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# LIMIT THEOREMS FOR SUBCRITICAL AGE-DEPENDENT BRANCHING PROCESSES WITH TWO TYPES OF IMMIGRATION

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□ *For the classical subcritical age-dependent branching process the effect of the following two-type immigration pattern is studied. At a sequence of renewal epochs a random number of immigrants enters the population. Each subpopulation stemming from one of these immigrants or one of the ancestors is revived by new immigrants and their offspring whenever it dies out, possibly after an additional delay period. All individuals have the same lifetime distribution and produce offspring according to the same reproduction law. This is the Bellman-Harris process with immigration at zero and immigration of renewal type (BHPIOR). We prove a strong law of large numbers and a central limit theorem for such processes. Similar conclusions are obtained for their discrete-time counterparts (lifetime per individual equals one), called Galton-Watson processes with immigration at zero and immigration of renewal type (GWPIOR). Our approach is based on the theory of regenerative processes, renewal theory and occupation measures and is quite different from those in earlier related work using analytic tools.*

**Keywords** Bellman-Harris process; Galton-Watson process; Immigration at zero; Immigration of renewal type; Regenerative process.

**Mathematics Subject Classification** 60J80; 60K10.

## 1. INTRODUCTION

How does immigration at recurrent random epochs affect the long-term behavior of populations that would otherwise become extinct because their reproductive pattern is subcritical? This question will be investigated hereafter for some classical branching processes, namely

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simple Galton-Watson processes (discrete time) and Bellman-Harris processes (continuous time), and for a certain immigration pattern. Thus, individuals of the considered populations have i.i.d. lifetimes (identically 1 in the discrete-time case) and produce independent numbers of offspring at their death with a common subcritical distribution. Immigration is assumed to occur at an independent sequence of renewal epochs, the number of immigrants being i.i.d., and further whenever a subpopulation stemming from one of these immigrants or one of the ancestors dies out, possibly after a delay period. The number of immigrants at these extinction epochs as well as the delay periods are each sequences of i.i.d. random variables, also.

If only the second type of state-dependent immigration occurs then, by subcriticality, the resulting branching process is easily seen to be a strongly regenerative process (see Ref.[16]) whose successive extinction times constitute regeneration epochs with finite mean. It therefore converges in distribution to a limiting variable with positive mean (see Proposition 2.1). Because additional immigration at successive renewal epochs leads to a compound of such processes, a linear growth behavior is to be expected, at least under some mild regularity conditions. Our main results are a confirmation of this conjecture and a central limit theorem for the considered branching processes. The focus will be on the continuous-time case because corresponding results in discrete time are then obtained by almost trivial adjustments of the arguments. Essential tools will be the theory of regenerative processes, renewal theory and occupation measures. This is in contrast to earlier related work using the "classical" analytic approach towards such processes based upon generating functions, Laplace transforms and integral equations.

The described immigration patterns for Bellman-Harris or Galton-Watson processes have been discussed in a number of papers. The Galton-Watson process with immigration at 0 (Foster-Pakes model) was first studied by Foster[4] and Pakes[10–12], later by Mitov and Yanev[9] under varying additional assumptions. Its continuous time analog was studied by Yamazato[18] and Mitov and Yanev[9]. Jagers[6] and Pakes and Kaplan[13] provided results for Bellman-Harris processes with immigration of the second type (at renewal epochs). Results for both immigration types appeared in Weiner[17], but a combination of them was first investigated by the second author of this article in Refs.[14,15]. The last reference proves Theorem 2.2, below, under stronger conditions and by analytic means. Some of the aforementioned articles deal with the case of critical reproduction. Immigration at 0 then still entails that the branching process is strongly regenerative but with cycles of infinite mean length. This, in turn, causes a drastic change as to its asymptotic behavior that will not be an issue here.

Following Mitov and Yanev[9] and the above informal description, a *Bellman-Harris process with immigration at 0* (BHPIO) $(Z(t))_{t\geq 0}$ is a continuous-time age-dependent branching process whose model parameters are an individual lifetime distribution $G$ with $G(0) = 0$, an offspring distribution $(p_j)_{j\geq 0}$ with p.g.f. $f(s)$, a number of immigrants distribution $(g_j)_{j\geq 0}$ with p.g.f. $g(s)$, and finally a distribution $D$ of the delay times elapsing after extinction epochs before new immigrants enter the population. The discrete-time variant $(Z(n))_{n\geq 0}$, where $t \in [0, \infty]$ is replaced with $n \in \mathbb{N}_0$, and where $G = \delta_1$ (Dirac measure at 1) and $D$ is a distribution on $\mathbb{N}_0$, will be called a *Galton-Watson process with immigration at 0* (GWPIO).

In order to extend the previous model by an additional immigration pattern at renewal epochs, let $Z_{ij} = (Z_{ij}(t))_{t\geq 0}$ for $i \geq 0$, $j \geq 1$ be independent BHPIO with one ancestor and the same model parameters as $(Z(t))_{t\geq 0}$. Let $(\sigma_n)_{n\geq 0}$ be a zero-delayed renewal process with increment distribution $F$ and $(Y_n)_{n\geq 1}$ a sequence of i.i.d. integer-valued random variables with common distribution $(h_j)_{j\geq 0}$ and p.g.f. $h(s)$. The $Y_n$ are supposed to be the numbers of individuals entering the population at times $\sigma_n$. A further integer-valued random variable $Y_0$ gives the number of ancestors of the considered population. It is assumed that $(\sigma_n)_{n\geq 0}$, $(Y_n)_{n\geq 1}$, $Y_0$ and all $Z_{ij}$ are mutually independent. A *Bellman-Harris process with immigration at zero and immigration of renewal type* (BHPIOR) $(X(t))_{t\geq 0}$ is then obtained as

$$X(t) \stackrel{\text{def}}{=} \sum_{i=0}^{N(t)} Z_i(t - \sigma_i), \quad t \geq 0, \tag{1.1}$$

where $Z_i(t) \stackrel{\text{def}}{=} 0$ for $t < 0$, $N(t) \stackrel{\text{def}}{=} \sup\{n \geq 0 : \sigma_n \leq t\}$, and

$$Z_i(t) \stackrel{\text{def}}{=} \sum_{j=1}^{Y_i} Z_{ij}(t), \quad t \geq 0, \tag{1.2}$$

is a BHPIO with $Y_i$ ancestors. Its discrete time variant, where the $Z_i$ are GWPIO and $(\sigma_n)_{n\geq 0}$ forms a discrete renewal process, is called a *Galton-Watson process with immigration at zero and immigration of renewal type* (GWPIOR).

## 2. RESULTS

In order to formulate our results some further notation is needed. Let $(Z(t))_{t\geq 0}$ be a BHPIO (or GWPIO with $t \in \mathbb{N}_0$) as described in the

Introduction. Define

$$m \stackrel{\text{def}}{=} \sum_{k\geq 1} kp_k = f'(1), \quad m_G \stackrel{\text{def}}{=} \int_0^\infty t\,G(dt),$$

and similarly $m_F$ and $m_D$. Let the $p$th moments of $(p_k)_{k\geq 0}, G, F, D$ be denoted as $m_p, m_{G,p}$, $m_{F,p}$ and $m_{D,p}$, respectively. Put $\mathbb{P}_k \stackrel{\text{def}}{=} \mathbb{P}(\cdot|Z(0)=k)$ for $k\geq 0$ and $\mathbb{P}^* \stackrel{\text{def}}{=} \sum_{k\geq 0} g_k \mathbb{P}_k$, so that the initial distribution of $(Z(t))_{t\geq 0}$ under $\mathbb{P}^*$ is $(g_k)_{k\geq 0}$. We will simply write $\mathbb{P}$ in assertions where the distribution of $Z(0)$ does not matter. Let $T_1$ be the first extinction epoch of $(Z(t))_{t\geq 0}$ after 0, defined as

$$T_1 \stackrel{\text{def}}{=} \inf\{t>0 : Z(t-)>0 \text{ and } Z(t)=0\}$$

in continuous time (and as $\inf\{n\geq 1 : Z(n)=0\}$ in discrete time). Note that, under each $\mathbb{P}_k$ with $k\geq 1$, $(\widehat{Z}(t))_{t\geq 0} \stackrel{\text{def}}{=} (Z(t)\mathbf{1}_{\{T_1>t\}})_{t\geq 0}$ is an ordinary BHP with lifetime distribution $G$ (or GWP with $G=\delta_1$), offspring distribution $(p_j)_{j\geq 0}$ and extinction time $T_1$, which has finite mean under every $\mathbb{P}_k$. Let $\Phi(s,t) \stackrel{\text{def}}{=} \mathbb{E}_1 s^{\widehat{Z}(t)}$ be the p.g.f. of $\widehat{Z}(t)$ under $\mathbb{P}_1$ and $m(t) \stackrel{\text{def}}{=} \mathbb{E}_1 \widehat{Z}(t)$ for $t\geq 0$. Put also $\Lambda(t) \stackrel{\text{def}}{=} \mathbb{E}^* Z(t)$ and $\Lambda_2(t) = \mathbb{E}^* Z(t)^2$ for $t\geq 0$. When moving to the process $(X(t))_{t\geq 0}$ defined in (1.1) we put $Z(t) \stackrel{\text{def}}{=} Z_0(t)$ for $t\geq 0$ and retain the previous notation.

**Proposition 2.1.** *Let $(Z(t))_{t\geq 0}$ be a subcritical* BHPIO *with arbitrary ancestor distribution, $g'(1)<\infty$, and $m_G<\infty$. Suppose also $m_D<\infty$, and that the convolution $G*D$ is nonarithmetic. Then $Z(t) \stackrel{d}{\to} Z(\infty)$, $t\to\infty$, for an integer-valued random variable $Z(\infty)$ satisfying*

$$\mathbb{P}(Z(\infty)=n) = \begin{cases} \dfrac{m_D}{\beta}, & \text{if } n=0, \\ \dfrac{1}{\beta}\displaystyle\int_0^\infty \mathbb{P}^*(\widehat{Z}(t)=n)\,dt, & \text{if } n\geq 1, \end{cases} \tag{2.1}$$

*where $\beta \stackrel{\text{def}}{=} \mathbb{E}^* T_1 + m_D$ is finite. $Z(\infty)$ has p.g.f.*

$$\Phi(s,\infty) = \frac{m_D}{\beta} + \frac{1}{\beta}\int_0^\infty (g(\Phi(s,t)) - g(\Phi(0,t)))dt \tag{2.2}$$

*and mean $\Lambda(\infty)$ given by*

$$\Lambda(\infty) = \frac{g'(1)m_G}{(1-m)\beta}. \tag{2.3}$$

*Moreover,*

$$\lim_{t\to\infty} \mathbb{E}_k Z(t) = \lim_{t\to\infty} \Lambda(t) = \Lambda(\infty) \tag{2.4}$$

*for all* $k \geq 0$. *If* $f''(1)$, $m_{G,2}$ *and* $m_{D,2}$ *are all finite, then also*

$$\lim_{t\to\infty} \mathbb{E}_k Z(t)^2 = \lim_{t\to\infty} \Lambda_2(t) = \Lambda_2(\infty) \stackrel{\text{def}}{=} \mathbb{E}Z(\infty)^2 \tag{2.5}$$

*holds for each* $k \geq 0$, *and*

$$\Lambda_2(\infty) = \frac{g'(1)m_G}{(1-m)\beta} + \frac{1}{\beta}\left(\frac{g'(1)f''(1)}{1-m} + g''(1)\right)\int_0^\infty m(t)^2\,dt < \infty. \tag{2.6}$$

Turning to the BHPIOR $(X(t))_{t\geq 0}$ formally introduced by (1.1), let us first point out for later use the following almost trivial consequences of the previous proposition. Each $(Z_i(t))_{t\geq 0}$ defined by (1.2) is the random sum of $Y_i$ i.i.d. BHPIO $(Z_{ij}(t))_{t\geq 0}$ with one ancestor, and $Y_i$ is independent of these processes. Suppose that the conditions of Proposition 2.1 ensuring $Z_{ij}(t) \stackrel{d}{\to} Z(\infty)$ are satisfied and let $Z^1(\infty), Z^2(\infty), \ldots$ be i.i.d. copies of $Z(\infty)$, which are also independent of $Y$, a generic copy of $Y_1, Y_2, \ldots$ Then we infer for each $i \geq 1$ that

$$Z_i(t) \stackrel{d}{\to} Z^*(\infty) \stackrel{\text{def}}{=} \sum_{j=1}^{Y} Z^j(\infty), \quad t \to \infty, \tag{2.7}$$

and

$$\lim_{t\to\infty} \mathbb{E}Z_i(t) = \mathbb{E}Z^*(\infty) = h'(1)\Lambda(\infty). \tag{2.8}$$

For the last result it should be recalled that $h$ denotes the p.g.f. of $Y$. If in addition to the previous assumptions $f''(1)$, $m_{G,2}$, $m_{D,2}$, $h'(1)$ and $h''(1)$ are all finite, then Proposition 2.1 further implies

$$\lim_{t\to\infty} \mathbb{E}Z_i(t)^2 = \mathbb{E}Z^*(\infty)^2 = h''(1)\Lambda(\infty)^2 + h'(1)(\Lambda_2(\infty) - \Lambda(\infty)^2) < \infty \tag{2.9}$$

for each $i \geq 1$.

**Theorem 2.2.** *Let* $(X(t))_{t\geq 0}$ *be a subcritical* BHPIOR *with arbitrary ancestor distribution,* $g'(1) < \infty$, $h'(1) < \infty$ *and* $m_G < \infty$. *Suppose also* $m_F < \infty$, $m_D < \infty$, *and that* $G * D$ *is nonarithmetic. Then*

$$\frac{X(t)}{t} \stackrel{\mathbb{P}}{\to} \frac{g'(1)h'(1)m_G}{(1-m)m_F\beta}, \quad t \to \infty. \tag{2.10}$$

As already mentioned in the Introduction, the previous result was proven under stronger conditions and by different means in Ref.[14]. In fact, the conditions imposed there will lead us now to the following central limit theorem.

**Theorem 2.3.** *Let $(X(t))_{t\geq 0}$ be a subcritical* BHPIOR *with arbitrary ancestor distribution, $g'(1) < \infty$, $f''(1) < \infty$, $h''(1) < \infty$ and $m_{G,2} < \infty$. Suppose also $m_F < \infty$, $m_{D,2} < \infty$, and that at least one of G or D is spread out. Then*

$$\frac{X(t) - (N(t)+1)h'(1)\Lambda(\infty)}{t^{1/2}} \xrightarrow{d} N(0, m_F \Xi(\infty)^2) \tag{2.11}$$

*where*

$$\Xi(\infty)^2 \stackrel{\text{def}}{=} (h''(1) - h'(1)^2)\Lambda(\infty)^2 + h'(1)(\Lambda_2(\infty) - \Lambda(\infty)^2) \tag{2.12}$$

*denotes the variance of $Z^*(\infty)$, the limiting variable defined in* (2.7).

As one can easily see from the proofs in the next section, all previous results persist to hold in discrete time, i.e., for GWPIO and GWPIOR. Minor adjustments are only caused by the fact that the renewal process $(\sigma_n)_{n\geq 0}$ as well as the delay periods are now integer-valued, which entails that nonarithmetic renewal limits must be replaced with their arithmetic counterpart. The following results are therefore stated without proof. The attribute "1-arithmetic" is used as a shorthand expression for "arithmetic with lattice-span 1."

**Proposition 2.4.** *Let $(Z(k))_{k\geq 0}$ be a subcritical* GWPIO *with arbitrary ancestor distribution and $g'(1) < \infty$. Suppose also $m_D < \infty$ and that $G * D = \delta_1 * D$ is 1-arithmetic. Then $Z(k) \xrightarrow{d} Z(\infty)$, $t \to \infty$, for an integer-valued random variable $Z(\infty)$ satisfying*

$$\mathbb{P}(Z(\infty) = n) = \begin{cases} \dfrac{m_D}{\beta}, & \text{if } n = 0, \\ \dfrac{1}{\beta}\displaystyle\sum_{k\geq 0} \mathbb{P}^*(\widehat{Z}(k) = n), & \text{if } n \geq 1, \end{cases} \tag{2.13}$$

*where $\beta \stackrel{\text{def}}{=} \mathbb{E}^* T_1 + m_D$. $Z(\infty)$ has p.g.f.*

$$\Phi(s, \infty) = \frac{m_D}{\beta} + \frac{1}{\beta}\sum_{k\geq 0}(g(\Phi(s,k)) - g(\Phi(0,k))) \tag{2.14}$$

*and mean $\Lambda(\infty)$ given by* (2. 3) *with $m_G = 1$. Moreover,*

$$\lim_{k\to\infty} \mathbb{E}_j Z(k) = \lim_{k\to\infty} \Lambda(k) = \Lambda(\infty) \tag{2.15}$$

*for all* $j \geq 0$. *If* $f''(1)$ *and* $m_{D,2}$ *are finite, then also*

$$\lim_{k\to\infty} \mathbb{E}_j Z(k)^2 = \lim_{k\to\infty} \Lambda_2(k) = \Lambda_2(\infty) \tag{2.16}$$

*holds for each* $j \geq 0$, *and*

$$\Lambda_2(\infty) = \frac{g'(1)}{(1-m)\beta} + \frac{1}{\beta(1-m^2)}\left(\frac{g'(1)f''(1)}{1-m} + g''(1)\right). \tag{2.17}$$

**Theorem 2.5.** *Let* $(X(k))_{k\geq 0}$ *be a subcritical* GWPIOR *with arbitrary ancestor distribution and* $g'(1) < \infty$, $h'(1) < \infty$. *Suppose also* $m_F < \infty$, $m_D < \infty$, *and that* $G * D$ *is 1-arithmetic. Then (2.10) remains true with* $m_G = 1$ *and* $t \to \infty$ *through the integers.*

**Theorem 2.6.** *Let* $(X(k))_{k\geq 0}$ *be a subcritical* BHPIOR *with arbitrary ancestor distribution,* $g'(1) < \infty$, $h''(1) < \infty$ *and* $f''(1) < \infty$. *Suppose also* $m_F < \infty$, $m_{D,2} < \infty$ *and that* $G * D$ *is 1-arithmetic. Then* (2.11) *remains true as* $t \to \infty$ *through the integers.*

## 3. PROOFS

***Proof of Proposition*** 2.1. Let $T_1, T_2, \ldots$ and $X_1, X_2, \ldots$ denote the successive extinction epochs and delay times of $(Z(t))_{t\geq 0}$. Obviously, $(Z(t))_{t\geq 0}$ is a classical regenerative process (see Ref.[1], Chpt. 5) with regeneration times $T_n + X_n$, $n \geq 1$. So, cycles start (and end) at successive immigration epochs (the first one at 0). They are independent and for $n \geq 2$ also identically distributed with mean $\beta$ the finiteness of which we will show at the end of this proof. Because, by assumption, $G * D$ and thus the distribution of $T_1 + X_1$ are nonarithmetic, the ergodic theorem for regenerative processes [1, Thm. V.1.2] gives $Z(t) \xrightarrow{d} Z(\infty)$ with

$$\mathbb{P}(Z(\infty) = n) = \frac{1}{\beta}\mathbb{E}^*\left(\int_0^{T_1+X_1} \mathbf{1}_{\{Z(t)=n\}}\, dt\right) \tag{3.1}$$

for $n \in \mathbb{N}_0$. It follows directly from (3.1) that

$$\mathbb{P}(Z(\infty) = 0) = \frac{m_D}{\beta}.$$

For $n \geq 1$, we obtain

$$\begin{aligned}\mathbb{P}(Z(\infty) = n) &= \frac{1}{\beta}\mathbb{E}^*\left(\int_0^{T_1} \mathbf{1}_{\{\widehat{Z}(t)=n\}}\, dt\right)\\ &= \frac{1}{\beta}\int_0^{\infty} \mathbb{P}^*(\widehat{Z}(t) = n)\, dt\end{aligned}$$

completing the proof of (2.1). Now

$$\begin{aligned}\Phi(s,\infty) &= \frac{m_D}{\beta} + \frac{1}{\beta}\sum_{n\geq 1}\int_0^\infty s^n \mathbb{P}^*(\widehat{Z}(t)=n)dt\\ &= \frac{m_D}{\beta} + \frac{1}{\beta}\int_0^\infty \sum_{k\geq 0} g_k \sum_{n\geq 1} s^n \mathbb{P}_k(\widehat{Z}(t)=n)dt\\ &= \frac{m_D}{\beta} + \frac{1}{\beta}\int_0^\infty \sum_{k\geq 0} g_k\big(\Phi(s,t)^k - \Phi(0,t)^k\big)dt\\ &= \frac{m_D}{\beta} + \frac{1}{\beta}\int_0^\infty (g(\Phi(s,t)) - g(\Phi(0,t)))dt\end{aligned}$$

for $s\in[0,1)$. As to $\Lambda(\infty)=\Phi'(1,\infty)$, where the prime means of course differentiation with respect to the first argument of $\Phi$, we get

$$\Lambda(\infty) = \frac{1}{\beta}\int_0^\infty g'(1)\Phi'(1,t)dt = \frac{g'(1)}{\beta}\int_0^\infty m(t)dt$$

and then further for $I \stackrel{\text{def}}{=} \int_0^\infty m(t)dt$, by conditioning upon the ancestor's death time $v$ and offspring number,

$$\begin{aligned} I &= \mathbb{E}_1\bigg(\int_0^\infty \widehat{Z}(t)dt\bigg)\\ &= m_G + \mathbb{E}_1\bigg(\int_v^\infty \widehat{Z}(t)dt\bigg)\\ &= m_G + \sum_{k\geq 0} p_k \int_0^\infty \mathbb{E}_k\widehat{Z}(t)dt\\ &= m_G + mI.\end{aligned}$$

We used $\mathbb{E}_k\widehat{Z}(t) = k\,m(t)$ for the final equality. This shows $I=\frac{m_G}{1-m}$ and thus $\Lambda(\infty)=\frac{g'(1)m_G}{(1-m)\beta}$, i.e., (2.3).

Turning to (2.4), put $S_0 \stackrel{\text{def}}{=} 0$, $S_n \stackrel{\text{def}}{=} T_n + X_n$ for $n\geq 1$, and notice that $(S_n)_{n\geq 0}$ is a zero-delayed nonarithmetic renewal process under $\mathbb{P}^*$. Denote by $\mathbb{U}$ the associated renewal measure and put further

$$\begin{aligned} Q_1 &\stackrel{\text{def}}{=} \sum_{k\geq 1} kp_k \mathbb{P}_1(v\in\cdot|\widehat{Z}(v)=k),\\ Q_2 &\stackrel{\text{def}}{=} \sum_{k\geq 2} k(k-1)p_k \mathbb{P}_1(v\in\cdot|\widehat{Z}(v)=k),\end{aligned}$$

$$m_2(t) \stackrel{\text{def}}{=} \mathbb{E}_1\widehat{Z}(t)^2,$$
$$v(t) \stackrel{\text{def}}{=} Var_1\widehat{Z}(t) = m_2(t) - m(t)^2.$$

Note that $\mathbb{E}^*\widehat{Z}(t) = g'(1)m(t)$. It then follows that

$$\begin{aligned}\Lambda(t) &= \sum_{n\geq 1} \mathbb{E}^* Z(t)\mathbf{1}_{\{S_{n-1}\leq t\leq T_n\}} \\ &= \int_{[0,t]} \mathbb{E}^* Z(t-x)\mathbf{1}_{\{T_1>t-x\}}\mathbb{U}(dx) \\ &= \int_{[0,t]} \mathbb{E}^*\widehat{Z}(t-x)\mathbb{U}(dx) \\ &= \int_{[0,t]} g'(1)m(t-x)\mathbb{U}(dx).\end{aligned}$$

Because $m(t) = \int_{[0,t]} \mathbb{P}_1(\nu > t-x)\mathbb{V}(dx)$ (see Ref.[2], p. 151), where $\nu$ is as given above and $\mathbb{V} = \sum_{n\geq 0} Q_1^{*n}$ denotes the *defective* renewal measure associated with $Q_1$, we further infer

$$\Lambda(t) = \int_{[0,t]}\int_{[0,t-y]} g'(1)\mathbb{P}_1(\nu > t-x-y)\mathbb{U}(dx)\mathbb{V}(dy). \tag{3.2}$$

The function $t \mapsto \mathbb{P}_1(\nu > t)$ is clearly directly Riemann integrable ($m_G < \infty$). Consequently, a combination of the key renewal theorem and the dominated convergence theorem ($\mathbb{V}[0,\infty]) = \frac{1}{1-m} < \infty$) yields

$$\lim_{t\to\infty} \Lambda(t) = \frac{\mathbb{V}([0,\infty])}{\beta}\int_0^\infty g'(1)\mathbb{P}_1(\nu > x)dx = \frac{g'(1)m_G}{(1-m)\beta} = \Lambda(\infty)$$

and thus proves (2.4) for $\mathbb{P} = \mathbb{P}^*$. The same result is then obtained for $\mathbb{P} = \mathbb{P}_k$, $k \geq 0$, because

$$\mathbb{E}_k Z(t) = \mathbb{E}_k\widehat{Z}(t)\mathbf{1}_{\{T_1>t\}} + \int_{[0,t]} \Lambda(t-x)\mathbb{P}_k(T_1 \in dx) \tag{3.3}$$

for all $t \geq 0$ and $k \geq 0$, and

$$\mathbb{E}_k\widehat{Z}(t)\mathbf{1}_{\{T_1>t\}} = \mathbb{E}_k\widehat{Z}(t) = k\,m(t) \to 0,$$

as $t \to \infty$.

We proceed with the proof of (2.5) and (2.6) and assume from now on that $m_{G,2}$ and $m_{D,2}$ are both finite. Notice that $Q_2$ has total mass $f''(1)$.

A standard renewal argument using $Var_k\widehat{Z}(t) = kv(t)$ leads to

$$m_2(t) = \int_{[0,t]} \mathbb{P}_1(v > t-x)\mathbb{V}(dx) + \int_{[0,t]} m(t-x)^2 Q_2 * \mathbb{V}(dx), \tag{3.4}$$

an with this identity a straightforward calculation yields

$$\int_0^\infty m_2(t)\,dt = \frac{m_G}{1-m} + \frac{f''(1)}{1-m}\int_0^\infty m(t)^2\,dt. \tag{3.5}$$

We also compute

$$\mathbb{E}^*\widehat{Z}(t)^2 = \sum_{k\geq 1} g_k \mathbb{E}_k\widehat{Z}(t)^2 = \sum_{k\geq 1} g_k\big(kv(t) + k^2 m(t)^2\big) = g'(1)m_2(t) + g''(1)m(t)^2$$

for any $t \geq 0$. It is not hard to verify that $m(t)^2$ and $m_2(t)$ are directly Riemann integrable. Using these facts we get

$$\begin{aligned}
\mathbb{E}^*Z(t)^2 &= \sum_{n\geq 1} \mathbb{E}^*Z(t)^2 \mathbf{1}_{\{S_{n-1}\leq t\leq T_n\}} \\
&= \int_{[0,t]} \mathbb{E}^*Z(t-x)^2 \mathbf{1}_{\{T_1 > t-x\}} \mathbb{U}(dx) \\
&= \int_{[0,t]} \mathbb{E}^*\widehat{Z}(t-x)^2 \mathbb{U}(dx) \\
&= \int_{[0,t]} \big(g'(1)m_2(t-x) + g''(1)m(t-x)^2\big)\mathbb{U}(dx)
\end{aligned}$$

and then by appealing to the key renewal theorem

$$\lim_{t\to\infty} \mathbb{E}^*Z(t)^2 = \frac{g'(1)}{\beta}\int_0^\infty m_2(t)dt + \frac{g''(1)}{\beta}\int_0^\infty m(t)^2\,dt. \tag{3.6}$$

By computing $\Phi''(1,\infty) = \mathbb{E}Z(\infty)(Z(\infty)-1)$ from (2.2) one can check that the right hand side of (3.6) also equals $\Lambda_2(\infty)$. An equation similar to (3.3) holds for $\mathbb{E}_k Z(t)^2$ and leads to the conclusion that $\lim_{t\to\infty} \mathbb{E}_k Z(t)^2 = \lim_{t\to\infty} \mathbb{E}^*Z(t)^2 = \Lambda_2(\infty)$ for each $k \geq 0$. This completes the proof of (2.5). By plugging (3.5) into (3.6), we further obtain (2.6).

We finish this proof by showing that $\beta = \mathbb{E}^*T_1 + m_D$ is finite. Because $m_D < \infty$ by assumption we must actually verify $\mathbb{E}^*T_1 < \infty$. Note that $\mathbb{E}_k T_1 \leq k\,\mathbb{E}_1 T_1$ for each $k \geq 1$ because $T_1$ is distributed under $\mathbb{P}_k$ as the maximum of $k$ independent variables with the same distribution as $T_1$

under $\mathbb{P}_1$. Consequently,

$$\mathbb{E}^* T_1 = \sum_{k \geq 0} g_k \mathbb{E}_k T_1 \leq \mathbb{E}_1 T_1 \sum_{k \geq 0} k g_k = g'(1) \mathbb{E}_1 T_1 < \infty$$

as desired.

*Proof of Theorem* 2.2. By (1.1), $X(t) = \sum_{i=0}^{N(t)} Z_i(t - \sigma_i)$. It suffices to prove (2.10) with $\mathbb{P} = \mathbb{P}^*$ because only $Z_0(t)$ in the previous sum depends on the initial distribution and clearly satisfies $t^{-1} Z_0(t) \xrightarrow{\mathbb{P}} 0$ regardless of that distribution (choice of $\mathbb{P}$). Thus, fixing $\mathbb{P} = \mathbb{P}^*$, (2.7) after Proposition 2.1 gives $Z_i(t) \to^d Z^*(\infty)$ for each $i \geq 0$. Because the $(Z_i(t))_{t \geq 0}$ are càdlàg and independent of $(\sigma_n)_{n \geq 0}$, the Skorohod-Dudley coupling theorem (see Ref.[7], Theorem 3.3) ensures the existence of processes $(\widetilde{Z}_i(t))_{t \geq 0}$ and random variables $\widetilde{Z}_i(\infty)$, $i \geq 0$, such that

(1) $\widetilde{Z}_i(t) \stackrel{d}{=} Z_i(t)$ for all $t \in [0, \infty)$ and $i \geq 0$;
(2) $\widetilde{Z}_0(\infty) \stackrel{d}{=} \widetilde{Z}_1(\infty) \stackrel{d}{=} \cdots \stackrel{d}{=} Z^*(\infty)$;
(3) $\widetilde{Z}_i(t) \to \widetilde{Z}_i(\infty)$ a.s.
(4) the $(\widetilde{Z}_i(t))_{t \in [0,\infty]}$ are mutually independent and also independent of $(\sigma_n)_{n \geq 0}$.

As an immediate consequence, we get

$$\widetilde{X}(t) \stackrel{\text{def}}{=} \sum_{i=0}^{N(t)} \widetilde{Z}_i(t - \sigma_i) \stackrel{d}{=} X(t)$$

for each $t \in [0, \infty)$, whence it suffices to prove $t^{-1} \widetilde{X}(t) \xrightarrow{\mathbb{P}} \frac{g'(1) h'(1) m_G}{(1-m) m_F \beta}$. To this end, write

$$\frac{\widetilde{X}(t)}{t} = \frac{N(t)}{t} \left( \frac{1}{N(t)} \sum_{i=0}^{N(t)} (\widetilde{Z}_i(t - \sigma_i) - \widetilde{Z}_i(\infty)) + \frac{1}{N(t)} \sum_{i=0}^{N(t)} \widetilde{Z}_i(\infty) \right). \tag{3.7}$$

Note that $\frac{N(t)}{t} \to m_F^{-1}$ a.s. by the elementary renewal theorem and that $\frac{1}{N(t)} \sum_{i=0}^{N(t)} \widetilde{Z}_i(\infty) \to h'(1) \Lambda(\infty) = \frac{g'(1) h'(1) m_G}{(1-m)\beta}$ a.s. by the strong law of large numbers and (2.8). We are thus left with the proof of

$$\frac{1}{t} \sum_{i=0}^{N(t)} (\widetilde{Z}_i(t - \sigma_i) - \widetilde{Z}_i(\infty)) \xrightarrow{\mathbb{P}} 0, \quad t \to \infty. \tag{3.8}$$

Put $n(t) \stackrel{\text{def}}{=} \lfloor \frac{2t}{m_F} \rfloor$ for $t \geq 0$. Then $\mathbb{P}(N(t) > n(t)) \to 0$. Because $\widetilde{Z}_i(t - \sigma_i) \to \widetilde{Z}_i(\infty)$ a.s., and $\mathbb{E}_k \widetilde{Z}_i(t - \sigma_i) \to \mathbb{E}_k \widetilde{Z}_i(\infty)$ by (2.4) of Proposition 2.1

($\sigma_i$ independent of $(\widetilde{Z}_i(t))_{t\geq 0}$), a generalization of Scheffé's lemma (see Ref.[3], p. 94) implies

$$\lim_{t\to\infty} \mathbb{E}_k\big|\widetilde{Z}_i(t-\sigma_i) - \widetilde{Z}_i(\infty)\big| = 0$$

for any $i \geq 1$ and $k \geq 0$. It follows that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{n(t)} \mathbb{E}_k\big|\widetilde{Z}_i(t-\sigma_i) - \widetilde{Z}_i(\infty)\big|\mathbf{1}_{\{N(t)\leq n(t)\}} = 0,$$

which in combination with $\mathbb{P}(N(t) > n(t)) \to 0$ gives (3.8) for $\mathbb{P} = \mathbb{P}_k$ for any $k \geq 0$.

In order to prove Theorem 2.3 we need the following auxiliary result.

**Lemma 3.1.** *Under the conditions of Theorem* 2.3*,*

$$\Lambda(t) = \Lambda(\infty) + o(t^{-1}), \quad t \to \infty. \tag{3.9}$$

***Proof.*** We use the same notation as in the proof of Proposition 2.1. Note that $M \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}} \Lambda(t)$ is finite and that (3.2) may be rewritten as

$$\Lambda(t) = g'(1)(\mathbb{V} * \mathbb{U}(t) - G * \mathbb{V} * \mathbb{U}(t)), \quad t \geq 0, \tag{3.10}$$

because $\mathbb{P}_1(v \in \cdot) = G$. Consequently, for each $s, t \geq 0$,

$$\Lambda(t+s) - \Lambda(t) = g'(1)(\mathbb{V} * \mathbb{U}\left((t, t+s]\right) - G * \mathbb{V} * \mathbb{U}\left((t, t+s]\right))$$

Now use $\lim_{t\to\infty} \Lambda(t) = \Lambda(\infty)$ to infer

$$\begin{aligned}
|\Lambda(t) - \Lambda(\infty)| &\leq \sup_{s\geq 0}|\Lambda(t+s) - \Lambda(t)| \\
&\leq g'(1)\sup_{s\geq 0}|\mathbb{V} * \mathbb{U}\left((t, t+s]\right) - G * \mathbb{V} * \mathbb{U}((t, t+s])| \\
&\leq g'(1)\|\mathbb{V} * \mathbb{U}(t+\cdot) - G * \mathbb{V} * \mathbb{U}(t+\cdot)\|,
\end{aligned}$$

where $\|\cdot\|$ denotes total variation norm. The latter expression is indeed of the required order $o(t^{-1})$ because, by assumption, the increments of $(S_n)_{n\geq 0}$ (the renewal process associated with $\mathbb{U}$) are spread out and square integrable, see Ref.[8], (6.8)(i), p.86. This completes the proof of the lemma.

*Proof of Theorem* 2.3. Again it suffices to verify (2.11) under $\mathbb{P} = \mathbb{P}^*$ as one can easily check. Put $\Lambda^*(\infty) \stackrel{\text{def}}{=} \mathbb{E}Z^*(\infty) = g'(1)\Lambda(\infty)$. With this and the notation from the proof of Theorem 2.2, we have the decomposition

$$\begin{aligned}\frac{\widetilde{X}(t)-(N(t)+1)\Lambda^*(\infty)}{t^{1/2}} &= \frac{1}{t^{1/2}}\sum_{i=0}^{N(t)}(\widetilde{Z}_i(t-\sigma_i)-\widetilde{Z}_i(\infty)-\Lambda^*(t-\sigma_i)+\Lambda^*(\infty))\\ &\quad+\frac{1}{t^{1/2}}\sum_{i=0}^{N(t)}(\Lambda^*(t-\sigma_i)-\Lambda^*(\infty))\\ &\quad+\frac{1}{t^{1/2}}\sum_{i=0}^{N(t)}(\widetilde{Z}_i(\infty)-\Lambda^*(\infty)),\end{aligned}$$

and denote the three terms on the right hand side as $A_1(t)$,$A_2(t)$ and $A_3(t)$. The last expression $A_3(t)$ consists of i.i.d. random variables with mean zero and finite variance $\Xi(\infty)^2$, which together with $t^{-1}N(t)\to m_F^{-1}$ implies by a version of Anscombe's theorem[5]

$$\frac{1}{t^{1/2}}\sum_{i=0}^{N(t)}(\widetilde{Z}_i(\infty)-\Lambda^*(\infty))\stackrel{d}{\to}N(0,m_F\Xi(\infty)^2).$$

So it remains to show that $A_1(t)\stackrel{\mathbb{P}}{\to}0$ and $A_2(t)\stackrel{\mathbb{P}}{\to}0$. Let $\mathbb{U}_F$ denote the renewal measure associated with $(\sigma_n)_{n\geq 0}$. Starting with $A_1(t)$, we obtain by conditioning upon $(\sigma_n)_{n\geq 0}$ that

$$\begin{aligned}\mathbb{E}A_1(t)^2 &= \frac{1}{t}\mathbb{E}\left(\sum_{i=0}^{N(t)}\mathbb{E}((\widetilde{Z}_i(t-\sigma_i)-\widetilde{Z}_i(\infty)-\Lambda^*(t-\sigma_i)+\Lambda^*(\infty))^2|\sigma_i)\right)\\ &= \frac{1}{t}\mathbb{E}\left(\sum_{i\geq 0}\mathbb{E}((\widetilde{Z}_i(t-\sigma_i)-\widetilde{Z}_i(\infty)-\Lambda^*(t-\sigma_i)+\Lambda^*(\infty))^2|\sigma_i)\mathbf{1}_{\{\sigma_i\leq t\}}\right)\\ &= \frac{1}{t}\int_{[0,t]}\mathbb{E}(\widetilde{Z}_0(t-s)-\widetilde{Z}_0(\infty)-\Lambda^*(t-s)+\Lambda^*(\infty))^2\,\mathbb{U}_F(ds).\end{aligned}$$

A combination of $\widetilde{Z}_0(t)\to\widetilde{Z}_0(\infty)$ a.s. with (2.5) yields

$$\lim_{t\to\infty}\mathbb{E}\left(\widetilde{Z}_0(t)-\widetilde{Z}_0(\infty)\right)^2=0$$

by another appeal to the generalization of Scheffé's lemma and thus also

$$\lim_{t\to\infty}\mathbb{E}(\widetilde{Z}_0(t)-\widetilde{Z}_0(\infty)-\Lambda^*(t)+\Lambda^*(\infty))^2=\lim_{t\to\infty}Var(\widetilde{Z}_0(t)-\widetilde{Z}_0(\infty))=0.$$

Put $C_b \stackrel{\text{def}}{=} \sup_{t\geq b} \mathbb{E}(\widetilde{Z}_0(t)-\widetilde{Z}_0(\infty)-\Lambda^*(t)+\Lambda^*(\infty))^2$ for $b\geq 0$. Using

$$\lim_{t\to\infty} t^{-1}\mathbb{U}_F\big((t-b,t]\big)=0 \quad \textit{and} \quad \lim_{t\to\infty} t^{-1}\mathbb{U}_F([0,t-b])=m_F^{-1}$$

for all $b>0$, we now infer

$$\lim_{t\to\infty}\frac{1}{t}\int_{[0,t]}\mathbb{E}\Big(\widehat{Z}_0(t-s)-\widehat{Z}_0(\infty)-\Lambda^*(t-s)+\Lambda^*(\infty)\Big)^2\mathbb{U}_F(ds)$$
$$\leq \lim_{b\to\infty}\lim_{t\to\infty}\left(\frac{\mathbb{U}_F\big([0,t-b]\big)}{t}C_b+\frac{\mathbb{U}_F\big((t-b,t]\big)}{t}C_0\right)=0$$

and thus, $\mathbb{E}A_1(t)^2\to 0$, which in turn shows $A_1(t)\xrightarrow{\mathbb{P}}0$.

As to $A_2(t)$, we first note that Lemma 3.1 implies

$$|\Lambda^*(t)-\Lambda^*(\infty)|=h'(1)|\Lambda(t)-\Lambda(\infty)|\leq R(t)(t+2)^{-1}$$

for some bounded decreasing function $R$ with supremum $\|R\|_\infty$. By further using the well-known fact that $\gamma \stackrel{\text{def}}{=} \sup_{t\geq 0}\mathbb{U}_F\big([t,t+1]\big)<\infty$, we now infer

$$\begin{aligned}
\mathbb{E}|A_2(t)| &\leq \frac{1}{t^{1/2}}\mathbb{E}\left(\sum_{i=0}^{N(t)}|\Lambda^*(t-\sigma_i)-\Lambda^*(\infty)|\right)\\
&= \frac{1}{t^{1/2}}\int_{[0,t]}|\Lambda^*(t-s)-\Lambda^*(\infty)|\mathbb{U}_F(ds)\\
&\leq \frac{1}{t^{1/2}}\int_{[0,t]}\frac{R(t-s)}{t-s+2}\mathbb{U}_F(ds)\\
&\leq \frac{\|R\|_\infty}{t^{1/2}}\left(\mathbb{U}_F([0,1])+\sum_{n=0}^{\lfloor t\rfloor-1}\frac{1}{n+1}\mathbb{U}_F([t-n-1,t-n])\right)\\
&\leq \frac{\gamma\|R\|_\infty}{t^{1/2}}(1+\log(t+1)).
\end{aligned}$$

Because the latter expression converges to 0 as $t\to\infty$, we conclude $\mathbb{E}|A_2(t)|\to 0$ and particularly $A_2(t)\xrightarrow{\mathbb{P}}0$, as desired.

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