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The Bisexual Galton-Watson Process with Promiscuous Mating: Extinction Probabilities in the Supercritical Case

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We consider the bisexual Galton-Watson process (BGWP) with promiscuous mating, that is a branching process which behaves like an ordinary GWP as long as at least one male is produced in each generation. For the case of Poissonian reproduction, it was pointed out by Daley et al.(1986) that the extinction probability of such a BGWP apparently behaves like a constant times the respective probability of its asexual counterpart (where males do not matter) providing the number of ancestors grows to infinity. They further mentioned that they had no theoretical justification for this phenomenon. In the present article we will prove upper and lower bounds for the ratio between both extinction probabilities and introduce a recursive algorithm that can easily be implemented on a computer to produce very accurate approximations for that ratio. The final section contains a number of numerical results that have been obtained by use of this algorithm.

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1. Introduction

The bisexual Galton-Watson process with various mating functions has been introduced by Daley(1968) as a modification of the ordinary asexual one so as to allow for sexual reproduction. The underlying model can be described as follows, see Daley, Hull and Taylor(1986): We are given a two-type population process, whose n-th generation consists of Z_n^F females and Z_n^M males which form $Z_n = \zeta(Z_n^F, Z_n^M)$ mating units. Each mating unit reproduces independently of all other units according to the same bivariate distribution for each generation. Thus (Z_{n+1}^F, Z_{n+1}^F) can be defined by

$$(Z_{n+1}^F, Z_{n+1}^M) \; = \; \sum_{j=1}^{Z_n} (\xi_{n,j}, \eta_{n,j}) \tag{1.1}$$

where $(\xi_{n,j}, \eta_{n,j})_{n\geq 0, j\geq 1}$ forms a family of i.i.d. nonnegative integer-valued random variables. Plainly, the empty sum is defined as $(0,0)$ in (1.1). We assume that the mating function ζ is nondecreasing in each argument, integer-valued for integer-valued arguments and satisfies $\zeta(0,0) = 0$. Under these conditions $(Z_n)_{n\geq 0}$ is called a *bisexual Galton-Watson process (BGWP) with mating function* ζ.

It was pointed out by Hull(1982) that mating functions ζ likely to occur in real-life are *superadditive* in the sense that

$$\zeta(x_1 + x_2, y_1 + y_2) \; \geq \; \zeta(x_1, y_1) \; + \; \zeta(x_2, y_2) \tag{1.2}$$

for all $x_1, x_2, y_1, y_2 \in [0, \infty)$. Examples of this type are

(M1) $\zeta(x, y) = x \min\{1, y\}$ for all $x, y \geq 0$, known as *promiscuous mating*,

(M2) $\zeta(x, y) = \min\{x, y\}$ for all $x, y \geq 0$, known as *mating with fidelity*, or

(M3) $\zeta(x, y) = x$ for all $x, y \geq 0$, which is the mating function for the ordinary GWP with asexual reproduction.

Ultimate extinction for BGWP with mating functions like (M1) and (M2) but also others has been examined by Daley(1968), Bruss(1984), Hull(1982,1984) and most recently by Daley, Hull and Taylor(1986), the latter under no further assumption on ζ than superadditivity. Indeed, excluding the trivial case $\xi_{n,j} \equiv 1, \eta_{n,j} \equiv 1$, their main result reads as follows:

Theorem 1.1. *(Daley, Hull and Taylor) Let* $\boldsymbol{q}(j) = P(Z_n = 0 \text{ for some } n | Z_0 = j)$ *denote the extinction probability of* $(Z_n)_{n\geq 0}$ *given* j *initial mating units and* $\boldsymbol{m}(j) = j^{-1}E(Z_1 | Z_0 = j)$ *the respective mean reproduction rate. Then for every BGWP* $(Z_n)_{n\geq 0}$ *with superadditive mating function* $\boldsymbol{q}(j) = 1$ *for each* $j \geq 1$ *holds iff* $\lim_{j\to\infty} \boldsymbol{m}(j) \leq 1$.

The present work has been motivated by a discussion at the end of the paper by Daley, Hull and Taylor(1986). Numerical calculations they have done for a BGWP with supercritical Poissonian reproduction, more precisely $(\xi_{n,j}, \eta_{n,j}) \sim Poi(1.2) \otimes Poi(1.2)$, and promiscuous mating show that, as $j \to \infty$, $\boldsymbol{q}(j)$ quickly tends to a value about 1.33 times the respective extinction probability for the ordinary GWP with j ancestors and the same reproduction law, given by some q^j. They point out that they have no theoretical justification for that phenomenon and a little reflection shows that it is in fact a nontrivial problem.

In the present article we want to compare more deeply the extinctive behavior of a promiscuous BGWP $(Z_n)_{n\geq 0}$ with that of its asexual counterpart, henceforth denoted by $(F_n)_{n\geq 0}$, where females reproduce without mating. We always assume that $(Z_n)_{n\geq 0}$ has a product reproduction law $\boldsymbol{p}^F \otimes \boldsymbol{p}^M$, where $\boldsymbol{p}^F = (p_n^F)_{n\geq 0}$ and $\boldsymbol{p}^M = (p_n^M)_{n\geq 0}$ are probability distributions on $\mathbb{N}_0$. Let P_j be such that $P_j(Z_0 = F_0 = j) = 1$ for each $j \geq 1$ and define the extinction probability function

$$\boldsymbol{q}(j) \stackrel{\text{def}}{=} P_j(Z_n = 0 \text{ for some } n \geq 0), \quad j \in \mathbb{N}_0,$$

pertaining to $(Z_n)_{n\geq 0}$. Plainly, the reproduction law of the ordinary GWP $(F_n)_{n\geq 0}$ is $\boldsymbol{p}^F$, its extinction probability function q^j for some $q \in [0,1]$.

Suppose we are in the supercritical case defined as noncertain extinction, that is $\boldsymbol{q}(j) < 1$ for each $j \in \mathbb{N}$. Given our assumptions, an equivalent condition is that $\boldsymbol{p}^F$ has mean $\mu > 1$ and $p_0^M < 1$. Our main result, Theorem 3.1, then provides upper and lower bounds for $\boldsymbol{q}(j)$ in terms of q^j, if $p_0^M \leq q$, and of $(p_0^M)^j$, if $p_0^M > q$. In the latter case, exponential convergence to 1, as $j \to \infty$, of the ratio $\boldsymbol{q}(j)/(p_0^M)^j$ yields as a simple consequence (Corollary 3.2). Otherwise, however, convergence of $\boldsymbol{q}(j)/q^j$ turns out to be a very difficult question that will be dealt with in another article because the necessary arguments are of a totally different nature than those presented here. They involve potential theory and particularly the Martin boundary of an ordinary GWP. A further brief discussion is given at the end of Section 3.

Section 4 contains the proof of Theorem 3.1 while Section 5 provides upper and lower envelopes for $\boldsymbol{q}(j)$ that lead to very accurate numerical approximations based upon iteration. Some results for the Poissonian case are presented in Section 6. In particular, we have reproduced the approximations by Daley et al.(1986) mentioned above. Surprisingly, all our numerical results (and that includes further ones not given here for the binary splitting and the linear fractional case) strongly indicate rapid convergence of $\boldsymbol{q}(j)/q^j$ if $p_0^M < q$, thus being in contrast to the afore-mentioned problems with a theoretical justification.

Whenever dealing with the extinction probability function of a branching process $(Z_n)_{n\geq 0}$, the natural thing to start with is a look at generating functions. So let us take a brief look at the explicit form of the generating function $h_{k,n}$ of Z_n, given $Z_0 = k$, i.e.

$$h_{k,n}(s) = E_k s^{Z_n} \quad \text{for } k \geq 1,\ n \geq 0 \text{ and } s \in [-1,1],$$

Let f be the generating function of p^F, i.e. $f(s) = \sum_{n\geq 0} p_n^F s^n$. Then one can show by induction that

$$h_{k,n}(s) = 1 + \sum_{\pi \in \{0,1\}^n} (-1)^{|\pi|} \Big((f^{(\pi_1)} \circ \ldots \circ f^{(\pi_n)})^k(s) - (f^{(\pi_1)} \circ \ldots \circ f^{(\pi_n)})^k(1) \Big) \qquad (1.3)$$

for all $k, n \geq 1$, where $f^{(0)}(s) = f(s)$, $f^{(1)}(s) = p_0^M f(s)$, $\pi = (\pi_1, ..., \pi_n)$ and $|\pi| = \sum_{j=1}^n \pi_j$. With f_n denoting the n-fold iteration of f, (1.3) may be rewritten as

$$h_{k,n}(s) \;=\; f_n^k(s) + \sum_{\pi \neq (0,...,0)} (-1)^{|\pi|} \Big((f^{(\pi_1)} \circ ... \circ f^{(\pi_n)})^k(s) - (f^{(\pi_1)} \circ ... \circ f^{(\pi_n)})^k(1) \Big) \quad (1.4)$$

which is noteworthy because $f_n^k(s)$ is nothing but the generating function of F_n under P_k. In particular, we have

$$\boldsymbol{q}(k) \;=\; q^k + \lim_{n\to\infty} \sum_{\pi \neq (0,...,0)} (-1)^{|\pi|} \Big((f^{(\pi_1)} \circ ... \circ f^{(\pi_n)})^k(q) - (f^{(\pi_1)} \circ ... \circ f^{(\pi_n)})^k(1) \Big), \quad (1.5)$$

since $\boldsymbol{q}(k) = \lim_{n\to\infty} h_{k,n}(q)$ and $f_n(q) = q$ for each $n \geq 0$. Unfortunately, the occurring sum in (1.5) does not appear to be amenable to a further analysis of $\boldsymbol{q}(k)$ and we have thus turned to an alternative approach.

2. A FUNCTIONAL EQUATION FOR THE EXTINCTION PROBABILITY *q(k)*

For a moment we adopt a more general viewpoint to be described next. Define the ordinary GWP $(F_n)_{n\geq 0}$ by

$$F_{n+1} \;=\; \sum_{j=1}^{F_n} \xi_{n,j} \quad \text{for } n \geq 0 \quad (2.1)$$

and denote by $\boldsymbol{R} = (r_{i,j})_{i,j\geq 0}$ its transition matrix. Let $\kappa : I\!N_0 \to [0,1]$ be a function, called *killing rate*. We kill the process F_n at state i with probability $\kappa(i)$, which means we send it to a grave, for simplicity taken as 0.

The killed process $(Z_n)_{n\geq 0}$ is a Markov chain with transition matrix $\boldsymbol{P} = (p_{i,j})_{i,j\geq 0}$, where

$$p_{i,j} \;=\; \mathbf{1}_{\{0\}}(j)\kappa(i) + (1-\kappa(i)) r_{i,j}.$$

Define again the extinction probability function

$$\boldsymbol{q}(j) \;\stackrel{\text{def}}{=}\; P_j(Z_n = 0 \text{ for some } n \geq 0), \quad j \in I\!N_0.$$

A function $h : I\!N_0 \to I\!R$ is called (right) harmonic for $\boldsymbol{P}$ if it is nonnegative and satisfies $h = \boldsymbol{P}h$. We then have the obvious

LEMMA 2.1. *The extinction probability function* $\boldsymbol{q}$ *is a harmonic function for* $\boldsymbol{P}$. *In detail,*

$$\boldsymbol{q}(j) \;=\; \kappa(j) \;+\; (1-\kappa(j))\, E_j\, \boldsymbol{q}(F_1) \quad \textit{for all } j \geq 0. \quad (2.2)$$

$\boldsymbol{q}$ *is uniquely determined as the smallest harmonic solution with* $\boldsymbol{q}(0) = 1$ *and given by* $\boldsymbol{q} = \lim_{n\to\infty} \boldsymbol{P}^n \delta$ *where* $\delta = \mathbf{1}_{\{0\}}$.

Since $\boldsymbol{q}(j) = 1 - P_j(Z_n \geq 1 \text{ for all } n \geq 0)$, it is easily verified that

$$\boldsymbol{q}(j) \;=\; 1 \;-\; E_j\left(\prod_{n\geq 0}\Big(1-\kappa(F_n)\Big)\right). \tag{2.3}$$

Returning to the situation of a BGWP $(Z_n)_{n\geq 0}$ with promiscuous mating, we are obviously given a killed Markov chain with killing rate $\kappa(j) = (p_0^M)^j$ for $j \in I\!N_0$. Indeed, if $F_n = j$, this process is killed if no males are produced which happens with probability $(p_0^M)^j$.

With the help of (2.2) we will derive upper and lower bounds for $\boldsymbol{q}(k)$ in the following section (Theorem 3.1). Let us close the present one with a further lemma that shows $\boldsymbol{q}(k)$ to be the unique solution of (2.2) that satisfies $\boldsymbol{q}(0) = 1$ and $\boldsymbol{q}(\infty) \stackrel{\text{def}}{=} \lim_{k\to\infty} \boldsymbol{q}(k) = 0$, for the latter see at the beginning of Section 3.

LEMMA 2.2. *There is exactly one solution* $\boldsymbol{q}$ *of (2.2) with* $\boldsymbol{q}(0) = 1$ *and* $\boldsymbol{q}(\infty) = 0$.

PROOF. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two solutions of (2.2) having the stated properties. Put $\kappa = p_0^M$ and $\boldsymbol{d} = \boldsymbol{q}_1 - \boldsymbol{q}_2$, which clearly satisfies the equation $\boldsymbol{d}(j) = (1-\kappa^j)E_j\,\boldsymbol{d}(F_1)$ for all $j \geq 0$ as well as $\boldsymbol{d}(0) = \boldsymbol{d}(\infty) = 0$. By iterating this equation we obtain for every $j \geq 1$

$$\begin{aligned} \boldsymbol{d}(j) \;&\leq\; E_j\,\boldsymbol{d}(F_1) \;=\; E_j(1-\kappa^{F_1})\boldsymbol{d}(F_2) \;\leq\; E_j\mathbf{1}_{\{F_1>0\}}\boldsymbol{d}(F_2) \\ &\leq\; \dots \;\leq\; E_j(1-\kappa^{F_{n-1}})\boldsymbol{d}(F_n) \;\leq\; E_j\mathbf{1}_{\{F_{n-1}>0\}}\boldsymbol{d}(F_n) \;\to\; 0, \end{aligned}$$

as $n \to \infty$, since $\mathbf{1}_{\{F_{n-1}>0\}}\boldsymbol{d}(F_n) \to 0$ P_j-a.s.

3. UPPER AND LOWER BOUNDS FOR $\boldsymbol{q}(k)$

From now on we simplify our notation in that we put $p_n = p_n^F$ for $n \geq 0$ and $\kappa = p_0^M$. The particular choice of the other p_n^M will be of no relevance for the subsequent analysis.

We consider a BGWP $(Z_n)_{n\geq 0}$ obtained by killing at rate $\kappa(j) = \kappa^j$ an ordinary GWP $(F_n)_{n\geq 0}$ of females if no mating occured. According to (2.3) the extinction probability function $\boldsymbol{q}$ is then given by

$$\boldsymbol{q}(k) \;=\; 1 - E_k\left(\prod_{n\geq 0}(1-\kappa^{F_n})\right). \tag{3.1}$$

Let q be the extinction probability of $(F_n)_{n\geq 0}$ given $F_0 = 1$, that is under P_1. Consequently, q^k gives the respective probability under P_k.

It is a trivial consequence of the model assumptions that q^k always forms a lower bound for $\boldsymbol{q}(k)$ for each $k \geq 0$. However, under which conditions is the latter of the same order of magnitude as the former, i.e. when does

$$1 \leq \liminf_{k\to\infty} \frac{\boldsymbol{q}(k)}{q^k} \;\leq\; \limsup_{k\to\infty} \frac{\boldsymbol{q}(k)}{q^k} \;<\; \infty \tag{3.2}$$

hold true? Of course, this is an interesting question only in the supercritical case, in the present setup equivalent to

$$\kappa < 1 \quad \text{und} \quad \mu \stackrel{\text{def}}{=} \sum_{j\geq 1} jp_j > 1, \tag{3.3}$$

which are therefore standing assumptions throughout.

It is next readily seen from (3.1) that $\boldsymbol{q}(1) < 1$, and that $\boldsymbol{q}(k) \leq \boldsymbol{q}^k(1) \to 0$, as $k \to \infty$. Use the fact that a BGWP with promiscuous mating and k ancestors stochastically dominates the sum of k independent BGWP with the same reproductive behavior but one ancestor. By Lemma 2.2, $\boldsymbol{q}(k)$ is thus the unique solution of (2.2).

Theorem 3.1 below is the main result of this article and shows that (3.2) is indeed valid unless $\kappa \geq q$. Recall that $f(s) = \sum_{j\geq 0} p_j s^j$ denotes the generating function of $\boldsymbol{p}^F = (p_n)_{n\geq 0}$ and f_n its n-fold iterate for each $n \geq 0$, in particular $f_0(s) = s$.

THEOREM 3.1. *Assuming (3.3), the following assertions hold for all $k \geq 1$:*

(i) *If $\kappa < p_0$ then*

$$1 \;\leq\; \frac{\boldsymbol{q}(k)}{q^k} \;\leq\; 1 + \frac{\kappa}{p_0}. \tag{3.4}$$

(ii) *If $\kappa = p_0$ then*

$$1 + \frac{1-q}{1+q-p_0} \;\leq\; \frac{\boldsymbol{q}(k)}{q^k} \;\leq\; 2 \tag{3.5}$$

(iii) *If $p_0 < \kappa < q$ then*

$$1 + \frac{\kappa(1-q)}{\kappa q + (1-\kappa)p_0} \;\leq\; \frac{\boldsymbol{q}(k)}{q^k} \;\leq\; (n+2)\left(\frac{1}{1-\kappa} + \frac{p_0}{\kappa}\right) \tag{3.6}$$

where n is determined through $f_n(p_0) < \kappa \leq f_{n+1}(p_0)$.

(iv) *If $\kappa = q$ then*

$$\frac{1-q}{q(a_1-q)+(1-q)} \;\leq\; \frac{\boldsymbol{q}(k)}{a_k q^k} \;\leq\; \frac{1}{1-q} + \frac{p_0}{q} \tag{3.7}$$

where $a_k \stackrel{\text{def}}{=} E_k(\tau|\tau < \infty)$, $\tau = \inf\{n \geq 0 : F_n = 0\}$.

(v) *If $\kappa > q$ then*

$$1 \;\leq\; \frac{\boldsymbol{q}(k)}{\kappa^k} \;\leq\; 1 + \frac{f(\kappa)}{\kappa - f(\kappa)}. \tag{3.8}$$

The proof of Theorem 3.1 will be given in the Section 4. Observe that (3.6) indeed completely covers the case $p_0 < \kappa < q$ because $f_n(p_0)$ strictly increases to q. Note also that $p_0 = \kappa$ holds in particular when $\boldsymbol{p}^F = \boldsymbol{p}^M$, that is when male and female offspring are produced according to the same distribution. Since a_k evidently tends to infinity as $k \to \infty$, (3.7) implies $\sup_{k\geq 1} q^{-k}\boldsymbol{q}(k) = \infty$ if $\kappa = q$. Figure 1 gives an illustration of the obtained bounds for the case considered by Daley et al.(1986) where $\boldsymbol{p}^F$ is Poisson with mean $\mu = 1.2$.

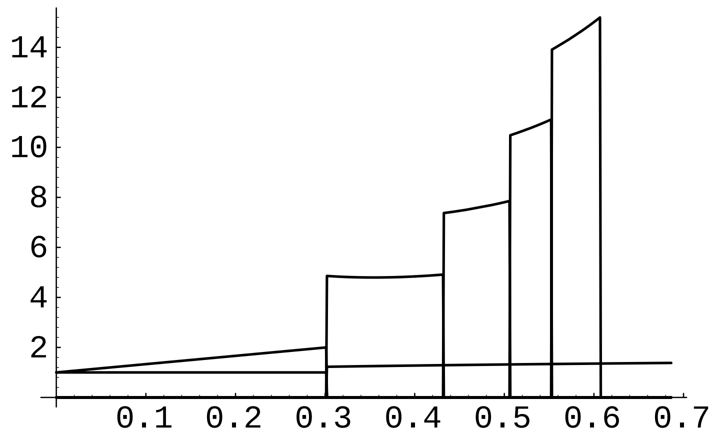

FIGURE 1. Lower and upper bounds for $\boldsymbol{r}(k)$ in the Poisson case with $\mu = 1.2$.

Define next $\boldsymbol{r}(k) = q^{-k}\boldsymbol{q}(k)$ for $k \geq 0$ and

$$\boldsymbol{r}^*(k) \;=\; \begin{cases} \boldsymbol{r}(k), & \text{if } \kappa < q \\ \boldsymbol{r}(k)/a_k, & \text{if } \kappa = q \\ \boldsymbol{q}(k)/\kappa^k, & \text{if } \kappa > q \end{cases} \tag{3.9}$$

It is natural to ask next whether or not $\boldsymbol{r}^*(k)$ converges as $k \to \infty$. Unfortunately, the answer is simple only in case $\kappa > q$ where even exponential convergence holds true ($f(\kappa) < \kappa$).

COROLLARY 3.2. *If $\kappa > q$ then for all $k \geq 0$*

$$1 \;\leq\; \frac{\boldsymbol{q}(k)}{\kappa^k} \;\leq\; 1 \;+\; C\left(\frac{f(\kappa)}{\kappa}\right)^k, \tag{3.10}$$

where C is the upper bound in (3.8).

PROOF. By (3.8) in Theorem 3.1, $\boldsymbol{q}(k) \leq C\kappa^k$ for all $k \geq 0$ whence by (2.2)

$$1 \;\leq\; \boldsymbol{r}(k) \;=\; 1 \;+\; \frac{1-\kappa^k}{\kappa^k} E_k \boldsymbol{q}(F_1) \;\leq\; 1 \;+\; \frac{C(1-\kappa^k)}{\kappa^k} E_k \kappa^{F_1} \;\leq\; 1 \;+\; \frac{Cf^k(\kappa)}{\kappa^k}$$

which is the asserted result.

The case $\kappa \leq q$ is much more difficult because, in contrast to the previous case, the bounding functions of Theorem 3.1 do not provide any insight into the asymptotic behavior of $\boldsymbol{q}(k)$ apart from the crude information that $\boldsymbol{r}^*(k)$ remains bounded. Before discussing this further, we state the following result on $\boldsymbol{r}$

Let $\hat{P}_k = P_k(\cdot|F_n \to 0)$ with expectation operator $\hat{E}_k$. It is well-known that $(F_n)_{n\geq 0}$ forms again an ordinary (subcritical) GWP under $\hat{P}_k$ with k ancestors, offspring generating function $\hat{f}(s) = q^{-1}f(sq)$ and reproduction mean $\hat{\mu} = f'(q) < 1$, see Athreya and Ney(1972, p.47f).

LEMMA 3.3. *The function* $\boldsymbol{r}(k) = q^{-k}\boldsymbol{q}(k)$ *satisfies*

$$\boldsymbol{r}(k) \;=\; \left(\frac{\kappa}{q}\right)^k \;+\; (1-\kappa^k)\,\hat{E}_k\boldsymbol{r}(F_1) \tag{3.11}$$

for each $k \geq 0$.

PROOF. The identity is a direct consequence of (2.2) if we note that $\hat{P}_k(F_1 = j) = P_k(F_1 = j)\; q^{j-k}$ holds for all $j, k \geq 0$ and thus

$$q^{-k}E_k\boldsymbol{r}(F_1) \;=\; \sum_{j\geq 0} P(F_1 = j)q^{k-j}\boldsymbol{r}(j) \;=\; \hat{E}_k\boldsymbol{r}(F_1).$$

Iterating equation (3.11) leads to the basic identity

$$\boldsymbol{r}(k) \;=\; \left(\frac{\kappa}{q}\right)^k \;+\; \hat{E}_k\left(\sum_{j=1}^{\tau}\left(\frac{\kappa}{q}\right)^{F_j}\prod_{i=0}^{j-1}\left(1-\kappa^{F_i}\right)\right) \tag{3.12}$$

where τ is here and for all the extinction time of $(F_n)_{n\geq 0}$. As one can see from this identity, the limiting behavior of $\boldsymbol{r}(k)$ for $k \to \infty$ is related to that of the time reversion at τ of $(F_n)_{n\geq 0}$ under $\hat{P}_k$. For an analysis of the latter potential theoretic arguments have to be employed involving the Martin boundary of $(F_n)_{n\geq 0}$. Since such arguments cannot be given shortly and are of a totally different nature than those given here we have decided to present them in a separate article. We finally note, however, that in contrast to the picture conveyed by the numerical results of Section 6 we have good reasons to believe that $\boldsymbol{r}(k)$ does *not* generally converge.

4. PROOF OF THEOREM 3.1

Recall that $\boldsymbol{P}$ denotes the transition matrix of $(Z_n)_{n\geq 0}$ and forms an operator that maps a function $h : \mathbb{N}_0 \to \mathbb{R}$ on $\boldsymbol{P}h$ given by $\boldsymbol{P}h(j) = \kappa^j + (1-\kappa^j)E_jh(F_1)$. Notice that $\boldsymbol{P}$ is order-preserving. The subsequent proof makes frequent use of the following two conclusions that can be drawn from Lemmata 2.1 and 2.2.

(1) If h is a superharmonic function for $\boldsymbol{P}$ ($h \geq \boldsymbol{P}h$) and $h(0) = 1$ then $\boldsymbol{P}^n h$ decreases to a harmonic limit $h_\infty \geq \boldsymbol{q}$ whence $h \geq \boldsymbol{q}$.

(2) If g is a subharmonic function for $\boldsymbol{P}$ with $g(0) = 1$ and if g is upper bounded by some superharmonic function h with $h(\infty) = \lim_{k\to\infty} h(k) = 0$ then $\boldsymbol{P}^n g$ increases to $\boldsymbol{q}$ implying $g \leq \boldsymbol{q}$.

Now consider the functions $h_c(0) = 1,\ h_c(k) = cq^k$ for $k \geq 1$ where $c \geq 1$ is to be suitably chosen below. Notice that $h_c \leq h_d$ if $c \leq d$ and that $h_c(\infty) = 0$. It follows

$$\boldsymbol{P}h_c(k) \;=\; \kappa^k + (1-\kappa^k)(p_0^k + cE_k\mathbf{1}_{\{F_1>0\}}q^{F_1}) \;=\; \kappa^k + (1-\kappa^k)(p_0^k + c(q^k - p_0^k)) \tag{4.1}$$

for all $k \geq 0$ whence $h_c \leq (\geq) \boldsymbol{P} h_c$ is equivalent to

$$c \ \leq (\geq) \ c_k \ \stackrel{\text{def}}{=} \ \frac{\kappa^k + (1-\kappa^k)p_0^k}{\kappa^k q^k + (1-\kappa^k)p_0^k} \ = \ 1 + \frac{\kappa^k(1-q^k)}{\kappa^k q^k + (1-\kappa^k)p_0^k}.$$

If $\kappa < p_0$ it is easily seen that $1 \leq c_k \to 1$, as $k \to \infty$, and that $\sup_{k\geq 1} c_k \leq 1 + \kappa/p_0$. Hence (3.4) follows from (1) by taking $h = h_{1+\kappa/p_0}$.

If $\kappa = p_0$ then

$$c_1 \ \leq \ c_k \ = \ 1 \ + \ \frac{1-q^k}{q^k + 1 - \kappa^k} \ \leq \ \lim_{k\to\infty} c_k \ = \ 2, \tag{4.2}$$

which implies (3.5) by first using (1) with $h = h_2$ and then (2) with $g = h_{c_1}$ and the same h. We note for the first inequality in (4.2) that

$$\begin{aligned} c_k - 1 \ &= \ \left(\frac{q^k - \kappa^k}{1-q^k} + \frac{1}{1-q^k}\right)^{-1} \ = \ \left(\frac{q-\kappa}{1-q} \cdot \frac{\sum_{j=0}^{k-1} \kappa^{k-1-j} q^j}{\sum_{j=0}^{k-1} q^j} + \frac{1}{1-q^k}\right)^{-1} \\ &\geq \ \left(\frac{q-\kappa}{1-q} + \frac{1}{1-q}\right)^{-1} \ = \ c_1 - 1. \end{aligned}$$

If $p_0 < \kappa < q$ then the same approach yields

$$c_1 \ \leq \ c_k \ = \ 1 \ + \ \frac{1-q^k}{q^k + (1-\kappa^k)(p_0/\kappa)^k} \ \leq \ \lim_{k\to\infty} c_k \ = \ \infty, \tag{4.3}$$

where the left inequality in (4.3) follows by a similar estimation as that leading to (4.2). We infer the lower bound in (3.6) by another appeal to (2) after having provided an upper superharmonic bound. Unfortunately, the latter requirement as well as the upper bound in (3.6) must be derived from another function class because the c_k's are now unbounded. We will finish the proof of the lower bound after (4.7).

Let us introduce the function

$$g(s,k) \ = \ 1 + \sum_{j\geq 0} \left(f_j^k(s) - f_j(p_0)^k\right) q^{-k}$$

for $k \geq 0$ and $s \in [0,1)$, hence $g(s,0) = 0$, which in case $f_n(p_0) \leq s \leq f_{n+1}(p_0)$, $n \geq 0$, satisfies

$$\begin{aligned} g(s,k) \ &\geq \ 1 + \sum_{j\geq 0} \left(f_{j+n}^k(p_0) - f_j^k(p_0)\right) q^{-k} \\ &= \ 1 + \sum_{i=0}^{n-1} \sum_{j\geq 0} \left(f_{j+i+1}^k(p_0) - f_{j+i}^k(p_0)\right) q^{-k} \\ &= \ (n+1) - \sum_{i=0}^{n-1} f_j^k(p_0) q^{-k} \end{aligned} \tag{4.4}$$

and, by a similar estimation,

$$g(s,k) \ \le\ (n+2) - \sum_{i=0}^{n} f_i^k(p_0) q^{-k}. \tag{4.5}$$

Recalling that $\boldsymbol{R}$ denotes the transition operator of $(F_n)_{n\ge 0}$, the important feature of this function can be stated as

$$E_k g(s,F_1) q^{F_1} \ =\ \boldsymbol{R}(g(s,\cdot)q^{\cdot})(k) \ =\ g(s,k)q^k - s^k.$$

for every $s \in [0,1]$ and $k \ge 0$. Namely,

$$\begin{aligned} E_k g(s,F_1)q^{F_1} \ &=\ E_k \mathbf{1}_{\{F_1>0\}} q^{F_1} \ +\ \sum_{j\ge 0} E_k \left(f_j^{F_1}(s) - f_j^{F_1}(p_0) \right) \\ &=\ (q^k - (p_0)^k) \ +\ \sum_{j\ge 0} \big(f_{j+1}^k(s) - f_{j+1}^k(p_0) \big) \\ &=\ q^k \left(1 - \left(\frac{p_0}{q}\right)^k + \sum_{j\ge 0} \big(f_j^k(s) - f_j^k(p_0) \big) q^{-k} - \left(\frac{s}{q}\right)^k + \left(\frac{p_0}{q}\right)^k \right) \\ &=\ g(s,k)q^k \ -\ s^k. \end{aligned} \tag{4.6}$$

Consider now the functions $\hat{h}_c(0) = 1$, $\hat{h}_c(k) = cg(\kappa,k)q^k$ for $k \ge 1$ and $c > 0$. Again, $\hat{h}_c \le \hat{h}_d$ if $c \le d$ and $\hat{h}_c(\infty) = 0$. Then

$$\begin{aligned} \boldsymbol{P}\hat{h}_c(k) \ &=\ \kappa^k + (1-\kappa^k)\big(p_0^k + cE_k g(\kappa, F_1)q^{F_1}\big) \\ &=\ \kappa^k + (1-\kappa^k)\Big(p_0^k + c\big(g(\kappa)q^k - \kappa^k\big)\Big), \end{aligned}$$

so that $\hat{h}_c \ge \boldsymbol{P}\hat{h}_c$ holds iff

$$c \ \ge\ \hat{c}_k \ \stackrel{\text{def}}{=}\ \frac{\kappa^k + (1-\kappa^k)(p_0)^k}{\kappa^k g(\kappa,k)q^k + (1-\kappa^k)\kappa^k} \ =\ \frac{1 + (1-\kappa^k)(p_0/\kappa)^k}{g(\kappa,k)q^k + (1-\kappa^k)}$$

for all $k \ge 1$. In order to show the upper bound in (3.6) for $\sup_{k\ge 1} \boldsymbol{q}(k)q^{-k}$ it suffices by (1) to verify that this bound is also an upper one for $\sup_{k\ge 1} g(\kappa,k)\hat{c}_k$. Let $f_n(p_0) < \kappa \le f_{n+1}(p_0)$ for an arbitrary $n \ge 0$, in which case $\sup_{k\ge 1} g(\kappa,k) \le n+2$ by (4.5). Now the desired result follows from

$$\begin{aligned} g(\kappa,k)\hat{c}_k \ &\le\ g(\kappa,k)\,\frac{1 + (1-\kappa^k)(p_0/\kappa)^k}{1-\kappa^k} \\ &=\ g(\kappa,k)\left\{ \left(\frac{1}{1-\kappa^k}\right) + \left(\frac{p_0}{\kappa}\right)^k \right\} \\ &\le\ (n+2)\left\{ \frac{1}{1-\kappa} + \frac{p_0}{\kappa} \right\} \ \stackrel{\text{def}}{=}\ \hat{c} \end{aligned} \tag{4.7}$$

for every $k \ge 1$.

By noting that $h_{c_1} \le \hat{h}_{\hat{c}}$ the proof of the lower bound in (3.6) is also settled.

For the case $\kappa = q$ we proceed in the same manner, this time choosing $h_c(0) = 1$ and $h_c(k) = cg(q,k)q^k$ for $k \geq 1$. Recall from before Lemma 3.3 $\hat{P}_k = P_k(\cdot|F_n \to 0)$ with expectation operator $\hat{E}_k$. $(F_n)_{n\geq 0}$ forms a subcritical GWP under $\hat{P}_k$ with k ancestors and reproduction generating function $\hat{f}(s) = q^{-1}f(qs)$. Furthermore, its n-th iterate $\hat{f}_n$ takes the form $\hat{f}_n(s) = q^{-n}f_n(qs)$ for each $n \geq 0$. Consequently, for each $k \geq 0$,

$$\begin{aligned} g(q,k) &= 1 + \sum_{j\geq 0} \left(f_j^k(q) - f_j^k(p_0)\right) q^{-k} \;=\; 1 + \sum_{j\geq 0} \left(1 - \hat{f}_j^k(p_0/q)\right) \\ &= 1 + \sum_{j\geq 1} \hat{P}_k(F_j > 0) \;=\; \sum_{j\geq 0} \hat{P}_k(\tau > j) \;=\; \hat{E}_k\tau \;=\; a_k. \end{aligned}$$

For inequality (4.9) below, we note that by the same argument which proved (4.2)

$$a_k q^k \;=\; q^k + (1-q^k)\sum_{j\geq 0} \frac{q^k - f_j(p_0)}{1-q^k} \;\leq\; q + \sum_{j\geq 0} \frac{q - f_j(p_0)}{1-q} \;=\; \frac{q(a_1 - q)}{1-q} \tag{4.8}$$

for all $k \geq 1$. Now $h_c \leq (\geq) \boldsymbol{P} h_c$ again holds iff $c \leq (\geq) c_k$ for all $k \geq 1$ where c_k here takes the form

$$c_k \;=\; \frac{1 + (1-q^k)(p_0/q)^k}{a_k q^k + (1-q^k)}.$$

The asserted inequality (3.7) thus follows from

$$c_k \;\leq\; \frac{1 + (1-q^k)(p_0/q)^k}{1-q^k} \;\leq\; \frac{1}{1-q} + \frac{p_0}{q}$$

and

$$c_k \;\geq\; \frac{1}{a_k q^k + (1-q^k)} \;\geq\; \frac{1}{\dfrac{q(a_1-q)}{1-q} + 1} \;=\; \frac{1-q}{q(a_1-q) + (1-q)} \tag{4.9}$$

for each $k \geq 1$, of course, by a further appeal to (1) and (2).

We finally have to consider the case $\kappa > q$ and put $h_c(0) = 1$, $h_c(k) = c\kappa^k$ for $k \geq 1$. Then $h_c \leq (\geq) \boldsymbol{P} h_c$ holds iff

$$c \;\leq (\geq)\; c_k \;\stackrel{\text{def}}{=}\; \frac{\kappa^k + (1-\kappa^k)p_0^k}{\kappa^k - (1-\kappa^k)(f(\kappa)^k - p_0^k)}$$

for all $k \geq 1$. (3.8) now follows from $\lim_{k\to\infty} c_k = 1$ (notice $f(\kappa) < \kappa$) and the inequality

$$\begin{aligned} 1 \leq\; c_k &= 1 + \frac{(1-\kappa^k)f^k(\kappa)}{\kappa^k - (1-\kappa^k)(f^k(\kappa) - p_0^k)} \\ &\leq 1 + \frac{f^k(\kappa)}{\kappa^k - f^k(\kappa)} \;=\; 1 + \frac{1}{(\kappa/f(\kappa))^k - 1} \\ &\leq 1 + \frac{1}{(\kappa/f(\kappa)) - 1} \;=\; 1 + \frac{f(\kappa)}{\kappa - f(\kappa)} \end{aligned}$$

for all $k \geq 1$. The proof of Theorem 3.1 is herewith complete.

5. Lower and upper envelopes for $q(k)/q^k$

The bounds given in Theorem 3.1 may clearly fail to be very accurate as being valid for all $k \geq 1$. But the bounds we have provided in the previous sections can be used as initializations for a recursive scheme that successively leads to sharpened upper and lower bounds for $\boldsymbol{q}$. The procedure is easily implemented on a computer and gives numerical results in those cases where the distribution of F_1 is known under each P_k. Such a case, namely where reproduction laws are Poissonian, is presented further below after having introduced the iteration scheme and its relevant properties.

Let $\boldsymbol{q}_0$ be any given approximation of $\boldsymbol{q}$ with $\boldsymbol{q}_0(0) = 1$ and superharmonic upper bound h satisfying $h(\infty) = 0$ (standing assumption throughout). $\boldsymbol{q}_0$ itself, however, need *not* be sub- or superharmonic. From a theoretical standpoint it would then be natural to approximate $\boldsymbol{q}$ by $\boldsymbol{P}^n\boldsymbol{q}_0$ which indeed converges pointwise to $\boldsymbol{q}$, as one can easily verify (see Lemma 5.1 below). On the other hand, this iteration would require to compute $\boldsymbol{P}^n\boldsymbol{q}_0(k)$ for every $k \geq 0$ and thus involve infinitely many computations. For that reason we have used another, though similar iteration scheme which, at each step, updates the current approximation only within a finite window, however of increasing size. More precisely, we define the n-th iteration $\boldsymbol{q}_n$ recursively by

$$\boldsymbol{q}_n(k) \;=\; \begin{cases} \boldsymbol{P}\boldsymbol{q}_{n-1}(k), & \text{if } 0 \leq k < n \\ \boldsymbol{q}_{n-1}(k), & \text{if } k \geq n \end{cases} \tag{5.1}$$

for each $n \geq 1$. A simple induction shows that

$$\boldsymbol{q}_n(k) = \boldsymbol{P}^{(n-k)^+}\boldsymbol{q}_0(k) \tag{5.2}$$

for all $n \geq 0$ and $k \geq 0$. Notice that $\boldsymbol{q}_n(0) = 1$ for all $n \geq 0$. The relevant properties of this recursive scheme arev stated as

Lemma 5.1. *$\boldsymbol{q}_0 \leq (\geq)\boldsymbol{q}$ implies $\boldsymbol{q}_n \leq (\geq)\boldsymbol{q}$ as well as $\lim_{n\to\infty} \boldsymbol{q}_n = \boldsymbol{q}$. If $\boldsymbol{q}_0$ is further sub(super)harmonic for $\boldsymbol{P}$, then even*

$$\boldsymbol{q}_0 \leq (\geq)\boldsymbol{q}_1 \leq (\geq)\boldsymbol{q}_2 \leq (\geq)... \uparrow (\downarrow)\boldsymbol{q} \tag{5.3}$$

holds true.

Proof. Since $\boldsymbol{P}$ is order-preserving, we have$\boldsymbol{q}_n(k) = \boldsymbol{P}^{(n-k)^+}\boldsymbol{q}_0(k) \leq (\geq)\boldsymbol{q}$ for all $n \geq 1$ and $k \geq 0$ if $\boldsymbol{q}_0 \leq (\geq)\boldsymbol{q}$. For the same reason we infer (5.3) in case of sub(super)harmonic $\boldsymbol{q}_0$.

In view of (5.2) the convergence of $\boldsymbol{q}_n$ to $\boldsymbol{q}$ clearly follows if we prove $\boldsymbol{P}^n\boldsymbol{q}_0 \to \boldsymbol{q}$, as $n \to \infty$. But the uniform boundedness of the $\boldsymbol{P}^n\boldsymbol{q}_0$ implies that each subsequence $(n_k)_{k\geq 1}$ contains a further subsequence $(n'_k)_{k\geq 1}$ such that $\boldsymbol{P}^{n'_k}\boldsymbol{q}_0$ converges to a harmonic limit $\boldsymbol{q}_\infty$, $\boldsymbol{q}_\infty(0) = 1$. Now $\boldsymbol{q}_0 \leq h$ for some superharmonic which vanishes at ∞ implies $\boldsymbol{q}_\infty(\infty) = 0$. Consequently, $\boldsymbol{q}_\infty = \boldsymbol{q}$ by Lemma 2.2 and the proof of Lemma 5.1 is complete.

6. Numerical Results

Let us finally turn to the question of how the recursive scheme in (5.1) can be used to provide numerical results. Suppose we are given a function $\boldsymbol{q}_0^{(0)}$ whose iteration $\boldsymbol{q}_1^{(0)}$ can be easily computed. If $\kappa < q$ such functions are naturally given by Theorem 3.1 and of the simple form $\boldsymbol{q}_0^{(0)}(0) = 1$, $\boldsymbol{q}_0^{(0)}(k) = cq^k$ for $k \geq 1$ and an appropriate $c \geq 1$. However, the reader should recall from the proof of Theorem 3.1, that these functions need not necessarily be sub- or superharmonic in case $p_0 < \kappa < q$. It is for that reason we have given the more general convergence result in Lemma 5.1.

Now (5.3) is perfectly designed for recursive calculations whenever $\boldsymbol{R}$ is known. This includes the cases of Poissonian and linear fractional reproduction as well as the case of binary splitting, to mention the probably most popular ones. Let us rewrite (5.3) as a recursive equation for $\boldsymbol{r}_1^{(n)}(k) \stackrel{\text{def}}{=} q^{-k}\boldsymbol{q}_1^{(n)}(k)$ since we are interested in approximations for $\boldsymbol{r}(k) = q^{-k}\boldsymbol{q}(k)$. We have

$$\boldsymbol{r}_1^{(n)}(k) \;=\; \boldsymbol{r}_1^{(0)}(k) + (1-\kappa^k)\sum_{j=0}^{n-1} r_{j,k}\Big(\boldsymbol{r}_1^{(n-1)}(j) - \boldsymbol{r}_0^{(0)}(j)\Big)q^{j-k}. \tag{6.1}$$

A collection of numerical results we obtained for the Poissonian case are reported below. When it turned out that corresponding results in the other afore-mentioned cases look qualitatively very similar we decided to refrain from their presentation here.

	$\boldsymbol{r}(k)$	
k	(5.4)	DHT
1	1.2439	1.2439
2	1.3161	1.3161
3	1.3302	1.3300
4	1.3310	1.3308
5	1.3301	1.3300
6	1.3296	1.3292
10	1.3295	1.3296
20	1.3295	1.3295
40	1.3295	1.3296
60	1.3295	1.3293
100	1.3295	
200	1.3295	

Table 1

A comparision of numerical values for $\boldsymbol{r}(k)$ obtained from (5.4) with those by Daley et al.(1986)

We first take a look at the situation that has been examined by Daley et al.(1986) So let $\boldsymbol{p}^F$ be a Poisson distribution with mean $\mu = 1.2$ which yields $q = 0.6863$. Also let $p_0 = \kappa$. Table 1 compares, for various k, the approximated values for $\boldsymbol{r}(k)$ obtained from our recursive algorithm with those by Daley et al.(DHT) who used a method based upon truncation of the transition matrix of $(Z_n)_{n\geq 0}$. We used 400 iterations and, in order to reduce the number of computations, a stopping rule that would keep fixed at any k the values of upper and lower envelope as soon as their difference would fall below 10^{-5}. As one can see, both methods lead to almost identical numbers.

We then performed similar calculations for $p_0 = \kappa$ and varying q. The respective graphs of $\boldsymbol{r}(k)$ for $q = 0.01,\ 0.2,\ 0.5$ and 0.8 within ranges of k that provided satisfactory precision are shown in Figures 2–5. It seems that for $q < 0.5$ the graph of $\boldsymbol{r}$ always behaves like a damped oscillation that eventually settles at a limiting value between 1 and 2, whereas for $q \geq 0.5$ such a limit point is rapidly approached in a non-oscillatory manner. However, we have no theoretical justification for this apparent phenomenon.

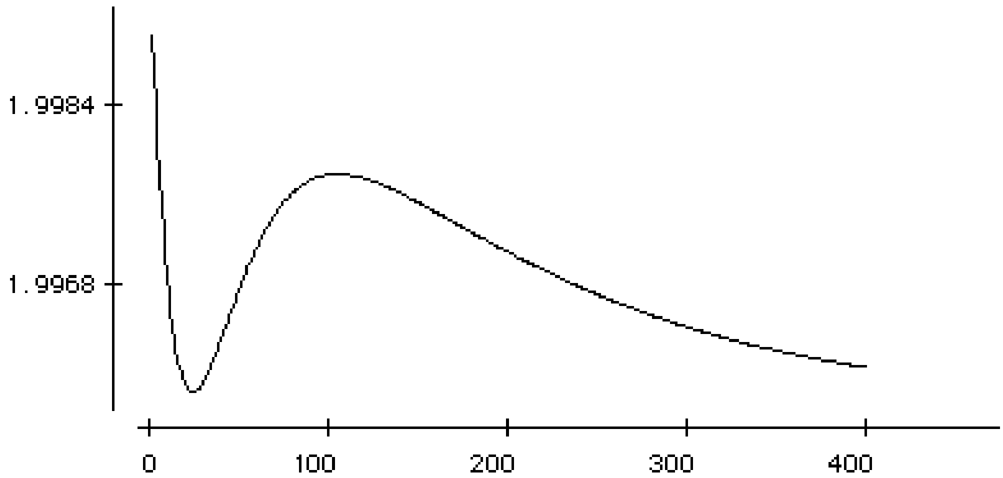

FIGURE 2. The case $q = 0.01$ and $\kappa = p_0$.

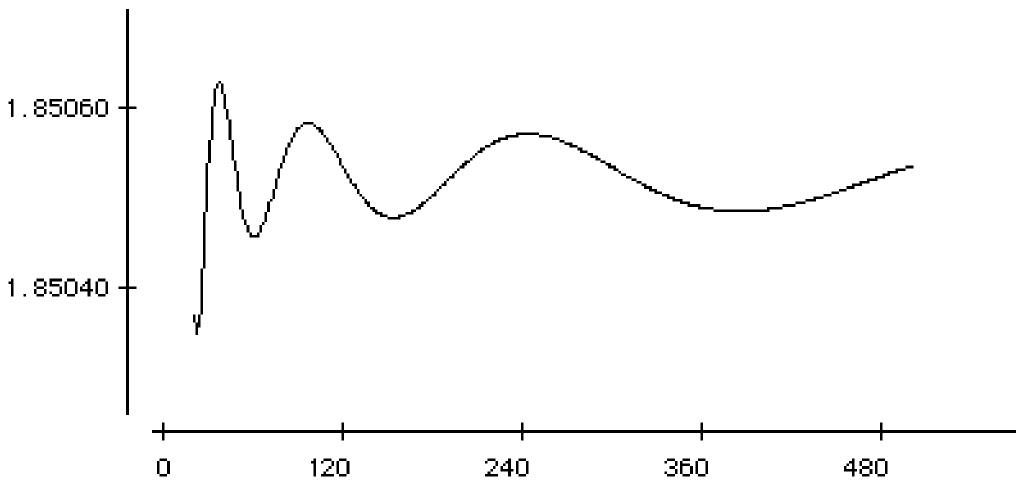

FIGURE 3. The case $q = 0.2$ and $\kappa = p_0$

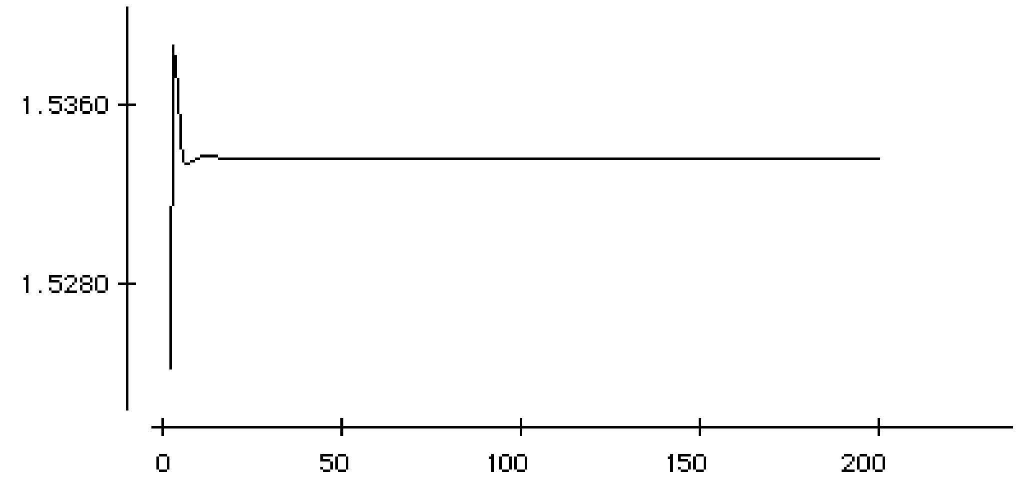

FIGURE 4. The case $q = 0.5$ and $\kappa = p_0$.

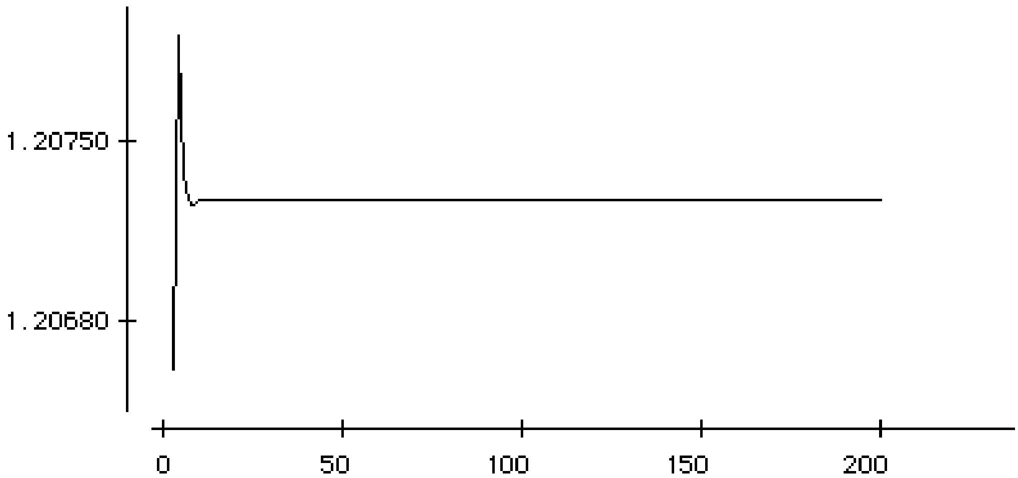

FIGURE 5. The case $q = 0.8$ and $\kappa = p_0$.

Finally, we looked at the graphs of $\boldsymbol{r}$ for fixed q but varying κ between values much smaller than the pertinent p_0 up to $\kappa = f_{10}(p_0)$. The results for $q = 0.2$ (implying $p_0 = 0.13375$ and $\mu = 2.0118$) and $\kappa = 0.05,\ 0.01,\ f(p_0),\ f_2(p_0),\ f_5(p_0)$ and $f_{10}(p_0)$ may be found in Figures 6–10. The case $\kappa = f_{10}(p_0)$ can be viewed as a good approximation of the case $\kappa = q$ within the range shown in the picture. In fact, we obtained the same curve modulo deviations of order $< 10^{-5}$ for $\kappa = f_{20}(p_0)$. Although the graphs of $\boldsymbol{r}$ for $\kappa = f_5(p_0)$ and $f_{10}(p_0)$ appear as increasing functions (with some finite limiting value according to our theorem), we suspect an oscillatory behavior of $\boldsymbol{r}$ for all $\kappa < q$, however, with rapidly decreasing amplitudes as κ increases to q, and with intervals between consecutive amplitudes that are too long to be visible in the given range chosen in the picture.

For $q \geq 0.5$ the graphs of $\boldsymbol{r}$ as κ varies look very much the same as those shown in Figures 4 and 5 where $\kappa = p_0$. We have thus refrained from displaying them here.

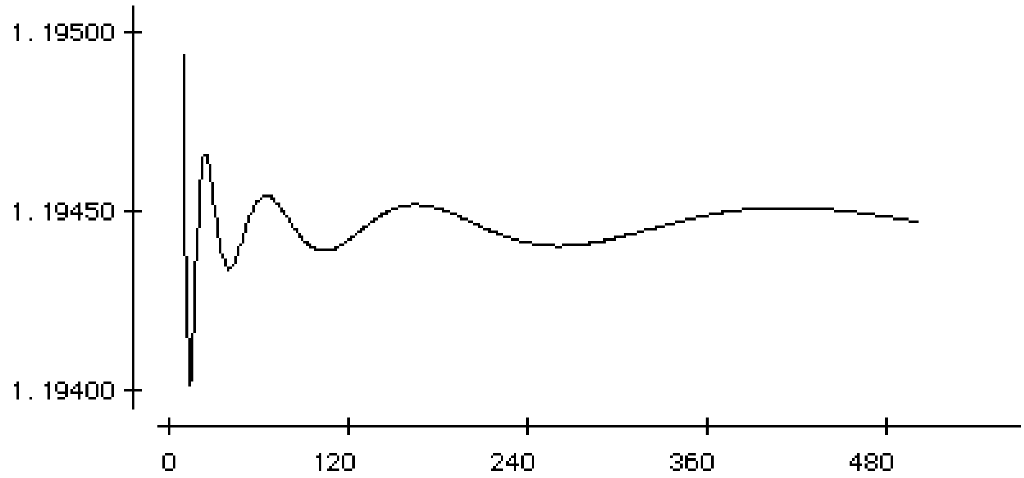

FIGURE 6. The case $q = 0.2$ and $\kappa = 0.05$.

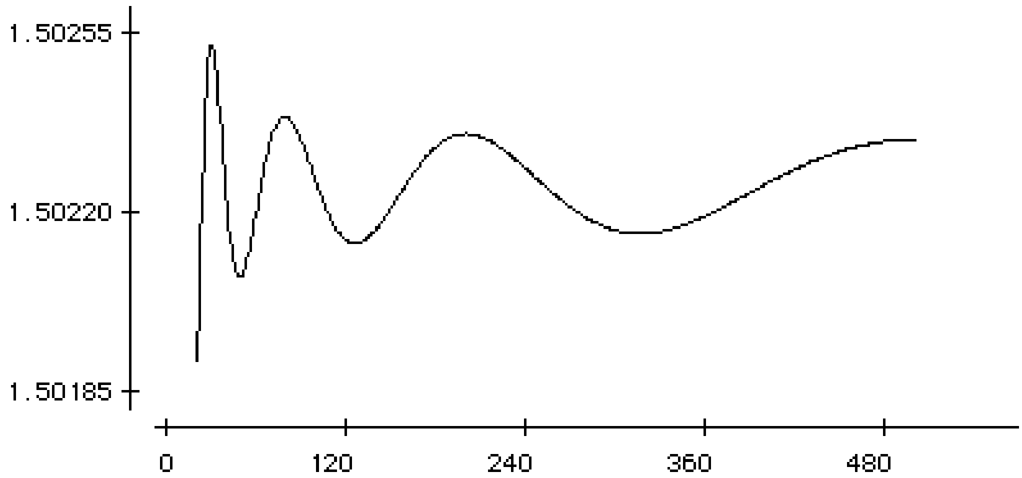

FIGURE 7. The case $q = 0.2$ and $\kappa = 0.1$.

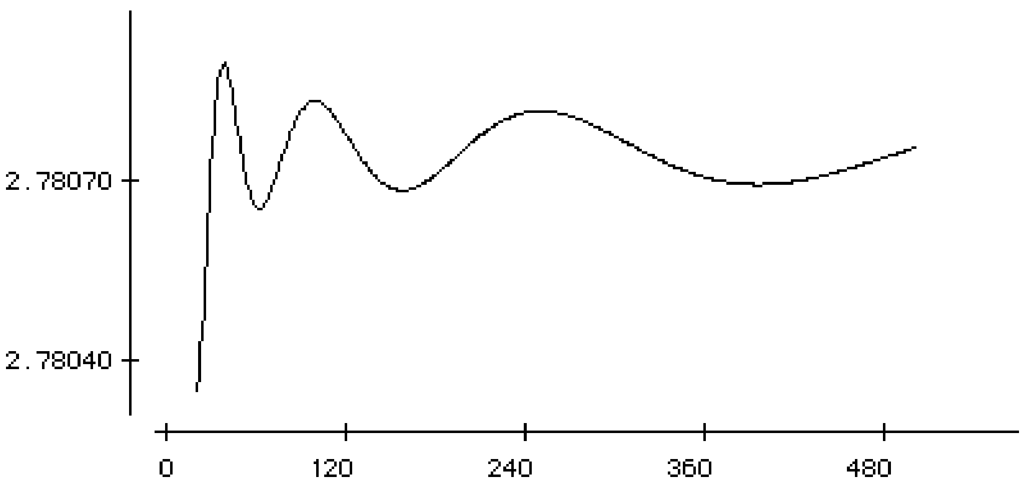

FIGURE 8. The case $q = 0.2$ with $\kappa = f(p_0)$.

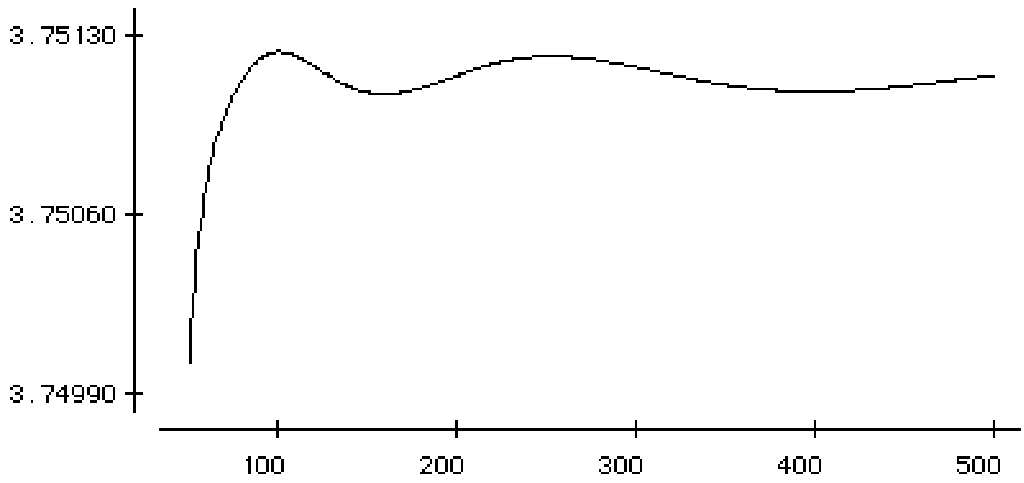

FIGURE 9. The case $q = 0.2$ with $\kappa = f_2(p_0)$.

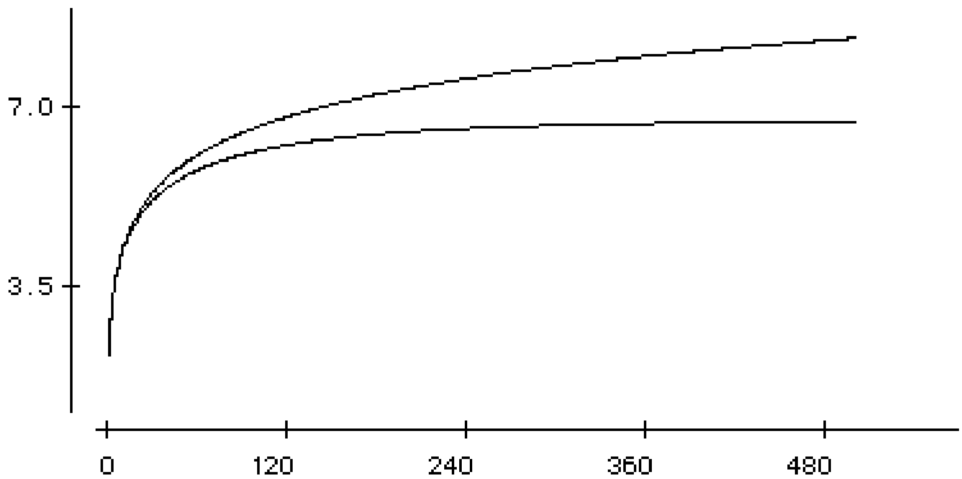

FIGURE 10. The case $q = 0.2$ with $\kappa = f_5(p_0)$ (lower curve) and $\kappa = f_{10}(p_0)$ (upper curve).

The speed of convergence of our recursive algorithm appears to be very fast for small values of q (not bigger than 0.4) but increasingly poor as q increases beyond 0.5. We always chose an iteration number between 400 and 500 and computed approximations for $\boldsymbol{r}(k)$ for k less than this iteration number, as suggested by (6.1). It then appeared for $q > 0.5$ that not only computation times would exponentially grow, but simultaneously the distance between lower and upper bounds of $\boldsymbol{r}(k)$ would be outside a satisfactory range (chosen as 10^{-4} or smaller) for k greater than about half the iteration number.

Despite the computational problems just mentioned for large values of k the grand picture conveyed by our numerical results is that $\boldsymbol{r}(k)$ always converges as k tends to infinity. It is therefore to be emphasized once more that we have theoretical reasons to conjecture the latter be generally false. On the other hand, if $\boldsymbol{r}(k)$ indeed diverges, then its variation for large values of k seems to be in a range of poor numerical interest, a "near-constancy" phenomenon also encountered for the so-called Harris function of certain supercritical ordinary GWP, see e.g.

Biggins and Nadarajah(1993) and the references therein.

References

ATHREYA, K.B. and NEY, P. (1972). *Branching Processes.* Springer, New York.

BIGGINS, J.D. and NADARAJAH, S. (1993). Near-constancy of the Harris function in the simple branching process. *Comm. Statist. - Stoch. Models* **9**, 435-444.

BRUSS, T. (1984). A note on extinction criteria for bisexual Galton-Watson processes. *J. Appl. Probab.* **21**, 915-919.

DALEY, D. (1968). Extinction probabilities for certain bisexual Galton-Watson branching processes. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **9**, 315-322.

DALEY, D., HULL, D.M. and TAYLOR, J.M. (1986). Bisexual Galton-Watson branching processes with superadditive mating functions. *J. Appl. Probab.* **23**, 585-600.

HULL, D.M. (1982). A necessary condition for extinction in those bisexual Galton-Watson branching processes governed by superadditive mating functions. *J. Appl. Probab.* **19**, 847-850.

HULL, D.M. (1984). Conditions for extinction in certain bisexual Galton-Watson branching processes. *J. Appl. Probab.* **21**, 414-418.