The Functional Equation of the Smoothing Transform

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Abstract

Given a sequence $T = (T_i)_{i\geq 1}$ of non-negative random variables, a function f on the positive halfline can be transformed to $\mathbb{E}\prod_{i\geq 1} f(tT_i)$. We study the fixed points of this transform within the class of decreasing functions. By exploiting the intimate relationship with general branching processes, a full description of the set of solutions is established without the moment conditions that figure in earlier studies. Since the class of functions under consideration contains all Laplace transforms of probability distributions on $[0,\infty)$, the results provide the full description of the set of solutions to the fixed-point equation of the smoothing transform, $X \stackrel{d}{=} \sum_{i\geq 1} T_i X_i$, where $\stackrel{d}{=}$ denotes equality of the corresponding laws and $X_1, X_2, \ldots$ is a sequence of i.i.d. copies of X independent of T. Moreover, since left-continuous survival functions are covered as well, the results also apply to the fixed-point equation $X \stackrel{d}{=} \inf\{X_i/T_i : i \geq 1, T_i > 0\}$.

Keywords: Branching random walk; Choquet-Deny type functional equation; fixed point; general branching process; multiplicative martingale; smoothing transformation; stochastic fixed-point equation; Weibull distribution; weighted branching; branching process

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1 Introduction

Let $T := (T_i)_{i\geq 1}$ be a sequence of non-negative random variables and consider the mapping $f \mapsto \mathbb{E}\prod_{i\geq 1} f(tT_i)$ for suitable functions $f : \mathbb{R} \to \mathbb{R}$. Then it is natural to call f a fixed point of this transformation if

$$f(t) \ = \ \mathbb{E}\prod_{i\geq 1} f(tT_i). \tag{1.1}$$

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The main objective here is to identify all fixed points within certain classes of functions, which becomes an increasingly challenging task as the available class gets bigger. There is a substantial literature, [10, 19, 29, 27, 24, 14, 7], and relatively complete results, improved here, when f must be the Laplace transform of a non-negative random variable. Much less was known up to now [26, 8, 6] when f is from the larger class of survival functions of non-negative random variables (or simply monotone decreasing functions with range $[0, 1]$). Solving the problem in this case is one of the main achievements of this paper. In fact the ideas also allow the available class to include suitable non-monotonic continuous functions, as will be indicated in the final section.

When f is the Laplace transform of a non-negative variable X the equation (1.1) can be rewritten in terms of random variables as

$$X \overset{d}{=} \sum_{i \geq 1} T_i X_i, \tag{1.2}$$

where $X_1, X_2, \dots$ are i.i.d. copies of X independent of T and $\overset{d}{=}$ means equality in distribution. An X, or its distribution, satisfying this is often called a fixed point of the smoothing transform (going back to Durrett and Liggett [19]). If instead f is the survival function of a non-negative variable X the equation (1.1) can be rewritten as

$$X \overset{d}{=} \inf\left\{\frac{X_i}{T_i} : i \geq 1,\, T_i > 0\right\}, \tag{1.3}$$

where the infimum over the empty set is defined to be ∞. (Notice that the inversion $x \mapsto x^{-1}$ turns this 'inf-type' equation into a 'sup-type' one, so the theory will cover these too.) Both (1.2) and (1.3) are examples of *stochastic fixed-point equations* (also called *recursive distributional equations* in [2]). Thus, for these two cases, characterising fixed points for equation (1.1) in the appropriate class corresponds to identifying the X which can arise in these stochastic fixed-point equations. In considering (1.2), the relevant class of functions (Laplace transforms) is quite restricted and so the problem is correspondingly easier. It turns out that solutions to (1.2) are intimately related to solutions to (1.3), which allows the characterisation of the latter using results for the former.

There is considerable interest in, and literature on, stochastic fixed-point equations like (1.2) and (1.3). They occur in various areas of applied probability: probabilistic combinatorial optimization [1], stochastic geometry [34], the analysis of recursive algorithms and data structures [37, 22, 18, 38, 31] and also in connection with branching particle systems [12, 23]. Inhomogeneous versions of (1.2) and the sup-type version of (1.3) arise in the average-case and worst-case analysis of divide-and-conquer algorithms and Rüschendorf [39, Theorem 3.1 and Theorem 4.2] showed, in a more restricted setting, that the solutions to the inhomogeneous versions are in one-to-one correspondence with the solutions of their homogeneous counterparts. In theoretical probability, they are of relevance in connection with the central limit problem [16] and in extreme value theory [36], where they can be interpreted as generalizations of the distributional equations of stability and min-stability, respectively. For further information we refer to the survey by Aldous and Bandyopadhyay [2].

Without loss of generality, suppose that the number $N = \sum_{i\geq 1} \mathbb{1}_{\{T_i>0\}}$ of positive terms satisfies

$$N \;=\; \sup\{i \geq 1 : T_i > 0\} \tag{1.4}$$

and define the function

$$m : [0,\infty) \to [0,\infty], \quad \theta \;\mapsto\; \mathbb{E}\sum_{i=1}^{N} T_i^{\theta}. \tag{1.5}$$

Its canonical domain, $\{m < \infty\}$, is an interval $\subseteq [0,\infty)$, for m may be viewed as the Laplace transform of the intensity measure of the point process $\mathcal{Z} := \sum_{i=1}^{N} \delta_{S(i)}$. Here $S(i) := -\log T_i$ ($i \in \mathbb{N}$) (and $S(i) := \infty$ if $T_i = 0$). The following assumptions will be in force throughout.

$$\mathbb{P}\left(T \in \{0,1\}^{\mathbb{N}}\right) \;<\; 1. \tag{A1}$$

$$\mathbb{E}N \;>\; 1. \tag{A2}$$

$$\text{There exists some } \alpha > 0 :\; 1 \;=\; m(\alpha) \;<\; m(\beta) \text{ for all } \beta \in [0,\alpha). \tag{A3}$$

This number α is called the *characteristic exponent (of T)*. Previous studies [29, 7, 8] show that a satisfactory characterisation will typically entail the existence of some $\alpha > 0$ such that $m(\alpha) = 1$, as in (A3), though [26] and [6] provide a study of a case where this fails. The discussions in [29] for Eq. (1.2) and [6] for Eq. (1.3) imply that only simple cases are ruled out by (A1) and (A2). Let $r > 1$ be the smallest number such that the strictly positive elements of T are concentrated almost surely on $r^{\mathbb{Z}}$, and let r=1 otherwise, that is, when the smallest closed multiplicative group containing the strictly positive elements of T is $\mathbb{R}^+$. The former is called the *r-geometric* (or *lattice*) case, the latter the *non-geometric* (or *continuous*) case. There are a few more technicalities to deal with before the main results can be stated in the next section, but a special case can now be given quite easily as illustration.

Theorem 1.1. *Suppose that* (A1)–(A3) *hold true, that there is a $\theta \in [0,\alpha)$ with $m(\theta) < \infty$ and T is non-geometric. Then there is a non-negative random variable W satisfying*

$$W \;\stackrel{d}{=}\; \sum_{i\geq 1} T_i^{\alpha} W_i, \tag{1.6}$$

(where $W_1, W_2, \ldots$ are i.i.d. copies of W independent of T) such that survival functions satisfying (1.1) *are given by the family, parametrized by $h \in \mathbb{R}^+$,*

$$f(t) \;=\; \mathbb{E}\exp(-Wht^{\alpha}). \tag{1.7}$$

Note that (1.6) is just (1.2) with T replaced by $T^{(\alpha)} := (T_1^{\alpha}, T_2^{\alpha}, \ldots)$. It is already known, under mild conditions that are relaxed a little in Theorem 2.3 here, that the solutions to (1.6) are unique up to a scale factor. Therefore, in

(1.7), the same family will result whichever solution to (1.6) is selected. The form (1.7) is a mixture (with mixing variable W) of Weibull survival functions. This form is not surprising in the light of results for deterministic T described in [8] nor the corresponding results for (1.2) going back to Durrett and Liggett [19]. In the latter case, f has to be a Laplace transform and (1.7) expresses it as a W-mixture of positive α-stable transforms (necessitating also that $\alpha \le 1$).

It is natural to deploy iteration to study a functional equation. A key aspect of the approach here is to remove the expectation on the right of (1.1) and then iterate. Suitably formulated, this iteration derives naturally from a branching process based on T. Solutions to (1.1) correspond to certain (multiplicative) martingales. Studying these, and their limits, delivers information on the form of the solutions. This basic idea goes back at least to Neveu [33] and is used more recently in [12, 14] and [6]. This technique is a kind of disintegration of (1.1), since it considers the stochastic processes obtained by removing the expectation in it and its iterates. For fixed t, under the iteration of the disintegration, the conditions imply that the arguments of the function f on the right of the equation become small. Hence, the properties of the whole function will be implicit in its behaviour for small arguments.

2 Main results

We continue with a specification of two further assumptions on T, viz.

$$\mathbb{E}\sum_{i\ge 1} T_i^\alpha \log T_i \;\in\; (-\infty, 0) \text{ and } \mathbb{E}\Big(\sum_{i\ge 1} T_i^\alpha\Big)\log^+\Big(\sum_{i\ge 1} T_i^\alpha\Big) \;<\; \infty. \tag{A4a}$$

$$\text{There exists some } \theta \in [0,\alpha) \text{ satisfying } m(\theta) \;<\; \infty. \tag{A4b}$$

In order to prove our main results, we need at least one of the assumptions (A4a), (A4b) to be true, in other words, we need the assumption

$$\text{(A4a) or (A4b) holds.} \tag{A4}$$

It is worth mentioning that (A4) is fairly weak compared to the assumptions in earlier works on fixed points of the smoothing transform, that is, on solutions to (1.2). For ease of reference to earlier results, when (A3) holds let

$$m'(\alpha) = \mathbb{E}\sum_{i\ge 1} T_i^\alpha \log T_i,$$

even when m is finite only at α; whenever we refer to $m'(\alpha)$ we will be assuming the expectation exists, which it certainly does when (A4) holds.

The r-geometric case involves some complications that require additional notation. A function h is multiplicatively r-periodic if $h(x) = h(rx)$ for all x. Given $r > 1$, let $\mathfrak{H}_r$ be the set of multiplicatively r-periodic functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $t \mapsto h(t)t^\alpha$ is non-decreasing (where α comes from (A3)). To deal with all cases together, let $\mathfrak{H}_1$ be the positive constant functions in the non-geometric case (when $r = 1$). In the corresponding result for (1.2), stated here as a corollary, it is further assumed that $\alpha \in (0,1]$. Then, let $\mathfrak{P}_r$ be the set of multiplicatively r-periodic functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(t)t^\alpha$ has a completely monotone derivative, and let $\mathfrak{P}_1$ be the positive constant functions in the non-geometric case — these functions were introduced in [19].

Henceforth, let $\mathcal{S}(\mathcal{M})$ be the set of solutions to the functional equation (1.1) within the class

$$\begin{aligned}\mathcal{M} \;=\; & \{f : [0,\infty) \to [0,1] : \; f \text{ is decreasing with} \\ & f(0) = f(0+) = 1 \text{ and } f(t) \in (0,1) \text{ for some } t > 0\}.\end{aligned}$$

Here, $f(0+)$ denotes the right limit of f at 0. Now we are ready to state our first main result.

Theorem 2.1. *Suppose that* (A1)–(A4) *hold. Then there is a non-negative random variable W satisfying* (1.6) *such that $\mathcal{S}(\mathcal{M})$ is given by the family, parametrized by $h \in \mathfrak{H}_r$,*

$$f(t) \;=\; \mathbb{E}\exp(-Wh(t)t^\alpha) \quad (t \geq 0). \tag{2.1}$$

Calling a $[0,\infty]$-valued solution X to (1.3) degenerate if $\mathbb{P}(X = 0 \text{ or } X = \infty) = 1$, one can deduce from Theorem 2.1 that the set of survival functions of non-degenerate solutions to (1.3) is given by the family (2.1) parametrized by the left-continuous $h \in \mathfrak{H}_r$.

Let $\mathcal{S}(\mathcal{L})$ be the set of solutions to (1.1) within the class $\mathcal{L}$ of Laplace transforms of probability distributions $\neq \delta_0$ on $[0,\infty)$.

Corollary 2.2. *Suppose that conditions* (A1)–(A4) *hold and that $\alpha \leq 1$. Then there is a non-negative W satisfying* (1.6) *such that $\mathcal{S}(\mathcal{L})$ is given by the family in* (2.1) *when parametrized by $h \in \mathfrak{P}_r$.*

From the formulations of Theorem 2.1 and Corollary 2.2 it is obvious that solutions to (1.1) based on the sequence $T^{(\alpha)} = (T_i^\alpha)_{i\geq 1}$ play a critical role since they appear as mixing distributions in all other cases. We need information on these fixed points at an early stage of our analysis. Hence, we continue with the following result.

Theorem 2.3. *Assuming* (A1)–(A4)*, the following assertions hold.*

- *(a) There is a non-trivial (i.e. non-zero) non-negative solution to* (1.6).
- *(b) Let W with Laplace transform φ be a solution to* (1.6)*. Then $1-\varphi(t)$ is regularly varying with index* 1 *as $t \downarrow 0$ and φ is unique up to a positive scaling factor in its argument.*

Part (a) of this Theorem follows directly from Theorem 1.1 in [29]. Part (b) can be concluded from existing literature if we additionally assume (A4b). In this case, the claimed uniqueness of φ follows from [14, Theorem 3]. The regular variation of $1-\varphi$ follows from [12, Theorem 1.4] in the case when $m'(\alpha) < 0$ and from [27, Theorem 1] if $m'(\alpha) = 0$. We postpone the proof of the result as stated here until Section 10. There we will see that for uniqueness up to scaling (A4a) can be replaced by $\mathbb{E}\,W < \infty$, which is in fact weaker.

Understanding the behaviour of solutions to (1.1) near zero is an essential step in proving the above results but also interesting in its own right because it allows the derivation of moment results for the corresponding distributions by classical Tauberian theorems in the case of Eq. (1.2) and by elementary calculations for the sup-type analogue of (1.3).

For a solution f, the near-zero behaviour is best considered in terms of D_α defined by

$$D_\alpha(t) \;=\; \frac{1-f(t)}{t^\alpha}, \qquad (t>0). \tag{2.2}$$

When $\alpha = 1$ and f is a Laplace transform, the convexity of f forces $D_1(t) = (1-f(t))/t$, to be decreasing in t, and then $D_1(0+)$ is finite exactly when the corresponding random variable has a finite mean.

Theorem 2.4. *Assume* (A1)–(A4)*, T is non-geometric and $f \in \mathcal{S}(\mathcal{M})$. Then $D_\alpha(t)$ is slowly varying as $t \downarrow 0$.*

A corresponding result for the geometric case is stated next, although it is tangential to the development of the results.

Theorem 2.5. *Assume* (A1)–(A4)*, T is r-geometric and $f \in \mathcal{S}(\mathcal{M})$. Then there exists a function $h \in \mathfrak{H}_r$ such that $D_\alpha(t)/h(t)$ is slowly varying as $t \downarrow 0$.*

It is often possible to say more about the form of the slowly varying functions. We omit the details but give an indication of the results. When $\alpha \neq 1$, Theorem 2.1 ensures that any solution $f \in \mathcal{S}(\mathcal{M})$ is of the form $f(t) = \varphi(h(t)t^\alpha)$ for some $h \in \mathfrak{H}_r$, where φ denotes the unique (up to a positve scale factor in its argument) solution of (1.1) with $T^{(\alpha)}$ instead of T. Considering (1.1) with $T^{(\alpha)}$ instead of T amounts to taking $\alpha = 1$ and then the function D_1, which comes from the Laplace transform φ, is slowly varying in very specific ways under fairly mild moment conditions. Roughly speaking, if $m'(1) < 0$, then $D_1(t)$ usually converges to a finite constant while, if $m'(1) = 0$, $D_1(t)$ usually looks like $-\log t$. See [10, Lemmas 2 and 4], [30] and [17], for information on the first case and [14, Theorems 4 and 5] for information on the second. It is easy to translate such results on φ to corresponding results on asymptotic behaviour of $f \in \mathcal{S}(\mathcal{M})$ at 0.

The rest of this paper is organised as follows. In Section 3, we prove the simple inclusions of our main results, Theorem 2.1 and Corollary 2.2. Sections 6–12 are dedicated to the proof of the converse direction of these two results. As indicated in the Introduction, iteration of (1.1) naturally leads to a branching model (variously known as weighted branching, branching random walk and multiplicative cascade) which we formally define in Section 4. Section 5 is devoted to the property of *endogeny* first introduced in [2]. It refers to special solutions to (1.2) having the property that all their randomness can be expressed in terms of the branching process with no further randomness needed. Section 6 collects some (known) connections between the branching model and random walk theory. The key object derived from solutions to (1.1), called their disintegration, is described in Section 7. With the help of this notion we are able to formulate a further result (Theorem 7.2) from which the proofs of Theorem 2.1 and Corollary 2.2 are easily completed. We then argue in Section 8, a technical interlude, that we can restrict ourselves to the analysis of (1.1) under the additional assumption that all $T_i < 1$. This allows us to make use of the theory of general (CMJ) branching processes in Section 9. The assertions on regular variation at 0 of $t^{-1}(1-\varphi(t))$, φ a solution to Eq. (1.6), and of $t^{-\alpha}(1-f(t))$, f a solution to (1.1), are then provided in Section 10 and Section 11, respectively. Based on these results, we are finally able to prove Theorem 2.3(b) (Section 10) and Theorem 7.2 (Section 12). The final section briefly addresses allowing the possibility of non-monotonic solutions to (1.1).

3 The simple inclusions

Lemma 3.1. *Let* (A1)–(A3) *hold and let W denote a non-negative random variable satisfying* (1.6)*. Then $f \in \mathcal{S}(\mathcal{M})$ for any f which is defined by* (2.1)*. If, moreover, $\alpha \leq 1$ and the parameter function h in* (2.1) *is chosen from $\mathfrak{P}_r$, then $f \in \mathcal{S}(\mathcal{L})$.*

Proof. Let W be a non-negative random variable satisfying (1.6). We denote its Laplace transform by φ. Then, with $W_1, W_2, \ldots$ denoting i.i.d. copies of W independent of T, using that $h(t) = h(tT_i)$ a.s. since $h \in \mathfrak{H}_r$,

$$\begin{aligned} f(t) = \varphi(h(t)t^\alpha) &= \mathbb{E}\exp(-Wh(t)t^\alpha) \\ &= \mathbb{E}\exp\Big(-\sum_{i\geq 1} T_i^\alpha W_i h(t)t^\alpha\Big) \\ &= \mathbb{E}\Big(\mathbb{E}\Big[\prod_{i\geq 1}\exp(-W_i h(tT_i)(tT_i)^\alpha)\,\Big|\,T\Big]\Big) \\ &= \mathbb{E}\prod_{i\geq 1}\varphi(h(tT_i)(tT_i)^\alpha) = \mathbb{E}\prod_{i\geq 1} f(tT_i). \end{aligned}$$

Therefore, f solves the functional equation. Then it is easily verified that $f \in \mathcal{S}(\mathcal{M})$. Now, moreover, suppose that $h \in \mathfrak{P}_r$. Then $f(t) = \varphi(h(t)t^\alpha) \in \mathcal{L}$ by [20, Criterion 2 on p. 441] and Bernstein's theorem. ☐

4 The associated branching model

A key tool for the further analysis of Eq. (1.1) is an associated weighted branching model (or multiplicative cascade, or branching random walk) which arises upon iteration of (1.1) and which we now describe.

Let $\mathbb{V} := \bigcup_{n\in\mathbb{N}_0}\mathbb{N}^n$ be the infinite Ulam-Harris tree, where $\mathbb{N} := \{1, 2, ...\}$ and $\mathbb{N}^0 = \{\varnothing\}$. Abbreviate $v = (v_1, ..., v_n)$ by $v_1...v_n$ and write $v|k$ for the restriction of v to the first k entries, that is, $v|k := v_1...v_k$, $k \leq n$. If $k > n$, put $v|k := v$. Write vw for the vertex $v_1...v_nw_1...w_m$ where $w = w_1, ..., w_m$. In this situation, we say that v is an ancestor of vw. The length of a node v is denoted by $|v|$, thus $|v| = n$ iff $v \in \mathbb{N}^n$. Next, let $\mathbf{T} := (T(v))_{v\in\mathbb{V}}$ denote a family of i.i.d. copies of T, where $T(\varnothing) = T = (T_i)_{i\geq 1}$. We interpret $T_i(v)$

as a weight attached to the edge (v, vi) in the infinite tree $\mathbb{V}$ and then define $L(\varnothing) := 1$ and, recursively,

$$L(vi) \; := \; L(v)T_i(v) \tag{4.1}$$

for $v \in \mathbb{V}$ and $i \in \mathbb{N}$. Thus $L(v)$ is the product of the weights along the unique path from the root $\varnothing$ to v. With this branching model, n-fold iteration of (1.1) gives:

$$f(t) \; = \; \mathbb{E} \prod_{|v|=n} f(tL(v)) \quad (t \geq 0). \tag{4.2}$$

For $n \in \mathbb{N}_0$, let $\mathcal{A}_n$ denote the σ-algebra generated by the sequences $T(v)$, $|v| < n$ and put $\mathcal{A}_\infty := \sigma(\mathcal{A}_n : n \geq 0) = \sigma(T(v) : v \in \mathbb{V})$. For $\theta \geq 0$, define

$$W_n^{(\theta)} \; := \; \sum_{|v|=n} L(v)^\theta, \quad n \geq 0. \tag{4.3}$$

Then $N_n := W_n^{(0)} = \sum_{|v|=n} \mathbb{1}_{\{L(v)>0\}}$ counts the positive branch weights in generation n. If $N = N_1 < \infty$ a.s., then $(N_n)_{n\geq 0}$ forms an ordinary Galton-Watson process with offspring distribution $\mathbb{P}(N \in \cdot)$. Assuming (A3) and thus $m(\alpha) = 1$, the sequence $(W_n^{(\alpha)})_{n\geq 0}$ constitutes a non-negative martingale with respect to $(\mathcal{A}_n)_{n\geq 0}$ and hence converges a.s. to $W^{(\alpha)} := \liminf_{n\to\infty} W_n^{(\alpha)}$ satisfying $\mathbb{E}\, W^{(\alpha)} \leq 1$ by Fatou's lemma. Let φ_α denote its Laplace transform. The martingale has been studied, in several disguises, by numerous authors. Further information on $W^{(\alpha)}$ will be given in the next section.

Let us further introduce the shift operators $[\cdot]_u$, $u \in \mathbb{V}$. Given any function $\Psi = \psi(\mathbf{T})$ of the weight family $\mathbf{T} = (T(v))_{v\in\mathbb{V}}$ pertaining to $\mathbb{V}$, define

$$[\Psi]_u \; := \; \psi((T(uv))_{v\in\mathbb{V}})$$

to be the very same function but for the weights pertaining to the subtree rooted at $u \in \mathbb{V}$. Any branch weight $L(v)$ can be viewed as such a function and thus we have $[L(v)]_u = T_{v_1}(u) \cdot ... \cdot T_{v_n}(uv_1...v_{n-1})$ if $v = v_1...v_n$, that is $[L(v)]_u = L(uv)/L(u)$ whenever $L(u) > 0$.

5 Endogeneous fixed points of the smoothing transformation

For our purposes, the relevance of the martingale limit $W^{(\alpha)}$, defined through (4.3), with α given by (A3), stems from the fact that, unless it is zero a.s., $W^{(\alpha)}$ is a solution to (1.6) and thus a possible mixing variable in our main results. In the following we will briefly dwell upon an additional property of $W^{(\alpha)}$ called *endogeny*, a term coined by Aldous and Bandyopadhyay [2, Definition 7], and which may be phrased as follows in the present context:

Definition 5.1. A random variable W with $\mathbb{P}(W > 0) > 0$ is called an *endogeneous fixed point of the smoothing transform with respect to (w.r.t.)*

$T^{(\alpha)}$ (or an *endogeneous solution to* (1.6)) if there exists a measurable function $g : [0,\infty)^{\mathbb{V}} \to [0,\infty]$ such that $W = g(\mathbf{T})$ and

$$W \;=\; \sum_{|v|=n} L(v)^{\alpha}\,[W]_v \quad \text{a.s.} \tag{5.1}$$

for all $n \geq 0$.

That $W^{(\alpha)}$, if non-degenerate at 0, is indeed endogeneous is a commonplace in the study of the martingale, see for example [10]. Lyons [30] showed that (under (A1)–(A3)) condition (A4a) is sufficient for $W^{(\alpha)}$ to be non-degenerate at 0 and, thus, to be an endogeneous fixed point. A complete characterisation of the non-degeneracy of $W^{(\alpha)}$ can be found in [3]. Therefore, under (A1)–(A4), $W^{(\alpha)}$ can only be degenerate if (A4a) fails and, thus, (A4b) holds.

Even if $W^{(\alpha)} = 0$ a.s., an endogeneous fixed point may exist. Namely, under suitable conditions, the limits of the Seneta-Heyde normed version of $W_n^{(\alpha)}$, see [12] and [23], are endogeneous fixed points. Furthermore, Biggins and Kyprianou [14, p. 623f.] showed that if $m'(\alpha) = 0$, under additional moment conditions, the so-called derivative martingale converges a.s. to a non-degenerate random variable $\partial W^{(\alpha)}$ which is an endogenous fixed point. In both these situations the fixed points must have infinite mean, which follows from the next Proposition.

Proposition 5.2. *Suppose that* (A1)–(A3) *hold and let W be an integrable endogenous fixed point w.r.t. $T^{(\alpha)}$. Then $\mathbb{E}\,W^{(\alpha)} = 1$ and $W = cW^{(\alpha)}$ a.s. with $c = \mathbb{E}\,W$. If, furthermore,* (A4) *holds true, then condition* (A4a) *is satisfied.*

Proof. By (5.1), the integrability of W and the martingale convergence theorem, we infer with $c = \mathbb{E}\,W > 0$

$$\begin{aligned} W &= \lim_{n\to\infty} \mathbb{E}(W \mid \mathcal{A}_n) = \lim_{n\to\infty} \mathbb{E}\Big(\sum_{|v|=n} L(v)^{\alpha}[W]_v \mid \mathcal{A}_n\Big) \\ &= c \lim_{n\to\infty} W_n^{(\alpha)} = c\,W^{(\alpha)} \quad \text{a.s.} \end{aligned}$$

and particularly $\mathbb{E}\,W^{(\alpha)} = 1$. This proves the first part of the proposition. Now suppose additionally that (A4) holds true. Let $(S_n)_{n\geq 0}$ be the zero-delayed random walk associated with $T^{(\alpha)}$, defined by having increment distribution

$$\mathbb{P}(S_1 \in B) \;:=\; \mu_{\alpha}(B) \;:=\; \mathbb{E}\sum_{i\geq 1} T_i^{\alpha}\,\mathbb{1}_B(-\log T_i)$$

for Borel subsets B of $\mathbb{R}$. Note that, by (A4) the definition of α, and of $m'(\alpha)$, $\mathbb{E}\,S_1 = m'(\alpha) \in [0,\infty)$. Now [30] implies (A4a) holds. ☐

We finish the section with a uniqueness result that sharpens Proposition 2.3 in the case of endogeneous fixed points.

Proposition 5.3. *Suppose that* (A1)–(A4) *hold. Let W and W^* be endogeneous fixed points w.r.t. $T^{(\alpha)}$. Then $W^* = cW$ a.s. for some $c > 0$.*

Proof. Let φ and φ^* denote the Laplace transforms of W and W^*, respectively. By Proposition 2.3(b), we already know that $\varphi(t) = \varphi^*(ct)$, and it is no loss of generality to assume $c = 1$. Using endogeny, the bounded and thus integrable random variable $\exp(-W)$ can be written in the form

$$\begin{aligned}\exp(-W) &= \lim_{n\to\infty} \mathbb{E}\left(\exp\left(-\sum_{|v|=n} L(v)^\alpha [W]_v\right) \middle| \mathcal{A}_n\right) \quad \text{a.s.} \\ &= \lim_{n\to\infty} \prod_{|v|=n} \varphi(L(v)^\alpha)\end{aligned}$$

and a similar result holds for $\exp(-W^*)$ with φ^* instead of φ on the right-hand side. Now $\varphi = \varphi^*$ implies $\exp(-W) = \exp(-W^*)$ a.s. □

This result is first used in Section 12. The only ingredient to the proof of the previous result which has not yet been verified is Proposition 2.3(b) and that will be proved in Section 10, so there is no circularity in the argument.

6 Renewal arguments

Let $S(v) := -\log L(v)$ $(v \in \mathbb{V})$ and $-\log 0 := \infty$. It is then easily verified (see [12, Lemma 4.1]) that

$$\mathbb{P}(S_n \in \cdot) \;=\; \mu_\alpha^{*n} \;=\; \mathbb{E}\sum_{|v|=n} e^{-S(v)\alpha}\delta_{S(v)} \quad (n \in \mathbb{N}_0). \tag{6.1}$$

Importantly, this connection between the branching model and its associated random walk is preserved under certain stopping schemes. To make this precise in the present context, let $\sigma : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{N}_0 \cup \{\infty\}$ denote a formal stopping rule, *i.e.*,

$$\sigma((s_n)_{n\geq 0}) \;=\; \inf\{n \geq 0 : (s_0, ..., s_n) \in B_n\}$$

where B_n is a Borel subset of $\mathbb{R}^{n+1}$, $n \geq 0$. For $n \in \mathbb{N}_0$, let σ_n denote the nth consecutive application of σ, which means that $\sigma_0 := 0$ and

$$\sigma_n \;:=\; \inf\{k > \sigma_{n-1} : (0, s_{\sigma_{n-1}+1} - s_{\sigma_{n-1}}, ..., s_k - s_{\sigma_{n-1}}) \in B_{k-\sigma_{n-1}}\}$$

for $n \in \mathbb{N}$. Then, for any $x = (v_i)_{i\geq 1} \in \mathbb{N}^{\mathbb{N}} =: \partial V$, the boundary of the Ulam-Harris tree $\mathbb{V}$, we can apply these formal stopping rules to the random walk along the infinite path $\varnothing \to v_1 \to v_1v_2 \to ...$ from the root to the boundary of $\mathbb{V}$, that is, we can consider $\sigma_n((S(x|k))_{k\geq 0})$, $n \in \mathbb{N}_0$. The set of all vertices in $\mathbb{V}$ in which σ_n stops any random walk from the root to the boundary of $\mathbb{V}$ is denoted by $\mathcal{T}_{\sigma_n}$, *i.e.*,

$$\mathcal{T}_{\sigma_n} \;:=\; \{x|\sigma_n((S(x|k))_{k\geq 0}) : x \in \partial\,\mathbb{V}\}.$$

We refer to the (random) sets $\mathcal{T}_{\sigma_n}$ as *homogeneous stopping lines (HSLs).* This notion indicates that the above defined random sets are special optional lines in the sense of Jagers [25], Kyprianou [28], and Biggins and Kyprianou [13], but where, additionally, stopping along any path of the infinite tree $\mathbb{V}$ follows the same stopping rule. By some obvious changes in the proof of Lemma 3.2 in [5], we infer

$$\mathbb{E} \sum_{v \in \mathcal{T}_{\sigma_n}} e^{-S(v)\alpha} \delta_{S(v)} \;=\; \mathbb{P}(S_{\sigma_n} \in \cdot, \sigma_n < \infty) \;=:\; (\mu_\alpha^\sigma)^{*n}, \tag{6.2}$$

where in slight abuse of notation we write σ_n instead of $\sigma_n((S_k)_{k\geq 0})$. We have thus established the analogue of (6.1) for the embedded branching model based upon $(\sigma_n)_{n\geq 0}$. Here we make use of the HSLs associated with the first ascending ladder epoch defined by $\sigma^> := \inf\{k \geq 0 : s_k > 0\}$. When applied to $(S_n)_{n\geq 0}$, this ladder epoch will again be denoted by $\sigma^>$, whereas $\mu_\alpha^{\sigma^>}$ will be abbreviated to $\mu_\alpha^>$.

Lemma 6.1. *If* (A1)–(A4) *hold, then* $\limsup_{n\to\infty} S_n = \infty$ *a.s.*

Proof. Under (A4) $\mathbb{E}\, S_1 \geq 0$ and the result follows from standard random walk theory. □

Lemma 6.2. *If* (A1)–(A3) *and* (A4a) *hold, then* $\sigma^> < \infty$ *a.s. and* $\mathbb{E}\, S_{\sigma^>} < \infty$.

Proof. $\sigma^> < \infty$ a.s. follows directly from Lemma 6.1, while (A4a), ensures $\mathbb{E}\, S_1 > 0$. Thus, from standard random walk theory, we infer integrability of $\sigma^>$ and then $\mathbb{E}\, S_{\sigma^>} = \mathbb{E}\, \sigma^> \, \mathbb{E}\, S_1 < \infty$ by Wald's equation. □

The following lemma has been proved as part (c) of Theorem 10 in [14].

Lemma 6.3. *If* (A1)–(A3) *hold, then, for any* $0 \leq \theta \leq \alpha$,

$$\mathbb{E} \sum_{v \in \mathcal{T}_{\sigma^>}} L(v)^\theta < \infty \quad \textit{if, and only if,} \quad \mathbb{E} \sum_{i\geq 1} T_i^\theta < \infty. \tag{6.3}$$

Proof. Using (6.1), (6.2) and $\mathbb{P}(\sigma^> < \infty) = 1$, we infer that (6.3) is equivalent to the assertion

$$\mathbb{E}\, e^{(\alpha-\theta)S_{\sigma^>}} < \infty \quad \text{if, and only if,} \quad \mathbb{E}\, e^{(\alpha-\theta)S_1} < \infty, \tag{6.4}$$

which in turn can be deduced from results in standard random walk theory, see, for instance, [20, Section XII.3]. □

7 Disintegration

Our analysis of Eq. (1.1) will be built on a pathwise counterpart of Eq. (4.2). For this purpose, define

$$M_n(t) \;:=\; \prod_{|v|=n} f(tL(v)), \quad n \geq 0 \tag{7.1}$$

for $f \in \mathcal{S}(\mathcal{M})$. Neveu [33] studied the *multiplicative martingales* $(M_n(t))_{n\geq 0}$ in the context of the KPP equation. More recently, the multiplicative martingales have been considered in the study of the functional equation of the smoothing transform [12, 14]. We state the fact that $(M_n(t))_{n\geq 0}$ is a martingale in the following lemma [12, Theorem 3.1].

Lemma 7.1. *Let $f \in \mathcal{S}(\mathcal{M})$ and $t \geq 0$. Then $(M_n(t))_{n\geq 0}$ forms a bounded non-negative martingale with respect to $(\mathcal{A}_n)_{n\geq 0}$ and thus converges a.s. and in $\mathcal{L}^1$ to a random variable $M(t)$ satisfying*

$$\mathbb{E}\, M(t) \;=\; f(t). \tag{7.2}$$

In the situation of Lemma 7.1, we call the stochastic process $M = (M(t))_{t\geq 0}$ the *disintegration of* f and also a *disintegrated fixed point*. By Lemma 7.1, we can calculate any solution to the functional equation (1.1) from its associated disintegrated fixed point.

Theorem 7.2. *If* (A1)–(A4) *hold, there exists an endogeneous fixed point W w.r.t. $T^{(\alpha)}$ such that for any $f \in \mathcal{S}(\mathcal{M})$ with disintegration M, there is a function $h \in \mathfrak{H}_r$ such that*

$$M(t) \;=\; e^{-Wh(t)t^\alpha} \quad a.s. \qquad (t \geq 0). \tag{7.3}$$

The proof of this is postponed until Section 12. Here we show how it allows us to complete the proofs of Theorem 2.1 and Corollary 2.2.

Proof of Theorem 2.1. By Lemma 3.1, we have $f \in \mathcal{S}(\mathcal{M})$ for any f given by (2.1) and parametrized with $h \in \mathfrak{H}_r$. For the reverse inclusion, pick any $f \in \mathcal{S}(\mathcal{M})$. Theorem 7.2 shows the existence of an endogeneous fixed point W w.r.t. $T^{(\alpha)}$ and a disintegration M of f. Denote the Laplace transform of W by φ. Now (7.2) and (7.3) give $f(t) = \varphi(h(t)t^\alpha)$ for $t > 0$, as required. ☐

Proof of Corollary 2.2. Let $\alpha \leq 1$. Again, Lemma 3.1 gives one inclusion. For the other, pick any $\varphi \in \mathcal{S}(\mathcal{L})$. As in the proof of Theorem 2.1, we obtain the existence of an endogeneous solution W to (1.6) such that for some $h \in \mathfrak{H}_r$ we have

$$\varphi(t) \;=\; \varphi(h(t)t^\alpha) \quad \text{a.s.} \qquad (t \geq 0)$$

for the Laplace transform φ of W. It remains to show that $h \in \mathfrak{P}_r$. For this it suffices to show that $t \mapsto h(t)t^\alpha$ has a completely monotone derivative in the r-geometric case. Without loss of generality, we assume $h(1) = 1$ and use the regular variation of $1 - \varphi$ (see Proposition 2.3) to infer

$$\frac{1-\varphi(tr^{-n})}{1-\varphi(r^{-n})} \;=\; \frac{1-\varphi(h(t)t^\alpha r^{-\alpha n})}{1-\varphi(r^{-\alpha n})} \;\to\; h(t)t^\alpha \quad (n \to \infty).$$

Thus $t \mapsto h(t)t^\alpha$ is the limit of a sequence of functions with completely monotone derivatives and therefore has a completely monotone derivative itself. ☐

We finish this section with a series of results that will be useful in the proof of Theorem 7.2.

Lemma 7.3 (see Lemma 5.2 in [6])**.** *Let $f \in \mathcal{S}(\mathcal{M})$ with disintegration M. Then, for all $t \geq 0$ and $n \in \mathbb{N}_0$, we have*

$$M(t) \;=\; \prod_{|v|=n} [M]_v(tL(v)) \quad a.s. \tag{7.4}$$

Lemma 7.3 provides us with a quick proof of the fact that $\mathcal{S}(\mathcal{M})$ is contained in the set of solutions to the functional equation (1.1) with the sequence T replaced by the family $(L(v))_{v\in\mathcal{T}}$, where $\mathcal{T}$ is an *a.s. dissecting HSL*. The last notion was introduced in [28] for general stopping lines. For a HSL $\mathcal{T}$ it means that a.s. there exists a positive integer n such that for any $v \in \mathbb{N}^n$ there is some $u \in \mathcal{T}$ satisfying $u = v|k$ for some $k < |v|$. In other words, with probability one $\mathcal{T}$ cuts through the tree prior to some (random) generation n.

Lemma 7.4. *Let $f \in \mathcal{S}(\mathcal{M})$ with disintegration M and let $\mathcal{T}$ denote an a.s. dissecting HSL. Then*

$$M(t) \;=\; \prod_{v\in\mathcal{T}} [M]_v(tL(v)) \quad \textit{a.s.} \tag{7.5}$$

and thus

$$f(t) \;=\; \mathbb{E}\prod_{v\in\mathcal{T}} f(tL(v)) \qquad (t \geq 0). \tag{7.6}$$

In particular, any $f \in \mathcal{S}(\mathcal{M})$ is also a solution to (1.1) *with the sequence $(T_i)_{i\geq 1}$ replaced by the family $(L(v))_{v\in\mathcal{T}}$.*

The subsequent proof of Lemma 7.4 (after some minor changes) also works for the more general *very simple lines* defined in [13, Section 6]. These are stopping lines where for any $v \in \mathbb{V}$ whether v is on the line or not is determined by the ancestry of v. But along different ancestral lines the stopping rules may be different.

Proof. Let $\mathcal{T}$ denote an a.s. dissecting HSL and fix $t \geq 0$. Define B to be the set where $[M]_v(tL(v)) = \prod_{i\geq 1}[M]_{vi}(tL(vi))$ for all $v \in \mathbb{V}$. In view of Eq. (7.4), the invariance of $\mathbb{P}(\mathbf{T} \in \cdot)$ under the shift $[\cdot]_v$ and the independence of $[\mathbf{T}]_v$ and $L(v)$, we have $\mathbb{P}(B) = 1$. Since $\mathcal{T}$ is a HSL, there exists some formal stopping rule σ such that $\mathcal{T} = \mathcal{T}_\sigma$. Putting $\mathcal{T}_n := \mathcal{T}_{\sigma\wedge n}$ we have that $\mathcal{T}_n$ is the HSL where along each path from the root to the boundary the stopping vertices are chosen according to the stopping rule $\sigma \wedge n$. By induction over n, we infer that on B

$$M(t) \;=\; \prod_{v\in\mathcal{T}_n} [M]_v(tL(v))$$

for all $n \geq 0$. Passing to the limit $n \to \infty$ yields the assertion since $\mathcal{T}$ is a.s. dissecting so that $\mathcal{T} = \mathcal{T}_n$ for some (random) n. □

Now we wish to approximate a disintegrated fixed point M not only by the sequence $M_n(t)$, $n \geq 0$, where the product is built over a fixed generation, but also by terms like $M_{\mathcal{T}}(t)$, where the product is built over all vertices in a HSL $\mathcal{T}$. Here, as was done in [12], we focus on special HSLs, namely, first exit lines $\mathcal{T}_t$ based on the first exit times $\tau(t)$, viz. $\tau(t) := \inf\{k \geq 0 : s_k > t\}$ and

$$\mathcal{T}_t \;:=\; \mathcal{T}_{\tau(t)} \;=\; \{v \in \mathbb{V} : S(v) > t \text{ and } S(v|k) \leq t \text{ for } k = 0, \ldots, |v|-1\}.$$

Lemma 7.5. *Given $f \in \mathcal{S}(\mathcal{M})$ with disintegration M, the following assertions hold for all $t \geq 0$:*

(a) $\lim_{n\to\infty} \sum_{|v|=n} 1 - f(tL(v)) \;=\; -\log M(t)$ *a.s.*

(b) $\lim_{u\to\infty} \prod_{v\in\mathcal{T}_u} f(tL(v)) \;=\; M(t)$ *a.s.*

(c) $\lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} 1 - f(tL(v)) \;=\; -\log M(t)$ *a.s.*

(d) $\lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} 1 - f(L(v)) \;=\; W$ *a.s., where* $W := -\log M(1)$.

Proof. (a) By [11, Theorem 3], $\sup_{|v|=n} L(v) \to 0$ a.s. Using this, $f(0+) = 1$, and $-\log x \sim 1 - x$ as $x \to 1$, we infer for arbitrary $t > 0$

$$-\log M(t) \;=\; -\log \lim_{n\to\infty} \prod_{|v|=n} f(tL(v)) \;=\; \lim_{n\to\infty} \sum_{|v|=n} 1 - f(tL(v)) \quad \text{a.s.}$$

(b) For $u \geq 0$, denote by $\mathcal{A}_{\mathcal{T}_u} := \sigma(T(v) : v \prec \mathcal{T}_u)$ the pre-$\mathcal{T}_u$ σ-algebra. Here, $v \prec V$ for $v \in \mathbb{V}$ and $V \subseteq \mathbb{V}$ means that v has no ancestor in V, in particular, $v \notin V$ (see [25] for a full discussion). More precisely, $\mathcal{A}_{\mathcal{T}_u}$ is defined as

$$\mathcal{A}_{\mathcal{T}_u} \;=\; \sigma\left(\{T(v) \in A\} \cap \{v \prec \mathcal{T}_u\} : v \in \mathbb{V}, A \in \mathfrak{B}([0,\infty)^{\mathbb{N}})\right), \tag{7.7}$$

where $\mathfrak{B}$ denotes the Borel-σ-algebra. $\mathcal{A}_{\mathcal{T}_u}$ increases as u increases. The proof of [12, Lemma 6.1] applies in the current context to give

$$M_{\mathcal{T}_u}(t) := \prod_{v\in\mathcal{T}_u} f(tL(v)) = \mathbb{E}[M(t) \,|\, \mathcal{A}_{\mathcal{T}_u}] \quad \text{a.s.}$$

Now let $\mathcal{G} := \sigma(\mathcal{A}_{\mathcal{T}_u} : u \geq 0)$. Standard theory implies that $M_{\mathcal{T}_u}(t) \to \mathbb{E}[M(t) \,|\, \mathcal{G}]$ a.s. as $u \uparrow \infty$. It remains to show that $M(t)$ is measurable w.r.t. $\mathcal{G}$. Since $M(t)$ is a function of the weight ensemble $(L(v))_{v\in\mathbb{V}}$, it suffices to show that any $L(v)$, $v \in \mathbb{V}$ is $\mathcal{G}$-measurable. To this end, fix $v = v_1 \ldots v_n \in \mathbb{N}^n$. If $L(v) = 0$ and thus $S(v) = \infty$, we have $v \not\prec \mathcal{T}_u$ for all $u \geq 0$. If, on the other hand, $L(v) > 0$, then $v \in \mathcal{T}_u$ for all $u > \max_{k=0,\ldots,n} S(v|k)$. In both cases, $L(v) = \lim_{u\to\infty} L(v)\, \mathbb{1}_{\{v \prec \mathcal{T}_u\}}$. For any fixed u,

$$L(v)\, \mathbb{1}_{\{v\prec\mathcal{T}_u\}} \;=\; \mathbb{1}_{\{v\prec\mathcal{T}_u\}} \prod_{k=0}^{n-1} T_{v_{k+1}}(v|k)\, \mathbb{1}_{\{v|k\prec\mathcal{T}_u\}}\,.$$

Clearly, $\mathbb{1}_{\{v\prec\mathcal{T}_u\}}$ is $\mathcal{A}_{\mathcal{T}_u}$-measurable. Further, elementary verifications show that the $T_{v_{k+1}}(v|k)\, \mathbb{1}_{\{v|k\prec\mathcal{T}_u\}}$ are also $\mathcal{A}_{\mathcal{T}_u}$-measurable. Thus, $M(t)$ is $\mathcal{G}$-measurable. Finally, we should remark that the formulation of the convergence in Lemma 7.5 indicates that the convergence holds outside a $\mathbb{P}$-null set for any sequence $u \uparrow \infty$. This is indeed true, for it can be shown that the martingale $(M_{\mathcal{T}_u}(t))_{u\geq 0}$ a.s. has right-continuous paths. (This follows basically from the fact that a.s. the positions $S(v)$, $v \in \mathbb{V}$ do not accumulate in finite intervals (a, b), $-\infty < a < b < \infty$.) Since we only need convergence along a fixed subsequence in what follows, we omit further details.

(c) This follows by combining assertion (b) with the arguments given in (a), where the simple observation that $L(v) \le e^{-u}$ for any $v \in \mathcal{T}_u$ replaces the application of $\sup_{|v|=n} L(v) \to 0$ a.s. as $n \to \infty$.

(d) Assertion (d) is a special case of (c), namely, the case $t = 1$. ☐

Lemma 7.6. *Let $f \in \mathcal{S}(\mathcal{M})$ and M be its disintegration. Suppose further that $1 - f$ is regularly varying of index α at 0 in the non-geometric case, while in the r-geometric case $(1 - f(ut))/(1 - f(t)) \to u^\alpha$ whenever $u \in r^{\mathbb{Z}}$ and t approaches 0 through a fixed residue class $sr^{\mathbb{Z}}$, $s > 0$. Then the following assertions hold:*

(a) $W_t := -\log M(t)$ is an endogeneous fixed point w.r.t. $T^{(\alpha)}$ for any $t > 0$.

(b) If $1 - f$ is regularly varying of index α at 0, then $M(t) = e^{-t^\alpha W}$ a.s. for all $t \ge 0$, where $W := W_1$. Furthermore, W is a.s. finite.

Proof. (a) Fix $t > 0$ and let $W_t := -\log M(t)$. By the proof of Lemma 6.2 in [6], $\mathbb{E}\, M(t) = f(t) < 1$ and thus $\mathbb{P}(W_t > 0) > 0$. For any $v \in \mathbb{V}$ and $s \in r^{\mathbb{Z}}$, a combination of $\sup_{|u|=n} L(u) \to 0$ a.s., our assumptions on the behaviour of f at 0 and Lemma 7.5(a) gives

$$\begin{aligned} -\log[M]_v(st) &= \lim_{n\to\infty} \sum_{|u|=n} 1 - f(st[L(u)]_v) && (7.8) \\ &= \lim_{n\to\infty} \sum_{|u|=n} \frac{1 - f(st[L(u)]_v)}{1 - f(t[L(u)]_v)} (1 - f(t[L(u)]_v)) \\ &= s^\alpha \lim_{n\to\infty} \sum_{|u|=n} 1 - f(t[L(u)]_v) \\ &= s^\alpha(-\log[M]_v(t)) \quad \text{a.s.} && (7.9) \end{aligned}$$

Use this with $s = L(v)$ for $|v| = n$ and recall (7.4) to obtain

$$\begin{aligned} W_t &= -\log \prod_{|v|=n} [M]_v(tL(v)) \\ &= \sum_{|v|=n} -\log[M]_v(tL(v)) = \sum_{|v|=n} L(v)^\alpha [W_t]_v \quad \text{a.s.,} \end{aligned}$$

where in the r-geometric case $L(v) \in r^{\mathbb{Z}}$ a.s. for all $v \in \mathbb{V}$ has been utilized. We have thus proved that W_t is an endogeneous fixed point.

(b) By an application of Eqs (7.8) and (7.9), which are valid for all $t, u > 0$ if $1 - f$ is regularly varying of index α at 0, we infer with $t = 1$, $v = \varnothing$ and arbitrary $s > 0$

$$M(s) = e^{-s^\alpha W} \quad \text{a.s.}$$

In particular, $f(t) = \varphi(t^\alpha)$ for all $t \ge 0$ where φ denotes the Laplace transform of W. This implies that $\varphi(t) \to 1$ as $t \downarrow 0$ and thus $W < \infty$. ☐

8 Reduction to the case $T_i < 1$ a.s. for all $i \geq 1$

As in [14, Section 3], one element in the approach here is the reduction to the simpler case when the weights T_i are bounded from above by 1. Recall from Section 6 the definition of $\sigma^>$ and put $\mathcal{T}^> = \mathcal{T}_{\sigma^>}$ and $N^> := |\mathcal{T}^>|$. Denote by $T^> := (T_i^>)_{i\geq 1}$ the enumeration of the family $(L(v))_{v\in\mathcal{T}^>}$ in decreasing order where $T_i^> := 0$ if $i > |\mathcal{T}^>|$.

Lemma 8.1. *If T satisfies* (A1)–(A3)*, then so does $T^> = (T_i^>)_{i\geq 1}$, thus*

$$\mathbb{P}(T_i^> \in \{0,1\} \textit{ for any } i \geq 1) < 1, \quad \mathbb{E}\, N^> > 1 \quad \textit{and}$$

$$1 = m^>(\alpha) < m^>(\beta) \quad \textit{for all } \beta \in [0,\alpha).$$

Moreover, if T satisfies (A4a) *or* (A4b)*, then the same holds true for $T^>$, respectively. Finally, $\mathbb{G}(T) = \mathbb{G}(T^>)$, where $\mathbb{G}(T)$ and $\mathbb{G}(T^>)$ denote the minimal closed multiplicative subgroups $\subset \mathbb{R}^+$ generated by T and $T^>$ respectively.*

Proof. Under the given assumptions, we can apply Lemma 6.1 to infer that $\limsup_{n\to\infty} S_n = \infty$ a.s. or, equivalently, $\mathbb{P}(\sigma^> < \infty) = 1$. Hence Proposition 5.1 in [4] yields that the sequence $((T_i^>)^\alpha)_{i\geq 1}$ satisfies conditions (A1)–(A3). Further, if also (A4a) is assumed for T, then again Proposition 5.1 in [4] yields the validity of (A4a) for $T^>$. If T satisfies (A4b), *i.e.*, if $m(\theta) < \infty$ for some $\theta < \alpha$, then Lemma 6.3 yields $m^>(\theta) < \infty$ for the same θ. It remains to prove that $\mathbb{G}(T) = \mathbb{G}(T^>)$. To this end, notice that $-\log \mathbb{G}(T) = \mathbb{G}(\mu_\alpha)$ and $-\log \mathbb{G}(T^>) = \mathbb{G}(\mu_\alpha^>)$, where $\mathbb{G}(\mu_\alpha)$ and $\mathbb{G}(\mu_\alpha^>)$ denote the minimal closed additive subgroups of $\mathbb{R}$ generated by the distributions μ_α and $\mu_\alpha^>$, respectively. Now, $\mu_\alpha = \mathbb{P}(S_1 \in \cdot)$ while by Eq. (6.2), $\mu_\alpha^> = \mathbb{P}(S_{\sigma^>} \in \cdot)$. From classical renewal theory (see *e.g.* [9, Section 2]) we know that the minimal closed subgroups generated by a distribution and an associated ladder height distribution coincide if the associated ladder index is a.s. finite. ☐

Now the reduction to the case where $T_i < 1$ a.s. for all $i \geq 1$ is easily justified: First, by Lemma 7.4, $f \in \mathcal{S}(\mathcal{M})$ entails that f also solves (1.1) with T replaced by $T^>$. Second, the previous lemma ensures that validity of (A1)–(A3) for T carries over to $T^>$ with the same characteristic exponent α, and the same inheritance holds true for (A4a), (A4b) and the minimal closed subgroup $\mathbb{G}(T)$, respectively. Therefore, imposing the additional assumption (besides (A1)–(A4))

$$T_i < 1 \text{ a.s. for all } i \geq 1 \tag{A5}$$

constitutes no loss of generality.

9 Results for general branching processes

The weighted branching model introduced in Section 4 gives rise to the definition of a related general (CMJ) branching process. This is a critical connection here and in [12, 14]. Recall that $S(v) := -\log L(v)$ for $v \in \mathbb{V}$. Then the reproduction point process $\mathcal{Z}^>$ of this general branching process is given by

$$\mathcal{Z}^> \;:=\; \sum_{v \in \mathcal{T}^>} \delta_{S(v)}, \tag{9.1}$$

and its nth generation by

$$\mathcal{Z}_n^> \;:=\; \sum_{v \in \mathcal{T}_n^>} \delta_{S(v)}, \tag{9.2}$$

where $\mathcal{T}_n^>$ denotes the HSL associated with the stopping rule $\sigma_n^>$, the nth strictly ascending ladder index. In particular, $\mathcal{T}_1^> = \mathcal{T}^>$. Lemma 8.1 establishes properties of $\{e^{-S(v)} : v \in \mathcal{T}^>\}$ that are inherited from T, or from the corresponding point process $\mathcal{Z}$, defined by

$$\mathcal{Z} \;:=\; \sum_{i=1}^{N} \delta_{S(i)}.$$

These properties can easily be reinterpreted as properties of the corresponding intensity measure $\mu^>$ of the reproduction point process $\mathcal{Z}^>$. The key reference for CMJ processes is [32], where $\mu^>$ is assumed not to be concentrated on a centred lattice (which corresponds exactly to what is here called the continuous or non-geometric case) but 'all results could be modified to the lattice case'. The details of the lattice case (at least concerning a.s. convergence results) have been supplied in [21].

Keep in mind that $\mathcal{T}_t$ is defined to be the HSL associated with the first exit time $\tau(t)$. Define $W_{\mathcal{T}_t}^{(\alpha)} := \sum_{v \in \mathcal{T}_t} L(v)^\alpha$.

Proposition 9.1. *$(W_{\mathcal{T}_t}^{(\alpha)})_{t \geq 0}$ is a non-negative martingale with a.s. limit $W^{(\alpha)}$.*

Source. The assertion follows from [32, Proposition 2.4]. □

Let T_t be the number of births in the general branching process up to and including time t, *i.e.*,

$$T_t \;=\; |\{v \in \mathbb{V} : v \in \mathcal{T}_n^> \text{ for some } n \in \mathbb{N}_0 \text{ and } S(v) \leq t\}|.$$

Then the survival set S of the process $(N_n)_{n \geq 0}$ satisfies

$$S \;=\; \{T_t \to \infty\} \;=\; \{\mathcal{T}_t \neq \emptyset \text{ for all } t \geq 0\} \quad \text{a.s.}$$

S has positive probability iff $\mathbb{E}\, N^> > 1$ where $N^> := |\mathcal{T}^>|$. Moreover, $S = \{W^{(\alpha)} > 0\}$ a.s. if $\mathbb{P}(W^{(\alpha)} > 0) > 0$, which in turn is guaranteed by (A4a).

The next result provides us with sufficient conditions for *ratio convergence* of certain general branching processes on S. More precisely, it focuses on the asymptotic behaviour of the ratio

$$\frac{\sum_{v \in \mathcal{T}_t} e^{-\beta(S(v)-t)}\, \mathbb{1}_{\{S(v)-t>c\}}}{\sum_{v \in \mathcal{T}_t} e^{-\alpha(S(v)-t)}} \tag{9.3}$$

with $\beta > 0$. The formulation of the next result is adapted to apply to both lattice (r-geometric) and continuous (non-geometric) cases.

Proposition 9.2. *Assume* (A1)–(A3) *and let* $\varepsilon > 0$. *Then the following two assertions hold:*

(a) If (A4a) *is satisfied, then for* $\beta = \alpha$ *and all sufficiently large* c

$$\frac{\sum_{v \in \mathcal{T}_t} e^{-\beta(S(v)-t)} \, \mathbb{1}_{\{S(v)-t>c\}}}{\sum_{v \in \mathcal{T}_t} e^{-\alpha(S(v)-t)}} \;\to\; \varepsilon(c) \;\leq\; \varepsilon \text{ on } S \text{ as } t \to \infty \tag{9.4}$$

in probability.

(b) If (A4b) *is satisfied, then* (9.4) *holds true in the a.s. sense for any* $\beta \geq \theta$ *and all sufficiently large* c *(depending on* β*).*

Proof. The result follows from Theorems 3.1 and 6.3 in [32] and the corresponding lattice-case results if we check that the appropriate conditions are fulfilled. In what follows we restrict ourselves to the continuous case, the lattice case being similar.

First note that in the situation of both assertions (a) and (b), (A1)–(A4) are fulfilled. Thus, by the results in Section 8, we can further assume the validity of (A5). The last condition means that we are in the situation of general branching processes while (A2) implies supercriticality. Since we are in the continuous case, Condition (i) in the introduction of [32] is satisfied. (A3) implies the existence of a *Malthusian parameter* (namely, α), which is Nerman's condition (ii), and the validity of (A4) ensures the validity of Nerman's condition (iii) (this is immediate if (A4a) holds whereas it follows from the fact that m is strictly decreasing (by (A5)) and convex on $[\theta, \infty)$ in the case that (A4b) holds).

Now, following Nerman's notation, the numerator in (9.3) derives from

$$\begin{aligned}
\phi(t) &= \mathbb{1}_{[0,\infty)}(t) \sum_{i \geq 1} e^{-\beta(S(i)-t)} \, \mathbb{1}_{\{S(i)>t+c\}} \\
&\leq \mathbb{1}_{[0,\infty)}(t) \sum_{i \geq 1} e^{-\beta(S(i)-t)} \, \mathbb{1}_{\{S(i)>t\}}
\end{aligned}$$

and the denominator from

$$\psi(t) = \mathbb{1}_{[0,\infty)}(t) \sum_{i \geq 1} e^{-\alpha(S(i)-t)} \, \mathbb{1}_{\{S(i)>t\}} \,.$$

$e^{-\beta t}\phi(t)$ and $e^{-\alpha t}\psi(t)$ are decreasing in $t \geq 0$. Thus, ϕ and ψ have paths in the Skorohod D-space and $\mathbb{E}\,\phi(t)$ and $\mathbb{E}\,\psi(t)$ are continuous almost everywhere w.r.t. Lebesgue measure.

Now we prove part (a) of the proposition. To this end, assume that (A4a) holds. Then ϕ and ψ satisfy Nerman's condition (3.2) because

$$\begin{aligned}
e^{-\alpha t}\phi(t) &\leq e^{-\alpha t}\psi(t) \\
&= e^{-\alpha t} \, \mathbb{1}_{[0,\infty)}(t) \sum_{i \geq 1} e^{-\alpha(S(i)-t)} \, \mathbb{1}_{\{S(i)>t\}} \;\leq\; \sum_{i \geq 1} e^{\alpha S(i)}
\end{aligned}$$

for all $t \geq 0$. Moreover,

$$\int_0^\infty e^{-\alpha t}\,\mathbb{E}\,\phi(t)\,\mathrm{d}t \;\leq\; \int_0^\infty e^{-\alpha t}\,\mathbb{E}\,\psi(t)\,\mathrm{d}t \;=\; \int_0^\infty \mathbb{E}\sum_{i\geq 1} e^{-\alpha S(i)}\,\mathbb{1}_{\{S(i)>t\}}\,\mathrm{d}t$$
$$= \int_0^\infty \mathbb{P}(S_1 > t)\,\mathrm{d}t \;=\; \mathbb{E}\,S_1,$$

where we have used (6.1). Now (A4a) yields that $\mathbb{E}\,S_1$ is positive and finite. Since $e^{-\alpha t}\psi(t)$ is decreasing, the integral criterion ensures the validity of Nerman's condition (3.1) for both, ϕ and ψ. Hence, by [32, Theorem 3.1] and another use of (6.1), we get

$$e^{-\alpha t}\sum_{v\in\mathcal{T}_t} e^{-\alpha(S(v)-t)}\,\mathbb{1}_{\{S(v)-t>c\}} \;\to\; W^{(\alpha)}\,\frac{\int_0^\infty e^{-\alpha s}\,\mathbb{E}\,\phi(s)\,\mathrm{d}s}{\mathbb{E}\,S_1}$$
$$= W^{(\alpha)}\,\frac{\int_c^\infty \mathbb{P}(S_1 > s)\,\mathrm{d}s}{\mathbb{E}\,S_1}$$

in probability as $t \to \infty$. For the denominator, Proposition 9.1 shows that

$$e^{-\alpha t}\sum_{v\in\mathcal{T}_t} e^{-\alpha(S(v)-t)} \;=\; W^{(\alpha)}_{\mathcal{T}_t} \;\to\; W^{(\alpha)} \quad \text{a.s.}$$

Thus, the ratio tends to $\varepsilon(c) := (\mathbb{E}\,S_1)^{-1}\int_c^\infty \mathbb{P}(S_1 > s)\,\mathrm{d}s$ in probability on the set of survival S as $t \to \infty$. Finally, integrability of S_1 ensures that $\varepsilon(c)$ can be made arbitrarily small.

Turning to the proof of part (b), suppose that (A4b) holds which gives

$$\mathbb{E}\sum_{i\geq 1} e^{-\theta S(i)} \;=\; m(\theta) \;<\; \infty.$$

This implies the validity of Nerman's Condition 6.1. As for his Condition 6.2, fix $\beta \geq \theta$ and observe that $e^{-\beta(S(i)-t)} \leq e^{-\theta(S(i)-t)}$ on $\{S(i) > t\}$. Thus,

$$e^{-\theta t}\,\mathbb{1}_{[0,\infty)}(t)\sum_{i\geq 1} e^{-\beta(S(i)-t)}\,\mathbb{1}_{\{S(i)>t\}} \;\leq\; \sum_{i\geq 1} e^{-\theta S(i)},$$

which is integrable by (A4b). Therefore, ϕ and ψ satisfy Nerman's Condition 6.2. Hence, by Theorem 6.3 in [32], we infer that the ratio in the proposition tends to $\varepsilon(c)$ a.s. on S where $\varepsilon(c)$ is defined as in the proof of part (a). By the same reasoning as above, $\varepsilon(c)$ tends to 0 as c tends to ∞ which completes our argument. □

Proposition 9.2 is an essential ingredient to the proof of the next result, which is in the spirit of Theorem 8.6 in [12] and is designed to give conditions which allow (9.6) to be deduced from (9.7).

Theorem 9.3. *Suppose that* (A1)–(A3) *and* (A4b) *hold. Assume also that the following three conditions hold for a sequence $t_n \uparrow \infty$, which in the r-geometric case takes values in $d\,\mathbb{Z}$ ($d := \log r$) only for all $n \geq 1$:*

(i) There are a non-negative function H and a random variable W such that

$$\sum_{v\in\mathcal{T}_{t_n}} e^{-\alpha S(v)} H(S(v)) \ \to\ W \quad a.s.\ as\ n\to\infty. \tag{9.5}$$

(ii) For some $h<\infty$,

$$\varepsilon_n(a) \ =\ \left(\frac{H(a+t_n)}{H(t_n)} - h\right) \ \to\ 0 \quad as\ n\to\infty$$

uniformly in a on compact subsets of $[0,\infty)$.

(iii) For a finite K, all $a\geq 0$, and all sufficiently large $n\geq 1$

$$\frac{H(a+t_n)}{H(t_n)} \ \leq\ Ke^{(\alpha-\theta)a}. \tag{9.6}$$

Then

$$H(t_n)\sum_{v\in\mathcal{T}_{t_n}} e^{-\alpha S(v)} \ \to\ hW \quad a.s. \quad (n\to\infty), \tag{9.7}$$

where in the r-geometric case it suffices that (ii) holds for $a\in d\,\mathbb{Z}$ only and uniform convergence on compact subsets of $[0,\infty)$ can be dropped.

Proof. Note first that, by increasing K if necessary, condition (iii) implies that for all large n

$$|\varepsilon_n(a)| \ \leq\ Ke^{(\alpha-\theta)a} \quad (a\geq 0).$$

Clearly, the limits in (9.5) and (9.7) are both zero when $\mathcal{T}_{t_n}$ is eventually empty, and so attention can centre on the survival set S. For this proof let $\sum$ be the sum over $v\in\mathcal{T}_{t_n}$. Then, considering the ratio of the terms on the left-hand sides of (9.5) and (9.7),

$$\begin{aligned}
\frac{\sum e^{-\alpha S(v)}H(S(v))}{H(t_n)\sum e^{-\alpha S(v)}} \ &=\ \frac{\sum e^{-\alpha S(v)}H(S(v))/H(t_n)}{\sum e^{-\alpha S(v)}} \\
&=\ \frac{\sum e^{-\alpha S(v)}(h+\varepsilon_n(S(v)-t_n))}{\sum e^{-\alpha S(v)}} \\
&=\ h+\frac{\sum e^{-\alpha(S(v)-t_n)}\varepsilon_n(S(v)-t_n)}{\sum e^{-\alpha(S(v)-t_n)}}.
\end{aligned}$$

Fix $c>0$ and note that $\delta_n := \sup\{|\varepsilon_n(a)| : 0\leq a\leq c\}$ tends to 0 by condition (ii). Then

$$\left|\frac{\sum e^{-\alpha(S(v)-t_n)}\varepsilon_n(S(v)-t_n)}{\sum e^{-\alpha(S(v)-t_n)}}\right| \ \leq\ \delta_n + \frac{\sum e^{-\theta(S(v)-t_n)}K\,\mathbb{1}_{\{S(v)-t_n>c\}}}{\sum e^{-\alpha(S(v)-t_n)}}.$$

Using Proposition 9.2, the right-hand side goes to zero as n and then c tends to infinity. In the r-geometric case the same argument works with $\delta_n := \max\{|\varepsilon_n(a)| : a\in[0,c]\cap d\,\mathbb{Z}\}$, which converges to zero when the convergence in (ii) holds for $a\in d\,\mathbb{Z}$. □

10 Uniqueness: proof of Theorem 2.3(b)

Lemma 10.1. *Assume that* (A1)–(A3) *hold and that* $\limsup_{n\to\infty} S_n = \infty$ *a.s. Further, assume that* φ *solves* (1.1) *with* T *replaced by* $(T_i^\alpha)_{i\geq 1}$. *Then* $1-\varphi(t)$ *is regularly varying of index* 1 *at* 0.

Note that in the situation of the Lemma, condition (A4) is sufficient for $\limsup_{n\to\infty} S_n = \infty$ a.s. to hold. The following proof is based on the proofs of Theorems 1.4 in [12] and 1 in [27].

Proof. For fixed $t > 0$, $u^{-1}(1-\varphi(ut))$ is the Laplace transform of a σ-finite measure on $[0,\infty)$ (see *e.g.* [20, Section XIII.2, Eq. (2.7)]) and thus so is $u^{-1}(1-\varphi(ut))/(1-\varphi(t))$. The latter is bounded by $u^{-1} \vee 1$ for $u > 0$. A standard selection argument shows that any sequence decreasing to 0 contains a subsequence $(t_n)_{n\geq 1}$ such that

$$u^{-1}\frac{1-\varphi(ut_n)}{1-\varphi(t_n)} \underset{n\to\infty}{\longrightarrow} l(u) \qquad (u>0)$$

for some decreasing and convex function $l : (0,\infty) \to (0,\infty)$. Now fix any such $(t_n)_{n\geq 1}$ with corresponding limiting function l. Then, by reproducing the following telescoping sum from [12, p. 345], which is obtained from the fact that φ satisfies the functional equation (1.1) with T_i^α instead of T_i, we get

$$\begin{aligned}
l(u) &= \lim_{n\to\infty} \frac{1-\varphi(ut_n)}{u(1-\varphi(t_n))} \\
&= \lim_{n\to\infty} \mathbb{E}\sum_{i\geq 1} T_i^\alpha \frac{1-\varphi(uT_i^\alpha t_n)}{uT_i^\alpha(1-\varphi(t_n))}\prod_{k<i}\varphi(ut_nT_k^\alpha) \\
&\geq \mathbb{E}\sum_{i\geq 1} \liminf_{n\to\infty} T_i^\alpha \frac{1-\varphi(uT_i^\alpha t_n)}{uT_i^\alpha(1-\varphi(t_n))}\prod_{k<i}\varphi(ut_nT_k^\alpha) \\
&= \mathbb{E}\sum_{i\geq 1} T_i^\alpha l(uT_i^\alpha) \;=\; \mathbb{E}\, l(ue^{-\alpha S_1}),
\end{aligned}$$

where the inequality follows from a double application of Fatou's Lemma and the last equality stems from (6.1) with $n = 1$. Thus $(l(ue^{-\alpha S_n}))_{n\geq 0}$ is a non-negative supermartingale and a.s. convergent to some finite limiting variable $g(u)$. Here,

$$g(u) \;=\; \lim_{n\to\infty} l(ue^{-\alpha S_n}) \;=\; \limsup_{n\to\infty} l(ue^{-\alpha S_n}) \;=\; l(0+),$$

using the assumption that $\limsup_{n\to\infty} S_n = \infty$ a.s. Hence, g is constant. Clearly, $g(1) = 1$ and thus $l(0+) = 1$, which in view of $l(1) = 1$ and the monotonicity of l immediately implies that $l(u) = 1$ for all $u \in (0,1]$. Since this limit is independent of the choice of $(t_n)_{n\geq 1}$, $1-\varphi$ is regularly varying of index 1 at 0. ☐

Theorem 10.2. *Suppose that* (A1)–(A3) *hold and that* $\mathbb{E}\, W^{(\alpha)} = 1$ *or* (A4b) *holds. Let* φ *be a solution to* (1.1) *with* $T^{(\alpha)}$ *instead of* T *and let* Φ *be its disintegration. Then*

$$\lim_{t\to\infty} e^t(1-\varphi(e^{-t}))\sum_{v\in\mathcal{T}_t} L(v)^\alpha \;=\; -\log\Phi(1) \quad a.s.$$

Note that the theorem also holds under (A1)–(A4), for, from [30], $\mathbb{E}\, W^{(\alpha)} = 1$ is slightly weaker than (A4a).

Proof. Let $W := -\log\Phi(1)$. We first consider the easier case that $\mathbb{E}\, W^{(\alpha)} = 1$. Then

$$W^{(\alpha)}_{\mathcal{T}_t} \;=\; \sum_{v\in\mathcal{T}_t} L(v)^\alpha \;\to\; W^{(\alpha)} \quad \text{a.s. as } t\to\infty. \tag{10.1}$$

Now suppose that $(1-\varphi(t))/t \to \infty$ as $t \downarrow 0$. Then, for any $K > 0$, using Lemma 7.5(c) and (10.1),

$$W \;=\; \lim_{t\to\infty}\sum_{v\in\mathcal{T}_t}(1-\varphi(L(v)^\alpha)) \;\geq\; \lim_{t\to\infty}\sum_{v\in\mathcal{T}_t} KL(v)^\alpha \;=\; KW^{(\alpha)} \quad \text{a.s.}$$

Letting $K\uparrow\infty$ yields $\mathbb{P}(W=\infty) \geq \mathbb{P}(W^{(\alpha)} > 0) > 0$ contradicting Lemma 7.6(b). Thus, $c := \mathbb{E}\, W < \infty$. By Proposition 5.2 we have $W = cW^{(\alpha)}$ and so $(1-\varphi(t))/t \to c$ as $t\downarrow 0$, which together with (10.1) give the result.

Now suppose (A4b) holds. Then $D_1(x) := x^{-1}(1-\varphi(x))$ is slowly varying at the origin by Lemma 10.1. In particular, $|D_1(xy)/D_1(y) - 1| \to 0$ as $y \downarrow 0$ uniformly in x on compact subsets of $(0,\infty)$ and for any $\varepsilon > 0$ there exists a finite K and a $C > 0$ such that $D_1(xy)/D_1(y) \leq Kx^{-\varepsilon}$ for all $x \leq 1$ and $y \leq C$. (These statements follow from Theorem 1.2.1 in [15] and from the integral representation of slowly varying functions given in Theorem 1.3.1 in [15].) Let $H(t) := D_1(e^{-t})$. Then assumptions (ii) and (iii) of Theorem 9.3 hold, and (i) follows from a calculation similar to that at the start of the proof of Lemma 7.6. Therefore, Theorem 9.3 gives the result. □

Proof of Theorem 2.3(b). Regular variation is given in Lemma 10.1. It remains to show uniqueness up to a scale factor. If $\mathbb{E}\, W^{(\alpha)} = 1$, the result follows already from the proof of Theorem 10.2, where we showed that $(1-\varphi(t))/t$ tends to $c \in (0,\infty)$ as $t \downarrow 0$. In the general case, let W and $\hat{W}$ be solutions to (1.6) with Laplace transforms φ and $\hat{\varphi}$, respectively. Let $D_1(t) := t^{-1}(1-\varphi(t))$ and $\hat{D}_1(t) := t^{-1}(1-\hat{\varphi}(t))$, $t > 0$. Then, by Theorem 10.2,

$$\lim_{t\to\infty} D_1(e^{-t})\sum_{v\in\mathcal{T}_t} L(v)^\alpha \;=\; -\log\Phi(1) \quad \text{a.s.}$$

where Φ denotes the disintegration of W. An analogous result holds for $\hat{W}$ and its disintegration $\widehat{\Phi}$. On the other hand,

$$\lim_{t\to\infty}\frac{D_1(e^{-t})\sum_{v\in\mathcal{T}_t} L(v)^\alpha}{\hat{D}_1(e^{-t})\sum_{v\in\mathcal{T}_t} L(v)^\alpha} \;=\; \lim_{t\to\infty}\frac{D_1(e^{-t})}{\hat{D}_1(e^{-t})},$$

that is, the limit of the ratios is a deterministic non-negative constant $c \in [0,\infty]$, say. This implies $-\log\Phi(1) = c(-\log\widehat{\Phi}(1))$ a.s. Now, by Lemma 7.6, $-\log\Phi(1)$ and $-\log\widehat{\Phi}(1)$ are both a.s. finite and positive with positive probability which implies $c \in (0,\infty)$ and hence the asserted uniqueness. □

11 Regular variation of fixed points at 0

The key to the proof of Theorem 7.2 is the verification that, for any $f \in \mathcal{S}(\mathcal{M})$, if (A4) holds $1-f$ is regularly varying at 0 with index α in the continuous case, and it is 'nearly' regularly varying otherwise.

Theorem 11.1. *Under* (A1)–(A4)*, any* $f \in \mathcal{S}(\mathcal{M})$ *satisfies*

$$\lim_{t\downarrow 0} \frac{1-f(ut)}{1-f(t)} \;=\; u^{\alpha} \tag{11.1}$$

for all $u \in (0,\infty)$ *in the continuous case and all* $u \in r^{\mathbb{Z}}$ *in the* r*-geometric case, where in the latter case the limit* $t \downarrow 0$ *is restricted to some arbitrary fixed residue class* $sr^{\mathbb{Z}}$*,* $s \in [1,r)$*.*

The rest of this section is devoted to the proof of this theorem, which is divided into four steps: The first step is a standard selection argument that guarantees that, for any solution $f \in \mathcal{S}(\mathcal{M})$ and any sequence $t \downarrow 0$, the ratio $(1-f(st))/(1-f(t))$ as a function of $s \in [0,1]$ has a convergent subsequence. In the second step we introduce $\mathcal{S}(\mathcal{M})^{\beta}$, a subset of the set of fixed points containing only fixed points which show a sufficiently regular behaviour at 0. Further, we prove that if $f \in \mathcal{S}(\mathcal{M})^{\beta}$ where $\beta := \theta$ if (A4b) holds and $\beta = \alpha$, otherwise, any limiting function as obtained in Step 1 satisfies a Choquet-Deny type equation. An appeal to the theory of Choquet-Deny type functional equations as presented in [35] provides us with a fairly good description of the behaviour of f at 0. The idea of utilizing a Choquet-Deny type equation was taken from the proof of Theorem 2.12 in [19]. Step 3 provides a proof of Theorem 11.1 based on the previous two steps and under the additional assumption that $f \in \mathcal{S}(\mathcal{M})^{\beta}$. In the final Step 4, we prove that $\mathcal{S}(\mathcal{M})^{\beta} = \mathcal{S}(\mathcal{M})$.

Using the results in Section 8, we can and will assume for the rest of this section that (A5) holds, *i.e.*, that $T_i < 1$ a.s. for all $i \geq 1$.

Step 1: The selection argument

Lemma 11.2. *Assume* (A1)–(A5) *hold and let* $f \in \mathcal{S}(\mathcal{M})$*. Then any sequence decreasing to zero contains a subsequence* $(t_n)_{n\geq 1}$ *such that, for an increasing function* $g : (0,1] \to [0,1]$ *satisfying* $g(1) = 1$*,*

$$\frac{1-f(ut_n)}{1-f(t_n)} \underset{n\to\infty}{\longrightarrow} g(u) \tag{11.2}$$

for all $u \in (0,1]$*.*

Proof. It follows from the proof of Lemma 6.2 in [6] that $1 - f(t) > 0$ for all $t > 0$. Thus, the ratio in (11.2) is well defined. Now starting with an initial sequence decreasing to zero we choose a subsequence giving convergence for each rational $u \in (0,1]$. This is possible since $(1 - f(ut))/(1 - f(t)) \in [0,1]$ by the monotonicity of f. This defines an increasing limit, which can have only countably many discontinuities. Now select further subsequences to get convergence at any discontinuity and define the resulting limit to be g. Obviously $g(1) = 1$. □

Step 2: An application of the theory of Choquet-Deny equations

We introduce a subset of $\mathcal{S}(\mathcal{M})$ with members that behave more regularly at 0. Recall that, for $f \in \mathcal{S}(\mathcal{M})$, $D_\beta(t)$ is $(1 - f(t))/t^\beta$. With this notation,

$$\mathcal{S}(\mathcal{M})^\beta \ := \ \{f \in \mathcal{S}(\mathcal{M}) : \sup_{u \le 1, t \le c} D_\beta(ut)/D_\beta(t) < \infty \text{ for some } c > 0\}. \quad (11.3)$$

For the rest of this section let $\beta := \theta$ if (A4b) holds and $\beta := \alpha$, otherwise.

Lemma 11.3. *Assume* (A1)–(A5) *and let* $f \in \mathcal{S}(\mathcal{M})^\beta$. *Then, for any sequence decreasing to zero, there exist a subsequence* $(t_n)_{n\ge1}$ *and a function* h *satisfying*

$$\lim_{n\to\infty} \frac{1 - f(ut_n)}{1 - f(t_n)} \ = \ h(u)u^\alpha \quad (11.4)$$

for all $u \in (0,1]$. *In the continuous case,* h *is a positive constant, while in the lattice case,* h *is strictly positive and multiplicatively* r*-periodic.*

Proof. Let $f \in \mathcal{S}(\mathcal{M})^\beta$. Then, for any given sequence decreasing to zero choose a subsequence according to Lemma 11.2, that is, a subsequence $(t_n)_{n\ge1}$ such that the fraction $(1-f(ut_n))/(1-f(t_n))$ converges to $g(u)$ for some increasing function $g : (0,1] \to [0,1]$ satisfying $g(1) = 1$. Then, as in the proof of Lemma 10.1,

$$\frac{1 - f(ut_n)}{u^\alpha(1 - f(t_n))} \ = \ \mathbb{E}\sum_{i\ge1} T_i^\alpha \frac{1 - f(uT_it_n)}{(uT_i)^\alpha(1 - f(t_n))} \prod_{k<i} f(ut_nT_k). \quad (11.5)$$

Since $f \in \mathcal{S}(\mathcal{M})^\beta$, we have

$$T_i^\alpha \frac{1 - f(uT_it_n)}{(uT_i)^\alpha(1 - f(t_n))} \ \le \ KT_i^\alpha(uT_i)^{\beta-\alpha} \ = \ Ku^{\beta-\alpha}T_i^\beta$$

for sufficiently large n, some deterministic constant $K < \infty$ and all i. By the definition of β, $m(\beta)$ is finite and thus the dominated convergence theorem yields upon letting $n \to \infty$ in (11.5)

$$g(u)/u^\alpha \ = \ \mathbb{E}\sum_{i=1}^N T_i^\alpha \frac{g(uT_i)}{(uT_i)^\alpha} \qquad (u \in (0,1]).$$

Equivalently (see (6.1)), $\tilde{g}(x) := e^{\alpha x}g(e^{-x})$ $(x \ge 0)$ satisfies the following Choquet-Deny type functional equation:

$$\tilde{g}(x) \ = \ \mathbb{E}\,\tilde{g}(x + S_1) \qquad (x \ge 0). \quad (11.6)$$

Since g is increasing and bounded, $\tilde{g}$ is locally bounded on $[0, \infty)$ and thus locally integrable w.r.t. Lebesgue measure. Moreover, since $1 = \tilde{g}(0) = \mathbb{E}\,\tilde{g}(S_1)$, we obtain that $\mathbb{P}(\tilde{g}(S_1) \geq 1) > 0$, which immediately implies that $\tilde{g}(x_0) \geq 1$ for some $x_0 > 0$. This in combination with $\tilde{g}$ being the product of a decreasing function and an increasing function gives $\tilde{g} > 0$ on $[0, x_0]$. Now assume first that we are in the continuous case.Then an application of Theorem 2.2.2 in [35] shows that $\tilde{g}$ equals a constant c almost everywhere w.r.t. Lebesgue measure. Utilizing $\tilde{g} > 0$ on $[0, x_0]$ yields $c > 0$. Rewriting this in terms of g gives $g(u) = cu^\alpha$ almost everywhere w.r.t. Lebesgue measure. From this we can conclude that $g(u) = cu^\alpha$ for all $u \in (0, 1)$ since g is known to be increasing. Furthermore, since $g(1) = 1$, $c \leq 1$, but to establish that $c = 1$ needs an additional argument given in the next step. In the lattice case we have that S_1 is confined to $\mathbb{Z}^d$ with $d := \log r$. Then Corollary 2.2.3 in [35] yields $\tilde{g}(x + nd) = \tilde{g}(x)$ for all $x \geq 0$ and $n \in \mathbb{N}_0$, that is, $\tilde{g}$ is d-periodic. This immediately provides us with the identity $g(u) = \tilde{g}(-\log u)u^\alpha =: h(u)u^\alpha$ $(u \in (0, 1])$ where $h(u) = \tilde{g}(-\log u)$ is multiplicatively r-periodic. The fact that h is strictly positive follows from the monotonicity of g in combination with $g(1) = 1$ and the periodicity of h. □

Step 3: Proof of Theorem 11.1 for $f \in \mathcal{S}(\mathcal{M})^\beta$

Let $f \in \mathcal{S}(\mathcal{M})^\beta$. It suffices to show that for any sequence $t_n \downarrow 0$ (where t_n is chosen from a fixed residue class of $\mathbb{R}^+$ mod $r^\mathbb{Z}$ in the r-geometric case) there exists a subsequence such that the convergence in (11.1) holds along this subsequence on $\mathbb{G}(T) \cap (0, 1]$ ($\mathbb{G}(T)$ is the closed multiplicative subgroup generated by T).

In the lattice (r-geometric) case, this follows without further ado from Lemma 11.3 since in this case the limiting function h appearing in the lemma satisfies $h(r^{-n}) = 1$ for all $n \geq 0$.

In the continuous case ($\mathbb{G}(T) = \mathbb{R}^+$), for any given sequence decreasing to 0, Lemma 11.3 provides a subsequence $(t_n)_{n\geq1}$ and a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{1 - f(st_n)}{1 - f(t_n)} = cs^\alpha \tag{11.7}$$

for all $s \in (0, 1]$. For fixed $u \in (0, 1]$, Lemma 11.3 can again be applied to the sequence $(u^{-1}t_n)_{n\geq1}$ and provides us with a further subsequence $(u^{-1}t'_n)_{n\geq1}$ such that for some $c' \in (0, 1]$

$$\lim_{n\to\infty} \frac{1 - f(su^{-1}t'_n)}{1 - f(u^{-1}t'_n)} = c's^\alpha \tag{11.8}$$

holds for all $s \in (0, 1]$. For the remaining argument it constitutes no loss of generality to assume that $(t_n)_{n\geq1} = (t'_n)_{n\geq1}$. A combination of (11.7) and (11.8) then yields for $s \in (0, 1]$:

$$\begin{aligned} c'(su)^\alpha &= \lim_{n\to\infty} \frac{1 - f((su)u^{-1}t_n)}{1 - f(u^{-1}t_n)} \\ &= \lim_{n\to\infty} \frac{1 - f(st_n)}{1 - f(t_n)} \frac{1 - f(uu^{-1}t_n)}{1 - f(u^{-1}t_n)} = cs^\alpha c'u^\alpha = cc'(su)^\alpha. \end{aligned}$$

The fact that $c' > 0$ implies $c = 1$, and thus that Theorem 11.1 holds for $f \in \mathcal{S}(\mathcal{M})^\beta$.

Step 4: Proof that $\mathcal{S}(\mathcal{M})^\beta = \mathcal{S}(\mathcal{M})$

In the fourth step, we fix $f \in \mathcal{S}(\mathcal{M})$ with disintegration M and show that $D_\beta(t) = t^{-\beta}(1 - f(t))$ satisfies the growth condition in the definition of the set $\mathcal{S}(\mathcal{M})^\beta$ in Eq. (11.3) and, thus, that $f \in \mathcal{S}(\mathcal{M})^\beta$. To this end, let $\overline{W} := -\log M(1)$. Then, by Lemma 7.5(d),

$$\lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)}) \;=\; \overline{W} \quad \text{a.s.} \tag{11.9}$$

Proposition 2.3, proved in Section 10, implies that (1.6) has a scale-unique solution the Laplace transform of which we denote by φ. Then $\tilde{D}_1(t) := t^{-1}(1 - \varphi(t))$ is slowly varying at 0. Applying Lemma 7.5(d) again,

$$\lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)}) \;=\; W \tag{11.10}$$

with $W := -\log \Phi(1)$, where Φ denotes the disintegration of W. W has Laplace transform φ and is an endogeneous solution to (1.6) (see Lemma 7.6). The same lemma implies that W is a.s. finite and not identically zero.

The idea now is to bound D_α using $\tilde{D}_1$ and thereby to bound the behaviour of D_α at zero. Let

$$\mathrm{K}_\mathrm{l} \;:=\; \liminf_{t\to\infty} \frac{D_\alpha(e^{-t})}{\tilde{D}_1(e^{-\alpha t})} \qquad \text{and} \qquad \mathrm{K}_\mathrm{u} \;:=\; \limsup_{t\to\infty} \frac{D_\alpha(e^{-t})}{\tilde{D}_1(e^{-\alpha t})}.$$

The next lemma the only property of D_α in addition to (11.9) that is relevant for the subsequent results in the fourth step.

Lemma 11.4. *For any $c > 0$ there is a $\delta > 0$ such that*

$$\frac{D_\alpha(e^{-(x+a)})}{D_\alpha(e^{-x})} \;\le\; e^{\delta} \qquad \text{and} \qquad \frac{D_\alpha(e^{-(x-a)})}{D_\alpha(e^{-x})} \;\ge\; e^{-\delta}$$

for all $x \in \mathbb{R}$ and $0 \le a \le c$.

Proof. Recall that $1 - f(t) > 0$ for all $t > 0$ by [5, Lemma 6.2]. Since $e^{-\alpha x} D_\alpha(e^{-x}) = 1 - f(e^{-x})$ decreases,

$$\frac{D_\alpha(e^{-(x+a)})}{D_\alpha(e^{-x})} \;=\; \frac{e^{-\alpha(x+a)} D_\alpha(e^{-(x+a)})}{e^{-\alpha(x+a)} D_\alpha(e^{-x})} \;\le\; \frac{e^{-\alpha x} D_\alpha(e^{-x})}{e^{-\alpha(x+a)} D_\alpha(e^{-x})} \;=\; e^{\alpha a} \;\le\; e^{\alpha c}$$

for any $0 \le a \le c$. The second estimation is just the reciprocal of the first. $\square$

Lemma 11.5. *Under* (A1)–(A5)*, the following assertions are true:*

(a) $0 < \mathrm{K}_\mathrm{l} \le \mathrm{K}_\mathrm{u} < \infty$.

(b) $\varphi(\mathrm{K}_\mathrm{u}\, t^\alpha) \le f(t) \le \varphi(\mathrm{K}_\mathrm{l}\, t^\alpha)$ *for all $t \ge 0$.*

(c) $\mathrm{K_l}\ \tilde{D}_1(\mathrm{K_l}\ t^\alpha) \le D_\alpha(t) \le \mathrm{K_u}\ \tilde{D}_1(\mathrm{K_u}\ t^\alpha)$ *for all* $t \ge 0$.

Proof. Since $\tilde{D}_1$ is slowly varying, Lemma 7.5(d) and Theorem 10.2 imply that

$$\lim_{t\to\infty} \tilde{D}_1\left(e^{-\alpha(t+c)}\right) \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \;=\; W \;=\; \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1\left(e^{-\alpha S(v)}\right) \quad \text{a.s.}$$

for any $c \ge 0$. For any $\varepsilon > 0$, by Proposition 9.2 with $\beta = \alpha$, for c large enough

$$\begin{aligned} &\lim_{t\to\infty} \frac{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})\, \mathbb{1}_{\{S(v)\le t+c\}}}{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})} \\ &\quad= 1 - \lim_{t\to\infty} \frac{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})\, \mathbb{1}_{\{S(v)-t>c\}}}{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})} \\ &\quad= 1 - \lim_{t\to\infty} \frac{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}\, \mathbb{1}_{\{S(v)-t>c\}}}{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}} \\ &\quad\ge 1-\varepsilon \quad \text{in probability on } S, \end{aligned}$$

where we used that $\tilde{D}_1$ is slowly varying at 0. Now, using Lemma 11.4 and that $\tilde{D}_1(e^{-x})$ is increasing in x,

$$\begin{aligned} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)}) &\ge \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)})\, \mathbb{1}_{\{S(v)\le t+c\}} \\ &\ge e^{-\delta} D_\alpha(e^{-(t+c)}) \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}\, \mathbb{1}_{\{S(v)\le t+c\}} \\ &\ge e^{-\delta} \frac{D_\alpha(e^{-(t+c)})}{\tilde{D}_1(e^{-\alpha(t+c)})} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})\, \mathbb{1}_{\{S(v)\le t+c\}} \end{aligned}$$

for some $\delta > 0$. Therefore, letting $t \to \infty$ along an appropriate sequence,

$$\overline{W} \;\ge\; e^{-\delta}\,\mathrm{K_u}(1-\varepsilon) W \quad \text{a.s.} \tag{11.11}$$

Since $\mathbb{E}\,\Phi(1) = \varphi(1) < 1$, we have $q := \mathbb{P}(W > 0) > 0$. On the other hand, as a consequence of the regular variation of $1-\varphi$ at 0, finiteness of $\mathrm{K_u}$ is not affected by rescaling f, although the numerical value of $\mathrm{K_u}$ may change. Thus we can assume that $f(1) > 1-q$ for the rescaled f. Then, $f(1) = \mathbb{E}\, e^{-\overline{W}} > 1-q$ and so $\mathbb{P}(\overline{W} = \infty) < q$. Consequently, $\mathbb{P}(W > 0, \overline{W} < \infty) > 0$. We now conclude from (11.11) that $\mathrm{K_u}$ is finite for the rescaled f and thus also for the original f. Then, using Lemma 7.5(c) and the slow variation of $\tilde{D}_1$ at 0, for any $t > 0$,

$$\begin{aligned} -\log M(t) &= \lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} t^\alpha e^{-\alpha S(v)} D_\alpha(te^{-S(v)}) \\ &\le \lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} t^\alpha e^{-\alpha S(v)}\, \mathrm{K_u}\, \tilde{D}_1(te^{-S(v)}) \\ &= t^\alpha\, \mathrm{K_u}\, W \quad \text{a.s.} \end{aligned}$$

After an appeal to (7.2), we deduce that $f(t) \ge \varphi(\mathrm{K_u}\ t^\alpha)$, where we used that W has Laplace transform φ. This proves the second half of each of (a) and (b).

In a similar way, using Lemma 11.4 and the fact that $\tilde{D}_1(e^{-x})$ is increasing in x,

$$\begin{aligned}
\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)}) \;&\le\; e^\delta D_\alpha(e^{-t}) \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}\, \mathbb{1}_{\{S(v)\le t+c\}} \\
&\quad + \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)})\, \mathbb{1}_{\{S(v)>t+c\}} \\
&\le\; e^\delta \frac{D_\alpha(e^{-t})}{\tilde{D}_1(e^{-\alpha t})} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \tilde{D}_1(e^{-\alpha S(v)})\, \mathbb{1}_{\{S(v)\le t+c\}} \\
&\quad + \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(e^{-S(v)})\, \mathbb{1}_{\{S(v)>t+c\}} \,.
\end{aligned}$$

Letting t tend to infinity along an appropriate sequence, we obtain with the help of Proposition 9.2

$$\overline{W} \;\le\; e^\delta\, \mathrm{K_l}\, W + \mathrm{K_u}\, \varepsilon W \;=\; (e^\delta\, \mathrm{K_l} + \mathrm{K_u}\, \varepsilon) W \quad \text{a.s.}$$

where $\varepsilon > 0$ depends on the choice of c. Since $\mathbb{E}\, M(1) = f(1) < 1$ by [5, Lemma 6.2], we have $\mathbb{P}(\overline{W} > 0) > 0$. On the other hand, $W < \infty$ a.s. by Lemma 7.6. Then, since ε can be made arbitrarily small, $\mathrm{K_l} > 0$ follows, for otherwise $\overline{W} = 0$ a.s. Now arguing as in the first part of the proof, we obtain $f(t) \le \varphi(\mathrm{K_l}\, t^\alpha)$, $t > 0$. Finally, part (c) is just a rearrangement of part (b). ☐

Lemma 11.6. *Under* (A1)–(A5), *we have* $\mathcal{S}(\mathcal{M})^\beta = \mathcal{S}(\mathcal{M})$.

Proof. By Lemma 11.5, $\varphi(\mathrm{K_u}\, t^\alpha) \le f(t) \le \varphi(\mathrm{K_l}\, t^\alpha)$ for all $t \ge 0$. Thus,

$$\frac{1-f(ut)}{1-f(t)} \;\le\; \frac{1-\varphi(\mathrm{K_u}\,(ut)^\alpha)}{1-\varphi(\mathrm{K_l}\, t^\alpha)} \tag{11.12}$$

for all $u \ge 0$ and $t > 0$.

Suppose first that (A4a) holds. Then we can assume w.l.o.g. that $W = W^{(\alpha)}$. Then φ is differentiable at 0 with derivative -1 so that

$$\frac{1-\varphi(\mathrm{K_u}\,(ut)^\alpha)}{1-\varphi(\mathrm{K_l}\, t^\alpha)} \;=\; \frac{1-\varphi(\mathrm{K_u}\,(ut)^\alpha)}{\mathrm{K_u}\,(ut)^\alpha} \frac{\mathrm{K_l}\, t^\alpha}{1-\varphi(\mathrm{K_l}\, t^\alpha)} \frac{\mathrm{K_u}}{\mathrm{K_l}} u^\alpha \;\le\; C \frac{\mathrm{K_u}}{\mathrm{K_l}} u^\alpha$$

for some $C < \infty$, all $u \le 1$ and all sufficiently small $t > 0$.

The situation is more delicate if (A4b) is assumed instead of (A4a). We show that for $f \in \mathcal{S}(\mathcal{M})$ and arbitrary $\varepsilon > 0$, there exist $K, c > 0$ such that

$$\frac{1-f(ut)}{u^\alpha(1-f(t))} \;\le\; K u^{-\varepsilon} \tag{11.13}$$

for all $u \le 1$ and all $t \le c$. We deduce from (11.12) that (keep in mind that $\tilde{D}_1(s) = (1-\varphi(s))/s$ is decreasing in s)

$$\frac{1-f(ut)}{u^\alpha(1-f(t))} \;\le\; \frac{\mathrm{K_u}}{\mathrm{K_l}} \frac{\tilde{D}_1(\mathrm{K_u}\, t^\alpha)}{\tilde{D}_1(\mathrm{K_l}\, t^\alpha)} \frac{\tilde{D}_1(\mathrm{K_u}\,(ut)^\alpha)}{\tilde{D}_1(\mathrm{K_u}\, t^\alpha)} \;\le\; \frac{\mathrm{K_u}}{\mathrm{K_l}} \frac{\tilde{D}_1(\mathrm{K_u}\,(ut)^\alpha)}{\tilde{D}_1(\mathrm{K_u}\, t^\alpha)}.$$

An application of Theorem 1.3.1 in [15] to the slowly varying function $\tilde{D}_1$ shows that the last ratio can be bounded from above by a constant times u^ε in a right neighborhood of 0, in other words, we have established (11.13). Since we can choose $\varepsilon \le \alpha - \theta$, the proof is complete. ☐

12 Proof of Theorem 7.2

Proof of Theorem 7.2. Let $f \in \mathcal{S}(\mathcal{M})$ and M denote the corresponding disintegrated fixed point. Then, using (11.1), we obtain from (7.8) and (7.9) in the proof of Lemma 7.6 that for any $u > 0$ and $s = 1$ (non-geometric case) or $u \in r^{\mathbb{Z}}$ and $s \in (r^{-1}, 1]$ (r-geometric case)

$$-\log M(su) \;=\; u^{\alpha}(-\log M(s)) \quad \text{a.s.}$$

Moreover, $-\log M(s)$ is an endogeneous solution to (1.6) by Lemma 7.6. Putting $W = -\log M(1)$, we see that M solves (7.3) in the continuous case. In the r-geometric case, Proposition 5.3 comes into play because it ensures that for any $s \in (r^{-1}, 1]$ there exists a constant $h(s) > 0$ such that $-\log M(s) = h(s)s^{\alpha}W$ a.s. Now we define $h(us) := h(s)$ for $u \in r^{\mathbb{Z}}$ and $s \in (r^{-1}, 1]$. Thus, h is defined on the whole positive halfline $(0, \infty)$. Using $-\log M(su) = u^{\alpha}(-\log M(s))$ a.s. for $u \in r^{\mathbb{Z}}$ and $s \in (r^{-1}, 1]$, we see that M has a representation as in (7.3) in the r-geometric case as well. To see that $h \in \mathfrak{H}_r$ it remains to prove that $t \mapsto h(t)t^{\alpha}$ is increasing. But in view of (7.3) and (7.2), this immediately follows from the monotonicity of f.

We have shown so far that for any disintegrated fixed point M there exist an endogenous fixed point W and some function $h \in \mathfrak{H}_r$ such that (7.3) holds. Since endogeneous fixed points are unique up to scaling by Proposition 5.3 and $\mathfrak{H}_r$ is invariant under scaling with positive factors, it is clear that one can choose W independent of f. ☐

13 Solutions in other sets of functions

It is worth looking at the when the arguments characterising monotonic solutions can be modified to apply to other classes of functions. A function f is called eventually uniformly continuous if it is uniformly continuous on $[K, \infty)$ for some finite K. Then the new class is the set $\mathcal{U}$ consisting of all functions $f : [0, \infty) \to [0, 1]$ with $f(0) = 1$ and $f(t) \to 1$ as $t \downarrow 0$ such that $\log(1 - f(e^{-t}))$ is eventually uniformly continuous. Note that when $f \in \mathcal{U}$ it is automatic that $f(t) < 1$ for all small enough $t > 0$. We define $\mathcal{S}(\mathcal{U})$ to be the set of functions $f \in \mathcal{U}$ solving the functional equation (1.1). Much of the argument carries over without change but there are a number of critical points in extending the characterisation results to $\mathcal{S}(\mathcal{U})$.

The disintegration lemma 7.1 works for $f \in \mathcal{S}(\mathcal{U})$. Also Lemmas 7.3 and 7.4 carry over without any problems. Thus we can argue as in Section 8 that it constitutes no loss of generality to make the additional assumption (A5). Further, Lemma 7.5 remains valid in the case of solutions from $\mathcal{S}(\mathcal{U})$. Lemma 11.2 is the first result which needs substantial changes.

Lemma 13.1. *Lemma 11.2 holds for $f \in \mathcal{S}(\mathcal{U})$ with g continuous (rather than increasing).*

Proof. The functions $H_t(z) = \log(1 - f(te^{-z})) - \log(1 - f(t))$ ($z \geq 0$) are equicontinuous for all small enough t and uniformly bounded at $z = 0$. Hence, by the Arzela-Ascoli theorem, for any sequence, there is a subsequence $(t_n)_{n\geq 1}$ and a continuous function h such that

$$H_{t_n}(z) \;=\; \log\left(\frac{1-f(t_n e^{-z})}{1-f(t_n)}\right) \;\to\; h(z) \qquad (z \geq 0).$$

The asserted convergence follows with $g(u) := \exp(h(-\log u))$, $u \in (0,1]$. □

Using Lemma 13.1 it is readily seen that Lemma 11.3 also holds for $f \in \mathcal{S}(\mathcal{U})^{\beta}$, which has the natural definition. Only one change in the proof of Lemma 11.3 is necessary: continuity rather than monotonicity is used to show the limiting function f in (11.6) satisfies $f > 0$ on an interval including 0.

Lemma 13.2. *Let $f \in \mathcal{U}$. Then the conclusion of Lemma 11.4 holds.*

Proof. By uniform continuity, given c there is a B such that

$$|\log(1-f(te^{-a})) - \log(1-f(t))| \;\leq\; B$$

for $0 \leq a \leq c$ and all small positive t; then

$$\frac{1-f(te^{-a})}{1-f(t)} \;\in\; [e^{-B}, e^{B}].$$

Now recall that $D_\alpha(s) := s^{-\alpha}(1-f(s))$ so that

$$D_\alpha(te^{-a})/D_\alpha(t) \leq e^{B}e^{\alpha a} \leq e^{B}e^{\alpha c}.$$

□

This lemma together with Lemma 7.5 giving the analog of Eq. (11.9) are all that is needed to show that $\mathcal{S}(\mathcal{U})^{\beta} = \mathcal{S}(\mathcal{U})$, and therefore, to arrive at the following theorem. For it let $\mathfrak{C}_r$ be positive constants when $r = 1$ and positive, continuous, multiplicatively r-periodic functions otherwise.

Theorem 13.3. *Suppose that conditions* (A1)–(A4) *hold. Then there is a non-negative W satisfying* (1.6) *such that $\mathcal{S}(\mathcal{U})$ is given by the family in* (2.1) *when parametrized by $h \in \mathfrak{C}_r$.*

The motivation to consider solutions to (1.1) within the class $\mathcal{U}$ stems from the study of recursive algorithms. Suppose that X is a *real-valued* random variable satisfying (1.2). If X has a symmetric distribution, then the characteristic function ϕ of X is symmetric, $[0,1]$-valued and satisfies the functional equation (1.1). If it can be shown that $\phi \in \mathcal{U}$, then Theorem 13.3 applies.

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