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On the Markov Renewal Theorem

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Corrected Version of the Article in
Stoch. Proc. Appl. **50**, 37-56 (1994)

1998

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On the Markov Renewal Theorem

(Corrected version)

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Let $(\mathcal{S}, \mathfrak{S})$ be a measurable space with countably generated σ-field $\mathfrak{S}$ and $(M_n, X_n)_{n\geq 0}$ a Markov chain with state space $\mathcal{S} \times I\!R$ and transition kernel $I\!P : \mathcal{S} \times (\mathfrak{S} \otimes \mathfrak{B}) \to [0,1]$. Then $(M_n, S_n)_{n\geq 0}$, where $S_n = X_0 + \ldots + X_n$ for $n \geq 0$, is called the associated Markov random walk. Markov renewal theory deals with the asymptotic behavior of suitable functionals of $(M_n, S_n)_{n\geq 0}$ like the Markov renewal measure $\sum_{n\geq 0} P((M_n, S_n) \in A \times (t + B))$ as $t \to \infty$ where $A \in \mathfrak{S}$ and B denotes a Borel subset of $I\!R$. It is shown that the Markov renewal theorem as well as a related ergodic theorem for semi-Markov processes hold true if only Harris recurrence of $(M_n)_{n\geq 0}$ is assumed. This was proved by purely analytical methods by Shurenkov [16] in the one-sided case where $I\!P(x, [0, \infty)) = 1$ for all $x \in \mathcal{S}$. Our proof uses probabilistic arguments, notably the construction of regeneration epochs for $(M_n)_{n\geq 0}$ such that $(M_n, X_n)_{n\geq 0}$ is at least nearly regenerative and an extension of Blackwell's renewal theorem to certain random walks with stationary, 1-dependent increments.

AMS 1991 subject classifications. Primary 60K15 Secondary 60K05, 60J05, 60J15.

Keywords and phrases. Markov renewal theory, Markov random walk, semi-Markov process, Harris recurrence, regeneration epochs, Blackwell's renewal theorem, random walks with stationary, 1-dependent increments.

1. Introduction

Let $(\mathcal{S},\mathfrak{S})$ be a measurable space with countably generated σ-field $\mathfrak{S}$, $\mathfrak{B}$ the Borel σ-field on $I\!R$ and $I\!P:\mathcal{S}\times(\mathfrak{S}\otimes\mathfrak{B})\to[0,1]$ a transition kernel. Let further $(M_n,X_n)_{n\ge0}$ be an associated Markov chain, defined on a probability space $(\Omega,\mathcal{A},P)$, with state space $\mathcal{S}\times I\!R$, i.e.

$$P(M_{n+1}\in A,X_{n+1}\in B|M_n,X_n) \;=\; I\!P(M_n,A\times B) \quad \text{a.s.} \tag{1.1}$$

for all $n\ge0$ and $A\in\mathfrak{S},B\in\mathfrak{B}$. Thus (M_{n+1},X_{n+1}) depends on the past only through M_n. It is easily seen that $(M_n)_{n\ge0}$ forms a Markov chain with state space $\mathcal{S}$ and transition kernel $I\!P^*(x,A)\stackrel{\text{def}}{=}I\!P(x,A\times I\!R)$. Given $(M_j)_{j\ge0}$, the $X_n,n\ge0$ are conditionally independent with

$$P(X_n\in B|(M_j)_{j\ge0}) \;=\; Q(M_{n-1},M_n,B) \quad \text{a.s.} \tag{1.2}$$

for all $n\ge1,B\in\mathfrak{B}$ and a kernel $Q:\mathcal{S}^2\times\mathfrak{B}\to[0,1]$. Let throughout a canonical model be given with probability measures $P_{x,y},\ x\in\mathcal{S},y\in I\!R$, on $(\Omega,\mathcal{A})$ such that $P_{x,y}(M_0=x,X_0=y)=1$. If λ denotes any distribution on $\mathcal{S}\times I\!R$ put $P_\lambda(\cdot)=\int_{\mathcal{S}\times I\!R}P_{x,y}(\cdot)\lambda(dx,dy)$ in which case (M_0,X_0) has initial distribution λ under P_λ. Expectation under P_λ is denoted by E_λ. P and E are used for probabilities and expectations, respectively, which are independent of the initial distribution. Finally, for $x\in\mathcal{S}$ and probability measures ν on $\mathcal{S}$, we write for short E_x,E_ν instead of $E_{x,0},E_{\nu\otimes\delta_0}$, respectively, where δ_0 is Dirac measure at 0.

Markov renewal theory deals with the asymptotic behavior of functionals of the *Markov random walk (MRW)* $(M_n,S_n)_{n\ge0}$ and related processes where $S_n=X_0+...+X_n$ for $n\ge0$. Its main result is the Markov renewal theorem and states the following (in the nonarithmetic case): If $(M_n)_{n\ge0}$ has a unique stationary measure ξ (up to multiplicative constant), if $\mu(x)\stackrel{\text{def}}{=}E(X_1|M_0=x)$, $\mu\stackrel{\text{def}}{=}\int_{\mathcal{S}}\mu(x)\,\xi(dx)>0$ and if $\lambda\!\!\lambda_0$ denotes Lebesgue measure on $I\!R$, then

$$\lim_{t\to\infty}E_\lambda\left(\sum_{n\ge0}g(M_n,t-S_n)\right) \;=\; \mu^{-1}\int_{\mathcal{S}}\int_{I\!R}g(x,y)\ \lambda\!\!\lambda_0(dy)\ \xi(dx) \tag{1.3}$$

under appropriate assumptions on the initial distribution λ, the kernel $I\!P$ and the function g. A similar version can, of course, be formulated in the arithmetic case.

Closely related with the previous result is an ergodic theorem ((1.5) below) in the proper renewal case when all X_n's are positive, i.e. when $I\!P(x,\mathcal{S}\times(0,\infty))=1$ for all $x\in\mathcal{S}$. S_n is then usually interpreted as the n-th transition epoch for the chain $(M_n)_{n\ge0}$ where it moves from M_{n-1} to M_n and X_n consequently denotes the associated sojourn time in the former state. Under these assumptions we call $(M_n,S_n)_{n\ge0}$ a *Markov renewal process (MRP)*. In order to incorporate a transition to M_0 after a *positive* delay S_0 in definition (1.4) below we extend our chain by a further variable M_{-1} which denotes the current state at $t=0$ if $S_0>0$. This can be done in accordance with the previous definitions by letting P_λ be such that

$$P_\lambda((M_{n-1},S_n)_{n\ge0}\in\cdot) \;=\; \int_{\mathcal{S}\times[0,\infty)}P((M_{n-1},S_n)_{n\ge1}\in\cdot|M_1=x,S_1=y)\ \lambda(dx,dy).$$

Now put $S_{-1} = 0$ and $N(t) = \sup\{n \geq -1 : S_n \leq t\}$ for $t \geq 0$. Suppose $N(t) < \infty$ for all t (non-explosive case) and define

$$(Z(t), A(t)) \;=\; (M_{N(t)}, t - S_{N(t)}) \;=\; \sum_{n \geq -1} (M_n, t - S_n)\, \mathbf{1}_{\{S_n \leq t < S_{n+1}\}}, \quad t \geq 0, \tag{1.4}$$

where $\mathbf{1}_{\{.\}}$ denotes the indicator function. Observe that the latter summation extends over $n \geq 0$ only in case $S_0 = 0$. $(Z(t))_{t \geq 0}$ is called a *semi-Markov process (SMP)* with embedded chain $(M_n)_{n \geq -1}$ and sojourn times $(X_n)_{n \geq 0}$, $(A(t))_{t \geq 0}$ the *age process* associated with $(M_{n-1}, S_n)_{n \geq 0}$. As a consequence of (1.3), one can show

$$\lim_{t \to \infty} E_\lambda g(Z(t), A(t)) \;=\; \mu^{-1} \int_{\mathcal{S}} \int_{[0,\infty)} E_x\big(g(M_1, y)\mathbf{1}_{\{X_1 > y\}}\big)\, \lambda\!\!\lambda_0(dy)\, \xi(dx) \tag{1.5}$$

for suitable functions g.

(1.3) and (1.5) have been proved under varying assumptions by a number of authors, and we mention here Orey [14], Kesten [9], Jacod [8], Athreya, McDonald and Ney [6], and Athreya and Ney [5]. The most general result on (1.3) for MRP $(M_n, S_n)_{n \geq 0}$ is due to Shurenkov [16] who also deals with the arithmetic case including its proper definition. However, his proof is purely analytical and gives rise to the question whether there is a more probabilistic alternative. This has been the motivation for the present work.

In view of [5] and [6] it is obvious that a probabilistic proof should make use of a regeneration technique developed in [4] and in a slightly different form also in [13]. This technique is by now a standard tool in the limit theory for Markov chains with general state space. It is based upon a minorization condition on the transition kernel, see (1.6) below, which allows to reconstruct the considered process with an embedded sequence of regeneration points and then to apply classical renewal theory. It thus provides, e.g., a simple proof of the fundamental ergodic theorem for so-called Harris-recurrent Markov chains. Recall that the Markov chain $(M_n)_{n \geq 0}$ with n-step transition kernel $I\!\!P_n^*$ is called *Harris-recurrent* (or *Harris chain*) if there exists a set $\mathfrak{R} \in \mathfrak{S}$, some $r \geq 1$, $\alpha > 0$ and a probability measure φ on $\mathfrak{R}$ such that $P_x(M_n \in \mathfrak{R}$ i.o.$) = 1$ for all $x \in \mathcal{S}$ and furthermore

$$I\!\!P_r^*(x, A) \;\geq\; \alpha^{1/2} \varphi(A) \tag{1.6}$$

holds for all $x \in \mathfrak{R}$ and $A \in \mathfrak{S}$. $\mathfrak{R}$ is then called a regeneration set for $(M_n)_{n \geq 0}$ and the latter strongly aperiodic if $r = 1$ in (1.6). As being used later (see Section 3), note that (1.6) further implies

$$I\!\!P_r^* \otimes I\!\!P_r^*(x, A \times B) \;\geq\; \alpha \varphi(A) \varphi(B) \tag{1.7}$$

for all $x \in \mathfrak{R}$ and $A \in \mathfrak{S}$ where $I\!\!P_r^* \otimes I\!\!P_r^*(x, A \times B) \stackrel{\text{def}}{=} \int_A I\!\!P_r^*(y, B)\, I\!\!P_r^*(x, dy)$.

The application of the afore-mentioned regeneration technique to Markov renewal theory, though still being powerful and elegant, has had some limitations so far as the results in [5] and [6] show. Due to the conditional independence of the X_n's given $(M_n)_{n \geq 0}$ it is tempting to expect that Harris recurrence of the latter chain alone implies (1.3) and (1.5) for suitable functions g. Unfortunately, finding a regeneration scheme for $(M_n)_{n \geq 0}$ does not generally lead to one for $(M_n, X_n)_{n\geq 0}$. Indeed, if T is a regeneration epoch for $(M_n)_{n\geq 0}$ with $(M_{T+n})_{n\geq 0}$ being independent of $(M_n)_{0\leq n<T}$, then $(X_{T+n})_{n\geq 0}$ may still depend on the past through M_{T-1}. In this case the cycles, that are formed by splitting $(M_n, X_n)_{n\geq 0}$ into segments of random length with the help of consecutive regeneration epochs, are not independent but 1-dependent. In order to eliminate this problem, it has been assumed in [6] that the conditional distribution of X_n given M_{n-1}, M_n contains a part only depending on M_{n-1} which corresponds to the following minorization condition on $I\!P$: Let $(M_n)_{n\geq 0}$ be a strongly aperiodic Harris chain with regeneration set $\Re$ and minorizing distribution φ as given by (1.6). Suppose further the existence of a kernel ϕ such that

$$I\!P(x, A\times B) \;\geq\; \alpha\varphi(A)\phi(x, B) \tag{1.8}$$

for all $x\in\Re,\ A\in\mathfrak{S},\ B\in\mathfrak{B}$ and some $\alpha\in(0,1)$. Then $(M_n, X_n)_{n\geq 0}$ possesses a regeneration scheme with regeneration occuring at T if $M_{T-1}\in I\!R$, M_T is generated independent of M_{T-1} according to φ and finally X_T according to $\phi(M_{T-1},\cdot)$, thus depending on M_{T-1} only. Notice that X_T then belongs to the cycle of M_{T-1} and not to that of M_T. More general minorization conditions are introduced in [5] and [12], but they still contain an extra condition on the dependence between (M_{n-1}, M_n) and X_n beyond Harris recurrence of $(M_n)_{n\geq 0}$. By making use of a weaker regeneration scheme we will show that Harris recurrence alone suffices to prove (1.3) and (1.5), thus giving a new proof of Shurenkov's result and extending it to the two-sided case. The crucial part of the proof requires an extension of Blackwell's renewal theorem to certain random walks with 1-dependent increments.

The further organization of the paper is as follows: The results are presented in Section 2 and proved in Section 5. A description of the regeneration schemes to be employed can be found in Section 3 and the afore-mentioned extension of Blackwell's renewal theorem forms the content of Section 4.

2. The Results

Before we state the results let us give some further notation. We are always assuming $(M_n)_{n\geq 0}$ to be a Harris chain satisfying (1.6) above. Call $(T_n)_{n\geq 0}$ a *sequence of regeneration epochs* for this chain if it satisfies the following conditions:

(a) $0 = T_0 < T_1 < T_2 < ... < \infty$ a.s. under each P_λ.

(b) There is a filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $(M_n)_{n\geq 0}$ is adapted and Markov and each T_n a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(c) $(T_{n+j}-T_n, M_{T_n+j})_{j\geq 0}$ is independent of $T_0, ..., T_n$ for each $n\geq 0$;

(d) If $\zeta = P(M_{T_1}\in\cdot)$, then $P((T_{n+j}-T_n, M_{T_n+j})_{j\geq 0}\in\cdot) = P_\zeta((T_j, M_j)_{j\geq 0}\in\cdot)$ for all $n\geq 1$.

Conditions (c) and (d) make $(M_n)_{n\geq 0}$ what has been called in [17] a wide sense regenerative process, see also [3]. It follows that the cycle lengths $T_{n+1}-T_n$ are independent for $n\geq 0$ and further identically distributed for $n\geq 1$ under each P_λ. The latter is also true for the cycles $(M_j)_{T_n\leq j<T_{n+1}}, n\geq 1$ themselves, but these need not be independent. However, they can at

most be one-dependent as following from condition (b), more precisely, the fact that $(M_n)_{n\ge0}$ is Markov with respect to $(\mathcal{F}_n)_{n\ge0}$.

Next it is to be noted that

$$\xi(A) \ \stackrel{\text{def}}{=}\ E_\zeta\left(\sum_{j=0}^{T_1-1}\mathbf{1}_{\{M_j\in A\}}\right),\quad A\in\mathfrak{S}, \tag{2.1}$$

defines a σ-finite invariant measure for $(M_n)_{n\ge0}$ which is unique up to a multiplicative constant. In particular, $(M_n)_{n\ge0}$ has a unique stationary distribution ξ^* iff $E_\zeta T_1<\infty$. Note that the limits in (1.3) and (1.5) remain unaffected by the particular choice of ξ because it appears in the numerator as well as in the denominator. This is of importance later on in the proofs of the main results below when choosing a suitable sequence of regeneration epochs and then defining ξ by (2.1) for the latter. The construction of a basic sequence $(\tau_n)_{n\ge0}$ of regeneration epochs for $(M_n)_{n\ge0}$ with $\zeta=\varphi$ is given in [4] for $r=1$ and in [3] for general r, see the beginning of Section 3 for a brief description. $(M_n)_{n\ge0}$ is then called d-periodic, $d\in\mathbb{N}$, if τ_1 is d-arithmetic under P_φ, and aperiodic in case $d=1$. Validity of (1.6) with $r=1$ easily implies $d=1$. Note that by stationarity

$$E_\zeta S_{T_1}\ =\ \int_{\mathcal{S}}\mu(x)\,\xi(dx)\ =\ \mu \tag{2.2}$$

where $\mu(x)=E(X_1|M_0=x)$ should be recalled.

Let us finally define for any distribution λ on $\mathcal{S}\times\mathbb{R}$ the operator Γ_λ by

$$\Gamma_\lambda g(t)\ =\ E_\lambda\left(\sum_{j=0}^{T_1-1}g(M_j,t-S_j)\right),\quad t\in\mathbb{R}, \tag{2.3}$$

where $g:\mathcal{S}\times\mathbb{R}\to\mathbb{R}$ can be any function for which the right-hand side exists for all $t\in\mathbb{R}$ and T_1 is a regeneration epoch for $(M_n)_{n\ge0}$ to be suitably specified later.

As usual in renewal theory, we have to distinguish between two different cases according to whether the involved renewal measure is concentrated on a lattice or not. The appropriate definition in the present context is not immediately obvious but has been provided by Shurenkov [16]. We call $\mathbb{P}$ as well as an associated MRW $(M_n,S_n)_{n\ge0}$ *d-arithmetic*, if $d>0$ is the maximal number for which there exists a measurable function $\gamma:\mathcal{S}\to[0,d)$, called *shift function*, such that

$$P(X_1\in\gamma(x)-\gamma(y)+d\mathbb{Z}|M_0=x,M_1=y)\ =\ 1\quad \xi\otimes\mathbb{P}^*\text{-a.s.} \tag{2.4}$$

where $\xi\otimes\mathbb{P}^*$ is given through $\xi\otimes\mathbb{P}^*(A\times B)=\int_A\mathbb{P}^*(x,B)\xi(dx)$ for $A,B\in\mathfrak{S}$. $(M_n,S_n)_{n\ge0}$ and $\mathbb{P}$ are called *nonarithmetic* if no such d exists.

THEOREM 2.1. *Let $(M_n)_{n\ge0}$ be an aperiodic Harris chain with stationary measure ξ, and let $(M_n,X_n)_{n\ge0}$ be a Markov chain with state space $\mathcal{S}\times\mathbb{R}$, transition kernel $\mathbb{P}$ and associated MRW $(M_n,S_n)_{n\ge0}$ such that $\mu=\int_{\mathcal{S}}\mu(x)\,\xi(dx)\in(0,\infty)$.*

(a) If $\mathbb{P}$ is nonarithmetic, then (1.3) holds with $\lambda=\delta_{x,y}$ for ξ-almost all $x\in\mathcal{S}$, all $y\in\mathbb{R}$ and for every measurable function $g:\mathcal{S}\times\mathbb{R}\to\mathbb{R}$ which satisfies the following conditions:

$$g(x,\cdot)\ \text{is}\ \lambda\!\!\lambda_0\text{-a.e. continuous for each}\ x\in\mathcal{S}, \tag{2.5}$$

$$\int_{\mathcal{S}} \sum_{n\in\mathbb{Z}} \sup_{n\rho\le y<(n+1)\rho} |g(x,y)|\ \xi(dx)\ <\ \infty \quad \textit{for some } \rho>0. \tag{2.6}$$

(b) If $I\!P$ is d-arithmetic with shift function γ, then

$$\lim_{k\to\infty} E_{x,y}\left(\sum_{n\ge 0} g(M_n, kd+\gamma(x)-S_n)\right)\ =\ \frac{d}{\mu}\int_{\mathcal{S}}\sum_{n\in\mathbb{Z}} g(x, nd+\gamma(x))\ \xi(dx) \tag{2.7}$$

for ξ-almost all $x\in\mathcal{S}$, all $y\in d\mathbb{Z}$ and every measurable function $g:\mathcal{S}\times I\!R\to I\!R$ which satisfies

$$\int_{\mathcal{S}}\sum_{n\in\mathbb{Z}} |g(x,nd+\gamma(x))|\ \xi(dx)\ <\ \infty. \tag{2.8}$$

Corollary 2.2. *In the situation of Theorem 2.1, suppose additionally $I\!P(x,\mathcal{S}\times(0,\infty))=1$ for all $x\in\mathcal{S}$ and let $(Z(t),A(t))_{t\ge0}$ be as in (1.4).*

(a) If $I\!P$ is nonarithmetic, then (1.5) holds with $\lambda=\delta_{x,y}$ for ξ-almost all $x\in\mathcal{S}$, all $y\ge0$ and for every measurable function $g:\mathcal{S}\times[0,\infty)\to[0,\infty)$ which satisfies (2.5) and such that $f(x,y)\stackrel{\text{def}}{=}g(x,y)P_x(X_1>y)\mathbf{1}_{[0,\infty)}(y)$ satisfies (2.6).

(b) If $I\!P$ is d-arithmetic with shift function γ and $Y_1\stackrel{\text{def}}{=}X_1-\gamma(M_0)+\gamma(M_1)$, then

$$\lim_{k\to\infty} E_{x,y}g(Z(kd+\gamma(x)),A(kd+\gamma(x))\ =\ \frac{d}{\mu}\int_{\mathcal{S}}\sum_{n\ge0} g(x,nd+\gamma(x))\,P_x(Y_1>nd)\ \xi(dx). \tag{2.9}$$

for ξ-almost all $x\in\mathcal{S}$, all $y\in dI\!N_0$ and every measurable function $g:\mathcal{S}\times I\!R\to I\!R$ such that $f(x,y)\stackrel{\text{def}}{=}g(x,y+\gamma(x))P_x(Y_1>y)$ satsifies (2.8).

Remarks. (a) Theorem 2.1 can be extended to d-periodic Harris chains $(M_n)_{n\ge0}$, $d\ge2$ when using the decomposition

$$E_\lambda\left(\sum_{n\ge0} g(M_n,t-S_n)\right)\ =\ \sum_{r=0}^{d-1} E_\lambda\left(\sum_{n\ge0} g(M_{nd+r},t-S_{nd+r})\right)$$

and the fact that $(M_{nd+r})_{n\ge0}$ is aperiodic with an essentially unique stationary measure on each cyclic class.

(b) Assumptions (2.5) and (2.6) in the nonarithmetic case are needed to ensure direct Riemann integrability of $\Gamma_\psi g$, where ψ is given through the sequence of regeneration epochs to be specified later on. [6] and [5] use a stronger condition than (2.5), namely $g(x,\cdot)$ to be continuous except on a countable set not depending on x. We therefore give a proof of the asserted implication in Lemma 5.1 at the end of Section 5. Shurenkov gives an alternative direct Riemann integrability condition (see (16) in [16]) instead of (2.6) which implies (2.5). His condition (17) will not be needed here and can thus be omitted.

(c) Note that (2.6) forces $g(x,\cdot)$ to be bounded for ξ-almost all $x\in\mathcal{S}$. It further implies

$$\lim_{t\to\infty}|g(x,t)|\ =\ 0 \quad \text{for } \xi\text{-almost all } x\in\mathcal{S}, \tag{2.10}$$

$$\int_{\mathcal{S}}\sup_{t\in I\!R}|g(x,t)|\ \xi(dx)\ <\ \infty. \tag{2.11}$$

We will see later on, see Lemma 5.2 in Section 5, that these conditions imply $\lim_{t\to\infty} \Gamma_{x,y}g(t) = 0$ for ξ-almost all $x \in \mathcal{S}$ and all $y \in I\!R$ however the regeneration epoch T_1 is chosen in (2.3).

(d) Let us finally comment on the possibility of proving Theorem 2.1 also for general bounded kernels $I\!P$, i.e. $\sup_{x\in S} I\!P(x, \mathcal{S} \times I\!R) < \infty$, such that $I\!P^* = I\!P(\cdot, \cdot \times I\!R)$ possesses a unique (up to a constant factor) invariant σ-finite measure ξ and an everywhere positive, ξ-a.e. finite harmonic function h, i.e. $\int_{\mathcal{S}} h(y) I\!P^*(x, dy) = h(x)$. This has been done in the one-sided case by Shurenkov and can be reduced to the situation here when switching to the transformed probability kernel

$$\hat{I\!P}(x, C) \stackrel{\text{def}}{=} h(x)^{-1} \int \mathbf{1}_C(y,t)\, h(y)\; I\!P(x, dy \times dt),$$

$A \in \mathfrak{S} \otimes \mathfrak{B}$. Its projection kernel $\hat{I\!P}^*(x, dy) \stackrel{\text{def}}{=} \hat{I\!P}(x, dy \times I\!R) = h(x)^{-1} h(y) I\!P^*(x, dy)$ has the invariant measure $\hat{\xi}(dx) = h(x)\xi(dx)$ as one can easily check. The desired results follow by applying Theorem 2.1 to $\hat{I\!P}$ (which has the same lattice structure as $I\!P$ itself) and then rewriting the limits in terms of $I\!P$ and ξ. It turns out that they are the same up to an extra factor $h(x)$ (which is 1 in the proper probability setting here) and a change from μ to $\hat{\mu} = \int_{\mathcal{S}} \mu(x)\hat{\xi}(dx) = \int_{\mathcal{S}} \mu(x) h(x) \xi(dx)$. Further details are omitted.

3. Regeneration

We keep the notation of the previous sections and assume the conditions of Theorem 2.1 be given with $I\!P^*$ satsifying (1.6). Let $(M_n)_{n\ge 0}$ together with a sequence of regeneration epochs $(\tau_n)_{n\ge 0}$ be constructed as follows: Given any initial distribution λ, choose independently χ_1 as a geometric random number with parameter $\frac{1}{2}$, say, and M_0 with distribution λ. Go on by recursively generating M_j according to $I\!P^*(M_{j-1}, \cdot)$ until the first $n > \chi_1$ such that $M_n \in \Re$. Now toss a coin showing head with probability α (see (1.7)) and generate (M_{n+r}, M_{n+2r}) according to φ^2 if head comes up in which case $n + 2r = \tau_1$ is a regeneration epoch. Otherwise, let (M_{n+r}, M_{n+2r}) be such that the overall transition probability equals $I\!P^*_r \otimes I\!P^*_r(M_n, \cdot)$. If $r > 1$, finally generate $M_{n+1}, ..., M_{n+r-1}, M_{n+r+1}, ..., M_{n+2r-1}$ according to the prescribed conditional distribution given M_n, M_{n+2r}. This completes the construction of the first block (not necessarily cycle), and the successive ones are defined analogously. Given a full realization of $(M_n)_{n\ge 0}$, the sequence $(X_n)_{n\ge 0}$ can be generated in the desired form and independent of the coin flipping results and the values of the i.i.d. geometric random variables $\chi_1, \chi_2, ...$ We omit the formal definition of the τ_n as not being important in what follows, see [4] and [3] for more details. Note, however, that the use of geometric variables does not appear there and has been included here for technical reasons only, see right after (3.9) in the proof of Lemma 3.3 below. Note also that $\zeta = \varphi$ for this particular sequence of regeneration epochs.

In the following let $I\!P$ be nonarithmetic or d-arithmetic with shift function $\gamma \equiv 0$. The restriction on γ is only imposed for ease of exposition and can always be enforced by switching from $(M_n, X_n)_{n\ge 0}$ to $(M_n, X_n - \gamma(M_{n-1}) + \gamma(M_n))_{n\ge 0}$. The proof of Theorem 2.1 in the more diffcult nonarithmetic case relies upon the introduction of an appropriate family of "approximate" regeneration schemes which means to define sequences of regeneration epochs $(T_n)_{n\ge 0}$

for $(M_n)_{n\ge0}$, in fact subsequences of $(\tau_n)_{n\ge0}$ given above, such that all $X_{T_n-2r+1},...,X_{T_n}$ become almost constant, where the "almost constant" is controlled by an $\varepsilon > 0$ which can be made arbitrarily small. The cycles induced by each such $(T_n)_{n\ge0}$ are 1-dependent but become independent in the limit ($\varepsilon \downarrow 0$). This is enough for our purposes when combined with a renewal theorem for random walks with 1-dependent increments stated in the next section. In the arithmetic case the situation is easier because we can choose $X_{T_n-2r+1},...,X_{T_n}$ to be constant.

For $x, y \in \Re$ and $z \in I\!R^{2r}$, let

$$\begin{aligned} \mathbb{F}(x,y,z,\cdot) &= P((M_1,...,M_{2r-1}) \in \cdot | M_0 = x, M_{2r} = y, (X_1,...,X_{2r}) = z, \tau_1 = 2r), \\ Q^*(x,y,\cdot) &= P((X_1,...,X_{2r}) \in \cdot | M_0 = x, M_{2r} = y, \tau_1 = 2r) \end{aligned}$$

Note that, since $P(M_r \in \cdot | M_0 = x, M_{2r} = z, \tau_1 = 2r) = \varphi$,

$$Q^*(x,z,A\times B) = \int_{\mathcal{S}} Q_r(x,y,A) Q_r(y,z,B)\ \varphi(dy) \tag{3.1}$$

where $Q_r(x,y,\cdot) = P((X_1,...,X_r) \in \cdot | M_0 = x, M_r = y)$ for $x,y \in \mathcal{S}$. For $\varepsilon \ge 0$ and $z \in I\!R^{2r}$ denote further by $I_\varepsilon(z)$ the closed $2r$-dimensional cube with center z and edge length 2ε. Our construction is based on the following lemma the proof of which is very similar in nature to that of Orey's C-Set Theorem, see [11], p. 107ff. We postpone it to the Appendix at the end of the paper.

LEMMA 3.1. *There is some $c \in I\!R^{2r}$ and a version of Q^* such that for all $\varepsilon > 0$ there are measurable $C_\varepsilon, D_\varepsilon \subset \Re$ satisfying $P_\varphi((M_{\tau_1-2r}, M_{\tau_1}) \in C_\varepsilon \times D_\varepsilon) > 0$ and*

$$\eta(\varepsilon) \stackrel{\text{def}}{=} \inf_{(x,y)\in C_\varepsilon\times D_\varepsilon} Q^*(x,y,I_{\varepsilon/2r}(c)) > 0. \tag{3.2}$$

If $I\!P$ is d-arithmetic (with shift function 0), then (3.2) remains true for $\varepsilon = 0$ if $c \in (d\mathbb{Z})^{2r}$ is suitably chosen.

The regeneration scheme in case of nonarithmetic $I\!P$ now looks as follows: We fix an arbitrary $\varepsilon > 0$ and choose $c, C_\varepsilon, D_\varepsilon, \eta(\varepsilon)$ according to the previous lemma where the notational dependence on ε is hereafter suppressed. Let $(R_n)_{n\ge0}$ be a sequence of i.i.d. Bernoulli variables with success probability η which is also independent of $(M_n, \tau_n)_{n\ge0}$. Suppose a realization of the latter sequence is already given, but the X_n have not yet been generated. We then proceed as follows: Each time τ_j when $(M_{\tau_j-2r}, M_{\tau_j}) \in C\times D$ we generate $(X_{\tau_j-2r+1},...,X_{\tau_j})$ according to $Q^*(M_{\tau_j-2r}, M_{\tau_j}, I_{\varepsilon/2r}(c)\cap\cdot)/Q^*(M_{\tau_j-2r}, M_{\tau_j}, I_{\varepsilon/2r}(c))$ if $R_j = 1$, and otherwise such that the overall probability distribution is $Q^*((M_{\tau_j-2r}, M_{\tau_j}),\cdot)$. Next discard the old values of $M_{\tau_j-2r+1},...,M_{\tau_j-1}$ and re-generate new ones according to $\mathbb{F}(M_{\tau_j-2r}, M_{\tau_j}, (X_{\tau_j-2r+1},...,X_{\tau_j}),\cdot)$. At all remaining time points n we simply generate X_n according to $Q(M_{n-1}, M_n,\cdot)$ independent of all other variables generated so far. If $I\!P$ is arithmetic, we do exactly the same with $\varepsilon = 0$ so that $(X_{\tau_j-2r+1},...,X_{\tau_j}) = c$ if $(M_{\tau_j-2r}, M_{\tau_j}) \in$

$C \times D$ and $R_j = 1$. It is easily verified that $(M_n, X_n)_{n\geq 0}$ is indeed a Markov chain with transition kernel $I\!P$ under this construction. Let

$$T_0 = 0 \quad \text{und} \quad T_n = \inf\{\tau_j \geq T_{n-1} + 2r : (M_{\tau_j - 2r}, M_{\tau_j}, R_j) \in C \times D \times \{1\}\} \quad \text{for } n \geq 1 \quad (3.3)$$

which are randomized stopping times for $(M_n)_{n\geq 0}$ in the sense of Pitman and Speed [15]. Indeed, their definition only involves the τ_j's, which are themselves randomized stopping times for $(M_n)_{n\geq 0}$, and the Bernoulli variables $R_j, j \geq 1$, that are independent of $(M_n, \tau_n)_{n\geq 0}$.

LEMMA 3.2. *$(T_n)_{n\geq 0}$ is a sequence of regeneration epochs for $(M_n)_{n\geq 0}$ with $P(M_{T_1} \in \cdot) = \zeta = \varphi(\cdot \cap D)/\varphi(D)$. Furthermore, for each $n \geq 1$*

$$P((M_{T_n+j}, X_{T_n+j+1})_{j\geq 0} \in \cdot | (M_j, X_j)_{0\leq j \leq T_n - 2r}) \;=\; P_\zeta((M_j, X_{j+1})_{j\geq 0} \in \cdot) \quad a.s., \qquad (3.4)$$

i.e. $(M_{T_n+j}, X_{T_n+j+1})_{j\geq 0}$ and $(M_j, X_j)_{0\leq j\leq T_n-2r}$ are independent for all $n \geq 1$. Moreover, $(S_{T_n})_{n\geq 1}$ forms a random walk with stationary, 1-dependent increments under P_ζ and these increments are even i.i.d. in the arithmetic case.

PROOF. We have to check validity of the conditions (a)-(d) stated at the beginning of Section 2. For $n \geq 0$, let $\mathcal{F}_n = \sigma((M_j)_{0\leq j\leq n}, (\{T_j \leq k\})_{j\geq 0, 0\leq k\leq n})$ which is the smallest filtration to which $(M_n)_{n\geq 0}$ is adapted and such that the T_n's are stopping times with respect to it. Since the T_n's are also randomized stopping times for $(M_n)_{n\geq 0}$, the latter chain is even Markov adapted, as following from Proposition 2.5(iv) in [15]. This proves (b). As for (c) and (d) we obtain for all $A \in \mathfrak{S}$ and $n \geq 1$

$$\begin{aligned}
P(M_{T_n} \in A) &= \sum_{j\geq 1} P(T_n = \tau_j, M_{\tau_j} \in A) \\
&= \sum_{j\geq 1} P(T_n > \tau_{j-1}, (M_{\tau_j - 2r}, M_{\tau_j}) \in C \times D, R_j = 1, M_{\tau_j} \in A) \\
&= \sum_{j\geq 1} P(T_n > \tau_{j-1}, M_{\tau_1 - 2r} \in C, R_j = 1) \varphi(A \cap D) \\
&= \sum_{j\geq 1} P(T_n > \tau_{j-1}, M_{\tau_1 - 2r} \in C, R_j = 1)\, \varphi(D)\, \zeta(A) \\
&= \sum_{j\geq 1} P(T_n = \tau_j)\, \zeta(A)
\end{aligned}$$

which together with

$$P((T_{n+j} - T_n, M_{T_n+j})_{j\geq 0} \in \cdot | \mathcal{F}_{T_n}) \;=\; P((T_{n+j} - T_n, M_{T_n+j})_{j\geq 0} \in \cdot | M_{T_n}) \quad \text{a.s.},$$

yields the desired conclusions. As for the remaining assertions, we do not supply further details because they merely involve standard computations like the previous one.

Our final lemma of this section deals with the lattice-type of S_{T_1} under P_ζ as it relates to that of $I\!P$.

LEMMA 3.3. *If $I\!P$ has lattice-span $d \ge 0$, with shift function 0 in case of positive d, and if $(T_n)_{n\ge0}$ denotes the sequence of regeneration epochs for $(M_n)_{n\ge0}$ defined in (3.3), then for all $0 < |t| < 2\pi/d$*

$$\inf_{n\ge1} |E(e^{itS_{T_n}}|M_0, M_{T_n})| \;<\; 1 \quad P_\xi\text{-a.s.} \tag{3.5}$$

In particular, $P_\zeta(S_{T_1} \in \cdot)$ is of the same lattice-type.

PROOF. Suppose that (3.5) fails, i.e. $E(e^{itS_{T_n}}|M_0, M_{T_n}) = e^{it\vartheta_n(M_0,M_{T_n})}$ P_ξ-a.s. for some $t \in (0, \frac{2\pi}{d})$ and suitable functions $\vartheta_n : \mathcal{S}^2 \to [0, \frac{2\pi}{t})$. It is no loss of generality to assume $d < 1$ and $t = 2\pi$. We will show that under these assumptions $I\!P$ must already be 1-arithmetic with some shift function γ and thus produce a contradiction to $d < 1$.

Note first that for each $n > k \ge 1$

$$\rho_n(x, y, t) \;=\; E(\rho_k(x, M_{T_k}, t)\rho_{n-k}(M_{T_k}, y, t)|M_0 = x, M_{T_n} = y) \quad \xi \otimes \zeta\text{-a.s.} \tag{3.6}$$

where $\rho_n(x, y, t) \overset{\text{def}}{=} E(e^{itS_{T_n}}|M_0 = x, M_{T_n} = y)$. From this combined with the assumption above one can easily infer for $n \ge 2$

$$\vartheta_n(M_0, M_{T_n}) \;\equiv_{\mathbb{Z}}\; \vartheta_1(M_0, M_{T_1}) \;+\; \vartheta_{n-1}(M_{T_1}, M_{T_n}) \quad P_\xi\text{-a.s.},$$

where $\equiv_{\mathbb{Z}}$ means equality modulo some integer. But the $M_{T_n}, n \ge 0$ are further independent under P_ξ and for $n \ge 1$ also identically distributed according to ζ so that by conditioning

$$\vartheta_1(M_0, M_{T_1}) \;\equiv_{\mathbb{Z}}\; \gamma_n(M_0) - \gamma_{n-1}(M_{T_1}) \quad P_\xi\text{-a.s.} \tag{3.7}$$

where $\gamma_n(x) \overset{\text{def}}{=} \int \vartheta_n(x, y)\, \zeta(dy) \in [0, 1)$. In particular,

$$\gamma_2(M_0) - \gamma_1(M_{T_1}) \;\equiv_{\mathbb{Z}}\; \gamma_3(M_0) - \gamma_2(M_{T_1}) \quad P_\xi\text{-a.s.}$$

which is same as

$$\gamma_2(M_0) - \gamma_3(M_0) \;\equiv_{\mathbb{Z}}\; \gamma_1(M_{T_1}) - \gamma_2(M_{T_1}) \;\equiv_{\mathbb{Z}}\; c \;\in\; (-1, 1) \quad P_\xi\text{-a.s.} \tag{3.8}$$

where the latter equivalence is again due to the independence of M_0 and M_{T_1}. By combining (3.7) with $n = 3$ and (3.8), we obtain the crucial identity

$$\vartheta_1(M_0, M_{T_1}) \;\equiv_{\mathbb{Z}}\; \gamma_2(M_0) - \gamma_2(M_{T_1}) - c, \quad P_\xi\text{-a.s.}$$

Put $\gamma = \gamma_2$. Then the latter identity gives

$$\rho_1(M_0, M_{T_1}, 2\pi) \;=\; e^{2\pi i[\gamma(M_0)-\gamma(M_{T_1})-c]}. \tag{3.9}$$

By using the definition of τ_1 at the beginning of this section, in particular the memoryless-property of χ_1, one can easily see that

$$E(e^{2\pi i[S_{T_1}-S_n+c+\gamma(M_{T_1})]}|M_n, \chi_1 \ge n) \;=\; E_{M_n} e^{2\pi i[S_{T_1}+c+\gamma(M_{T_1})]} \;=\; e^{-2\pi i\gamma(M_n)} \quad P_\xi\text{-a.s.}$$

for each $n \geq 0$. It is this identity which has motivated us to additionally introduce the geometric variables $\chi_1, \chi_2, ...$ Namely, if ν denotes any probability measure equivalent to ξ, we now conclude with its help and with that of (3.9)

$$\begin{aligned}
1 &= E_\nu e^{2\pi i[S_{T_1}+c+\gamma(M_{T_1})-\gamma(M_0)]} \\
&= P_\nu(\chi_1 = 0) + \int_{\{\chi_1 \geq 1\}} e^{2\pi i[X_1-\gamma(M_0)]} E(e^{2\pi i[S_{T_1}+c+\gamma(M_{T_1})]}|M_1, \chi_1 \geq 1)\, dP_\nu \\
&= P_\nu(\chi_1 = 0) + \int_{\{\chi_1 \geq 1\}} e^{2\pi i[X_1-\gamma(M_0)+\gamma(M_1)]}\, dP_\nu \\
&= P_\nu(\chi_1 = 0) + P_\nu(\chi_1 \geq 1)\, E_\xi e^{2\pi i[X_1-\gamma(M_0)+\gamma(M_1)]},
\end{aligned}$$

hence $E_\nu e^{2\pi i[X_1-\gamma(M_0)+\gamma(M_1)]} = 1$. However, the latter clearly implies $I\!P$ to be 1-arithmetic with shift function γ and therefore the desired contradiction.

Remark. With the help of the identity

$$\begin{aligned}
&E(e^{itS_{T_1}}|M_0, M_{T_1}, \chi_1 \geq n) \\
&\quad = E(E(e^{itS_n}|M_0, M_n)\, E(e^{it(S_{T_1}-S_n)}|M_n, M_{T_1}, \chi_1 \geq n)|M_0, M_{T_1}, \chi_1 \geq n) \quad P_\xi\text{-a.s.}
\end{aligned}$$

it is not difficult to verify that

$$\inf_{n \geq 1} |E(e^{itS_n}|M_0, M_n)| < 1 \quad P_\xi\text{-a.s. for all } t \neq 0 \tag{3.10}$$

implies (3.5) in the nonarithmetic case. The latter has been used in [1] for proving Blackwell's renewal theorem for $(S_{T_n})_{n\geq 0}$ under each P_λ. The proof there is based on Fourier analysis, and we therefore provide an alternative one next for being interested in a probabilistic derivation of the Markov renewal theorem.

4. An Extension of Blackwell's Renewal Theorem

In the following we let $I\!P$ be nonarithmetic and keep $\varepsilon, c, C = C_\varepsilon, D = D_\varepsilon, \eta = \eta(\varepsilon)$ and $\psi = \psi(\varepsilon)$ as well as the definition of all further variables from the previous section fixed. Define $\hat{S}_n = S_{T_n}$ and $\hat{X}_n = \hat{S}_n - \hat{S}_{n-1}$. For the proof of Theorem 2.1 we need to show that $(\hat{S}_n)_{n\geq 0}$ satisfies Blackwell's renewal theorem under P_ζ for which a result from [2] combined with a coupling argument will be employed. Recall from Lemma 3.2 that $(\hat{S}_n)_{n\geq 0}$ has stationary, 1-dependent increments under P_ζ with mean $\mu \in (0, \infty)$ under the assumptions of Theorem 2.1, see (2.2). Define its renewal measure under P_λ through

$$\hat{U}_\lambda(B) = \sum_{n\geq 0} P_\lambda(S_{T_n} \in B), \quad B \in \mathfrak{B}. \tag{4.1}$$

Renewal theory for random walks with stationary increments has been developed by a number of authors, notably Berbee [7] and Lalley [10]. The following proposition confines to the special situation which is of interest here.

PROPOSITION 4.1. *Given the previous assumptions, for each bounded interval $I \subset I\!R$*

$$\lim_{t\to\infty} \hat{U}_\zeta(t+I) \;=\; (E_\zeta \hat{S}_1)^{-1}\lambda\!\!\lambda_0(I) \;=\; \mu^{-1}\lambda\!\!\lambda_0(I). \tag{4.2}$$

Some preparation for the proof has to be given first, and we thus leave the present context for a moment.

In [2], a sequence $(X_n)_{n\ge 0}$ of real-valued random variables with canonical filtration $(\mathcal{F}_n)_{n\ge 0}$ is called

— *stochastically bounded (s.b.)*, if there exist distributions F, G with finite mean such that

$$F(t) \;\le\; \sup_{n\ge 0} \|P(X_{n+1} \le t|\mathcal{F}_n)\|_\infty \;\le\; G(t) \quad \text{for all } t \in I\!R, \tag{4.3}$$

where $\|\cdot\|_\infty$ denotes the usual L_∞-norm.

— *stochastically stable (s.s.) with mean θ*, if it is s.b. and if

$$\lim_{k\to\infty} \|k^{-1}L_{n,k}^n \;-\; \theta\|_\infty \;=\; 0 \tag{4.4}$$

where $L_{n,k} \stackrel{\text{def}}{=} E(S_{n+k} - S_n|S_0, ..., S_n)$.

The important property of random walks $(S_n)_{n\ge 0}$ with s.s. increments with positive mean θ and with renewal measure U is that, for each $\varepsilon \in (0,\theta)$, there is a nonarithmetic distribution H such that

$$(\theta+\varepsilon)^{-1}\lambda\!\!\lambda_0(I) \;\le\; \liminf_{t\to\infty} H * U(t+I) \;\le\; \limsup_{t\to\infty} H * U(t+I) \;\le\; (\theta-\varepsilon)^{-1}\lambda\!\!\lambda_0(I), \tag{4.5}$$

for ach bounded interval I, see Proposition 5.1 in [2].

Returning to the present context, we have the following

LEMMA 4.2. *For each $\varepsilon > 0$, $(S_{T_n} - S_{T_{n-1}})_{n\ge 1}$ is s.s. with mean μ under P_ζ.*

PROOF. Put $c = (c_1, ..., c_{2r})$, $b = \sum_{j=1}^{2r} c_j$ and $Y_n = S_{T_n - 2r} - S_{T_{n-1}}$ for $n \ge 1$, the latter being a sequence of i.i.d. random variables, as one can easily see from the construction before Lemma 3.2. Of course, the $\hat{X}_n$ are not i.i.d. Furthermore $X_{T_n-2r+j} \in [c_{j+1} - \varepsilon/2r, c_{j+1} + \varepsilon/2r]$ for all $n \ge 1$ and $0 \le j \le 2r-1$. Consequently, $Y_n + b - \varepsilon \le \hat{X}_n \le Y_n + b + \varepsilon$, and we obtain upon setting $\mathcal{G}_n = \sigma(S_{T_j}, 0 \le j \le n)$ and using the independence of Y_n and $\mathcal{G}_{n-1}$

$$P_\zeta(Y_1 \le t - b - \varepsilon) \;\le\; P_\zeta(\hat{X}_n \le t|\mathcal{G}_{n-1}) \;\le\; P_\zeta(Y_1 \le t - b + \varepsilon) \quad P_\zeta\text{-a.s.}$$

for all $t \in I\!R$ and $n \ge 1$ where upper and lower distribution function clearly belong to integrable distributions. Finally, since by 1-dependence

$$E(\hat{X}_{n+j}|\mathcal{G}_n) \;=\; E_\zeta \hat{S}_1 \;=\; \mu \quad P_\zeta\text{-a.s. for all } j \ge 2,$$

the asserted mean stability condition (4.4) obviously holds true.

PROOF OF PROPOSITION 4.1. By combining Lemma 4.2 with (4.5), it is obviously enough to prove that $(\hat{S}_n)_{n\ge 0}$ can be successfully coupled with a delayed copy $(\hat{S}'_n)_{n\ge 0}$ where the delay $\hat{S}'_0$ must be suitably chosen. The latter means that we fix an arbitrary $\varepsilon > 0$

together with a nonarithmetic distribution H satisfying (4.5) and let $\hat{S}_0'$ have distribution H and be independent of all further occurring variables. For the remaining construction of $(\hat{S}_n' - \hat{S}_0')_{n\ge 0}$, we take an ordinary renewal process $(N_n)_{n\ge 0}$ independent of $(M_n, X_n, T_n)_{n\ge 0}$ whose increments are geometrically distributed with some parameter $p \in (0,1)$ which does not matter for our purposes. Put $V_n = T_{N_n}$ for each $n \ge 0$. It is then easily verified that the $Z_n \stackrel{\text{def}}{=} S_{V_n} - S_{V_{n-1}} = \hat{S}_{N_n} - \hat{S}_{N_{n-1}}, n \ge 1$ form again a stationary sequence with mean μ/p under P_ζ, that they are conditionally independent given $(M_{V_n})_{n\ge 0}$ and that the latter variables are still i.i.d. with joint distribution ζ. We define a copy $(Z_n')_{n\ge 0}$ of $(Z_n)_{n\ge 0}$ as follows: For even n, we put $Z_n' = Z_n$, while for odd n, we generate Z_n', given $M_{V_{n-1}}, M_{V_n}$, as a conditionally independent copy of Z_n. Let $F_j((M_{T_n}, Z_{n+1})_{n\ge 0}, \cdot)$ denote the conditional distribution of $\hat{X}_j$ given $(M_{T_n}, Z_{n+1})_{n\ge 0}$ for each $j \ge 1$. The complete sequence $(\hat{S}_n' - \hat{S}_0')_{n\ge 0}$ can now be obtained by generating each increment $\hat{X}_j'$ according to $F_j((M_{T_n}, Z_{n+1}')_{n\ge 0}, \cdot)$ and independent of all other $\hat{X}_k'$. It is readily checked that this leads indeed to a copy $(\hat{S}_n' - \hat{S}_0')_{n\ge 0}$ of $(\hat{S}_n)_{n\ge 0}$. We note instead the intrinsic feature of this construction, namely

$$\hat{S}_{N_n}' - \hat{S}_{N_n} \;=\; \hat{S}_0' \;+ \sum_{1\le 2k+1\le n} (Z_{2k+1}' - Z_{2k+1}) \quad \text{for all } n \ge 0,$$

where the $Z_{2k+1}' - Z_{2k+1}, k \ge 0$ are i.i.d. centered and nonarithmetic random variables under P_ζ. It is the latter property which requires Lemma 3.3 and the introduction of $(N_n)_{n\ge 0}$. In fact, (3.5) implies for all $t \ne 0$

$$E_\zeta e^{it(Z_1' - Z_1)} \;=\; \sum_{n\ge 0} P_\zeta(N_1 = n)\, E_\zeta |E(e^{itS_{T_n}} | M_0, M_{T_n})|^2 \;<\; 1.$$

We see from this argument that any renewal process $(N_n)_{n\ge 0}$ such that $P_\zeta(N_1 = n) > 0$ for all $n \ge 1$ could have served here equally well.

Given any $\eta > 0$, the η-coupling time τ, defined by

$$\tau \;=\; \inf\{N_n : |\hat{S}_{N_n}' - \hat{S}_{N_n}| < \eta\},$$

is now P_ζ-a.s. finite because $(\hat{S}_{N_n}' - \hat{S}_{N_n})_{n\ge 0}$ forms a η-recurrent random walk on $\mathbb{R}$. The associated coupling process

$$\hat{S}_n^* \;\stackrel{\text{def}}{=}\; \hat{S}_n \,\mathbf{1}_{\{\tau \ge n\}} \;+\; (\hat{S}_\tau + (\hat{S}_n' - \hat{S}_\tau'))\,\mathbf{1}_{\{\tau < n\}}$$

further defines a copy of $(\hat{S}_n)_{n\ge 0}$ due to the conditional independence of the increments given $(M_{T_n})_{n\ge 0}$. The remaining arguments for proving Proposition 4.1 are by now standard and will not be spelled out any further. They may e.g. be found in [2] in a context of similar type, see the proof of Theorem 3.1 there.

5. Proof of Theorem 2.1 and Corollary 2.2

Proof of Theorem 2.1. *Arithmetic case.* Suppose first $\mathbb{P}$ be d-arithmetic with shift function $\gamma = 0$, w.l.o.g. $d = 1$, and let $(T_n)_{n\ge 0}$ be as defined in Section 3. It then follows from Lemma 3.2 that $(M_{T_n+j}, S_{T_n+j+1} - S_{T_n})_{j\ge 0}$ and $S_{T_n} = S_{T_n-2r} + b,\ b = c_1 + ... + c_{2r}$, are independent under P_ζ, and that $(S_{T_n})_{n\ge 0}$ forms a zero-delayed, 1-arithmetic random walk with i.i.d. increments with mean $\mu\in(0,\infty)$. W.l.o.g. let $g\ge 0$. Recall from (2.3) the definition of $\Gamma_\lambda g$, of course with T_1 as just chosen, and put

$$g*U_\lambda(t) \;=\; E_\lambda\left(\sum_{n\ge 0} g(M_n,t-S_n)\right),\quad t\in I\!R. \tag{5.1}$$

It follows for all $k\in\mathbb{Z}$

$$\begin{aligned} g*U_\zeta(k) &= \Gamma_\zeta g(k) \;+\; \sum_{n\ge 1} E_\zeta\left(\sum_{j=T_n}^{T_{n+1}-1} g(M_j,k-S_j)\right)\\ &= \Gamma_\zeta g(k) \;+\; \sum_{n\ge 1} E_\zeta\left(\sum_{j=T_n}^{T_{n+1}-1} g(M_j,k-S_{T_n}-(S_j-S_{T_n}))\right)\\ &= \Gamma_\zeta g(k) \;+\; \sum_{n\ge 1}\sum_{j\in\mathbb{Z}} \Gamma_\zeta g(k-j)\,P_\zeta(S_{T_n}=j)\\ &= \sum_{j\in\mathbb{Z}} \Gamma_\zeta g(k-j)\,\hat U_\zeta(\{j\}) \;=\; (\Gamma_\zeta g)*\hat U_\zeta(k) \end{aligned} \tag{5.2}$$

where $\hat U_\zeta$ denotes the renewal measure associated with $(S_{T_n})_{n\ge 0}$ under P_ζ. Since the latter random walk has i.i.d. 1-arithmetic increments, (2.7) with $\gamma=0,\ d=1$, and with $E_{x,y}$ replaced by E_ζ follows from the key renewal theorem, (2.8) and

$$\sum_{n\in\mathbb{Z}}\Gamma_\zeta g(n) \;=\; E_\zeta\left(\sum_{j=0}^{T_1-1}\sum_{n\in\mathbb{Z}} g(M_j,n)\right) \;=\; \int_{\mathcal{S}}\sum_{n\in\mathbb{Z}} g(x,n)\ \xi(dx). \tag{5.3}$$

To prove (2.7) in the form as stated in Theorem 2.1 (of course, still with $\gamma\equiv 0$ and $d=1$), one can easily see that

$$g*U_{x,y}(k) \;=\; \Gamma_{x,y}g(k) \;+\; \sum_{j\in\mathbb{Z}} g*U_\zeta(k-j)\,P_{x,y}(S_{T_1}=j). \tag{5.4}$$

This implies the desired result by the first part, dominated convergence and $\lim_{k\to\infty}\Gamma_{x,y}g(k)=0$ for ξ-almost all $x\in\mathcal{S}$ and all $y\in\mathbb{Z}$, which in turn follows from Lemma 5.2 below when observing that $g(x,y)$ can always be chosen as $\sum_{n\in\mathbb{Z}} g(x,n)\,\mathbf{1}_{[n,n+1)}(y)$ and then obviously satsifies (2.6), hence (2.10) and (2.11) by Remark (c) in Section 2.

If $I\!P$ is 1-arithmetic with non-vanishing shift function γ, a simple transformation yields the asserted result as well. Namely,

$$E_{x,y}\left(\sum_{n\ge 0} g(M_n,k+\gamma(x)-S_n)\right) \;=\; E_{x,y}\left(\sum_{n\ge 0} g_\gamma(M_n,k-W_n)\right), \tag{5.5}$$

where $g_\gamma(x,y)\overset{\text{def}}{=}g(x,y+\gamma(x))$ and $W_n\overset{\text{def}}{=}S_n-\gamma(M_0)+\gamma(M_n)=S_0+\sum_{j=1}^n(X_j-\gamma(M_{j-1})+\gamma(M_j))$ for $n\ge 0$. But $(M_n,W_n)_{n\ge 0}$ is a 1-arithmetic MRW with shift function 0, so that we can apply the above arguments to the right-hand side in (5.5) giving the desired result.

Proof of Theorem 2.1. *Nonarithmetic case.* Here the situation is slightly more complicated by an additional approximation argument coming in. We fix an arbitrary $\varepsilon > 0$ and let then $c = (c_1, ..., c_{2r}) \in I\!\!R^{2r}, C, D, (T_n)_{n\geq 0}$ and ζ be as defined in Section 3 (for that ε). We further put again $b = \sum_{j=1}^{2r} c_j$ and define, for arbitrary $\delta > 0$, the piecewise constant functions $g_\delta, \tilde{g}_\delta, g^\delta, \tilde{g}^\delta$ through

$$\begin{aligned} g_\delta(x,y) &= \inf_{t\in[n\delta,(n+1)\delta]} g(x,t) \quad \text{und} \quad g^\delta(x,y) = \sup_{t\in[n\delta,(n+1)\delta]} g(x,t) \\ \tilde{g}_\delta(x,y) &= \inf_{t\in[(n-1)\delta,(n+2)\delta]} g(x,t) \quad \text{und} \quad \tilde{g}^\delta(x,y) = \sup_{t\in[(n-1)\delta,(n+2)\delta]} g(x,t) \end{aligned}$$

on $(n\delta, (n+1)\delta],\ n \in \mathbb{Z}$. These functions obviously satisfy the inequality

$$\tilde{g}_\delta(x,y') \leq g_\delta(x,y) \leq g(x,y) \leq g^\delta(x,y) \leq \tilde{g}^\delta(x,y') \tag{5.6}$$

for all $x \in \mathcal{S}, y \in I\!\!R$ and $y' \in [y-\delta, y+\delta]$.

Now use (5.6), $S_{T_n} - S_{T_n-2r} \in [b-\varepsilon, b+\varepsilon]$ and the independence of $(M_{T_n+j}, S_{T_n+j+1} - S_{T_n})_{j\geq 0}$ and S_{T_n-2r} under P_ζ (Lemma 3.2) to obtain

$$\begin{aligned} g{*}U_\zeta(t) &= \varGamma_\zeta g(t) + \sum_{n\geq 1} E_\zeta\left(\sum_{j=T_n}^{T_{n+1}-1} g(M_j, t-S_j)\right) \\ &\leq \varGamma_\zeta g(t) + \sum_{n\geq 1} E_\zeta\left(\sum_{j=T_n}^{T_{n+1}-1} g^\varepsilon(M_j, t - S_{T_n-2r} - b - (S_j - S_{T_n}))\right) \\ &= \varGamma_\zeta g(t) + \sum_{n\geq 1} \int_{I\!\!R} E_\zeta\left(\sum_{j=0}^{T_1-1} g^\varepsilon(M_j, t-s-b-S_j)\right) P_\zeta(S_{T_n-2r} \in ds) \qquad (5.7) \\ &\leq \varGamma_\zeta \tilde{g}^\varepsilon(t) + \sum_{n\geq 1} \int_{I\!\!R} E_\zeta\left(\sum_{j=0}^{T_1-1} \tilde{g}^\varepsilon(M_j, t-s-S_j)\right) P_\zeta(S_{T_n} \in ds) \\ &= \int_{I\!\!R} \varGamma_\zeta \tilde{g}^\varepsilon(t-s)\, \hat{U}(ds) = (\varGamma_\zeta \tilde{g}^\varepsilon) * \hat{U}_\zeta(t) \end{aligned}$$

and analogously

$$g * U_\zeta(t) \geq \int_{I\!\!R} \varGamma_\zeta \tilde{g}_\varepsilon(t-s)\, \hat{U}_\zeta(ds) = (\varGamma_\zeta \tilde{g}_\varepsilon) * \hat{U}_\zeta(t), \tag{5.8}$$

where here $\hat{U}_\zeta$ denotes the renewal measure of $(S_{T_n})_{n\geq 0}$ under P_ζ. We have shown in Section 4 that the latter random walk satisfies Blackwell's renewal theorem, whence we can apply the key renewal theorem to both inequalities provided $\varGamma_\zeta \tilde{g}_\varepsilon, \varGamma_\zeta \tilde{g}^\varepsilon$ are d.R.i. But this follows from Lemma 5.1 below when observing that $\tilde{g}_\varepsilon, \tilde{g}^\varepsilon$ do also satisfy (2.5) and (2.6) if g does. Recalling $\mu = \int_{\mathcal{S}} \mu(x)\ \xi(dx)$ and using

$$\int_{I\!\!R} \varGamma_\zeta f(t)\ \lambda\!\!\lambda_0(dt) = E_\zeta\left(\sum_{j=0}^{T_1-1} \int_{I\!\!R} f(M_j, t)\ \lambda\!\!\lambda_0(dt)\right) = \int_{\mathcal{S}} \int_{I\!\!R} f(x,t)\ \lambda\!\!\lambda_0(dt)\ \xi(dx) \tag{5.9}$$

whenever $\mathit{\Gamma}_\zeta f$ is d.R.i., we hence obtain

$$\limsup_{t\to\infty} g * U_\zeta(t) \ \le\ \mu^{-1} \int_{\mathcal{S}} \int_{\mathbb{R}} \tilde{g}^\varepsilon(x,t)\ \lambda\!\!\lambda_0(dt)\ \xi(dx)$$

and a reverse inequality for the liminf with $\tilde{g}^\varepsilon$ replaced by $g_{2\varepsilon}$. If we finally observe that by (2.5)

$$\tilde{g}_\varepsilon(x,\cdot) \uparrow g(x,\cdot) \quad \text{und} \quad \tilde{g}^\varepsilon(x,\cdot) \downarrow g(x,\cdot) \quad \lambda\!\!\lambda_0\text{-a.e., as } \varepsilon \downarrow 0$$

for all $x \in \mathcal{S}$, then (1.3) for $\lambda = \zeta$ follows by monotone convergence.

(1.3) for $g * U_{x,y}(t)$ for ξ-almost all $x \in \mathcal{S}$ and all $y \in \mathbb{R}$ yields after observing that, instead of (5.3), we here obtain

$$g * U_{x,y}(t) \ \le (\ge)\ \mathit{\Gamma}_{x,y}\tilde{g}^\varepsilon_{(\varepsilon)}(t) \ +\ \int_{\mathbb{R}} \tilde{g}^\varepsilon_{(\varepsilon)} * U_\zeta(t-s)\ P_{x,y}(S_{T_1} \in ds)$$

for all $\varepsilon > 0$. Indeed, Lemma 5.2 then gives $\mathit{\Gamma}_{x,y}\tilde{g}^\varepsilon_{(\varepsilon)}(t) \to 0$ for ξ-almost all $x \in \mathcal{S}$ and all $y \in \mathbb{R}$, and the result follows from the previous part, dominated convergence ($\mathit{\Gamma}_{x,y}\tilde{g}^\varepsilon_{(\varepsilon)}$ is bounded) and upon $\varepsilon \downarrow 0$. Further details can be omitted.

Proof of Corollary 2.2. If $\mathbb{P}$ is nonarithmetic, for all $x \in \mathcal{S}$ and $y \ge 0$

$$\begin{aligned} E_{x,y}g(Z(t),A(t)) &= \sum_{n\ge -1} E_{x,y}\left(g(M_n, t-S_n)\, \mathbf{1}_{\{S_n \le t < S_{n+1}\}} \right) \\ &= \sum_{n\ge -1} E_{x,y}\left(g(M_n, t-S_n)\, \mathbb{P}(M_n, \mathcal{S}\times(t-S_n,\infty))\mathbf{1}_{\{S_n\le t\}} \right) \qquad (5.10) \\ &= E_{x,y}g(M_{-1},t)\,\mathbf{1}_{\{S_0>t\}} \ +\ f * U_{x,y}(t), \\ \text{where } f(x,y) &= g(x,y)P_x(X_1>y)\mathbf{1}_{[0,\infty)}(y). \end{aligned}$$

Since f satisfies (2.5) and (2.6), the latter condition by assumption and the former because g does so, we infer (1.5) for ξ-almost all $x \in \mathcal{S}$ and all $y \in \mathbb{R}$ by applying Theorem 2.1 and observing that $E_{x,y}g(M_{-1},t)\mathbf{1}_{\{S_0>t\}} = 0$ for $t > y$.

If $\mathbb{P}$ is d-arithmetic with shift function γ, then we have for all $x \in \mathcal{S}$ and $y \in d\mathbb{N}_0$

$$\begin{aligned} &E_{x,y}g(Z(kd+\gamma(x)), A(kd+\gamma(x))) \\ &\quad= \sum_{n\ge -1} E_{x,y}\left(g(M_n, kd+\gamma(x)-S_n)\,\mathbf{1}_{\{S_n\le kd+\gamma(x)<S_{n+1}\}}\right) \\ &\quad= \sum_{n\ge -1} E_{x,y}\left(g_\gamma(M_n, kd-W_n)\,\mathbf{1}_{\{W_n\le kd<W_{n+1}\}}\right) \qquad (5.11) \\ &\quad= E_{x,y}g(M_{-1},kd+\gamma(x))\,\mathbf{1}_{\{S_0>kd+\gamma(x)\}} \ +\ f_\gamma * U_{x,y}(t), \\ \text{where } f_\gamma(x,y) &= g_\gamma(x,y)P_x(Y_1>y)\,\mathbf{1}_{[0,\infty)}(y), \end{aligned}$$

and $g_\gamma(x,y) = g(x,y+\gamma(x))$, $W_n = S_n - \gamma(M_0) + \gamma(M_n)$ should be recalled from the proof of Theorem 2.1 in the arithmetic case. Note that $0 \le \gamma(x) < d$ for all $x \in \mathcal{S}$ implies

$1_{\{S_n \le kd+\gamma(x) < S_{n+1}\}} = \mathbf{1}_{\{W_n \le kd < W_{n+1}\}}$. Again, the desired result (2.9) follows from Theorem 2.1 because f_γ satisfies (2.8) by assumption and $E_{x,y} g(M_{-1}, kd+\gamma(x))\, \mathbf{1}_{\{S_0 > kd+\gamma(x)\}} = 0$ for $kd+\gamma(x) > y$.

For completeness we finally have to state two lemmata which were used in the above proof of Theorem 2.1.

LEMMA 5.1. *Let $g : \mathcal{S} \times I\!R \to [0,\infty)$ be a measurable function which satisfies (2.5) and (2.6). Then $\Gamma_\zeta g$ is d.R.i. for each choice of regeneration epoch T_1 in (2.3), where $\zeta = P(M_{T_1} \in \cdot)$.*

PROOF. For arbitrary $f : I\!R \to [0,\infty)$ and $\delta > 0, n \in \mathbb{Z}$ let

$$B_n^\delta(f) \;=\; \sup_{n\delta \le t < (n+1)\delta} f(t), \quad \overline{\sigma}(f,\delta) \;=\; \sum_{n\in\mathbb{Z}} B_n^\delta(f)\, \mathbf{1}_{[n\delta,(n+1)\delta)},$$

i.e. $\overline{\sigma}(f,\delta)$ forms an upper step function for f. As one can easily see,

$$\sum_{n\in\mathbb{Z}} B_n^\delta(f(\cdot - s)) \;\le\; 2\sum_{n\in\mathbb{Z}} B_n^\delta(f) \quad \text{for all } \delta>0,\ s\in I\!R.$$

We infer for all $0<\delta<\rho$ with the help of (2.6)

$$\begin{aligned} \delta^{-1}\int_{-\infty}^{\infty} \overline{\sigma}(\Gamma_\zeta g,\delta)\, dt \;&=\; \sum_{n\in\mathbb{Z}} B_n^\delta(\Gamma_\zeta g) \;\le\; E_\zeta\left(\sum_{j=0}^{T_1-1}\sum_{n\in\mathbb{Z}} B_n^\delta(g(M_j,\cdot - S_j))\right) \\ &\le\; E_\zeta\left(\sum_{j=0}^{T_1-1}\sum_{n\in\mathbb{Z}} 2B_n^\delta(g(M_j,\cdot))\right) \;=\; 2\int_{\mathcal{S}}\sum_{n\in\mathbb{Z}} B_n^\delta(g(x,\cdot))\,\xi(dx) \;<\; \infty. \end{aligned} \tag{5.12}$$

It is therefore enough to prove, see Proposition IV.4.1(ii) in [3], the $\lambda\!\!\lambda_0$-almost everywhere continuity of $\Gamma_\zeta g$, which in turn can be inferred from its Riemann-integrability on each compact interval $[a,b]$. For the latter, we restrict ourselves to prove

$$\lim_{\delta\downarrow 0}\int_a^b \overline{\sigma}(\Gamma_\zeta g,\delta)(t)\, dt \;=\; \int_a^b \Gamma_\zeta g(t)\, dt,$$

i.e. convergence of the upper sums. For the lower sums one may proceed exactly the same way. Put $c_n^\delta = \lambda\!\!\lambda_0([a,b]\cap[n\delta,(n+1)\delta))$ for $n\in\mathbb{Z}$, $\delta>0$ and $G_j = g(M_j,\cdot - S_j)$ for $j\in I\!N_0$. Then

$$\begin{aligned} \int_a^b \Gamma_\zeta g(t)\, dt \;&\le\; \int_a^b \overline{\sigma}(\Gamma_\zeta g,\delta)(t)\, dt \;=\; \sum_{n\in\mathbb{Z}} c_n^\delta B_n^\delta(\Gamma_\zeta g) \\ &\le\; E_\zeta\left(\sum_{j=0}^{T_1-1}\sum_{n\in\mathbb{Z}} c_n^\delta B_n^\delta(G_j)\right) \;=\; E_\zeta\left(\sum_{j=0}^{T_1-1}\int_a^b \overline{\sigma}(G_j,\delta)(t)\, dt\right) \\ &\overset{\delta\downarrow 0}{\to}\; E_\zeta\left(\sum_{j=0}^{T_1-1}\int_a^b G_j(t)\, dt\right) \;=\; \int_a^b E_\zeta\left(\sum_{j=0}^{T_1-1} G_j(t)\right) dt \;=\; \int_a^b \Gamma_\zeta g(t)\, dt, \end{aligned}$$

where the last line holds by dominated convergence, because, by assumptions (2.5) and (2.6), $G_j(t)$ is bounded and $\lambda\!\!\lambda_0$-a.e. continuous, thus Riemann-integrable on $[a,b]$. (5.12) further yields

$$\sum_{j=0}^{T_1-1} \int_a^b \overline{\sigma}(G_j,\delta)(t)\ dt \ \leq\ 2\delta \sum_{j=0}^{T_1-1} \sum_{n\in\mathbb{Z}} B_n^\delta(g(M_j,\cdot)),$$

where the right-hand side is P_ζ-integrable by (2.6) for all $\delta<\rho$ (see second line of (5.12)).

LEMMA 5.2. *Let $g:\mathcal{S}\times\mathbb{R}\to[0,\infty)$ be a measurable function which satisfies (2.10) and (2.11) and T_1 be an arbitrary regeneration epoch for $(M_n)_{n\geq 0}$ in (2.3). Then $\lim_{t\to\infty}\Gamma_{x,y}g(t)=0$ for ξ-almost all $x\in\mathcal{S}$ and all $y\in\mathbb{R}$.*

PROOF. Apart from minor modifications, the proof coincides with that of Proposition 4.1 in [6]. We omit further details.

6. APPENDIX

PROOF OF LEMMA 3.1. We begin by reviewing some necessary facts from the proof of Orey's C-Set Theorem. Since $\mathfrak{S}$ is countably generated there is an increasing sequence $\mathfrak{S}_n$ of finite partitions of $\mathcal{S}$. Denote $\mathcal{S}_n(x)\in\mathfrak{S}_n$ the set containing $x\in\mathcal{S}$ and $\mathcal{S}_n(x,y)=\mathcal{S}_n(x)\times\mathcal{S}_n(y)$. If ψ is a maximal irreducibility measure for $(M_n)_{n\geq0}$ there exist jointly measurable functions $p^n(x,y)$, $n\geq1$, such that $\mathbb{P}_n^*(x,dy)=p^n(x,y)\psi(dy)$ and

$$\inf_{(x,y)\in\mathfrak{R}\times\mathfrak{R}} p^r(x,y)\ \geq\ \alpha^{1/2}$$

holds for some suitable $r\geq1$, $\alpha>0$ and ψ-positive $\mathfrak{R}$, w.l.o.g. $\psi(\mathfrak{R})=1$. The latter establishes (1.6) with $\varphi=\psi(\cdot\cap\mathfrak{R})$.

Given these settings, let $c=(c^1,c^2)\in\mathbb{R}^r\times\mathbb{R}^r$ be an arbitrary point of increase of ϱ, defined through

$$\varrho(A\times B)\ \stackrel{\text{def}}{=}\ \int_\mathfrak{R}\int_\mathfrak{R}\int_\mathfrak{R} Q_r((x,y),A)Q_r((y,z),B)\ \varphi(dx)\ \varphi(dy)\ \varphi(dz)$$

for $A,B\in\mathfrak{B}^r$. This means that

$$\varrho(I_{\varepsilon/2r}(c))\ =\ \int_\mathfrak{R}\int_\mathfrak{R}\int_\mathfrak{R} Q_r((x,y),I_{\varepsilon/2r}(c^1))Q_r((y,z),I_{\varepsilon/2r}(c^2))\ \varphi(dx)\ \varphi(dy)\ \varphi(dz)\ >\ 0$$

for all $\varepsilon>0$, respectively $\varepsilon=0$ if $\mathbb{P}$ is arithmetic with shift function 0. Consequently, for

$$A^i\ =\ A^i(\varepsilon,m)\ \stackrel{\text{def}}{=}\ \{(x,y)\in\mathfrak{R}^2: Q_r(x,y,I_{\varepsilon/2r}(c^i))>1/m\},$$

$i=1,2$, we have

$$\varphi^3(\{(x,y,z)\in\mathfrak{R}^3:(x,y)\in A^1,(y,z)\in A^2\})\ >\ 0$$

for sufficiently large m.

Keep ε, m fixed in the following. By the Basic Differentiation Theorem for measures there are φ^2-null sets N^1, N^2 such that for $i=1,2$ and all $(x,y)\in A^i-N^i$

$$\lim_{n\to\infty}\varphi^2(A^i\cap\mathcal{S}_n(x,y))/\varphi^2(\mathcal{S}_n(x,y)) \;=\; 1. \tag{6.1}$$

By proceeding similarly as in the proof of Orey's theorem, fix now any (u,v,w) from the set

$$\{(x,y,z):(x,y)\in A^1-N^1,(y,z)\in A^2-N^2\}$$

and then $n\geq 1$ so large that, by (6.1),

$$\begin{aligned}\varphi^2(A^1\cap\mathcal{S}_n(u,v)) \;&\geq\; (3/4)\varphi^2(\mathcal{S}_n(u,v)) \;>\; 0,\\ \varphi^2(A^2\cap\mathcal{S}_n(v,w)) \;&\geq\; (3/4)\varphi^2(\mathcal{S}_n(v,w)) \;>\; 0.\end{aligned} \tag{6.2}$$

Put $A^{i,x}=\{y\in\Re:(x,y)\in A^i\}$ and $\hat{A}^{i,y}=\{x\in\Re:(x,y)\in A^i\}$. Defining

$$\begin{aligned}C \;=\; C_\varepsilon \;&\stackrel{\text{def}}{=}\; \{x\in\mathcal{S}_n(u):\varphi(A^{1,x}\cap\mathcal{S}_n(v))\geq(3/4)\varphi(\mathcal{S}_n(v))\},\\ D \;=\; D_\varepsilon \;&\stackrel{\text{def}}{=}\; \{z\in\mathcal{S}_n(w):\varphi(\hat{A}^{2,z}\cap\mathcal{S}_n(v))\geq(3/4)\varphi(\mathcal{S}_n(v))\},\end{aligned}$$

we infer from (6.2) and Fubini's theorem

$$\begin{aligned}\varphi^2(A^1\cap\mathcal{S}_n(u,v)) \;&=\; \int_{\mathcal{S}_n(u)}\varphi(A^{1,x}\cap\mathcal{S}_n(v))\;\varphi(dx) \;\geq\; (3/4)\varphi^2(\mathcal{S}_n(u,v)) \;>\; 0,\\ \varphi^2(A^2\cap\mathcal{S}_n(v,w)) \;&=\; \int_{\mathcal{S}_n(w)}\varphi(\hat{A}^{2,y}\cap\mathcal{S}_n(v))\;\varphi(dx) \;\geq\; (3/4)\varphi^2(\mathcal{S}_n(v,w)) \;>\; 0\end{aligned}$$

and thus $\varphi(C_\varepsilon)>0$, $\varphi(D_\varepsilon)>0$. Moreover, for all $x\in C$ and $y\in D$,

$$\varphi(A^{1,x}\cap\hat{A}^{2,y}) \;\geq\; \varphi(A^{1,x}\cap\mathcal{S}_n(v))+\varphi(\hat{A}^{2,y}\cap\mathcal{S}_n(v))-\varphi(\mathcal{S}_n(v)) \;\geq\; \varphi(\mathcal{S}_n(v))/2 \;>\; 0.$$

Recalling (3.1), we finally obtain

$$\begin{aligned}Q^*(x,z,I_{\varepsilon/2r}(c)) \;&=\; \int_\Re Q_r(x,y,I_{\varepsilon/2r}(c^1))Q_r(y,z,I_{\varepsilon/2r}(c^2))\;\varphi(dy)\\ &\geq\; \int_{A^{1,x}\cap\hat{A}^{2,y}} Q_r(x,y,I_{\varepsilon/2r}(c^1))Q_r(y,z,I_{\varepsilon/2r}(c^2))\;\varphi(dy)\\ &\geq\; m^{-2}\varphi(A^{1,x}\cap\hat{A}^{2,y}) \;\geq\; \frac{\varphi(\mathcal{S}_n(v))}{2m^2} \;\stackrel{\text{def}}{=}\; \eta(\varepsilon)\end{aligned}$$

for all $x\in C$ and $y\in D$ which completes the proof of Lemma 3.1.

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