\varepsilon}\right)^N + \frac{|u-v|}{w} N \right) \tag{4.6}
$$

for all $\varepsilon > 0$, $0 < w < 1-\varepsilon$ and $N \in I\!N$.

Proof. By using $0 \leq \sum_{n\geq 0}(1 - f_n^k(u)) < \infty$ for all $k \in I\!N$ and $u < 1$ we obtain with $m = m(u)$

$$
\begin{aligned}
h(k,u) &= \sum_{n\geq 0} \Big(f_n^k(u) - 1 \Big) \;-\; \sum_{n\geq m} \Big(f_{n-m}^k(f_m(0)) - 1 \Big) \;-\; \sum_{n=0}^{m-1} \Big(f_n^k(0) - 1 \Big) \\
&= \sum_{n\geq 0} \Big(f_n^k(u) - f_n^k(f_m(0)) \Big) \;+\; \sum_{n=0}^{m-1} \Big(1 - f_n^k(0) \Big) \;\leq\; m
\end{aligned}
$$

because the first sum in the previous line is negative.

In order to prove (4.6), we note first that

$$
\begin{aligned}
h(k,u) &= \sum_{n\geq 0} E_k u^{Z_n} \mathbf{1}_{\{Z_n>0\}} \\
&= \sum_{i\geq 1} u^i \sum_{n\geq 0} P_k(Z_n = i) \;=\; \sum_{i\geq 1} u^i \mathbb{G}(k,i).
\end{aligned}
$$

Define $h_N(k,u) \stackrel{\text{def}}{=} \sum_{i=1}^N u^i \mathbb{G}(k,i)$ and choose an arbitrary $\varepsilon > 0$, w.l.o.g. $< 1-w$. Then, for all $k \in I\!N$ and $u \leq w$

$$
\begin{aligned}
h(k,u) - h_N(k,u) &= \sum_{i>N} u^i \mathbb{G}(k,i) \\
&\leq \left(\frac{u}{w+\varepsilon}\right)^N \sum_{i>N} (w+\varepsilon)^i \mathbb{G}(k,i) \qquad (4.7)\\
&= \left(\frac{u}{w+\varepsilon}\right)^N h(k, w+\varepsilon) \;\leq\; \left(\frac{w}{w+\varepsilon}\right)^N m(w+\varepsilon),
\end{aligned}
$$

where (4.5) has been used for the final inequality. Moreover, for all $u, v \le w$ and $N \ge 1$,

$$
\begin{aligned}
|h_N(k,u) - h_N(k,v)| &= |u-v| \sum_{i=1}^{N} \left(\frac{u^i - v^i}{u-v} \right) \mathbb{G}(k,i) \\
&= |u-v| \sum_{i=1}^{N} \sum_{j=0}^{i-1} u^j v^{i-1-j} \mathbb{G}(k,i) \qquad (4.8) \\
&\le \frac{|u-v|}{w} \sum_{i=1}^{N} i w^{i-1} \mathbb{G}(k,i) \;\le\; \frac{|u-v|}{w} N m(w),
\end{aligned}
$$

the last inequality again by (4.5). By combining (4.7) and (4.8) with a simple application of the triangle inequality, we finally obtain (4.6)

Going back into (4.1), we are now ready to prove

LEMMA 4.5. *For all $w < 1$,*

$$\lim_{N\to\infty} \sup_{k\ge 1; u,v \le w} |R(k,u,v) - R_N(k,u,v)| = 0.$$

PROOF. Indeed, we infer with the help of (4.1), (4.4) and the previous lemma

$$
\begin{aligned}
&\limsup_{N\to\infty} \sup_{k\ge 1; u,v\le w} |R(k,u,v) - R_N(k,u,v)| \\
&\quad \le 2 \limsup_{N\to\infty} N \sup_{k\ge 1; u,v\le w} |h(k, u\vee v) - h(k,(u\vee v) f_N(0))| \\
&\quad \le 2m(w+\varepsilon) \limsup_{N\to\infty} \left(2N \left(\frac{w}{w+\varepsilon} \right)^N + \frac{(u\vee v)(1-f_N(0))}{w} N^2 \right) = 0
\end{aligned}
$$

recalling that $1 - f_N(0)$ converges to 0 exponentially fast [3, I.11].

PROOF OF PROPOSITION 4.1. A combination of Lemma 4.3 and 4.5 clearly yields uniform compact convergence of the R_N to R. Since the R_N are further continuous by Lemma 4.2, we conclude the continuity of R as claimed.

In view of the main assertion of Theorem 2.1, namely the nonconstancy of the extinction probability ratio $\boldsymbol{r}(x)$ for suitable pairs (κ, q), two further lemmata are needed.

LEMMA 4.6. *The function $\eta_1(t)$, $t \in (-1, 0]$, is not a constant.*

PROOF. Note first that

$$\eta_1(t) = D_s U(s,t)_{|s=0} = Q'(0) \sum_{n\in\mathbb{Z}} \mu^{n-t} \exp\left(Q(0)\mu^{n-t}\right).$$

Since our assumptions in Section 2 guarantee η_1 to be everywhere positive, we particularly have $Q'(0) > 0$. We make the change of variables $x = \mu^{-t}$, i.e. $t = -\log_\mu x$. Defining

$$\Psi(x,y) \;\stackrel{\text{def}}{=}\; \sum_{n\in\mathbb{Z}} \left(e^{xy\mu^n} - e^{y\mu^n}\right), \quad y < 0 < x,$$

we obviously have

$$D_x\Psi(x,y) \;=\; \sum_{n\in\mathbb{Z}} y\mu^n e^{xy\mu^n}$$

and therefore

$$\eta_1(-\log_\mu x) \;=\; \frac{Q'(0)}{Q(0)} x D_x\Psi(x, Q(0)).$$

Now suppose η_1 be constant and infer

$$D_x\Psi(x, Q(0)) \;=\; \frac{c}{x}$$

for all $x \in [1, 1/\mu)$ and some constant $c < 0$. The equality extends to all $x > 0$ because both sides are evidently analytic functions on the half plane of complex numbers with positive real part. Integration together with $\Psi(1,\cdot) \equiv 0$ then implies

$$\Psi(x, Q(0)) \;=\; \int_1^x D_z\Psi(z, Q(0))\, dz \;=\; c\log x \tag{4.9}$$

for all $x > 0$. Next, the functional equation

$$\Psi(x,y) \;=\; \Psi\left(\frac{xy}{z}, z\right) \;+\; \Psi\left(\frac{z}{y}, y\right)$$

for all $y, z < 0 < x$ together with (4.9) leads to

$$\Psi(x,y) \;=\; \Psi\left(\frac{xy}{Q(0)}, Q(0)\right) \;+\; \Psi\left(\frac{Q(0)}{y}, y\right) \;=\; c\log\left(\frac{xy}{Q(0)}\right) \;+\; \Psi\left(\frac{Q(0)}{y}, y\right)$$

for all $y < 0 < x$. For $x = 1$, we get

$$\Psi\left(\frac{Q(0)}{y}, y\right) \;=\; c\log\left(\frac{Q(0)}{y}\right)$$

and thereby

$$\Psi(x,y) \;=\; c\log x$$

for all $y < 0 < x$. Rewriting this result for $U(s,t)$, we find that

$$U(s,t) \;=\; \Psi\left(\frac{Q(s)}{Q(0)}, \mu^{-t}Q(0)\right) \;=\; c\log\left(\frac{Q(s)}{Q(0)}\right),$$

which is impossible because the $U(\cdot,t)$ are pairwise distinct by Theorem 3.1.

LEMMA 4.7. *Given any $q \in (0,1)$, the function $R(\cdot, u/q, u)$ is not constant for all sufficiently small $u \in (0,q)$.*

PROOF. $R(x,u/q,u)$ is analytic in u and $R(\cdot,0,0)\equiv 1$. Writing

$$\begin{aligned} R(x,u/q,u) &= E\left(\prod_{i\ge 1}(1-u^{W_i(x)})\middle| W_0(x)=0\right) \\ &+ \frac{u}{q}E\left(\sum_{n\ge 1}\left(\frac{u}{q}\right)^{W_n(x)-1}\prod_{i>n}(1-u^{W_i(x)})\middle| W_0(x)=0\right) \end{aligned}$$

and noting that $W_n(x)\ge 1$ for all $x\in(-1,0]$ and $n\ge 1$, it is easily verified that

$$D_uR(x,u/q,u)_{|u=0} = \frac{1-q}{q}\eta_1(x).$$

Since $\eta_1(x)$ is not constant in x, the same holds true for $D_uR(x,u/q,u)$ at $u=0$. Consequently, picking two distinct values $x_1,x_2\in(-1,0]$ with $D_uR(x_1,u/q,u)_{|u=0}\ne D_uR(x_2,u/q,u)_{|u=0}$, we must also have $R(x_1,u/q,u)\ne R(x_2,u/q,u)$ for all sufficiently small $u\in(0,q)$ (using $R(x_1,0,0)=R(x_2,0,0)$ and the continuity of $D_uR(x,u/q,u)$ in u). This proves the lemma.

PROOF OF THEOREM 2.1. Suppose $0<\kappa<q$. Since $\boldsymbol{r}(k)=(\kappa/q)^k+R(k,\kappa/q,\kappa)$ for $k\in I\!N$ (see (2.3)) and $\boldsymbol{r}(x)=R(x,\kappa/q,\kappa)$ for $x\in(-1,0]$, assertion (2.5) follows directly from Proposition 4.1. Moreover, we infer from the previous lemma that $\boldsymbol{r}$ is not a constant for infinitely many, in fact all sufficiently small $\kappa\in(0,q)$. We have thus proved the theorem.

5. PROOF OF THEOREM 2.2

We begin with an auxiliary lemma which gives an asymptotic estimate of the expected extinction time $a_k=E_k\tau$ as $k\to\infty$.

LEMMA 5.1. *Let $(Z_n)_{n\ge 0}$ be a subcritical GWP with reproduction mean $\mu>0$ and $EZ_1\log Z_1<\infty$. Then*

$$\lim_{k\to\infty}\frac{a_k}{\log_{1/\mu}k} = 1.$$

PROOF. Recall from [3, I.11] that $f_n(0)=1-c_n\mu^n$ with positive constants $c_n\in[0,1]$ converging to some $c>0$. It is the positivity of c where the $(X\log X)$-condition enters. Since $P_k(\tau>n)=P_k(Z_n>0)=1-f_n^k(0)$, we infer

$$\begin{aligned} \frac{a_k}{\log_{1/\mu}k} &= \frac{1}{\log_{1/\mu}k}\sum_{n\ge 0}P_k(\tau>n) = \frac{1}{\log_{1/\mu}k}\sum_{n\ge 0}\Big(1-f_n^k(0)\Big) \\ &= \frac{1}{\log_{1/\mu}k}\sum_{n\ge 0}\Big(1-(1-c_n\mu^n)^k\Big) \end{aligned}$$

Fix any $\varepsilon\in(0,1)$, put $n_*=n_*(\varepsilon,k)\stackrel{\text{def}}{=}(1-\varepsilon)\log_{1/\mu}k$, $n^*=n^*(\varepsilon,k)\stackrel{\text{def}}{=}(1+\varepsilon)\log_{1/\mu}k$ and split up the sum into three parts $S_1(k),S_2(k),S_3(k)$ ranging from 0 to n_*-1, from n_* to n^*-1, and from n^* to ∞, respectively. Note that $\mu^{n_*} = k^{-(1-\varepsilon)}$ and $\mu^{n^*} = k^{-(1+\varepsilon)}$. The three sums will be considered separately.

Choose m such that $\inf_{n\geq m} c_n \geq c/2$. Then we have for $S_1(k)$

$$\begin{aligned}
\Big(1-\varepsilon-\frac{m}{\log_{1/\mu}k}\Big)&\Big(1-(1-c\mu^m/2)^k\Big)\\
&\leq \frac{1}{\log_{1/\mu}k}\sum_{n=m}^{n_*-1}\Big(1-(1-c\mu^n/2)^k\Big)\\
&\leq S_1(k)\\
&\leq \frac{1}{\log_{1/\mu}k}\sum_{n=0}^{n_*-1}\Big(1-(1-\mu^n)^k\Big)\\
&\leq (1-\varepsilon)\Big(1-(1-\mu^{n_*})^k\Big)\\
&= (1-\varepsilon)\Big(1-(1-k^{-(1-\varepsilon)})^k\Big)\\
&\leq (1-\varepsilon)\Big(1-\exp(-2k^{\varepsilon})\Big),
\end{aligned}$$

where the last inequality holds for sufficiently large k using $\log(1-x)\geq -2x$ for all positive x sufficiently close to 0. Consequently,

$$\lim_{k\to\infty} S_1(k) \;=\; 1-\varepsilon.$$

For $S_2(k)$ we just note $0\leq S_2(k)\leq 2\varepsilon$. Finally, we obtain for $S_3(k)$, if k is sufficiently large,

$$\begin{aligned}
0 \;\leq\; S_3(k) \;&\leq\; \frac{1}{\log_{1/\mu}k}\sum_{n\geq 0}\Big(1-(1-\mu^{n^*+n})^k\Big)\\
&= \frac{1}{\log_{1/\mu}k}\sum_{n\geq 0}\Big(1-(1-k^{-(1+\varepsilon)}\mu^n)^k\Big)\\
&\leq \frac{1}{\log_{1/\mu}k}\sum_{n\geq 0}\Big(1-\exp(-2k^{-\varepsilon}\mu^n)\Big)\\
&\leq \frac{2}{k^{\varepsilon}\log_{1/\mu}k}\sum_{n\geq 0}\mu^n
\end{aligned}$$

where $1-e^{-x}\leq x$ for all x has been used for the final inequality. Hence

$$\lim_{k\to\infty} S_3(k) \;=\; 0.$$

Putting the results together the assertion of the lemma easily follows because $\varepsilon\in(0,1)$ was arbitrary.

Proof of Theorem 2.2. We first note that

$$\frac{\boldsymbol{q}(k)}{a_k q^k} \;=\; \frac{\boldsymbol{r}(k)}{a_k} \;=\; \frac{1}{a_k}\left(1+E_k\left(\sum_{j=1}^{\tau}\prod_{i=0}^{j-1}\Big(1-q^{Z_i}\Big)\right)\right)$$

because $\kappa = q$. We thus have to show

$$\lim_{k\to\infty} \frac{1}{a_k} E_k\left(\sum_{j=1}^{\tau}\left(1-\prod_{i=0}^{j-1}\left(1-q^{Z_i}\right)\right)\right) \;=\; 0.$$

Fix an arbitrary $\varepsilon \in (0,1)$, put $N(k) \stackrel{\text{def}}{=} \lfloor \varepsilon a_k \rfloor$ and split up the sum under the expectation into the sum from 1 to $\tau - N(k) - 1$ (of course, equal to 0 if $\tau - N(k) \le 0$) and the sum from $\tau - N(k)$ to τ. As for the latter, we immediately have

$$0 \;\le\; \limsup_{k\to\infty} \frac{1}{a_k} E_k\left(\sum_{j=\tau-N(k)}^{\tau}\left(1-\prod_{i=0}^{j-1}\left(1-q^{Z_i}\right)\right)\right) \;\le\; \lim_{k\to\infty} \frac{N(k)+1}{a_k} \;=\; \varepsilon.$$

Turning to the first sum, we use once more the inequality $1-\prod_{i=0}^{n}(1-x_i) \le \sum_{i=0}^{n} x_i$ for numbers $x_1, ..., x_n \in [0,1]$ and obtain

$$\begin{aligned}
&\frac{1}{a_k} E_k\left(\sum_{j=1}^{\tau-N(k)-1}\left(1-\prod_{i=0}^{j-1}\left(1-q^{Z_i}\right)\right)\right)\\
&\quad\le\; \frac{1}{a_k} E_k\left(\sum_{j=1}^{\tau-N(k)-1}\sum_{i=0}^{j-1} q^{Z_i}\right)\\
&\quad\le\; \frac{1}{a_k} E_k\left(\sum_{j\ge 1}\sum_{i=0}^{j-1} q^{Z_i}\mathbf{1}_{\{Z_{j+N(k)}>0\}}\right)\\
&\quad=\; \frac{1}{a_k}\sum_{i\ge 0}\sum_{j>i} E_k\Big(q^{Z_i} P_{Z_i}(Z_{j-i+N(k)} > 0)\Big)\\
&\quad=\; \frac{1}{a_k}\sum_{i\ge 0}\sum_{j>N(k)} E_k\Big(q^{Z_i}(1-f_j^{Z_i}(0))\Big)\\
&\quad=\; \frac{1}{a_k}\sum_{i\ge 0}\sum_{j>N(k)} \Big(f_i^k(q) - f_i^k(qf_j(0))\Big).
\end{aligned}$$

Now a first order Taylor expansion of $f_i^k(qf_j(0))$ about q together with the monotonicity of f_i and f_i' gives for some z_{ij} between $qf_j(0)$ and q

$$\begin{aligned}
f_i^k(qf_j(0)) \;&=\; f_i^k(q) - kqf_i^{k-1}(z_{ij})f_i'(z_{ij})(1-f_j(0))\\
&\ge\; f_i^k(q) - kqf_i^{k-1}(q)f_i'(1)(1-f_j(0)).
\end{aligned}$$

Hence the above estimation can be continued as

$$\begin{aligned}
&\le\; \frac{kq}{a_k}\left(\sum_{i\ge 0} f_i^{k-1}(0)f_i'(1)\right)\left(\sum_{j>N(k)}(1-f_j(0))\right)\\
&\le\; \frac{kq}{a_k}\left(\sum_{i\ge 0} f_i^{k-1}(q)\mu^i\right)\left(\sum_{j>N(k)} 2c\mu^j\right). \qquad (5.1)
\end{aligned}$$

Now the second sum in (5.1) is clearly bounded by a constant times $\mu^{\varepsilon \log_{1/\mu} k} = k^{-\varepsilon}$ for all k (since $N(k) = \lfloor \varepsilon a_k \rfloor \simeq \varepsilon \log_{1/\mu} k$ by Lemma 5.1), while the first can be bounded by a constant times $k^{-(1-\varepsilon)}$ for sufficiently large k. To see the latter, split up the first sum as

$$\left(\sum_{i=0}^{\lfloor (1-\varepsilon) \log_{1/\mu} k \rfloor} + \sum_{i > \lfloor (1-\varepsilon) \log_{1/\mu} k \rfloor} \right) f_i^{k-1}(q) \mu^i .$$

Observe that

$$\sum_{i > \lfloor (1-\varepsilon) \log_{1/\mu} k \rfloor} f_i^{k-1}(q) \mu^i \; \leq \; \frac{\mu^{(1-\varepsilon) \log_{1/\mu} k}}{1-\mu} \; = \; \frac{k^{-(1-\varepsilon)}}{1-\mu} .$$

Since, for all $i_0 \leq i \leq (1-\varepsilon) \log_{1/\mu} k$, i_0 sufficiently large (independent of k), all k sufficiently large and some $Q(q) \in (-1, 0)$, see [3, I.11],

$$\begin{aligned} f_i^{k-1}(q) \; &\leq \; (1 + Q(q) \mu^i / 2)^{k-1} \\ &\leq \; (1 + Q(q) \mu^{(1-\varepsilon) \log_{1/\mu} k} / 2)^{k-1} \leq \; \exp((k-1)^{\varepsilon} Q(q)/2), \end{aligned}$$

we further have

$$\sum_{i=0}^{\lfloor (1-\varepsilon) \log_{1/\mu} k \rfloor} f_i^{k-1}(q) \mu^i \; \leq \; i_0 f_{i_0}^{k-1}(q) + (1-\varepsilon) \log_{1/\mu} k \, \exp((k-1)^{\varepsilon} Q(q)/2)$$

for all k sufficiently large.

Putting the pieces together, we finally conclude in (5.1)

$$\frac{kq}{a_k} \left(\sum_{i \geq 0} f_i^{k-1}(q) \mu^i \right) \left(\sum_{j > N(k)} 2c\mu^j \right) \; \leq \; \frac{\text{const}}{a_k}$$

which converges to 0 as $k \to \infty$.

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