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Two Comparison Theorems for Nonlinear First Passage Times and Their Linear Counterparts

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Two Comparison Theorems for Nonlinear First Passage Times and Their Linear Counterparts

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Let $(S_n)_{n\geq 0}$ be a zero-delayed nonarithmetic random walk with positive drift μ and $(\xi_n)_{n\geq 0}$ be a slowly varying perturbation process (see conditions (C.1-3) in the Introduction). The results of this note are two weak convergence theorems for the difference $\tau(t)-\nu(t)$, as $t\to\infty$, where $\tau(t)=\inf\{n\geq 1: S_n>t\}$ and $\nu(t)=\inf\{n\geq 1: S_n+\xi_n>t\}$ denotes its nonlinear counterpart. The main result (Theorem 1) states the existence of a limit distribution for $\tau(t)-\nu(t)$ providing the weak convergence of the ξ_n to a distribution Λ. Two applications in sequential statistics are also given.

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1. Introduction and Results

Given i.i.d. random variables $X, X_1, X_2, ...$ with common distribution $\mathbb{F}$ and finite positive mean μ, let $(S_n)_{n\geq 0}$ be the associated zero-delayed random walk, i.e. $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$. For $t \geq 0$, define the first passage time

$$\tau(t) \;=\; \inf\{n \geq 1 : S_n > t\} \tag{1.1}$$

and let $R_t = S_{\tau(t)} - t$ be the associated excess over the boundary. Both variables are of great interest in renewal theory. A basic result states that $t^{-1}\tau(t)$ converges to μ^{-1} a.s. and in L_1, as $t \to \infty$, see e.g. [1], Theorems III.4.1 and III.8.1. If X has finite positive variance σ^2, then furthermore (see [1], Theorem III.5.1)

$$\frac{\tau(t) - t/\mu}{\sigma\mu^{-3/2}t^{1/2}} \;\stackrel{d}{\to}\; N(0,1). \tag{1.2}$$

Denote by $\mathbb{F}_+$ the distribution of $R_0 = S_{\tau(0)}$, the first ascending ladder height, and by μ_+ its mean. Providing $\mathbb{F}$ is nonarithmetic, in which case $\mathbb{F}_+$ also is, R_t converges in distribution to an absolutely continuous distribution $\mathbb{F}_e$ with density $\mu_+^{-1}(1 - \mathbb{F}_+(x))\mathbf{1}_{(0,\infty)}(x)$ which is sometimes called the stationary renewal distribution associated with $\mathbb{F}$. Here we have used the common convention of identifying distribution and distribution function

Motivated by certain sequential statistical procedures, see [7] and [8], there has been much interest in the following nonlinear analogue of $\tau(t)$. Given a sequence of perturbations $(\xi_n)_{n\geq 1}$, define $Z_n = S_n + \xi_n$ for $n \geq 1$ and

$$\nu(t) \;=\; \inf\{n \geq 1 : Z_n > t\}. \tag{1.3}$$

If $n^{-1}\xi_n$ converges to 0 a.s., which reflects the idea that ξ_n is a small perturbation compared to S_n, then $t^{-1}\nu(t)$ has the same a.s. limit as $t^{-1}\tau(t)$, which is μ^{-1}, see [2]. Considering the nonlinear excess $Y_t = Z_{\nu(t)} - t$, Lai and Siegmund [4] showed that Y_t also converges to $\mathbb{F}_e$ given above provided that X be nonarithmetic and the following further conditions hold:

(C.1) $(X_k, \xi_k)_{1\leq k\leq n}$ is independent of $(X_k)_{k>n}$ for every $n \geq 1$;

(C.2) $n^{-1}\max\{|\xi_1|, ..., |\xi_n|\} \stackrel{P}{\to} 0$ as $n \to \infty$;

(C.3) for each $\delta > 0$ there exists $\varepsilon > 0$ such that

$$P(\max_{1\leq k\leq n\delta} |\xi_{n+k} - \xi_n| > \varepsilon) < \varepsilon$$

for all $n \geq 1$.

Under these assumptions a straightforward adaption of Gut's argument shows that instead of a.s. convergence we still have

$$\frac{\nu(t)}{t} \;\stackrel{P}{\to}\; \frac{1}{\mu} \quad (t \to \infty). \tag{1.4}$$

Under the name "nonlinear renewal theory" many further results related to $\nu(t)$ were derived by Lai and Siegmund [4],[5]. Roughly speaking, their approach is to approximate $\nu(t)$ by a linear first passage time $\tau(b(t))$ on a set of probability almost one as t and $b(t)$ get large and then to use ordinary renewal theory. In this note we take up this idea and give a comparison of $\nu(t)$ and $\tau(t)$ in form of two weak convergence results, Theorem 1 and 2 below, for their difference as t tends to infinity. This is also motivated by recent related results by Larsson-Cohn [6] who proved almost sure convergence and convergence in probability of suitable normalizations of $\nu(t)-\tau(t)$ under varying conditions on the perturbations ξ_n. We finally mention Gut's paper [2] which examines $\nu(t)$ under very general assumptions on $(\xi_n)_{n\geq 1}$.

For $a \geq 0$, let $\mathbb{G}_a$ be the distribution on $I\!N_0$, the set of nonnegative integers, with counting density

$$g_a(n) \;=\; \begin{cases} 1-\mathbb{F}_e(a), & \text{if } n=0 \\ \displaystyle\int_{[0,a]} P(\tau(a-x)=n)\;\mathbb{F}_e(dx), & \text{if } n\geq 1. \end{cases} \tag{1.5}$$

Hence, for all $r\in[0,\infty)$

$$\begin{aligned} \mathbb{G}_a(r) &= 1-\mathbb{F}_e(a)+\int_{[0,a]} P(\tau(a-x)\leq r)\;\mathbb{F}_e(dx) \\ &= 1-\mathbb{F}_e(a)+\int_{[0,a]} P(M_r > a-x)\;\mathbb{F}_e(dx) \\ &= 1-\mathbb{F}_e(a)+\int_{[0,\infty)} (\mathbb{F}_e(a)-\mathbb{F}_e((a-s)^+))\;P(M_r\in ds), \end{aligned} \tag{1.6}$$

where $M_r = \max_{0\leq k\leq r} S_k$. Note that $\mathbb{G}_0=\delta_0$, the Dirac measure at 0, and put $\mathbb{G}_{-a}(r) = 1-\mathbb{G}_a(-r-)$ for $r\in I\!R$.

We close this section by stating our main results, the proofs of which can be found in Section 3. Two examples are discussed in Section 4.

THEOREM 1. *Suppose $\mathbb{F}$ be nonarithmetic, conditions (C.1-3) and further $\xi_n \xrightarrow{d} \Lambda$ for a distribution Λ on $(I\!R,\mathfrak{B})$. Then the difference $\tau(t)-\nu(t)$ converges in distribution to Q, given by*

$$Q \;=\; \int_{I\!R} \mathbb{G}_a\;\Lambda(da).$$

THEOREM 2. *Suppose $\mathbb{F}$ be nonarithmetic, conditions (C.1-3) and $\varphi(n)^{-1}\xi_n \xrightarrow{d} \xi$ for an unbounded regularly varying function φ and a random variable ξ with a distribution continuous at 0. Then $\varphi(t/\mu)^{-1}(\tau(t)-\nu(t))$ converges in distribution to $\mu^{-1}\xi$.*

Note that in Theorem 2 $n^{-1}\xi_n \xrightarrow{P} 0$ implies $\varphi(t)=t^\alpha L(t)$ for some $\alpha\in[0,1]$ and some slowly varying function L. Moreover, $\lim_{t\to\infty} L(t)=0$ in case $\alpha=1$. On the other hand, if $\alpha=0$, then $P(\xi=0)=0$ implies $|\xi_n| \xrightarrow{P} \infty$ and thus $\lim_{t\to\infty} L(t)=\infty$. Assuming $\varphi(n)^{-1}\xi_n\to 0$ a.s. instead of our weak convergence condition, Larsson-Cohn [6] showed that $\varphi(t)^{-1}(\tau(t)-\nu(t))$ converges to 0 in probability.

Let us finally note that we have confined ourselves to the nonarithmetic case although correponding results may be formulated also for arithmetic random walks. However, they look more complicated because the nonlinear excess $S_{\nu(t)}-t$ no longer converges to the same distribution as $S_{\tau(t)}-t$ in general. A further discussion of the arising problems will be omitted.

2. Some Preparatory Lemmata

In this section we collect some preparatory lemmata on the distribution family $(\mathbb{G}_a)_{a\in\mathbb{R}}$.

Lemma 1. *If $\mathbb{F}$ is nonarithmetic, then $\lim_{b\to a}\mathbb{G}_b(r)=\mathbb{G}_a(r)$ for all $a,r\in\mathbb{R}$, so that in particular the family $(\mathbb{G}_a)_{a\in\mathbb{R}}$ is weakly continuous ($\mathbb{G}_b \xrightarrow{w} \mathbb{G}_a$ as $b\to a$). The convergence is uniform in the sense that*

$$\lim_{\varepsilon\downarrow 0}\sup_{r\in\mathbb{R}}\sup_{a,b:|b-a|\le\varepsilon}|\mathbb{G}_b(r)-\mathbb{G}_a(r)| \;=\; 0. \tag{2.1}$$

Proof. Since $\mathbb{G}_{-a}$ is the reflection of $\mathbb{G}_a$ at 0 it suffices to consider $a\ge 0$ and then further only $r\ge 0$. The subsequent inequalities follow from (1.6) and the convergence to 0 from the continuity of $\mathbb{F}_e$.

$$|\mathbb{G}_b(r)-\mathbb{G}_0(r)| \;\le\; \mathbb{F}_e(b)+\int_{[0,\infty)}(\mathbb{F}_e(b)-\mathbb{F}_e((b-s)^+))\;P(M_r\in ds) \xrightarrow{b\downarrow 0} 0.$$

If $a>0$, then

$$|\mathbb{G}_b(0)-\mathbb{G}_a(0)| \;\le\; |\mathbb{F}_e(a)-\mathbb{F}_e(b)| \xrightarrow{b\downarrow a} 0,$$

and for $r>0$

$$|\mathbb{G}_b(r)-\mathbb{G}_a(r)| \;\le\; \int_{[0,\infty)}|\mathbb{F}_e(b-s)-\mathbb{F}_e((a-s)^+)|\;P(M_r\in ds) \xrightarrow{b\downarrow a} 0.$$

Finally, we note that the uniformity assertion (2.1) also follows from these inequalities because $\mathbb{F}_e$ is even uniformly continuous.

Our next lemma is crucial for the proofs of Theorem 1 and 2.

Lemma 2. *If F is nonarithmetic, then, for each $a\in\mathbb{R}$, $\tau(t+a)-\tau(t)$ converges in distribution to $\mathbb{G}_a$, as $t\to\infty$. Moreover, the convergence is uniform in the sense that*

$$\lim_{t\to\infty}\sup_{\substack{a\ge-\delta t\\ r\in\mathbb{R}}}|P(\tau(t+a)-\tau(t)\le r)-\mathbb{G}_a(r)| \;=\; 0 \tag{2.2}$$

for each $\delta\in(0,1)$.

Proof. Put $M_n=\max_{0\le k\le n}S_k$ for $n\in\mathbb{N}_0$. We have for all $a\ge 0$ and $r\in\mathbb{N}_0$ that

$$\begin{aligned} P(\tau(t+a)-\tau(t)\le r) \;&=\; P(R_t>a)+\int_{[0,a]}P(\tau(a-x)\le r)\;P(R_t\in dx)\\ &=\; P(R_t>a)+\int_{[0,a]}P(M_r>a-x)\;P(R_t\in dx)\\ &=\; P(R_t>a)+\int_{[0,\infty)}P(R_t\in((a-s)^+,a])\;P(M_r\in ds), \end{aligned}$$

and for $r \in \mathbb{N}$

$$\begin{aligned} P(\tau(t-a)-\tau(t) \leq -r) &= P(R_{t-a} > a) + \int_{[0,a]} P(\tau(a-x) > r-1)\; P(R_{t-a} \in dx) \\ &= P(R_{t-a} > a) + \int_{[0,a]} P(M_{r-1} \leq a-x)\; P(R_{t-a} \in dx) \\ &= P(R_{t-a} > a) + \int_{[0,a]} P(R_{t-a} \leq a-s)\; P(M_{r-1} \in ds). \end{aligned}$$

From this the assertions are easily inferred. For the uniform convergence in $a \geq -\delta t$ and $r \in \mathbb{R}$ one has to use the fact that $P(R_t \leq r)$ converges uniformly in r to $\mathbb{F}_e(r)$ because $\mathbb{F}_e$ is continuous.

LEMMA 3. *Let $\mathbb{F}$ be nonarithmetic and $a(t), b(t)$ be such that $\lim_{t\to\infty} a(t) = \infty$ and $\lim_{t\to\infty} b(t)/a(t) = r$ for some $r \in \mathbb{R}$. Then*

$$\lim_{t\to\infty} \mathbb{G}_{a(t)}(b(t)) = \begin{cases} 1, & \text{if } r > 1/\mu \\ 0, & \text{if } r < 1/\mu \end{cases}. \tag{2.3}$$

and

$$\lim_{t\to\infty} \mathbb{G}_{-a(t)}(b(t)) = \begin{cases} 1, & \text{if } r > -1/\mu \\ 0, & \text{if } r < -1/\mu \end{cases}. \tag{2.4}$$

PROOF. Again it suffices to prove the first assertion because the second one then follows via $\mathbb{G}_{-a(t)}(b(t)) = 1 - \mathbb{G}_{a(t)}(-b(t)-)$. But the first line of (1.6) gives

$$\mathbb{G}_{a(t)}(b(t)) = 1 - \mathbb{F}_e(a(t)) + \int_{[0,a(t)]} P(a(t)^{-1}\tau(a(t)-x) \leq a(t)^{-1}b(t))\; \mathbb{F}_e(dx)$$

which together with $t^{-1}\tau(t) \to \mu^{-1}$ a.s. and the dominated convergence theorem implies the assertion.

3. PROOFS OF THEOREMS 1 AND 2

PROOF OF THEOREM 1. Fix an arbitrary $\varepsilon \in (0, \mu/2)$. By (C.3), we can choose $\eta \in (0, \varepsilon)$ so small that

$$P(\max_{n_1 \leq n \leq n_2} |\xi_n - \xi_{n_1}| > \varepsilon) < \varepsilon \tag{3.1}$$

for all $t \geq \mu + \eta$, where $n_1 = (\mu+\eta)^{-1}t$ and $n_2 = (\mu-\eta)^{-1}t$. Put $\varepsilon_1 = \frac{\mu+\eta/3}{\mu+\eta}$, $\varepsilon_2 = \frac{\eta/3}{\mu+\eta}$ and then

$$A_t = \{S_{n_1} \leq \varepsilon_1 t, \eta_{n_1} \leq \varepsilon_2 t\}.$$

for $t \geq 0$. Then $n^{-1}S_n \to \mu$ and $n^{-1}\xi_n \to 0$ a.s. imply

$$\lim_{t\to\infty} P(A_t) = 1.$$

Defining

$$\begin{aligned}\nu_1(t) &= \inf\{n \ge n_1 : Z_n > t\} \wedge n_2,\\ \nu_2(t) &= \inf\{n \ge n_1 : S_n + \xi_{n_1} > t\} \wedge n_2,\\ \nu'(t) &= \inf\{n \ge n_1 : S_n + \xi_{n_1} > t\},\\ \tau'(t) &= \inf\{n \ge n_1 : S_n > t\},\end{aligned}$$

we easily infer

$$\begin{aligned}\nu(t) - \nu_1(t) &\stackrel{P}{\to} 0;\\ P(\nu_2(t+\varepsilon) \le \nu_1(t) \le \nu_2(t-\varepsilon)) &> 1-\varepsilon;\\ \nu'(t) - \nu_2(t) &\stackrel{P}{\to} 0;\\ \tau'(t) - \tau(t) &\stackrel{P}{\to} 0\end{aligned}$$

from $t^{-1}\nu(t) \to \mu^{-1}, t^{-1}\nu'(t) \to \mu^{-1}, t^{-1}\tau(t) \to \mu^{-1}$ a.s. and (3.1). Consequently, setting

$$B_t = \{\nu'(t+\varepsilon) \le \nu(t) \le \nu'(t-\varepsilon), \tau'(t) = \tau(t)\} \cap A_t,$$

we have

$$\limsup_{t\to\infty} P(B_t) > 1 - 2\varepsilon$$

and further

$$\limsup_{t\to\infty} |P(\tau(t) - \nu(t) \le r) - P(\{\tau'(t) - \nu(t) \le r\} \cap B_t)| < 2\varepsilon \tag{3.2}$$

for every $r \in \mathbb{R}$. Now

$$\begin{aligned}P(\{\tau'(t) - \nu(t) \le r\} \cap B_t) &\le P(\{\tau'(t) - \nu'(t-\varepsilon) \le r\} \cap A_t)\\ &= \int_{(-\infty,\varepsilon_1 t]\times(-\infty,\varepsilon_2 t]} P(\tau(t-s) - \tau(t-s-\varepsilon-v) \le r)\ P(S_{n_1} \in ds, \xi_{n_1} \in dv)\\ &\le E\mathbb{G}_{\xi_{n_1}+\varepsilon}(r) + \sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge-\varepsilon_2 t}} |P(\tau(b+a) - \tau(a) \le r) - \mathbb{G}_a(r)|,\end{aligned}$$

and since, by Lemma 2, the final expression tends to 0, as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} P(\{\tau'(t) - \nu(t) \le r\} \cap B_t) \le \int \mathbb{G}_{v+\varepsilon}(r)\ \Lambda(dv) \tag{3.3}$$

for every $r \in \mathbb{N}_0$, because $\xi_{n_1} \stackrel{d}{\to} \Lambda$ and $\mathbb{G}_{v+\varepsilon}(r)$ is a bounded continuous function of v. A similar estimation gives

$$\begin{aligned}P(\{\tau'(t) - \nu(t) \le r\} \cap B_t) &\ge P(\{\tau'(t) - \nu'(t+\varepsilon) \le r\} \cap A_t)\\ &= \int_{(-\infty,\varepsilon_1 t]\times(-\infty,\varepsilon_2 t]} P(\tau(t-s) - \tau(t-s+\varepsilon-v) \le r)\ P(S_{n_1} \in ds, \xi_{n_1} \in dv)\\ &\ge E\mathbb{G}_{\xi_{n_1}-\varepsilon}(r) - P(A_t^c) - \sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge\varepsilon-\varepsilon_2 t}} |P(\tau(b+a) - \tau(a) \le r) - \mathbb{G}_a(r)|,\end{aligned}$$

and thereby

$$\liminf_{t\to\infty} P(\{\tau'(t) - \nu(t) \le r\} \cap B_t) \ge \int \mathbb{G}_{v-\varepsilon}(r)\ \Lambda(dv). \tag{3.4}$$

Finally, the assertion follows when combining (3.2)-(3.4) and letting ε tend to 0 which yields $\lim_{\varepsilon\to 0}\mathbb{G}_{v\pm\varepsilon}(r)=\mathbb{G}_v(r)$ for all $r\in I\!R$.

Proof of Theorem 2. We keep the notation of the previous proof, but this time we choose $\varepsilon=\varepsilon(t),\eta=\eta(t)$ as functions of t in such a way that $0<\eta(t)<\varepsilon(t)\downarrow 0$, as $t\uparrow\infty$, and

$$\lim_{t\to\infty}P(\sup_{n_1\le n\le n_2}|\xi_n-\xi_{n_1}|>\varepsilon(t)) \;=\; 0;$$
$$\lim_{t\to\infty}P(A_t) \;=\; 1;$$
$$\lim_{t\to\infty}\sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge-\varepsilon_2 t,\\ x\in I\!R}}|P(\tau(b+a)-\tau(a)\le r)-\mathbb{G}_a(r)| \;=\; 0.$$

It is then obvious that

$$\frac{\tau(t)-\tau'(t)}{\varphi(t/\mu)}\xrightarrow{P}0,\qquad \frac{\nu(t)-\nu'(t)}{\varphi(t/\mu)}\xrightarrow{P}0,$$

and therefore enough to show the assertion for $\varphi(t/\mu)^{-1}(\tau'(t)-\nu'(t))$. Recall that $\varphi(t)=t^\alpha L(t)$ for some $\alpha\in[0,1]$ and some slowly varying function L which converges to ∞ if $\alpha=0$. Consequently, $\lim_{t\to\infty}\varphi(t/\mu)/\varphi(n_1)=1$. A look at the subsequent inequalities immediately shows that we can assume there w.l.o.g. that

$$\zeta_n\;\stackrel{\text{def}}{=}\;\frac{\xi_n}{\varphi(n)}\;\to\;\frac{\xi}{\mu}\quad\text{a.s.}$$

as $n\to\infty$. Similar to the proof of Theorem 1, we obtain

$$\begin{aligned}
&P(\{\tau'(t)-\nu'(t)\le r\varphi(t/\mu)\}\cap A_t)\\
&=\int_{(-\infty,\varepsilon_1 t]\times(-\infty,\varepsilon_2 t]}P(\tau(t-s)-\tau(t-s-v)\le r\varphi(t/\mu))\;P(S_{n_1}\in ds,\xi_{n_1}\in dv)\\
&\le\int_{(-\infty,\varepsilon_2 t]}\mathbb{G}_v(r\varphi(t/\mu))\;P(\xi_{n_1}\in dv)+\sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge-\varepsilon_2 t,\\ x\in I\!R}}|P(\tau(b+a)-\tau(a)\le r)-\mathbb{G}_a(r)|\\
&\le\; E\mathbb{G}_{\zeta_{n_1}\varphi(n_1)}(r\varphi(t/\mu))\;+\;\sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge-\varepsilon_2 t,\\ x\in I\!R}}|P(\tau(b+a)-\tau(a)\le r)-\mathbb{G}_a(r)|
\end{aligned}$$

and conversely

$$\begin{aligned}
&P(\{\tau'(t)-\nu'(t)\le r\varphi(t/\mu)\}\cap A_t)\\
&\ge\; E\mathbb{G}_{\zeta_{n_1}\varphi(n_1)}(r\varphi(t/\mu))\;-\;P(A_t^c)\;-\;\sup_{\substack{b\ge(1-\varepsilon_1)t,\\ a\ge-\varepsilon_2 t,\\ x\in I\!R}}|P(\tau(b+a)-\tau(a)\le r)-\mathbb{G}_a(r)|.
\end{aligned}$$

By letting t tend to ∞ in both inequalities and invoking Lemma 3 (recall $P(\xi=0)=0$), we conclude

$$\limsup_{t\to\infty}P(\{\tau'(t)-\nu'(t)\le r\varphi(t/\mu)\}\cap A_t)\;\le\;P(\mu^{-1}\xi\le r)$$

and

$$\liminf_{t\to\infty} P(\{\tau'(t) - \nu'(t) \le r\varphi(t/\mu)\} \cap A_t) \ \ge\ P(\mu^{-1}\xi < r)$$

and therefore

$$\lim_{t\to\infty} P(\tau'(t) - \nu'(t) \le r\varphi(t/\mu)) \ =\ P(\mu^{-1}\xi \le r)$$

whenever $P(\mu^{-1}\xi \le \cdot)$ is continuous at r. This completes the proof of Theorem 2.

4. Examples.

Two examples shall demonstrate the applicability of Theorems 1 and 2. The first one arises as the stopping rule of a repeated significance test for normally distributed observations, see [8], Example 7.1 on p. 71.

Example 1. Let $(W_n)_{n\ge 0}$ be a zero-delayed nonarithmetic random walk with non-zero drift $\tilde{\mu}$ and finite positive variance $\tilde{\sigma}^2$. Consider the first passage times

$$\nu(t) \ \stackrel{\text{def}}{=}\ \inf\{n \ge 1 : |W_n| > \sqrt{tn}\}, \quad t \ge 0, \tag{4.1}$$

of $(W_n)_{n\ge 0}$ beyond a square root boundary. Squaring the stopping condition and rearranging terms leads to the required form

$$\nu(t) \ =\ \inf\{n \ge 1 : S_n + \xi_n > t\} \tag{4.2}$$

with

$$S_n \ \stackrel{\text{def}}{=}\ n\tilde{\mu}^2 + 2\tilde{\mu}(W_n - n\tilde{\mu}) \quad \text{and} \quad \xi_n \ \stackrel{\text{def}}{=}\ \frac{(W_n - n\tilde{\mu})^2}{n} \tag{4.3}$$

for $n \ge 1$. Obviously, $(S_n)_{n\ge 0}$ forms a zero-delayed nonarithmetic random walk with positive drift $\mu \stackrel{\text{def}}{=} \tilde{\mu}^2$ and finite positive variance $\sigma^2 \stackrel{\text{def}}{=} 4\tilde{\mu}^2\tilde{\sigma}^2$. Moreover, as mentioned in [3], p. 847, the ξ_n are easily seen to satisfy the conditions (C.1-3) as well as $\xi_n \stackrel{d}{\to} \Lambda$ where $\Lambda(t) = \chi(t/\tilde{\sigma})$, χ a chi-squared distribution with one degree of freedom. Hence Theorem 1 applies to this example and ensures that $\tau(t) - \nu(t)$ converges in distribution to a chi-squared mixture (up to the scaling factor $1/\tilde{\sigma}$) of the $\mathbb{G}_a$. Let us mention that the same result follows with a little more work for the one-sided stopping rules

$$\nu^+(t) \ \stackrel{\text{def}}{=}\ \inf\{n \ge 1 : W_n > \sqrt{tn}\}, \quad t \ge 0.$$

Our second example is also taken from [8] (Example 6.1 on p. 62) and appears as the stopping rule of an open ended test for the unknown mean of a normal distribution with unit variance under a normal prior.

Example 2. Let $(W_n)_{n\ge 0}$ be as in Example 1 and consider the stopping times

$$\nu(t) \ \stackrel{\text{def}}{=}\ \inf\{n \ge 1 : |W_n| > \sqrt{(n+1)[t + \log(n+1)]}\}, \quad t \ge 0. \tag{4.4}$$

These are of the form (4.2) with S_n defined as in (4.3) and perturbations

$$\xi_n \ \stackrel{\text{def}}{=}\ \frac{(W_n - n\tilde{\mu})^2}{n} - \frac{W_n^2}{n(n+1)} - \log(n+1) \tag{4.5}$$

for $n \geq 1$. As in Example 1, validity of (C.1-3) can easily be verified for these variables. Furthermore $\xi_n / \log n \to -1$ a.s. Hence we infer

$$\frac{\tau(t) - \nu(t)}{\log t} \quad \stackrel{P}{\to} \quad -\frac{1}{\tilde{\mu}^2} \quad (t \to \infty) \tag{4.6}$$

from Theorem 2 and the well-known fact that convergence in distribution and convergence in probability are equivalent in case of degenerate limit laws.

On the other hand, we can improve on (4.6) because the divergent part $-\log(n+1)$ of ξ_n is deterministic and slowly varying. Defining

$$\hat{\nu}(t) \quad \stackrel{\text{def}}{=} \quad \inf\{n \geq 1 : S_n + \hat{\xi}_n > t\}$$

for $t \geq 0$ where

$$\hat{\xi}_n \quad \stackrel{\text{def}}{=} \quad \xi_n + \log(n+1) \quad = \quad \frac{(W_n - n\tilde{\mu})^2}{n} - \frac{W_n^2}{n(n+1)}$$

for $n \geq 1$, we first obtain upon using $\log \nu(t) - \log t + 2 \log \tilde{\mu} = \log(\frac{\nu(t)}{t/\tilde{\mu}^2}) \to 0$ a.s. that

$$\nu(t) - \hat{\nu}(t + \log t - 2 \log \tilde{\mu}) \quad \stackrel{P}{\to} \quad 0. \tag{4.7}$$

Since further the $\hat{\xi}_n$ satisfy (C.1-3) and evidently converge in distribution to Λ given by $\Lambda(t) = \chi(t + \tilde{\mu}^2)$, χ as in Example 1 a chi-squared distribution with one degree of freedom, we conclude with Theorem 1 that $\tau(t) - \hat{\nu}(t)$ converges in distribution to a chi-squared mixture (up to the shift $-\tilde{\mu}^2$) of the $\mathbb{G}_a$. A combination with (4.7) finally leads to the result

$$\tau(t + \log t - 2 \log \tilde{\mu}) - \nu(t) \quad \stackrel{d}{\to} \quad \int \mathbb{G}_a \, \chi(da + \tilde{\mu}^2) \quad (t \to \infty). \tag{4.8}$$

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References

[1] GUT, A. (1988). *Stopped Random Walks. Limit Theorems and Applications.* Springer, New York.

[2] GUT, A. (1992). First passage times for perturbed random walks. *Sequential Analysis* **11**, 149-179.

[3] HAGWOOD, C. and WOODROOFE, M. (1982). On the expansion for expected sample size in non-linear renewal theory. *Ann. Probab.* **10**, 844-848.

[4] LAI, T.L. and SIEGMUND, D. (1977). A non-linear renewal theory with applications in sequential analysis I. *Ann. Statist.* **5**, 946-954.

[5] LAI, T.L. and SIEGMUND, D. (1979). A non-linear renewal theory with applications in sequential analysis II. *Ann. Statist.* **7**, 60-76.

[6] LARSSON-COHN, L. (2001). On the difference between two first passage times. *Studia Sci. Math. Hungar.* **37**, 53-67.

[7] SIEGMUND, D. (1985). *Sequential Analysis. Test and Confidence Intervals.* Springer, New York.

[8] WOODROOFE, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis.* CBMS-NSF Reg. Conf. Series in Appl. Math. **39**, SIAM, Philadelphia.