The Markov Renewal Theorem and Related Results

Gerold Alsmeyer

Institut für Mathematische Statistik
Fachbereich Mathematik
Westfälische Wilhelms-Universität Münster
Einsteinstraße 62
D-48149 Münster, Germany

We give a new probabilistic proof of the Markov renewal theorem for Markov random walks with positive drift and Harris recurrent driving chain. It forms an alternative to the one recently given in [1] and follows more closely the probabilistic proofs provided for Blackwell's theorem in the literature by making use of ladder variables, the stationary Markov delay distribution and a coupling argument. A major advantage is that the arguments can be refined to yield convergence rate results.

AMS 1991 subject classifications. 60J05, 60J15, 60K05, 60K15.

Keywords and phrases. Markov random walks, ladder variables, Harris recurrence, regeneration epochs, coupling.

1. Introduction

Random walks with Markov-modulated increments, called Markov random walks, form a natural generalization of those with i.i.d. increments. Due to their much greater variability without loosing too much regular structure, they have become increasingly popular for modelling applications, notably in queueing theory, risk theory and the theory of branching processes. As a consequence, there is strong interest in a thorough description of their intrinsic properties. This falls to a far extent in the range of Markov renewal theory which provides the necessary extension of classical renewal theory treating the case of i.i.d. increments. Since the latter is mainly built upon one result, called Blackwell's renewal theorem, it does not come by surprise that a corresponding master theorem exists in Markov renewal theory and emerges from the former one as a generalization by additionally taking care of the Markov modulation providing an appropriate recurrence property in order to retain a stationary pattern. Although a first version of that result, called Markov renewal theorem, goes back to an old paper by Smith [16] where increments are positive and modulation is with respect to an ergodic Markov chain with countable state space, general versions allowing for arbitrary state spaces and increments with positive mean (in a certain sense) have a much shorter history. A purely probabilistic proof under most general assumptions has been given only very recently by the author [1], but is still amenable for improvements. In view of the many proofs that have appeared for Blackwell's theorem this should not be surprising. The main purpose of the present paper is therefore to give yet another (probabilistic) proof of the Markov renewal theorem which, though not being shorter, has a number of advantages over that in [1]. First it stays closer to the nice and fairly simple ones for Blackwell's theorem in [8], [11] and [17] by making use of ladder variables, a stationary delay distribution and coupling. Second and more important, the arguments used here are sharper in that they can be refined to yield convergence rate results, again along similar lines as for Blackwell's theorem, see [3].

2. The Markov Renewal Setup and Preliminaries

We begin with a brief description of the Markov renewal setup to be considered throughout: Given a measurable space $(\mathcal{S},\mathfrak{S})$ with countably generated σ-field $\mathfrak{S}$, let $(M_n,X_n)_{n\ge0}$ be a temporally homogeneous Markov chain with state space $\mathcal{S}\times\mathbb{R}$ and transition kernel $\boldsymbol{P}:\mathcal{S}\times(\mathfrak{S}\otimes\mathfrak{B})\to[0,1]$, $\mathfrak{B}$ the Borel σ-field on $\mathbb{R}$. Hence

$$P(M_{n+1}\in A,X_{n+1}\in B|M_n,X_n) \;=\; \boldsymbol{P}(M_n,A\times B) \quad \text{a.s.} \tag{2.1}$$

which means that (M_{n+1},X_{n+1}) depends on the past only through M_n and that $(M_n)_{n\ge0}$ forms a Markov chain with state space $\mathcal{S}$ and transition kernel $\mathbb{P}(x,A)\overset{\text{def}}{=}\boldsymbol{P}(x,A\times\mathbb{R})$. Conditioned upon $(M_j)_{j\ge0}$, the $X_n,n\ge0$ are further conditionally independent with

$$P(X_n\in B|(M_j)_{j\ge0}) \;=\; Q(M_{n-1},M_n,B) \quad \text{a.s.} \tag{2.2}$$

for all $n \geq 1, B \in \mathfrak{B}$ and a kernel $Q : \mathcal{S}^2 \times \mathfrak{B} \to [0,1]$. Let throughout a canonical model be given with probability measures $P_{x,y},\ x \in \mathcal{S}, y \in I\!R$, on $(\Omega, \mathcal{A})$ such that $P_{x,y}(M_0 = x, X_0 = y) = 1$. For any distribution (or σ-finite measure) λ on $\mathcal{S} \times I\!R$ put $P_\lambda(\cdot) = \int_{\mathcal{S}\times I\!R} P_{x,y}(\cdot)\lambda(dx,dy)$ in which case (M_0, X_0) has initial distribution λ under P_λ. For $x \in \mathcal{S}$ and σ-finite measures ν on $\mathcal{S}$, we write for short E_x, E_ν instead of $E_{x,0}, E_{\nu\otimes\delta_0}$, respectively, where δ_0 is Dirac measure at 0. Finally, P and E are used for probabilities and expectations, respectively, that do not depend at all on initial conditions.

The *Markov random walk (MRW)* associated with $(M_n, X_n)_{n\geq 0}$ is given by $(M_n, S_n)_{n\geq 0}$, where $S_n = X_0 + ... + X_n$ for $n \geq 0$. We call it a *Markov renewal process (MRP)* when all X_n's are positive, i.e. when $\boldsymbol{P}(x, \mathcal{S} \times (0,\infty)) = 1$ for all $x \in \mathcal{S}$. Introducing the potential

$$U_\lambda \ = \ \sum_{n\geq 0} P_\lambda((M_n, S_n) \in \cdot) \tag{2.3}$$

we obtain the *Markov renewal measure* of $(M_n, S_n)_{n\geq 0}$ under P_λ which constitutes a natural extension of the renewal measure of a random walk with i.i.d. increments.

The basic condition on the driving chain $(M_n)_{n\geq 0}$ is it be Harris recurrent which means that there exist some $\alpha \in (0,1]$, $r \geq 1$, a recurrence set $\Re \in \mathfrak{S}$, i.e. $P_x(M_n \in \Re \text{ i.o.}) = 1$ for all $x \in \mathcal{S}$, and a probability measure φ on $(\mathcal{S}, \mathfrak{S})$, $\varphi(\Re) = 1$, such that

$$I\!P_r(x, \cdot) \ \geq \ \alpha\varphi \quad \text{for all } x \in \Re, \tag{2.4}$$

where $I\!P_r$ denotes the r-step transition kernel of $(M_n)_{n\geq 0}$. Under these conditions $\Re$ is called a *regeneration set* for $(M_n)_{n\geq 0}$, and one can redefine $(M_n)_{n\geq 0}$ on a possibly enlarged probability space together with a *sequence* $(\tau_n)_{n\geq 0}$ *of regeneration epochs* characterized through the following conditions (see [4], [5] or [12]):

(R.1) $0 = \tau_0 < \tau_1 < \tau_2 < ... < \infty$ a.s. under each P_λ.

(R.2) There is a filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $(M_n)_{n\geq 0}$ is Markov adapted and each τ_n a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(R.3) under each $P_x, x \in \mathcal{S}$, the cycles $(\tau_{n+1} - \tau_n, M_{\tau_n}, ..., M_{\tau_{n+1}-1}), n \geq 0$, are one-dependent and stationary for $n \geq 1$ with cycle debut distribution $\zeta \overset{\text{def}}{=} P(M_{\tau_1} \in \cdot)$.

It follows from (R.2) and the strong Markov property that

(R.4) $P((\tau_{n+j} - \tau_n, M_{\tau_n+j})_{j\geq 0} \in \cdot|\mathcal{F}_{\tau_n}) = P_{M_{\tau_n}}((\tau_j, M_j)_{j\geq 0} \in \cdot)$ P_x-a.s. for each $n \geq 0$ and $x \in \mathcal{S}$.

$(\tau_n)_{n\geq 0}$ can further be chosen such that

$$\xi \ \overset{\text{def}}{=} \ E_\zeta\left(\sum_{k=0}^{\tau_1-1} \mathbf{1}_{\{M_k\in\cdot\}}\right) \tag{2.5}$$

defines the unique (up to a multiplicative constant) σ-finite stationary measure, in which case we call the sequence *regular*. $(M_n)_{n\geq 0}$ is positive recurrent iff $E_\zeta\tau_1 < \infty$, and $\hat{\xi} = \xi/E_\zeta\tau_1$ then its unique stationary distribution. Note that

$$\int_{\mathcal{S}} h(x)\ \xi(dx) \ = \ E_\zeta \left(\sum_{k=0}^{\tau_1 - 1} h(M_k) \right) \tag{2.6}$$

for every ξ-integrable function h, implying in particular

$$E_\zeta S_{\tau_1} \ = \ \int_{\mathcal{S}} \mu(x)\ \xi(dx) \ = \ E_\xi X_1 \ \stackrel{\text{def}}{=} \ \mu \tag{2.7}$$

where $\mu(x) \stackrel{\text{def}}{=} E(X_1 | M_0 = x)$.

By conditional independence of $(X_n)_{n\geq 0}$ given $(M_n)_{n\geq 0}$ and property (2.2) the above construction can be done in such a way that (R.2) still holds for the bivariate chain $(M_n, X_n)_{n\geq 0}$ and (R.3), (R.4) for the extended cycles $(\tau_{n+1} - \tau_n, (M_k, X_{k+1})_{\tau_n \leq k < \tau_{n+1}}), n \geq 0$. As a consequence, the embedded process $(S_{\tau_n})_{n\geq 0}$ has one-dependent stationary increments (apart from the first one) under each P_x and thus diverges a.s. to ∞, as does $(S_n)_{n\geq 0}$ itself, providing $\mu > 0$. We call μ the *drift* of $(M_n, S_n)_{n\geq 0}$.

The usual distinction of lattice-types in renewal theory does also apply to MRW's but looks a bit more complicated. The right condition is due to Shurenkov [14] and looks as follows: $\boldsymbol{P}$ as well as $(M_n, S_n)_{n\geq 0}$ are called *d-arithmetic*, if $d \geq 0$ denotes the maximal number for which there exists a measurable function $\gamma : \mathcal{S} \to [0, d)$, called *shift function*, such that

$$P(X_1 \in \gamma(x) - \gamma(y) + d\mathbb{Z} | M_0 = x, M_1 = y) \ = \ 1 \quad \xi \otimes I\!\!P\text{-a.s.} \tag{2.8}$$

where $\xi \otimes I\!\!P$ is given through $\xi \otimes I\!\!P(A \times B) = \int_A I\!\!P(x, B)\xi(dx)$ for $A, B \in \mathfrak{S}$. $(M_n, S_n)_{n\geq 0}$ and $\boldsymbol{P}$ are called *nonarithmetic* if no such d exists. Notice that, if $(M_n, S_n)_{n\geq 0}$ is d-arithmetic with shift function γ, then $(M_n, d^{-1}(S_n - \gamma(M_0) + \gamma(M_n))_{n\geq 0}$ is 1-arithmetic with shift function 0.

Markov renewal theory deals with the asymptotic behavior of functionals of $(M_n, S_n)_{n\geq 0}$ and related processes. The Harris recurrence of $(M_n)_{n\geq 0}$ will always be assumed hereafter unless stated otherwise. Although the main thrust of the theory is for positive increments X_n, we also want to consider the more general case of positive drift μ. In order to do so we resort to the use of ladder variables which is a well-known technique in classical renewal theory. However, the details are somewhat different in the present context and also more difficult due to the additional Markov modulation that has to be taken into account. We refer to Section 4.

3. Main Results

We are now ready to formulate the Markov renewal theorem and a number of consequences. Earlier versions of the results under varying assumptions have appeared in [1], [6], [7], [9], [10], [13], [14], [15]. We restrict ourselves to the nonarithmetic case because the treatment of arithmetic MRW's requires only straightforward modifications of the subsequent arguments and is even simpler in places, see also [1].

THEOREM 1. *Let $(M_n,S_n)_{n\ge 0}$ be a nonarithmetic MRW with Harris recurrent driving chain $(M_n)_{n\ge 0}$ with stationary measure ξ. If $\mu=\int_{\mathcal S}\mu(x)\,\xi(dx)\in(0,\infty]$ and $g:\mathcal{S}\times I\!R\to I\!R$ is any measurable function satisfying*

$$g(x,\cdot)\ \textit{is}\ \ell_0\textit{-a.e. continuous for}\ \xi\textit{-almost all}\ x\in\mathcal{S}, \tag{3.1}$$

$$\int_{\mathcal S}\sum_{n\in\mathbb{Z}}\ \sup_{n\rho\le y<(n+1)\rho}|g(x,y)|\ \xi(dx)\ <\ \infty\quad \textit{for some}\ \rho>0, \tag{3.2}$$

then

$$g*U_x(t)\ \stackrel{\text{def}}{=}\ E_x\left(\sum_{n\ge 0}g(M_n,t-S_n)\right)\ \to\ \frac{1}{\mu}\int_{\mathcal S}\int_{I\!R}g(u,v)\ \ell_0(dv)\ \xi(du), \tag{3.3}$$

holds for ξ-almost all $x\in\mathcal{S}$ where ℓ_0 denotes Lebesgue measure. In particular,

$$\lim_{t\to\infty}U_x(A\times t+I)\ =\ \mu^{-1}\xi(A)\ell_0(I) \tag{3.4}$$

for all $A\in\mathfrak{S}$ with $\xi(A)<\infty$ and all finite intervals I. The occuring limits are interpreted as 0 if $\mu=\infty$.

Conditions (3.1) and (3.2) are the appropriate substitutes for the well-known direct Riemann integrability in the standard renewal setup. Note that (3.2) forces $g(x,\cdot)$ to be bounded for ξ-almost all $x\in\mathcal{S}$. It further implies

$$\lim_{t\to\infty}|g(x,t)|\ =\ 0\quad \text{for } \xi\text{-almost all } x\in\mathcal{S}, \tag{3.5}$$

$$\int_{\mathcal S}\sup_{t\in I\!R}|g(x,t)|\ \xi(dx)\ <\ \infty. \tag{3.6}$$

Note also that (3.2) implicitly assumes the measurability of $\inf_{a\le y<b}g(x,y)$ and $\sup_{a\le y<b}g(x,y)$ for all $-\infty<a<b<\infty$.

Closely related with the previous result is an ergodic theorem ((3.8) below) in the proper renewal case when $(M_n,S_n)_{n\ge 0}$ forms a MRP. We may then interpret S_n as the n-th transition epoch of $(M_n)_{n\ge 0}$ where it moves from M_{n-1} to M_n, X_n thus being the associated sojourn time. In order to incorporate a transition to M_0 after a *positive* delay S_0 in (3.7) below we extend the chain backwards by variable M_{-1} which denotes the current state at $t=0$ if $S_0>0$. Now put $S_{-1}=0$ and $N(t)=\sup\{n\ge -1: S_n\le t\}$ for $t\ge 0$. Suppose $N(t)<\infty$ for all t (non-explosive case) and define

$$\begin{aligned}(Y(t),A(t),R(t))\ &\stackrel{\text{def}}{=}\ (M_{N(t)},t-S_{N(t)},S_{N(t)+1}-t)\\ &=\ \sum_{n\ge -1}(M_n,t-S_n,S_{n+1}-t)\mathbf{1}_{\{S_n\le t<S_{n+1}\}},\quad t\ge 0.\end{aligned} \tag{3.7}$$

Notice that the latter summation extends only over $n\ge 0$ if $S_0=0$. $(Y(t))_{t\ge 0}$ is called a *semi-Markov process (SMP)* with embedded chain $(M_n)_{n\ge -1}$ and sojourn times $(X_n)_{n\ge 0}$, $(A(t))_{t\ge 0}$ the *age process* and $(R(t))_{t\ge 0}$ the *residual life process* associated with $(M_{n-1},S_n)_{n\ge 0}$.

Corollary 1. *In the situation of Theorem 1 suppose additionally* $\boldsymbol{P}(x, \mathcal{S}\times(0,\infty)) = 1$ *for all* $x \in \mathcal{S}$ *and let* $(Y(t), A(t), R(t))_{t\ge 0}$ *be as in (3.7). Then, as* $t \to \infty$,

$$E_x g(Y(t), A(t), R(t)) \;\to\; \frac{1}{\mu}\int_{\mathcal{S}}\int_{[0,\infty)}\int_{[0,w)} g(u,v,w-v)\ \ell_0(dv)\ \boldsymbol{P}(u,\mathcal{S}\times dw)\ \xi(du) \tag{3.8}$$

holds for ξ*-almost all* $x \in \mathcal{S}$ *and for every measurable function* $g : \mathcal{S}\times[0,\infty)^2 \to I\!R$ *such that* $f(x,y) \overset{\text{def}}{=} E_x g(x,y,X_1-y)\ \mathbf{1}_{\{X_1>y\}}$ *satisfies (3.1) and (3.2). In particular,*

$$\lim_{t\to\infty} P_x(Y(t)\in A, A(t)>s, R(t)>r) \;=\; \frac{1}{\mu}\int_A\int_{(r+s,\infty)} P_u(X_1>v)\ dv\ \xi(du) \tag{3.9}$$

for all $A \in \mathfrak{S}$ *with* $\xi(A)<\infty$ *and all* $r,s\ge 0$.

Corollary 1 is a direct consequence of Theorem 1 because by a simple renewal argument

$$E_x g(Y(t), A(t), R(t)) \;=\; E_x g(M_{-1}, t, S_0 - t)\mathbf{1}_{\{S_0>t\}} \;+\; f * U_x(t).$$

Further details of its proof will hence be omitted.

Next let $(M_n^*, S_n^*)_{n\ge 0} = (M_{\sigma_n}, S_{\sigma_n})_{n\ge 0}$ with $\sigma_0 = 0$ and $\sigma_n = \inf\{k > \sigma_{n-1} : S_k > S_{\sigma_{n-1}}\}$ for $n \ge 1$ being the strictly ascending ladder epochs. So S_n^* gives the ladder height associated with σ_n and M_n^* the pertinent driving chain state. For $t\ge 0$ consider the first passage times

$$T(t) \;=\; \inf\{n\ge 0 : S_n > t\} \quad \text{und} \quad T^*(t) \;=\; \inf\{n \ge 0 : S_n^* > t\}.$$

It is well-known that asymptotic properties of $T(t)$ are strongly tied to those of the *excess over the boundary* $R(t) = S_{T(t)} - t = S^*_{T^*(t)} - t$, which in case of positive X_n's indeed coincides with the equally denoted variable introduced further above. Let $Z(t) = M_{T(t)} = M^*_{T^*(t)}$. By a simple renewal argument

$$E_x g(Z(t), R(t)) \;=\; E_x g(M_0^*, S_0^*)\mathbf{1}_{\{S_0^*>t\}} \;+\; h * U_x^*(t)$$

where $h(x,y) \overset{\text{def}}{=} E_x g(M_1^*, X_1^* - y)\mathbf{1}_{\{X_1^*>y\}}$ and U_λ^* denotes the renewal measure associated with $(M_n^*, S_n^*)_{n\ge 0}$ under P_λ. Hence the following result is another direct consequence of Theorem 1 and again stated without proof. Let $\boldsymbol{P}^*$ be the transition kernel of $(M_n^*, X_n^*)_{n\ge 0}$, ξ^* the essentially unique stationary measure of $(M_n^*)_{n\ge 0}$ (existing by Theorem 2 in Section 4) and $\mu^* = E_{\xi^*} X_1^*$.

Corollary 2. *Given the situation of Theorem 1,*

$$\lim_{t\to\infty} E_x g(Z(t), R(t)) \;=\; \frac{1}{\mu^*}\int_{\mathcal{S}}\int_{\mathcal{S}\times(0,\infty)}\int_{[0,z)} g(v,w)\ \ell_0(dw)\ \boldsymbol{P}^*(u, dv\times dz)\ \xi^*(du) \tag{3.10}$$

holds for ξ^*-almost all $x \in \mathcal{S}$ and for every measurable function $g : \mathcal{S} \times [0,\infty)^2 \to I\!R$ such that h as defined above satisfies (3.1) and (3.2). In particular,

$$\lim_{t\to\infty} P_x(Z(t) \in A, R(t) > r) \;=\; \frac{1}{\mu^*}\int_{(r,\infty)} P_{\xi^*}(M_1^* \in A, X_1^* > z)\; \ell_0(dz) \tag{3.11}$$

for all $A \in \mathfrak{S}$ with $\xi^*(A) < \infty$ and all $r \geq 0$.

In the next section we briefly collect some important facts from [2] on the step-size a ladder epochs $\sigma_n(a)$ (see (4.1) below) and ladder heights which will then be used to reduce the proof of Theorem 1 to the case of very simple MRP's $(M_n, S_n)_{n\geq 0}$ where $(M_n)_{n\geq 0}$ consists of i.i.d. variables (thus in particular being a strongly aperiodic Harris chain) and where the X_n are lower bounded by 1. Moreover, that MRP is *completely nonarithmetic* in the sense that $E_x|E(e^{itX_1}|M_0, M_1)| < 1$ for every $t \neq 0$ and $x \in \mathcal{S}$. The reduction is obtained by a standard technique we call cyclic decomposition. In Section 6 we derive the unique measure λ^s satisfying $U_{\lambda^s} = \mu^{-1}\xi \otimes \ell_0^+$, $\ell_0^+ = \ell_0(\cdot \cap [0,\infty))$, which in turn furnishes the coupling proof of (3.3) given in Section 7. Finally, a number of purely technical lemmata are collected in an Appendix.

4. Ladder Variables

For $a \geq 0$, define $\sigma_0(a) = 0$ and recursively

$$\sigma_n(a) \;\stackrel{\text{def}}{=}\; \inf\{k > \sigma_{n-1}(a) : S_k - S_{\sigma_{n-1}} > a\}, \quad n \geq 1. \tag{4.1}$$

From the above we see that all $\sigma_n(a)$ are a.s. finite under each P_λ. If $a = 0$ we obtain the familiar (strictly ascending) ladder epochs. More convenient for our purposes, however, is the sequence $(\sigma_n(a))_{n\geq 0}$ with some $a > 0$ for having the advantage that the Markov renewal measure of the pertinent embedded sequence $(M_{\sigma_n(a)}, S_{\sigma_n(a)})_{n\geq 0}$ is trivially bounded by 1 on sets $A \times [t, t+h]$, $A \in \mathfrak{S}, t \in I\!R$ and $h \in (0, a)$, regardless of the underlying P_λ. That $(M_{\sigma_n(a)}, S_{\sigma_n(a)})_{n\geq 0}$ forms a MRP can be easily verified, but there seems to be no simple and short argument showing the Harris recurrence of its driving chain $(M_{\sigma_n(a)})_{n\geq 0}$. Yet a positive answer has been provided in [2] and is stated for reference below (Theorem 2).

Put $\sigma_n \stackrel{\text{def}}{=} \sigma_n(a)$, $(M_n^*, S_n^*) \stackrel{\text{def}}{=} (M_{\sigma_n}, S_{\sigma_n})$ for $n \geq 0$, and let ξ^* be the stationary measure of $(M_n^*)_{n\geq 0}$ (if it exists). Let us further define the sequence $(Z_n)_{n\geq 0}$ with state space $(-\infty, a] \cup \{\Delta\}$ through $Z_0 = \Delta\mathbf{1}_{\{S_0 > a\}} + S_0\mathbf{1}_{\{S_0 \leq a\}}$ and

$$Z_n \;=\; \begin{cases} \Delta, & \text{if } Z_{n-1} = \Delta,\ X_n > a \text{ or } Z_{n-1} \leq a < Z_{n-1} + X_n \\ X_n, & \text{if } Z_{n-1} = \Delta,\ X_n \leq a \\ Z_{n-1} + X_n, & \text{if } Z_{n-1} < a,\ Z_{n-1} + X_n \leq a \end{cases} \tag{4.2}$$

for $n \geq 1$. $Z_n = \Delta$ means that $\sigma_{k(n)} = n$, while $Z_n \leq a$ denotes the *excursion* of S_n from the previous ladder height $S_{\sigma_{k(n)}}$, $k(n) \stackrel{\text{def}}{=} \sup\{k : \sigma_k \leq n\}$. The recursive structure of the

Z_n implies immediately that $(M_n, Z_n)_{n\ge 0}$ is again Markov. Let us call it the pertinent *level a chain of excursions.*

THEOREM 2. *Given a MRW $(M_n, S_n)_{n\ge 0}$ with Harris recurrent driving chain $(M_n)_{n\ge 0}$ and drift $\mu = E_\xi X_1 > 0$, the following assertions hold true for $(M_n, Z_n)_{n\ge 0}$ and $(M_n^*, S_n^*)_{n\ge 0}$:*

(i) $(M_n, Z_n)_{n\ge 0}$ *forms a Harris chain which is positive recurrent if and only if the same holds true for $(M_n)_{n\ge 0}$. Furthermore there exists a regular sequence of regeneration epochs $(\tau_n)_{n\ge 0}$ for both chains such that $Z_{\tau_n} = \Delta$ for all $n \ge 1$.*

(ii) $(M_n^*)_{n\ge 0}$ *forms a Harris chain which is positive recurrent if the same holds true for $(M_n)_{n\ge 0}$. Moreover, a regular sequence of regeneration epochs for both chains exists.*

(iii) $(M_n^*, \sigma_n)_{n\ge 0}$ *and $(M_n^*, S_n^*)_{n\ge 0}$ are MRP's, their lattice-type being that of $(M_n, n)_{n\ge 0}$, and $(M_n, S_n)_{n\ge 0}$, respectively, with the same shift function if arithmetic.*

(iv) $E_\xi S_1^* < \infty$ *iff* $E_{\xi^*} S_1^* < \infty$ *iff* $\mu < \infty$.

(v) $E_{\xi^*}\sigma_1 < \infty$ *iff* $(M_n)_{n\ge 0}$ *is positive recurrent.*

We have already mentioned that Theorem 2 is proved in [2]. Although $\sigma_n = \sigma_n(0)$ there, the arguments carry over verbatim to the case $\sigma_n = \sigma_n(a)$ for any $a > 0$. Parts (iii)–(v) are stated for completeness but actually not needed here.

5. Reduction by Cyclic Decomposition

FIRST REDUCTION BY GEOMETRIC SAMPLING. Given a nonarithmetic MRW $(M_n, S_n)_{n\ge 0}$ with positive drift, Harris recurrent driving chain $(M_n)_{n\ge 0}$ and level 1 excursion chain $(M_n, Z_n)_{n\ge 0}$, let $(\eta_n)_{n\ge 0}$ be an independent (under each P_λ) zero-delayed renewal process with i.i.d. geometric(1/2) increments, that is $P(\eta_1 = n) = 1/2^n$ for each $n \ge 1$. Put $(\hat M_n, Y_n, \hat S_n) = (M_{\eta_n}, Z_{\eta_n}, S_{\eta_n})$ for $n \ge 0$ and note that $(\hat M_n, \hat S_n)_{n\ge 0}$ forms another MRW whose driving chain $(\hat M_n)_{n\ge 0}$ as well as $(\hat M_n, Y_n)_{n\ge 0}$ are strongly aperiodic Harris chains, the former with the same stationary measure ξ as $(M_n)_{n\ge 0}$ itself. As for strong aperiodicity, we only consider the latter chain and note first that, by Theorem 2, a regeneration set $\Re$ of $(M_n)_{n\ge 0}$ exists such that $\Re \times \Delta$ is one for $(M_n, Z_n)_{n\ge 0}$. Denote by $\widetilde{I\!\!P}$ and $I\!\!K$ the transition kernels of $(M_n, Z_n)_{n\ge 0}$ and $(\hat M_n, Y_n)_{n\ge 0}$, respectively. By (2.4),

$$\widetilde{I\!\!P}_r((x,\Delta),\cdot) \;\ge\; \alpha\,\varphi\otimes\delta_\Delta$$

for suitable α, r and φ, $\varphi(\Re) = 1$. Consequently,

$$I\!\!K((x,\Delta),\cdot) \;=\; \sum_{n\ge 1} 2^{-n}\,\widetilde{I\!\!P}_n((x,\Delta),\cdot) \;\ge\; 2^{-r}\,\widetilde{I\!\!P}_r((x,\Delta),\cdot) \;\ge\; \alpha 2^{-r}\varphi\otimes\delta_\Delta$$

for all $x \in \Re$.

Note that $(\hat M_n, Y_n)_{n\ge 0}$ is *not* the level 1 excursion chain associated with $(\hat M_n)_{n\ge 0}$, denoted by $(\hat M_n, \hat Z_n)_{n\ge 0}$ with transition kernel $\widehat{I\!\!P}$. On the other hand, we obviously have $\{Y_n = \Delta\} \subset$

$\{\hat{Z}_n = \Delta\}$ as well as

$$I\!K((x,\Delta),\cdot) \;=\; P((\hat{M}_1, Y_1) \in \cdot | \hat{M}_0 = x, Y_0 = \Delta) \;=\; P((\hat{M}_1, Y_1) \in \cdot | \hat{M}_0 = x, \hat{Z}_0 = \Delta)$$

for all $x \in \mathcal{S}$, whence

$$\begin{aligned} \widehat{I\!P}((x,\Delta), \cdot \times \{\Delta\}) &= P(\hat{M}_1 \in \cdot, \hat{Z}_1 = \Delta | \hat{M}_0 = x, \hat{Z}_0 = \Delta) \\ &\geq P(\hat{M}_1 \in \cdot, Y_1 = \Delta | \hat{M}_0 = x, \hat{Z}_0 = \Delta) \;=\; I\!K((x,\Delta), \cdot \times \{\Delta\}) \end{aligned}$$

follows and thereby that $(\hat{M}_n, \hat{Z}_n)_{n\geq 0}$ is also a strongly aperiodic Harris chain with regeneration set $\Re \times \Delta$.

We show in Lemma A.6 of the Appendix that $E_x|E(e^{it\hat{X}_1}|\hat{M}_0, \hat{M}_1)| < 1$ for all $t \neq 0$ and $x \in \mathcal{S}$, so that $(\hat{M}_n, \hat{S}_n)_{n\geq 0}$ is also completely nonarithmetic. Its drift $\hat{\mu} = E_\xi \hat{X}_1$ obviously equals $\sum_{n\geq 1} 2^{-n} E_\xi S_n = 2E_\xi X_1 = 2\mu$.

Denote by $\hat{U}_\lambda$ the Markov renewal measure of $(\hat{M}_n, \hat{S}_n)_{n\geq 0}$ given initial distribution λ. The relation between $\hat{U}_\lambda$ and U_λ is stated in

Lemma 1. *For all measurable $g : \mathcal{S} \times I\!R \to [0,\infty)$,*

$$g * U_\lambda \;=\; \hat{g} * \hat{U}_\lambda, \quad \textit{where} \quad \hat{g}(x,y) \stackrel{\text{def}}{=} E_x\left(\sum_{k=0}^{\eta_1 - 1} g(M_k, y - S_k)\right). \tag{5.1}$$

Proof. The assertion follows directly by cyclic decomposition. Namely,

$$\begin{aligned} g * U_\lambda(t) &= \sum_{n\geq 0} E_\lambda \left(\sum_{k=\eta_n}^{\eta_{n+1}-1} g(M_k, t - S_k)\right) \\ &= \sum_{n\geq 0} \int_{\mathcal{S}\times(0,\infty)} E_x\left(\sum_{k=0}^{\eta_1-1} g(M_k, t - s - S_k)\right) P_\lambda(\hat{M}_n \in dx, \hat{S}_n \in ds) \\ &= \int_{\mathcal{S}\times(0,\infty)} \hat{g}(x, t-s) \sum_{n\geq 0} P_\lambda(\hat{M}_n \in dx, \hat{S}_n \in ds) \;=\; \hat{g} * \hat{U}_\lambda(t). \end{aligned}$$

We will show in Lemma A.4 of the Appendix that if g satisfies (3.1) and (3.2), then so does $\hat{g}$. Now, if we can prove Theorem 1, i.e. (3.3), for $(\hat{M}_n, \hat{S}_n)_{n\geq 0}$, it must hold true also for $(M_n, S_n)_{n\geq 0}$, because a simple computation involving Fubini's theorem yields

$$\frac{1}{2\mu}\int_{\mathcal{S}}\int_{I\!R} \hat{g}(x,t)\; \ell_0(dt)\; \xi(dx) \;=\; \frac{1}{\mu}\int_{\mathcal{S}}\int_{I\!R} g(x,t)\; \ell_0(dt)\; \xi(dx).$$

Second reduction by ladder epochs and regeneration. In view of the previous considerations we may now assume w.l.o.g. $(M_n, S_n)_{n\geq 0}$ to be a completely nonarithmetic MRW with driving chain $(M_n)_{n\geq 0}$ and level 1 excursion chain $(M_n, Z_n)_{n\geq 0}$ both being Harris recurrent and strongly aperiodic. Given the latter sequence with regeneration set $\Re \times \{\Delta\}$, let

$(\tau_n)_{n\ge 0}$ be the regular sequence of regeneration epochs for $(M_n, Z_n)_{n\ge 0}$ as well as $(M_n)_{n\ge 0}$ as constructed by Athreya and Ney in [5] (the standard coin-tossing procedure). Due to the strong aperiodicity, in addition to (R.1)–(R.4)

(R.5) $\quad P(\tau_1 - r = k, (M_n, Z_n)_{n\ge r} \in \cdot|\mathcal{F}_r) = \hat{P}_{M_r,Z_r}(\tau_1 = k, (M_n, Z_n)_{n\ge 0} \in \cdot)$

holds a.s. on $\{\tau_1 > r\}$ for all $r, k \ge 0$, where $\hat{P}_{x,z} \stackrel{\text{def}}{=} P(\cdot|M_0 = x, Z_0 = z)$. This extra property is of essential importance for the proof of (5.3) below in Lemma A.5 and the reason for including consideration of the sequence $(Z_n)_{n\ge 0}$ here. For a more detailed discussion we refer to the Appendix.

Put $\hat{X}_0 = X_0$, $\hat{X}_n = S_{\tau_n} - S_{\tau_{n-1}}$ and $\hat{M}_n = M_{\tau_n}$. Since $Z_{\tau_n} = \Delta$ for all $n \ge 1$, the $\hat{X}_n$, $n \ge 1$, are all larger than 1. Moreover, $(\hat{M}_n, \hat{S}_n)_{n\ge 0}$ forms a MRP whose driving chain is not only ergodic but even consisting of i.i.d. variables for $n \ge 1$ with common (stationary) distribution ζ. It has the same drift as $(M_n, S_n)_{n\ge 0}$ itself because $E_\zeta \hat{X}_1 = E_\zeta S_{\tau_1} = \mu$ by (2.7). Finally, it is again completely nonarithmetic. Namely, for $t \ne 0$ and $x \in \mathcal{S}$ we infer

$$E_x|E(e^{it\hat{X}_1}|\hat{M}_0, \hat{M}_1)| \;=\; E_x|E(e^{itX_1}|M_0, M_1)E(e^{it(S_{\tau_1}-S_1)}|M_1, M_{\tau_1})| \;<\; 1$$

from $E_x|E(e^{itX_1}|M_0, M_1)| < 1$.

As before, denote by $\hat{U}_\lambda$ its Markov renewal measure given initial distribution λ and put $\hat{N}(C) = \sum_{n\ge 0} \mathbf{1}_{\{(\hat{M}_n,\hat{S}_n)\in C\}}$ for $C \in \mathfrak{S} \otimes \mathfrak{B}$, hence $\hat{U}_\lambda(C) = E_\lambda \hat{N}(C)$. Now $\hat{X}_n > 1$ for all $n \ge 1$ obviously implies $\hat{N}(A \times [t, t+a]) \le 1$ for all $A \in \mathfrak{S}$, $t \in \mathbb{R}$ and $0 < a \le 1$, thus inferring the uniform integrability of $\{\hat{N}(A \times [t, t+a]); t \in \mathbb{R}\}$ as well as

$$\hat{U}_\lambda(A \times [t, t+a]) \;\le\; 1. \tag{5.2}$$

Since Lemma 1 remains valid when replacing $(\eta_n)_{n\ge 0}$ by $(\tau_n)_{n\ge 0}$, we have $g * U_\lambda = \hat{g} * \hat{U}_\lambda$ with $\hat{g}(x, y) \stackrel{\text{def}}{=} E_x(\sum_{k=0}^{\tau_1-1} g(M_k, y - S_k))$. We will show in Lemma A.5 of the Appendix that if g satisfies (3.1) and (3.2), then

$$\lim_{y\to\infty} \hat{g}(x, y) \to 0 \quad \text{for } \xi\text{-almost all } x \in \mathcal{S}. \tag{5.3}$$

In order to see that it suffices to prove Theorem 1 for $(\hat{M}_n, \hat{S}_n)_{n\ge 0}$, we further need

Lemma 2. *For each function g satisfying (3.1) and (3.2) with ζ instead of ξ,*

$$\int_{\mathcal{S}} \sup_{t\in\mathbb{R}} g * \hat{U}_r(t)\; \zeta(dr) \;<\; \infty. \tag{5.4}$$

Proof. Writing

$$\hat{U}_x(A \times B) \;=\; \delta_{(x,0)}(A \times B) \;+\; \int_A V_x^r(B)\; \zeta(dr), \quad A \in \mathfrak{S}, B \in \mathfrak{B}, \tag{5.5}$$

where $V_x^r = \sum_{n\ge 1} P_x(\hat{S}_n \in \cdot|\hat{M}_n = r)$, we obtain upon setting $I_n^\rho = [n\rho, (n+1)\rho)$ for $n \in \mathbb{Z}$ and $G(x) = \sup_{t\in\mathbb{R}} |g(x, t)|$

$$\begin{aligned} g * \hat{U}_x(t) &= g(x,t) + \int_{\mathcal{S}} \int_{(0,\infty)} g(r, t-s)\, V_x^r(ds)\, \zeta(dr) \\ &\le G(x) + \int_{\mathcal{S}} \sum_{n\in\mathbb{Z}} \left(\sup_{s\in I_n^\rho} g(r,s) \right) V_x^r(t - I_n^\rho)\, \zeta(dr) \\ &\le G(x) + \int_{\mathcal{S}} \sum_{n\in\mathbb{Z}} \sup_{s\in I_n^\rho} g(r,s)\, \zeta(dr) \end{aligned} \tag{5.6}$$

for each $t \in I\!R$ and $0 < \rho \le 1$, because $\hat{X}_n > 1$ for all $n \ge 1$ clearly gives $V_x^r([t, t+a]) \le 1$ for all $t, y \in I\!R$, $0 < a < 1$ and $x, r \in \mathcal{S}$ by the same argument yielding (5.2). We have already mentioned that validity of (3.2) for g (with ζ instead of ξ) forces $g(x,\cdot)$ to be bounded for ζ-almost all $x \in \mathcal{S}$ and furthermore (use (2.6) and (3.6))

$$\int_{\mathcal{S}} G(x)\, \zeta(dx) < \infty. \tag{5.7}$$

By combining this with (5.6) and (3.2) for g, we conclude

$$\int_{\mathcal{S}} \sup_{t\in I\!R} g * \hat{U}_r(t)\, \zeta(dr) \le \int_{\mathcal{S}} \left(G(r) + \sum_{n\in\mathbb{Z}} \sup_{s\in I_n^\rho} g(r,s) \right) \zeta(dr) < \infty$$

for all sufficiently small $\rho > 0$.

Put $\hat{Q}_x(r,\cdot) = P_x(\hat{S}_1 \in \cdot | \hat{M}_1 = r)$. Now we see from

$$g * U_x(t) = \hat{g} * \hat{U}_x(t) = \hat{g}(x,t) + \int_{\mathcal{S}} \int_{I\!R} \hat{g} * \hat{U}_r(t-s)\, \hat{Q}_x(r, ds)\, \zeta(dr) \tag{5.8}$$

that validity of Theorem 1 for the MRP $(\hat{M}_n, \hat{S}_n)_{n\ge0}$, when combined with (5.3), (5.4) and the dominated convergence theorem, easily yields the same for the MRW $(M_n, S_n)_{n\ge0}$ itself.

6. The Stationary Markov Delay Distribution

It is well-known in standard renewal theory and obtained by solving a renewal equation that for the renewal measure $U = \sum_{n\ge0} F_0 * F^{*(n)}$ of a delayed renewal process with delay distribution F_0 and inter-renewal distribution F with finite positive mean ν the identity $U = \nu^{-1}\ell_0^+$ holds iff F_0 equals the *stationary delay distribution* with Lebesgue density $\mu^{-1}(1 - F(t))\mathbf{1}_{(0,\infty)}(t)$. This forms the key for the various coupling proofs of Blackwell's renewal theorem, see [8], [11] and [17]. A weaker but still useful result can be shown for the case $\mu = \infty$, see [17].

Along the same lines, i.e. by solving a Markov renewal equation (see (6.2) below), we will now derive an analogue in the present setup which in view of (3.4) means to determine a *stationary Markov delay measure* ν^s on $\mathcal{S} \times I\!R$ such that

$$U_{\nu^s} = \xi \otimes \ell_0^+. \tag{6.1}$$

We see from Theorem 3 below that ν^s is finite (with total mass μ) iff μ is finite. The normalization of ν^s is then denoted by λ^s and called *stationary Markov delay distribution.*

Given (not necessarily stochastic) kernels $\boldsymbol{Q}$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2 : \mathcal{S} \times \mathfrak{S} \otimes \mathfrak{B} \to [0,\infty]$ and a measure λ on $\mathcal{S} \times I\!R$, define $\lambda * \boldsymbol{Q}$ on $\mathcal{S} \times I\!R$ by

$$\lambda * \boldsymbol{Q}(C) \;=\; \int_{\mathcal{S}\times I\!R} \boldsymbol{Q}(x, C-y)\; \lambda(dx \times dy), \quad C \in \mathfrak{S} \otimes \mathfrak{B},$$

and the convolution kernel $\boldsymbol{Q}_1 * \boldsymbol{Q}_2 : \mathcal{S} \times \mathfrak{S} \otimes \mathfrak{B} \to [0,\infty]$ by

$$(\boldsymbol{Q}_1 * \boldsymbol{Q}_2)(x, A \times B) \;=\; \int_{\mathcal{S}} \int_{I\!R} \boldsymbol{Q}_2(r, A \times B - s)\; \boldsymbol{Q}_1(x, dr \times ds)$$

for $A \in \mathfrak{S}$, $B \in \mathfrak{B}$ and $x \in \mathcal{S}$. $\boldsymbol{Q}^{*(n)}$ shall denote the n-fold convolution of $\boldsymbol{Q}$ with itself for $n \geq 1$ and $Q^{*(0)}(x,\cdot) \stackrel{\text{def}}{=} \delta_{(x,0)}$. Note that each measure on $\mathcal{S} \times I\!R$ may also be viewed a kernel from $\mathcal{S}$ to $\mathcal{S} \times I\!R$. We then clearly have $U_\lambda = \lambda * \sum_{n\geq 0} \boldsymbol{P}^{*(n)}$ and further

$$U_\lambda \;=\; \lambda * \sum_{n\geq 0} \boldsymbol{P}^{*(n)} \;=\; \lambda \;+\; \left(\lambda * \sum_{n\geq 0} \boldsymbol{P}^{*(n)}\right) * \boldsymbol{P} \;=\; \lambda \;+\; U_\lambda * \boldsymbol{P} \tag{6.2}$$

which is the renewal equation needed in the following.

THEOREM 3. *Given a MRP $(M_n, S_n)_{n\geq 0}$ with positive drift μ, there is a unique σ-finite measure ν^s satisfying (6.1), namely*

$$\nu^s(A \times B) \;=\; \int_B P_\xi(M_1 \in A, X_1 > y)\; \ell_0^+(dy), \quad A \in \mathfrak{S},\; B \in \mathfrak{B}. \tag{6.3}$$

It is finite iff $\mu < \infty$, in which case $\lambda^s = \nu^s/\mu$ defines the unique distribution on $\mathcal{S} \times [0,\infty)$ such that $U_{\lambda^s} = \mu^{-1}\xi \otimes \ell_0^+$. If ξ is finite and $\lambda_a^s \stackrel{\text{def}}{=} \mu_a^{-1}\nu^s(\cdot \cap (\mathcal{S} \times [0,a]))$ for $a > 0$, where $\mu_a \stackrel{\text{def}}{=} \nu^s(\mathcal{S} \times [0,a]) = E_\xi(X_1 \wedge a)$, then

$$U_{\lambda_a^s} \;\leq\; \mu_a^{-1}\xi \otimes \ell_0^+ \tag{6.4}$$

for all $a > 0$ with equality holding for the restrictions of both measures to $\mathcal{S} \times [0,a]$.

Note that Theorem 3 does neither require $\boldsymbol{P}$ to be nonarithmetic, nor the Harris recurrence of the driving chain $(M_n)_{n\geq 0}$ as long as it possesses a stationary measure ξ. We have stated the result in this general form because of its interest in its own right. Regarding the proof of Theorem 1, we only need Theorem 3 for $(\hat{M}_n, \hat{S}_n)_{n\geq 0}$ introduced in the previous section, for which the subsequent proof would be simpler. We note also that λ^s is nothing but the stationary distribution of the continuous-time Markov process $(Z(t), R(t))_{t\geq 0}$ in Corollary 2, providing positive X_n's. Jacod [9] has shown the Harris recurrence of $(Z(t), R(t))_{t\geq 0}$ and also given a different proof of (6.1).

Proof of Theorem 3. By solving for $\lambda = \nu^s$ equation (6.2) with U_λ replaced by $\xi\otimes\ell_0^+$, we obtain upon using $\xi(A)t = \int_{\mathcal{S}}\int_0^t \boldsymbol{P}(x, A\times[0,\infty))\,dy\,\xi(dx)$

$$\begin{aligned} \nu^s(A\times[0,t]) &= \xi(A)t - \int_{\mathcal{S}}\int_0^t \boldsymbol{P}(x, A\times[0,t-y])\ dy\ \xi(dx) \\ &= \int_{\mathcal{S}}\int_0^t \boldsymbol{P}(x, A\times(y,\infty))\ dy\ \xi(dx) \\ &= \int_0^t P_\xi(M_1\in A, X_1>y)\ dy \end{aligned}$$

for all $t\ge 0$ and $A\in\mathfrak{S}$ with $\xi(A)<\infty$. But U_{ν^s} also solves that equation and is *locally finite*, which means that $U_{\nu^s}(\mathcal{S}_n\times[0,t])<\infty$ for all $t\ge 0$ and events $\mathfrak{S}\ni\mathcal{S}_n\uparrow\mathcal{S}$ with $\xi(\mathcal{S}_n)<\infty$ for all $n\ge 1$. This is shown in Lemma A.3 in the Appendix, but trivially holds for $\hat{U}_{\hat{\nu}^s}$ associated with $(\hat{M}_n,\hat{S}_n)_{n\ge 0}$ of the previous section (second reduction), $\hat{\nu}^s$ having the obvious meaning. Hence $\Psi \stackrel{\text{def}}{=} U_{\nu^s} - \xi\otimes\ell_0^+$ constitutes a finite signed measure when restricted to $\mathcal{S}_n\times[0,t]$ for arbitrary t,n. For all $k,n\ge 1$ and $t\ge 0$, we obtain upon iterating (6.2)

$$\begin{aligned} |\Psi(\mathcal{S}_n\times[0,t])| &= |\Psi * \boldsymbol{P}^{*(k)}(\mathcal{S}_n\times[0,t])| \\ &\le \int_{\mathcal{S}\times[0,t]} \boldsymbol{P}^{*(k)}(x,\mathcal{S}_n\times[0,t-y])\ (U_{\nu^s}+\xi\otimes\ell_0^+)(dx\times dy), \end{aligned}$$

and the final expression converges to 0, as $k\to\infty$, because

$$U_{\nu^s} * \boldsymbol{P}^{*(k)}(\mathcal{S}_n\times[0,t]) \;=\; \sum_{j\ge k}\nu^s * \boldsymbol{P}^{*(j)}(\mathcal{S}_n\times[0,t]) \;\to 0$$

by dominated convergence (with majorant $U_{\nu^s}(\mathcal{S}_n\times[0,t])$), and

$$(\xi\otimes\ell_0^+) * \boldsymbol{P}^{*(k)}(\mathcal{S}_n\times[0,t]) \;=\; \int_0^t P_\xi(M_k\in\mathcal{S}_n, S_k\le y)\ dy \;\to\; 0$$

for the same reason (with majorant $P_\xi(M_k\in\mathcal{S}_n)=\xi(\mathcal{S}_n)$).

Now suppose $\xi(\mathcal{S})<\infty$, let $(M_n,S_{a,n})_{n\ge 0}$ be the MRP with increments $X_n\wedge a$, $n\ge 0$, for $a>0$ and $U^a_{x,y}$ its Markov renewal measure given $M_0=x, X_0^a=y$. Since $E_\xi(X_1\wedge a)=\mu_a$, one can easily verify that λ_a^s equals the stationary Markov delay distribution of $(M_n,S_{a,n})_{n\ge 0}$. But $U^a_{\lambda_a^s}$ clearly equals $U_{\lambda_a^s}$ on $\mathcal{S}\times[0,a]$ because of the nonnegative increments whence by the first part we obtain $U_{\lambda_a^s}=\mu_a^{-1}\xi\otimes\ell_0^+$ on $\mathcal{S}\times[0,a]$. The proof of (6.4) is completed by combining (6.1) with the obvious inequality

$$U_{\lambda_a^s}(A\times B) \;=\; \frac{1}{\mu_a}\int_{\mathcal{S}\times[0,a]} U_x(A\times(B-y))\ \nu^s(dx\times dy) \;\le\; \frac{1}{\mu_a}U_{\nu^s}(A\times B).$$

An extension of Theorem 3 to the two-sided case is easily obtained with the help of ladder variables (here $\sigma_n=\sigma_n(0)$) and cyclic decomposition. For its counterpart in the standard renewal setup see [8]. Denote by U_λ^+ the restriction of U_λ to subsets of $\mathcal{S}\times[0,\infty)$.

Corollary 3. *Given a MRW $(M_n,S_n)_{n\ge 0}$ with positive drift μ and Harris recurrent driving chain $(M_n)_{n\ge 0}$, let $(M_n^*,S_n^*)_{n\ge 0}$ be its embedded MRP at the strictly ascending ladder epochs $(\sigma_n)_{n\ge 0}$ with stationary Markov delay measure ν^s, i.e.*

$$\nu^s(A\times B) \;=\; \int_B P_{\xi^*}(M_1^*\in A, X_1^*>y)\,\ell_0^+(dy),\quad A\in\mathfrak{S},\; B\in\mathfrak{B}, \tag{6.5}$$

ξ^* the stationary measure of $(M_n^*)_{n\ge 0}$. Then

$$U_{\nu^s}(A\times B) \;=\; E_{\xi^*}\left(\sum_{k=0}^{\sigma_1-1}\mathbf{1}_{\{M_k\in A\}}\ell_0^+(B-S_k)\right) \tag{6.6}$$

for all $A\in\mathfrak{S}, B\in\mathfrak{B}$, in particular

$$U_{\nu^s}^+ \;=\; \xi\otimes\ell_0^+, \tag{6.7}$$

providing $\xi=E_{\xi^*}(\sum_{k=0}^{\sigma_1-1}\mathbf{1}_{\{M_k\in\cdot\}})$.

Proof. It can easily be verified that ξ as defined above defines a stationary measure for $(M_n)_{n\ge 0}$. Given (6.6), we then immediately infer (6.7), because $S_k\le 0$ and the translation invariance of the Lebesgue measure yield $\ell_0^+(B-S_k)=\ell_0^+(B)$ for all measurable $B\subset[0,\infty)$ and all $0\le k<\sigma_1$.

So it remains to prove (6.6). Let U_λ^* be the Markov renewal measure of $(M_n^*,S_n^*)_{n\ge 0}$ under P_λ. Notice that $(M_0^*,S_0^*)=(M_0,S_0)$ under P_{ν^s}. Now a cyclic decomposition with respect to $(\sigma_n)_{n\ge 0}$ combined with $U_{\nu^s}^*=\xi^*\otimes\ell_0^+$ (Theorem 3) gives

$$\begin{aligned} U_{\nu^s}(A\times B) \;&=\; \int_{\mathcal{S}}\int_{[0,\infty)} E_x\left(\sum_{k=0}^{\sigma_1-1}\mathbf{1}_{\{M_k\in A, y+S_k\in B\}}\right)\,\ell_0(dy)\;\xi^*(dx)\\ &=\; \int_{[0,\infty)} E_{\xi^*}\left(\sum_{k=0}^{\sigma_1-1}\mathbf{1}_{\{M_k\in A\}}\mathbf{1}_{B-S_k}(y)\right)\,\ell_0(dy)\end{aligned}$$

from which (6.6) follows by another appeal to Fubini's theorem.

7. The Coupling (Proof of Theorem 1)

Proof of Theorem 1. We are now ready to give the proof of Theorem 1. After the considerations in Section 4 it suffices to show (3.3) for MRP's $(M_n,S_n)_{n\ge 0}$ which additionally have the following properties:

(1) $M_n,\ n\ge 1$ are i.i.d. with common distribution ζ under each P_λ;

(2) $X_n>1$ for all $n\ge 0$;

(3) $E_x|E(e^{itX_1}|M_0,M_1)|<1$ for all $t\ne 0$ and $x\in\mathcal{S}$.

As a consequence of (3) we infer upon setting $F\overset{\text{def}}{=}P_\zeta(X_1\in\cdot)$ and $F^-(B)=F(-B)$

(4) $F*F^-$, the symmetrization of F, is nonarithmetic.

Indeed, the Fourier transform ψ_{F*F^-} of $F*F^-$ satisfies

$$\psi_{F*F^-}(t) \;=\; |E_\zeta e^{itX_1}|^2 \;\le\; \int_{\mathcal{S}} |E(e^{itX_1}|M_0,M_1)|^2 \, dP_\zeta \;<\; 1.$$

We now use a coupling argument which combines the technique in [1] with a so-called Ornstein coupling. Recall that $\xi=\zeta$ and thus $\mu=E_\zeta X_1$ in the present context. By (4) we can choose a sufficiently large c such that $G \stackrel{\text{def}}{=} F*F^-(\cdot\cap[0,c])/F*F^-([0,c])$ is also nonarithmetic. Let λ^s,λ_a^s be defined as in Theorem 3. Because of the simple structure of $(M_n,S_n)_{n\ge 0}$ we do not supply details of its construction along with a further MRP $(M_n^s,S_n^s)_{n\ge 0}$ such that for every initial distribution λ on $\mathcal{S}\times I\!R$

(C.1) $P_\lambda((M_0,X_0,M_0^s,X_0^s)\in\cdot) = \lambda\otimes\lambda^s$, if $\mu<\infty$, and $=\lambda\otimes\lambda_a^s$ for sufficiently large $a>1$, if $\mu=\infty$;

(C.2) $M_n=M_n^s$ for all $n\ge 1$;

(C.3) $X_{2n-1}=X_{2n-1}^s$ and $P_\lambda(X_{2n}-X_{2n}^s\in\cdot)=G$ for all $n\ge 1$.

It follows that $S_n-S_n^s=\sum_{0\le j\le n/2}(X_{2j}-X_{2j}^s)$ which is a delayed symmetric random walk with *bounded* nonarithmetic increments and thus topologically recurrent on $I\!R$ by the famous Chung-Fuchs-Ornstein Theorem. Consequently, the ε-coupling time

$$T(\varepsilon) \;=\; \inf\{n\ge 1: |S_n-S_n^s|\le\varepsilon\}$$

is a.s. finite for each $\varepsilon>0$. It should be noticed that the driving chains are coupled anyway for $n\ge 1$. Denote by

$$(\hat M_n,\hat S_n) \;=\; (M_n,S_n)\mathbf{1}_{\{T(\varepsilon)\ge n\}} + (M_n^s, S_n^s+(S_{T(\varepsilon)}-S_{T(\varepsilon)}^s))\mathbf{1}_{\{T(\varepsilon)<n\}}, \quad n\ge 0$$

the associated ε-coupling process which, under each P_λ, defines a copy of $(M_n,S_n)_{n\ge 0}$ and differs by at most ε in the second component from its "stationary" alternative $(M_n^s,S_n^s)_{n\ge 0}$ after $T(\varepsilon)$.

Now choose any $\varepsilon>0$ and let U^s be the Markov renewal measure of $(M_n^s,S_n^s)_{n\ge 0}$, which by construction is the same under each P_λ. Let g be a function satisfying (3.1) and (3.2) (with $\xi=\zeta$), w.l.o.g. $g\ge 0$. For $\rho>0$, define $g_\rho(x,y)=\sum_{n\in\mathbb{Z}}\left(\inf_{(n-1)\rho\le s<(n+1)\rho} g(x,s)\right)\mathbf{1}_{I_n^\rho}(y)$ and $g^\rho(x,y)=\sum_{n\in\mathbb{Z}}\left(\sup_{(n-1)\rho\le s<(n+1)\rho} g(x,s)\right)\mathbf{1}_{I_n^\rho}(y)$. One can easily verify that (3.1) and (3.2) imply

$$\lim_{\rho\downarrow 0}\int_{\mathcal{S}}\int_{-\infty}^{\infty}(g^\rho(x,y)-g_\rho(x,y))\;dy\;\zeta(dx) \;=\; 0, \tag{7.1}$$

which will be used below.

Suppose first $\mu<\infty$. Writing $T=T(\varepsilon)$, a straightforward estimation gives

$$\begin{aligned}|g*U_x(t)-\mu^{-1}\textstyle\int_{\mathcal{S}}\int_{-\infty}^{\infty} g(x,y)\,dy\,\zeta(dx)| \;&=\; |g*U_x(t)-g*U^s(t)|\\ &\le\; E_x\left(\sum_{n=0}^{T-1} g(M_n,t-S_n)\right) \;-\; E_x\left(\sum_{n=0}^{T-1} g(M_n^s,t-S_n^s)\right)\end{aligned}$$

$$+ \ E_x \left(\sum_{n \geq T} \Big(g^\varepsilon(M_n^s, t - S_n^s) - g_\varepsilon(M_n^s, t - S_n^s) \Big) \right). \tag{7.2}$$

The final expression of (7.2) is evidently bounded by

$$E \left(\sum_{n \geq 0} \Big(g^\varepsilon(M_n^s, t - S_n^s) - g_\varepsilon(M_n^s, t - S_n^s) \Big) \right) = \frac{1}{\mu} \int_{\mathcal{S}} \int_{-\infty}^{\infty} (g^\varepsilon(x,y) - g_\varepsilon(x,y)) \, dy \, \zeta(dx)$$

which, by (7.1), tends to 0 as $\varepsilon \to 0$.

The remaining two expressions on the right side of inequality (7.2) can be further estimated in the same manner, whence we only consider the first one.

$$\begin{aligned} E_x \left(\sum_{n=0}^{T-1} g(M_n, t - S_n) \right) &= \sum_{n \geq 0} \int_{\{T > n\}} g(M_n, t - S_n) \, dP_x \\ &\leq E_x \left(\sum_{n=0}^{k-1} g(M_n, t - S_n) \right) + \int_{\{T > k\}} \sup_{s \in \mathbb{R}} g * U_{M_k}(s) \, dP_x \stackrel{\text{def}}{=} I_1 + I_2, \end{aligned}$$

where the strong Markov property has been used. For each $k \geq 1$, rewrite I_2 as

$$\int_{\mathcal{S}} \sup_{s \in \mathbb{R}} g * U_r(s) \int_{\mathbb{R}} P(T > k | M_k = r, S_k - S_k^s = y) P(S_k - S_k^s \in dy | M_k = r) \, \zeta(dr)$$

and recall from Lemma 2 that $\sup_{s \in \mathbb{R}} g * U_r(s)$ is ζ-integrable. Hence I_2 converges to 0, as $k \to \infty$, by the dominated convergence theorem. By (3.5), for all $k \geq 1$ and ζ-almost all $x \in \mathcal{S}$, the integrand of I_1 converges to 0 P_x-a.s., as $t \to \infty$. Moreover, recalling $G(x) = \sup_{y \in \mathbb{R}} g(x,y)$, we have $\sum_{n=0}^{k-1} g(M_n, t - S_n) \leq g(x,t) + \sum_{n=1}^{k-1} G(M_n)$ and

$$E_x \left(\sum_{n=0}^{k-1} g(M_n, t - S_n) \right) \leq g(x,t) + (k-1) \int_{\mathcal{S}} G(r) \, \zeta(dr) < \infty,$$

where $P_x(M_n \in \cdot) = \zeta$ for all $n \geq 1$ and (5.7) have been used. Hence, by another appeal to the dominated convergence theorem, we obtain $I_1 = E_x\left(\sum_{n=0}^{k-1} g(M_n, t - S_n)\right) \to 0$, as $t \to \infty$, for all $k \geq 1$ and ζ-almost all $x \in \mathcal{S}$.

Now the right-hand side of inequality (7.2) can be made arbitrarily small by first choosing $\varepsilon > 0$ sufficiently small, then fixing $k \geq 1$ sufficiently large and by finally letting t tend to ∞. This proves (3.3).

If $\mu = \infty$, the assertion follows in a similar manner when using (6.4) of Theorem 3 and the fact that $\mu_a = E_\zeta(X_1 \wedge a) \to \infty$ as $a \to \infty$. We omit further details.

8. Appendix

In the following we collect a number of basic lemmata, some of which have already appeared in the literature, possibly under more restrictive assumptions. It is always assumed that $(M_n,S_n)_{n\ge 0}$ forms a MRW with Harris recurrent driving chain $(M_n)_{n\ge 0}$ and positive drift $\mu=E_\xi X_1$, ξ the essentially unique stationary measure of $(M_n)_{n\ge 0}$. We further keep the notation of the previous sections.

LOCAL FINITENESS OF THE U_x.

We start by giving some local finiteness properties of U_x for $x\in\mathcal{S}$, which addresses the behavior of $U_x(A\times[t,t+a])$ as a function of x and t. The next three lemmata may be found in [9] for the case of nonnegative X_n's.

LEMMA A.1. *$U_x(A\times C)<\infty$ for all $A\in\mathfrak{S}$ with $\xi(A)<\infty$, all compact $C\subset I\!R$ and ξ-almost all $x\in\mathcal{S}$.*

PROOF. We follow once more the two-step reduction described in Section 5.

1ST REDUCTION. If $\hat U_\lambda$ denotes the Markov renewal measure under P_λ of $(\hat M_n,\hat S_n)_{n\ge 0}$ obtained by geometric sampling (see Section 5), we infer from Lemma 1

$$U_\lambda(A\times B)\;=\;\mathbf{1}_{A\times B}*U_\lambda(0)\;=\;\widehat{\mathbf{1}_{A\times B}}*\hat U_\lambda(0)\;\le\;E\eta_1\,\hat U_\lambda(A\times B)\;=\;2\hat U_\lambda(A\times B)$$

for all $A\in\mathfrak{S},B\in\mathfrak{B}$. It is therefore enough to prove Lemma A.1 for the case of strongly aperiodic $(M_n)_{n\ge 0}$ and $(M_n,Z_n)_{n\ge 0}$, the level 1 excursion chain.

2ND REDUCTION. Let $(\tau_n)_{n\ge 0}$ be the sequence of regeneration epochs for both chains used for the second reduction in Section 5 and keep all notation from there, i.e. $\hat U_\lambda$ denotes now the Markov renewal measure under P_λ of $(\hat M_n,\hat S_n)_{n\ge 0}=(M_{\tau_n},S_{\tau_n})_{n\ge 0}$.

It clearly suffices to consider $U_x(A\times[-t,t])=g*U_x(t)$, where $g\stackrel{\rm def}{=}\mathbf{1}_{A\times[0,2t]}$. By (5.8), Lemma 1 (with τ_1 instead of η_1) and Lemma 2 in Section 5,

$$\begin{aligned}U_x(A\times[-t,t])\;&=\;\hat g(x,t)\;+\;\int_{\mathcal{S}}\int_{I\!R}\hat g*\hat U_r(t-s)\;\hat Q_x(r,ds)\;\zeta(dr)\\ &\le\;\hat g(x,t)\;+\;\int_{\mathcal{S}}\sup_{s\in I\!R}\hat g*\hat U_r(s)\;\zeta(dr)\;<\;\infty\end{aligned}$$

for ξ-almost all $x\in\mathcal{S}$, because $\hat g$ is finite for these x (Lemma A.5) and satisfies (3.1) and (3.2) with ζ instead of ξ (Lemma A.4).

LEMMA A.2. *For all $A\in\mathfrak{S}$, $x\in\mathcal{S}$, $t\in I\!R$ and $a>0$*

$$U_x(A\times[t,t+a])\;\le\;\sup_{r\in A}U_r(A\times[-a,a]).\tag{8.1}$$

PROOF. We obtain upon setting $\tau=\inf\{n:(M_n,S_n)\in A\times[t,t+a]\}$ and by using the strong Markov property

$$U_x(A\times[t,t+a])\;=\;\int_{A\times[t,t+a]}U_r(A\times[t-s,t-s+a])\;P_x(\tau<\infty,(M_\tau,S_\tau)\in dr\times ds)$$

from which (8.1) directly follows because $[t-s, t-s+a] \subset [-a, a]$ for all $s \in [t, t+a]$.

Following Jacod [9] we call a subset $A \in \mathcal{S}$ *of bounded potential* if there exists some $a > 0$ such that $\sup_{x \in A} U_x(A \times [-a, a]) < \infty$.

LEMMA A.3. *There exists an increasing sequence $(\mathcal{S}_n)_{n \geq 1}$ of events of bounded potential such that $\xi(\mathcal{S}_n) < \infty$ for each $n \geq 1$ and $P_x(\cap_{n \geq 1} \mathcal{S}_n^c) = 0$ for ξ-almost all $x \in \mathcal{S}$.*

PROOF. Let $(A_n)_{n \geq 1}$ be a sequence of ξ-finite sets in $\mathfrak{S}$ which increases to $\mathcal{S}$. Define

$$B_n = \{x \in A_n : U_x(A_n \times [-1, 1]) \leq c_n\}$$

where, by Lemma A.1, c_n can be chosen so large that $\xi(A_n - B_n) \leq \frac{1}{n}$. It follows by Lemma A.2

$$U_x(B_n \times [-1, 1]) \leq \sup_{r \in B_n} U_r(A_n \times [-1, 1]) \leq c_n,$$

for all $x \in B_n$, so that B_n is of bounded potential. The proof is now complete because $\mathcal{S}_n \stackrel{\text{def}}{=} \cup_{k=1}^n B_k$ is also of bounded potential, has finite ξ-measure and increases to $\mathcal{S}$ for A_n does so.

CONSEQUENCES OF (3.1) AND (3.2)

Let $(\tau_n)_{n \geq 0}$ be either a discrete renewal process independent of $(M_n, S_n)_{n \geq 0}$ under each P_λ, or a regular sequence of regeneration epochs for $(M_n)_{n \geq 0}$. Let further $\zeta = \xi / E\tau_1$ in the first case and $\zeta = P(M_{\tau_1} \in \cdot)$ in the second. Then (2.6) holds in either of these.

LEMMA A.4. *Given a measurable function $g : \mathcal{S} \times I\!R \to I\!R$ satisfying (3.1) and (3.2), the same holds true for $\hat{g}(x, y) = E_x(\sum_{k=0}^{\tau_1 - 1} g(M_k, y - S_k))$ with ξ in (3.2) replaced by ζ.*

PROOF. Let $N = \{x \in \mathfrak{S} : g(x, \cdot)$ is discontinuous on a set of positive Lebesgue measure$\}$. By (3.1), for ξ-almost all x we have $P_x(M_k \in N) = 0$ for all $k \geq 0$. For arbitrary $a, b \in I\!R, a < b$, we have

$$\int_a^b |\hat{g}(x, y+z) - \hat{g}(x, y)| \, \ell_0(dy) \leq E_x \left(\sum_{k=0}^{\tau_1 - 1} \int_a^b |g(M_k, y+z-S_k) - g(M_k, y-S_k)| \, \ell_0(dy) \right).$$

Now the right-hand expression converges to 0, as $z \to 0$, for all $x \in N^c$ because:

(1) the sum in parentheses is bounded by $2(b-a) \sum_{k=0}^{\tau_1 - 1} G(M_k)$, $G(x) = \sup_{t \in I\!R} |g(x, t)|$,
(2) (3.6), resp. (5.7), give $E_\zeta(\sum_{k=0}^{\tau_1 - 1} G(M_k)) < \infty$ and thus $E_x(\sum_{k=0}^{\tau_1 - 1} G(M_k)) < \infty$ for ζ-almost all $x \in \mathcal{S}$,
(3) Under P_x, $x \in N^c$, $g(M_k, \cdot - S_k)$ is ℓ_0-a.e. continuous for all $k \geq 0$.

We thus infer ℓ_0-a.e. continuity of $\hat{g}(x, \cdot)$ on $[a, b]$ for all $x \in N^c$ and thereby (3.1) for $\hat{g}$ since a, b were arbitrarily chosen.

Turning to (3.2), we obtain

$$\begin{aligned}
&\int_{\mathcal{S}}\sum_{n\in\mathbb{Z}}\sup_{n\rho\le y<(n+1)\rho}|\hat g(x,y)|\ \zeta(dx)\\
&\qquad\le\ \int_{\mathcal{S}}\sum_{n\in\mathbb{Z}}E_x\left(\sum_{k=0}^{\tau_1-1}\sup_{n\rho\le y<(n+1)\rho}|g(M_k,y-S_k)|\right)\ \zeta(dx)\\
&\qquad=\ \sum_{k\ge 0}\int_{\{\tau_1>k\}}\sum_{n\in\mathbb{Z}}\sup_{n\rho\le y<(n+1)\rho}|g(M_k,y-S_k)|\ dP_\zeta\\
&\qquad\le\ 2\sum_{k\ge 0}\int_{\{\tau_1>k\}}\sum_{n\in\mathbb{Z}}\sup_{2n\rho\le y<2(n+1)\rho}|g(M_k,y)|\ dP_\zeta\\
&\qquad=\ 2\,E_\zeta\left(\sum_{k=0}^{\tau_1-1}\sum_{n\in\mathbb{Z}}\sup_{2n\rho\le y<2(n+1)\rho}|g(M_k,y)|\right)\\
&\qquad=\ 2\int_{\mathcal{S}}\sum_{n\in\mathbb{Z}}\sup_{2n\rho\le y<2(n+1)\rho}|g(x,y)|\ \xi(dx)\ <\ \infty
\end{aligned}$$

for sufficiently small $\rho>0$, where the final equality follows from (2.6).

LEMMA A.5. *For ξ-almost all $x\in\mathcal{S}$, $\hat g(x,t)<\infty$ for all $t\in\mathbb{R}$ and $\lim_{t\to\infty}\hat g(x,t)=0$.*

PROOF. W.l.o.g. let $(M_n)_{n\ge0}$ as well as $(M_n,Z_n)_{n\ge0}$ be strongly aperiodic Harris chains and $(\tau_n)_{n\ge0}$ be the regular sequence of regeneration epochs introduced in Section 5 (2nd reduction) which, in addition to (R.1)–(R.4), possesses property (R.5). Recall $\hat P_{x,z}=P(\cdot|M_0=x,Z_0=z)$ from there. Put $\hat\zeta\overset{\text{def}}{=}P((M_{\tau_1},Z_{\tau_1})\in\cdot)$ and $\hat\xi\overset{\text{def}}{=}\hat E_\zeta(\sum_{k=0}^{\tau_1-1}\mathbf{1}_{\{(M_k,Z_k)\in\cdot\}})$, the stationary measure of $(M_n,Z_n)_{n\ge0}$ its first marginal being ξ. By means of (R.5), we now infer from Proposition 4.1 in [7] that

$$\lim_{t\to\infty}\hat h(x,z,t)\ =\ 0,\quad \hat h(x,z,t)\ \overset{\text{def}}{=}\ \hat E_{x,z}\left(\sum_{k=0}^{\tau_1-1}g(M_k,t-S_k)\right)$$

for $\hat\xi$-almost all $(x,z)\in\mathcal{S}\times((-\infty,1]\cup\{\Delta\})$. Their arguments further show $\hat h(x,z,t)<\infty$ for these (x,z) and all $t\in\mathbb{R}$. But $Z_0=\mathbf{\Delta}\mathbf{1}_{\{S_0>1\}}+S_0\mathbf{1}_{\{S_0\le1\}}$ implies

$$\hat h(x,z,t)\ =\ \hat g(x,t)\mathbf{1}_{\{S_0>1\}}+\hat g(x,t-z)\mathbf{1}_{\{S_0\le1\}}$$

which together with $\xi=\hat\xi(\cdot\times((-\infty,1]\cup\{\Delta\}))$ yields the assertions of the lemma.

The innocent looking Lemma A.5 is a main crux in the proof of Theorem 1 because it does not generally hold for any regular sequence of regeneration epochs $(\tau_n)_{n\ge0}$ for $(M_n)_{n\ge0}$. The argument given in [7] works only in the strongly aperiodic case for the very particular sequences based on Athreya and Ney's coin-tossing procedure which then ensures the extra property (R.5). We have tried hard to come up with an alternative proof under more general conditions but have failed. As a consequence, the reduction by geometric sampling and an explicit consideration of the level 1 excursion chain had to be made. An improvement on this technical point could lead to a substantially shorter proof of Theorem 1.

Lattice-type

We still have to verify that geometric sampling of a nonarithmetic MRW leads to a completely nonarithmetic one. This is provided by

Lemma A.6. *Given a nonarithmetic MRW* $(M_n, S_n)_{n\geq 0}$ *with Harris recurrent driving chain* $(M_n)_{n\geq 0}$ *and an independent (under each* P_λ*) geometric*$(1/2)$ *variable* η

$$E_x|E(e^{itS_\eta}|M_0, M_\eta)| < 1$$

for all $t \neq 0$ *and* $x \in \mathcal{S}$.

Proof. Denote by ξ the invariant measure of $(M_n)_{n\geq 0}$ and note that

$$E(e^{itS_\eta}|M_0, M_\eta) \;=\; E\left(\sum_{n\geq 1} \mathbf{1}_{\{\eta=n\}} E(e^{itS_n}|M_0, M_n)\middle| M_0, M_\eta\right) \quad P_x\text{-a.s.} \tag{8.2}$$

for each $x \in \mathcal{S}$. Suppose that $E_z|E(e^{itS_\eta}|M_0, M_\eta)| = 1$ for some $t \neq 0$, w.l.o.g. $t = 2\pi$, and some $z \in \mathcal{S}$. Hence

$$E(e^{2\pi i S_\eta}|M_0, M_\eta) = e^{2\pi i \vartheta(M_0, M_\eta)} \quad P_z\text{-a.s.}$$

for suitable $\vartheta : \mathcal{S}^2 \to [0, 1)$ which together with (8.2) gives

$$E(e^{2\pi i S_n}|M_0, M_n) = e^{2\pi i \vartheta(M_0, M_n)} \quad P_z\text{-a.s.}$$

for all $n \geq 1$. Put $f(x, y) \overset{\text{def}}{=} E(e^{2\pi i X_1}|M_0 = x, M_1 = y)$ and use

$$1 \;=\; E_z(e^{2\pi i (S_n - \vartheta(M_0, M_n))} \;=\; E_z\left(e^{-2\pi i \vartheta(M_0, M_n)} \prod_{k=1}^{n} f(M_{k-1}, M_k)\right) \tag{8.3}$$

to infer the existence of a function $\theta : \mathcal{S}^2 \to [0, 1)$ such that

$$f(M_{n-1}, M_n) = e^{2\pi i \theta(M_{n-1}, M_n)} \quad P_z\text{-a.s.}$$

and therefore $f(M_0, M_1) = e^{2\pi i \theta(M_0, M_1)}$ P_ξ-a.s. because $P_\xi^{(M_0, M_1)}$ and $\sum_{n\geq 1} 2^{-n} P_x^{(M_{n-1}, M_n)}$ are equivalent measures for all $x \in \mathcal{S}$ (ξ-irreducibility). (8.3) then also yields

$$\vartheta(z, M_n) \;\equiv_{\mathbb{Z}}\; \sum_{k=1}^{n} \theta(M_{k-1}, M_k) \quad P_z\text{-a.s.},$$

where $\equiv_{\mathbb{Z}}$ means equivalence modulo integers, and thus

$$\vartheta(z, M_{n+1}) - \vartheta(z, M_n) \;\equiv_{\mathbb{Z}}\; \theta(M_n, M_{n+1}) \quad P_z\text{-a.s.} \tag{8.4}$$

for all $n \geq 1$. Now, by replacing n with $\tau + k$ where τ is any regeneration time such that $(M_{\tau+n})_{n\geq 0}$ is independent of M_0 under P_z, we obtain for every $k \geq 0$

$$\begin{aligned} 1 &= E_z e^{2\pi i(\vartheta(M_0,M_{\tau+k+1})-\vartheta(M_0,M_{\tau+k})-\theta(M_{\tau+k},M_{\tau+k+1}))} \\ &= \int_{\mathcal{S}} e^{2\pi i(\vartheta(z,y)-\vartheta(z,x)-\theta(x,y))}\ P_\zeta^{(M_k,M_{k+1})}(dx,dy), \end{aligned}$$

$\zeta \stackrel{\text{def}}{=} P(M_\tau \in \cdot)$, and thereby

$$\theta(x,y) \ \equiv_{\mathbb{Z}} \ \hat{\vartheta}(x) - \hat{\vartheta}(y) \quad \xi'\text{-a.s.} \tag{8.5}$$

where $\hat{\vartheta}(x) \stackrel{\text{def}}{=} (1-\vartheta(z,x))\mathbf{1}_{\{\vartheta(z,\cdot)>0\}}(x) \in [0,1)$ and $\xi' \stackrel{\text{def}}{=} \sum_{k\geq 0} P_\zeta^{(M_k,M_{k+1})}$. But since ξ' and $\xi \otimes I\!\!P = P_\xi^{(M_0,M_1)}$ are equivalent measures, (8.5) is equivalent to $f(M_0,M_1) = e^{2\pi i(\hat{\vartheta}(M_0)-\hat{\vartheta}(M_1))}$ P_ξ-a.s. which contradicts $(M_n,S_n)_{n\geq 0}$ be nonarithmetic.

Let us finally note that Lemma A.6 remains valid in case of a d-arithmetic $(M_n,S_n)_{n\geq 0}$ with shift function γ in the sense that $E_x|E(e^{2\pi it(S_\eta-\gamma(M_0)+\gamma(M_\eta)/d)}|M_0,M_\eta)| < 1$ for all $0 < |t| < 1$ and $x \in \mathcal{S}$.

References

[1] ALSMEYER, G. (1994). On the Markov renewal theorem. *Stoch. Proc. Appl.* **50**, 37-56.

[2] ALSMEYER, G. (1996). The ladder variables of a Markov random walk. *Technical Report, University of Münster.*

[3] ALSMEYER, G. (1996). Rates of convergence in the Markov renewal theorem. *In preparation.*

[4] ASMUSSEN, S. (1987). *Applied Probability and Queues.* Wiley, New York.

[5] ATHREYA, K.B. and NEY, P. (1978a). A new approach to the limit theory of recurrent Markov chains. *Trans. Amer. Math. Soc.* **245**, 493-501.

[6] ATHREYA, K.B. and NEY, P. (1978b). Limit theorems for semi-Markov processes. *Bull. Austral. Math. Soc.* **19** 283-294.

[7] ATHREYA, K.B., McDONALD, D. and NEY, P. (1978a). Limit theorems for semi-Markov processes and renewal theory for Markov chains. *Ann. Probab.* **6** 788-797.

[8] ATHREYA, K.B., McDONALD, D. and NEY, P. (1978b). Coupling and the renewal theorem. *Amer. Math. Monthly* **85**, 809-814.

[9] JACOD, J. (1971). Théorème de renouvellement et classification pour les chaines semi-Markoviennes. *Ann. Inst. H. Poincaré* **B 7** 355-387.

[10] KESTEN, H. (1974). Renewal theory for functionals of a Markov chain with general state space. *Ann. Probab.* **2** 355-387.

[11] LINDVALL, T. (1977). A probabilistic proof of Blackwell's renewal theorem. *Ann. Probab.* **5**, 482-485.

[12] NUMMELIN, E.(1978). A splitting-technique for Harris recurrent Markov chains. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **43**, 309-318.

[13] OREY, S. (1961). Change of time scale for Markov processes. *Trans. Amer. Math. Soc.* **99**, 384-390.

[14] SHURENKOV, V.M. (1984). On the theory of Markov renewal. *Theory Probab. Appl.* **29**, 247-265.

[15] SHURENKOV, V.M. (1992). Markov renewal theory and its applications to Markov ergodic processes. *Tech. Rept. No. 24, Dept. Mathematics, Univ. Göteborg.*

[16] SMITH, W.L. (1955). Regenerative stochastic processes. *Proc. Roy. Soc. London* **A 232**, 6-31.

[17] THORISSON, H. (1987). A complete proof of Blackwell's renewal theorem. *Stoch. Proc. Appl.* **26**, 87-97.