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Minimal Position and and Critical Martingale Convergence in Branching Random Walks

(On a paper by Yueyun Hu and Zhan Shi)

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For a branching random walk on the line, an unusual almost sure limit theorem for its minimal position is proved along with a Heyde-Seneta-type convergence result for the critical martingale. The latter answers a question posed by Biggins and Kyprianou [12]. Moreover, the employed method applies to the study of directed polymers on a disordered tree and provides a rigorous proof of a phase transition phenomenon for the partition function in the sense of convergence in probability. This was already described by Derrida and Spohn [17]. Surprisingly, this phase transition disappears when considering almost sure convergence.

This report, not meant for publication, follows to a large extent the original work by Hu and Shi [20] and grew out of the attempt to clarify their arguments in various places. Naturally, this has led to some modifications, the most notable ones being that we show how to reduce to the case of almost certain survival and make strong use of recent "generalized ballot theorems" due to Addario-Berry and Reed [2]. This simplifies a number of (still) technical arguments.

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1. Introduction

This work (not meant for publication) grew out of the attempt to understand a recent article by Y. Hu and Z. Shi [20] which, despite containing very interesting results, we found not very accessible to the reader because its quite technical proofs appear to be incomplete in places and sometimes obscured by the way they are presented. A major flaw that appeared in the first version has meanwhile been corrected. Although the present text may be viewed to some extent as a rewritten version of [20] it also differs in various aspects, two main differences being that we make multiple use of recent sharp probability estimates for random walks obtained by Addario-Berry and Reed [2] as "generalized ballot theorems" and that we provide an argument which allows us to reduce many proofs to the case of nonextinctive branching random walks.

1.1. Critical branching random walk and martingale convergence

We consider a branching random walk (BRW) on the line $\mathbb{R}$. Initially, a particle is sitting at the origin. It produces children that form the first generation, and their displacements from the origin are described by a point process $\mathcal{Z}$ on the line. These children produce independently offspring as well (forming the second generation) their relative displacements with respect to the mother's position being independent copies of $\mathcal{Z}$. This process continues indefinitely unless the population dies out.

For a more formal model description let each individual of the nth generation be represented by a node $v = v_1...v_n$ of length $|v| = n$ in the infinite Ulam-Harris tree $\mathbb{V} = \cup_{n\geq 0}\mathbb{N}^n$ where $\mathbb{N} = \{1, 2, ...\}$ denotes the set of positive integers and $\mathbb{N}^0 \stackrel{\text{def}}{=} \{\emptyset\}$ by convention. Each vertex is uniquely connected to the root $\emptyset$ by the path $v|0 \stackrel{\text{def}}{=} \emptyset \to v|1 \to v|2 \to ... \to v|n = v$, where $v|k \stackrel{\text{def}}{=} v_1...v_k$ for $1 \leq k \leq n$. Denote by $N(v)$ the random number of offspring of individual v and by $X(v) \stackrel{\text{def}}{=} (X_i(v))_{i\geq 1}$ the vector of relative displacements of its children $v1, v2, ...$ By convention, $X_i(v) \stackrel{\text{def}}{=} -\infty$ if $i > N(v)$. The fundamental assumption is that the pairs $(N(v), X(v))$, $v \in \mathbb{V}$, are i.i.d., and we let (N, X) with $X = (X_1, X_2, ...)$ denote a generic copy of these. Putting $S(\emptyset) \stackrel{\text{def}}{=} 0$ and recursively

$$S(vi) \stackrel{\text{def}}{=} S(v) + X_i(v)$$

for any $v \in \mathbb{V}$ and $i \geq 1$, we see that $S(v)$ denotes the absolute position of individual v, where $S(v) = -\infty$ means that v has actually not been realized. The model thus completely specified is called branching random walk in the literature. Its number of particles process $(N_n)_{n\geq 0}$, say, forms a Galton-Watson process (GWP). Let $\mathbb{T}$ denote the pertinent Galton-Watson tree, i.e. $\mathbb{T} \stackrel{\text{def}}{=} \{v \in \mathbb{V} : S(v) > -\infty\}$, and $\mathbb{T}_n \stackrel{\text{def}}{=} \{v \in \mathbb{T} : |v| = n\}$. Put $\mathbb{S} \stackrel{\text{def}}{=} \{(v, S(v)) : v \in \mathbb{T}\}$ and $\mathbb{S}_n \stackrel{\text{def}}{=} \{(v, S(v)) : v \in \mathbb{T}_n\}$ which are the corresponding weighted random trees. Let us further define $\mathcal{F}_n$ to be the σ-field generated by $\{(v, S(v)) : v \in \mathbb{T}, |v| \leq n\}$. Finally, we put $\mathbf{P} \stackrel{\text{def}}{=} \mathbb{P}(\cdot||\mathbb{T}| = \infty)$ and write $\mathbf{E}$ for its expectation operator which means conditioning upon the event of survival.

A number of standing assumptions will be made hereafter. First of all we suppose that the population is supercritical, i.e.

$$\mathbb{E}N > 1, \tag{$C1$}$$

and hence survives with positive probability. For technical reasons we also assume that the number of offspring has a finite second moment, so

$$\mathbb{E}N^2 \ < \ \infty, \tag{$C2$}$$

and that there exists a constant $\overline{x}$ such that

$$\sup_{1\le i\le N} |X_i| \ \le \ \overline{x}, \tag{$C3$}$$

that is displacements of realized particles are uniformly bounded. Defining the Laplace transform

$$\psi(\theta) \ \stackrel{\mathrm{def}}{=} \ \mathbb{E}\Bigg(\sum_{i\ge 1} e^{-\theta X_i}\Bigg) \ = \ \mathbb{E}\Bigg(\sum_{|v|=1} e^{-\theta S(v)}\Bigg), \quad \theta \ge 0,$$

with derivative $\psi'(\theta) = -\mathbb{E}\big(\sum_{i\ge 1} X_i e^{-X_i}\big)$, we further assume that

$$\psi(1) = 1 \quad \text{and} \quad \psi'(1) = 0. \tag{$C4$}$$

The condition $\psi(1) = 1$ implies that the sequence

$$Z_n \ \stackrel{\mathrm{def}}{=} \ \sum_{|v|=n} e^{-S(v)}, \quad n \ge 0 \tag{1.1}$$

constitutes martingale which, by nonnegativity, converges a.s. to a limit Z_∞. It is of fundamental importance in the study of the BRW. If $\psi'(1) < 0$ and

$$\mathbb{E}Z_1 \log^+ Z_1 < \infty \tag{$LlogL$}$$

then Z_∞ is nondegenerate and also the L_1-limit of Z_n, thus $\mathbb{E}Z_\infty = 1$. This extension of the famous Kesten-Stigum theorem for GWP goes back to Biggins [9]. If $\psi'(1) < 0$ but $(L \log L)$ is not assumed, then Biggins and Kyprianou [10] could show the existence of constants c_n, $n \ge 0$ such that Z_n/c_n converges in probability to a a nondegenerate limit which is (strictly) positive on the event of survival. The counterpart of this result for GWP is known as the Heyde-Seneta theorem, and the c_n are therefore called a Heyde-Seneta norming.

For the more delicate case $\psi'(1) = 0$, Biggins and Kyprianou [12] raised the question whether a Heyde-Seneta norming of $(Z_n)_{n\ge 0}$ exists as well. The positive answer is provided by the following theorem.

Theorem 1.1. *Under the stated assumptions, there exists a sequence $(c_n)_{n\ge 1}$ of positive constants satisfying*

$$0 \ < \ \liminf_{n\to\infty} \frac{c_n}{n^{1/2}} \ \le \ \limsup_{n\to\infty} \frac{c_n}{n^{1/2}} \ < \ \infty, \tag{1.2}$$

such that $c_n Z_n$ converges in distribution to a random variable W which is positive on the event of survival, i.e. $\mathbf{P}(W>0)=1$.

The almost sure behavior of Z_n is described in Theorem 1.3 below. In combination, the two theorems provide a clear picture of the asymptotics of Z_n.

Our analysis will also show that (the distribution of) W forms a solution to the stochastic fixed point equation

$$W \ \stackrel{d}{=} \ \sum_{|v|=n} e^{-S(v)} W(v) \tag{1.3}$$

for each $n \geq 1$, where the $W(v)$ are copies of W, mutually independent and independent of $(S(v))_{|v|=n}$. This solution is in fact unique up to a scaling factor and also obtained as the law of the a.s. limit of the *derivative martingale* $W_n \stackrel{\text{def}}{=} \sum_{|v|=n} S(v)e^{-S(v)}$, $n \geq 0$. This was shown by Biggins and Kyprianou [32, Thms. 3,5 and 17] under conditions that are easily inferred from (C1-3).

1.2. The minimal position in the branching random walk

The position of the minimal (leftmost) particle in the nth generation appears to be a natural question in the analysis of a BRW. Various authors have studied the concentration of $M_n \stackrel{\text{def}}{=} \min_{v\in\mathbb{T}_n} S(v)$ about its median, see, for example, Bachmann [7], Bramson and Zeitouni [15] and Section 5 of the survey article by Aldous and Bandyopadhyay [3]. An example of a BRW where $M_n - \text{median}(M_n)$ is tight but not convergent in distribution has recently been constructed by Lifshits [23].

Our interest is in the asymtptotic speed of M_n as $n \to \infty$. Under (C4), it is known that

$$\lim_{n\to\infty} \frac{M_n}{n} \ = \ 0 \quad \text{and} \quad \lim_{n\to\infty} M_n \ = \ \infty \quad \mathbf{P}\text{-a.s.} \tag{1.4}$$

The transience of M_n follows from the fact that $Z_n \to 0$ a.s. The law of large numbers in (1.4) is a classic result which may be found in Hammersley [18], Kingman [21] and Biggins [8]. A refinement was obtained by McDiarmid [26]: there exists a finite constant c such that

$$\limsup_{n\to\infty} \frac{M_n}{\log n} \ \leq \ c \quad \text{a.s.}$$

Our second result to be stated next provides the exact rate at which M_n tends to infinity.

Theorem 1.2. *Assume* (C1–4). *Then*

$$\frac{1}{2} \ = \ \liminf_{n\to\infty} \frac{M_n}{\log n} \ < \ \limsup_{n\to\infty} \frac{M_n}{\log n} \ = \ \frac{3}{2} \quad \mathbf{P}\text{-}a.s. \tag{1.5}$$

and

$$\frac{M_n}{\log n} \ \stackrel{\mathbf{P}}{\to} \ \frac{3}{2}, \tag{1.6}$$

as $n \to \infty$.

Some comments are in order. Without assuming (C4) the law of large numbers in (1.4) takes the form $n^{-1}M_n \to c$ **P**-a.s., where $c \stackrel{\text{def}}{=} \inf\{a \in \mathbb{R} : g(a) \geq 0\}$ with $g(a) \stackrel{\text{def}}{=} \inf_{\theta\geq 0}[\theta a + \log\psi(\theta)]$. If

$$\frac{\theta^*\psi'(\theta^*)}{\psi(\theta^*)} - \log\psi(\theta^*) \;=\; 0 \tag{1.7}$$

for some $\theta^* \in (0,\infty)$, then the BRW having $\widehat{S}(v) \stackrel{\text{def}}{=} \theta^* S(v) + \log\psi(\theta^*)|v|$ instead of $S(v)$ satisfies (C4). Consequently, an analysis of M_n can be reduced to the case where (C4) holds true as long as (1.7) has a solution. Our result for this situation takes the form

$$\frac{\theta^* M_n + \log\psi(\theta^*)n}{\log n} \xrightarrow{\mathbf{P}} \frac{3}{2}$$

conditionally on the system's survival. On the other hand, it is possible that (1.7) has no solution and Theorem 1.2 is not applicable. Such a situation was studied by Bramson [14] by giving a class of BRW for which M_n exhibits an exotic $\log\log n$ behavior.

Let us further mention recent work (doctoral thesis) by Addario-Berry [1] containing very precise asymptotic estimates of $\mathbb{E}M_n$ under suitable conditions. Some of the required probability estimates, phrased as generalized ballot theorems, are written up in [2] and will also enter our proofs further below in some key places. For better reference the result is given as Theorem A.4 in the Appendix in a form tailored to our needs.

Finally, for branching Brownian motion, Theorem 1.2 was proved by Bramson [13] by means of a very deep analysis in connection with the Kolmogorov-Petrovsky-Piscuonov (KPP) equation.

1.3. Directed polymers on a disordered tree

The following model is taken from a well-known paper by Derrida and Spohn [17]: Let $\mathbb{T}$ be a rooted Cayley tree and consider all self-avoiding random walks (= directed polymers) of n steps on $\mathbb{T}$ starting from the root. A random variable (= potential) is attached to each edge of the tree, and it is assumed that these random variables are i.i.d. For each walk ω, its energy $E(\omega)$ is the sum of the potentials of the edges visited by ω. So the partition function is

$$\mathcal{Z}_{n,\beta} \;\stackrel{\text{def}}{=}\; \sum_{\omega} e^{-\beta E(\omega)},$$

where the sum is over all n-step self-avoiding walks on $\mathbb{V}$ and $\beta > 0$ is the inverse temperature.

Returning to our model in which $\mathbb{T}$ is a Galton-Watson tree we see that the energy $E(\omega)$ corresponds to a partial sum of the BRW at level n. The associated partition function becomes

$$Z_{n,\beta} \;\stackrel{\text{def}}{=}\; \sum_{|v|=n} e^{-\beta S(v)}, \quad \beta > 0. \tag{1.8}$$

Plainly, $Z_{n,1}$ is just the Z_n defined in (1.1).

If $0 < \beta < 1$ the study of $Z_{n,\beta}$ boils down to the case $\psi'(1) < 0$ (when replacing ψ with $\psi_\beta(\theta) \stackrel{\text{def}}{=} \frac{\psi(\theta\beta)}{\psi(\beta)}$ and observing that $\psi_\beta'(1) = \frac{\beta\psi'(\beta)}{\psi(\beta)} < 0$). In this case, by Biggins' convergence theorem [9], the martingale $\frac{Z_{n,\beta}}{\mathbb{E}Z_{n,\beta}} = \frac{Z_{n,\beta}}{\psi(\beta)^n}$ converges a.s. to a limiting variable which is positive on the event of survival.

If $\beta \geq 1$, then $\psi_\beta'(1) \geq 0$, and it is this case we deal with in the present paper. First note that, as a direct consequence of Theorem 1.1, $\lim_{n\to\infty} \mathbf{P}(n^{-\varepsilon} \leq n^{1/2} Z_n \leq n^{\varepsilon}) = 1$ for all $\varepsilon > 0$ or, equivalently,

$$\frac{\log Z_n}{\log n} \stackrel{\mathbf{P}}{\to} -\frac{1}{2},$$

as $n \to \infty$. Our next result shows that this statement can be sharpened to almost sure convergence.

Theorem 1.3. *Assume* (C1–4). *Then*

$$\lim_{n\to\infty} \frac{\log Z_n}{\log n} = -\frac{1}{2} \quad \mathbf{P}\text{-}a.s. \tag{1.9}$$

Our next result shows that the partition function exhibits a phase transition at $\beta = 1$.

Theorem 1.4. *Assume* (C1-4) *and let* $\beta > 1$. *Then*

$$-\frac{3\beta}{2} = \liminf_{n\to\infty} \frac{\log Z_{n,\beta}}{\log n} < \limsup_{n\to\infty} \frac{\log Z_{n,\beta}}{\log n} = -\frac{\beta}{2} \quad \mathbf{P}\text{-}a.s. \tag{1.10}$$

and

$$\frac{\log Z_{n,\beta}}{\log n} \stackrel{\mathbf{P}}{\to} -\frac{3\beta}{2}, \qquad n \to \infty. \tag{1.11}$$

Let us note that the remarks after Theorem 1.2 as for condition (C4) apply here as well.

The further organization of this paper is as follows: Section 2 provides the necessary facts about two useful measure changes. The first one allows us often to consider only the case where the BRW survives almost surely, the second one based on size-biasing provides an important link to an associated zero-mean random walk via the specification of a particular path within the tree, called spine, at each finite level. Preliminary tail and moment results for the critical martingale $(Z_n)_{n\geq 0}$ are the content of Section 3, followed by a first result on the almost sure behavior of M_n in Section 4 that particularly sharpens McDiarmid's result stated before Theorem 1.2. Section 5 may be seen as the core part of this article and contains two crucial moment results that are fundamental for the proofs of the main results to be found in Section 6. A number of auxiliary results from fluctuation theory for random walks including the already mentioned crucial probability estimates by Addario-Berry and Reed [2] are presented in an appendix. Of particular importance is Lemma A.5 we derive from their results and which, in a slighly weaker form, is used in [20] without giving any proof.

2. Measure changes, weighted trees and spines

This section is devoted to a short discussion of two change of measure techniques for weighted trees. The first one will frequently allow us to restrict ourselves to the case of almost certain survival and thus to avoid the technical nuisance of conditioning with respect to the survival event. For simple supercritical GWP, the argument is well-known and may be found e.g. in Athreya and Ney's standard textbook [6]. The second change of measure technique, known under the key words "size-biasing" and "spinal trees", is a harmonic transform based on the mean one martingale $(Z_n)_{n\geq 0}$. It has been described in various branching contexts, see for example Chauvin et al. [16], Lyons et al. [25], Biggins and Kyprianou [11], Hardy and Harris [19]. Subsection 2.2 will summarize the relevant facts for the BRW as given here.

2.1. Reduction to the case of almost certain survival

Let us start by briefly stating the attempted result for the settled case of the simple supercritical GWP $(N_n)_{n\geq 0}$. Let $(p_j)_{j\geq 0}$ denote the pertinent offspring distribution, $f(s) = \mathbb{E}s^N$ its generating function (g.f.) and $q \in [0,1)$ the extinction probability. Recall that $\mathbb{T}(v)$ denotes the Galton-Watson tree of individuals stemming from $v \in \mathbb{T}$ and define

$$\widehat{\mathbb{T}} \stackrel{\text{def}}{=} \{v \in \mathbb{T} : |\mathbb{T}(v)| = \infty\}$$

to be the subtree of $\mathbb{T}$ of individuals that generate surviving subpopulations. Then the pertinent population size process $(\widehat{N}_n)_{n\geq 0}$ forms a GWP with offspring g.f. $\hat{f}(s) = \frac{f(q+(1-q)s)-q}{1-q}$ under $\mathbf{P}$. Moreover, $(\widehat{N}_n)_{n\geq 0}$ has extinction probability 0 and satisfies

$$\lim_{n\to\infty} \frac{\widehat{N}_n}{N_n} = 1-q \quad \mathbf{P}\text{-a.s.}$$

This allows to study asymptotic properties of $(N_n)_{n\geq 0}$ by looking at $(\widehat{N}_n)_{n\geq 0}$ instead.

In the present work, we want to use a similar strategy that can be outlined as follows:

(1) Prove the results of Section 1 under the additonal condition

$$\mathbb{P}(N = 0) = 0 \tag{C5}$$

which means almost certain survival of the considered population.

(2) If, for $\beta \geq 1$ and $n \geq 0$, $\widehat{Z}_n, \widehat{Z}_{n,\beta}$ and $\widehat{M}_n$ denote the counterparts of $Z_n, Z_{n,\beta}$ and M_n, respectively, for the pruned tree $\widehat{\mathbb{T}}$, that is

$$\widehat{Z}_{n,\beta} \stackrel{\text{def}}{=} \sum_{v\in\widehat{\mathbb{T}}_n} e^{-\beta S(v)}, \quad \widehat{Z}_n \stackrel{\text{def}}{=} \widehat{Z}_{n,1} \quad \text{and} \quad \widehat{M}_n \stackrel{\text{def}}{=} \min_{v\in\widehat{\mathbb{T}}_n} S(v),$$

then prove that, on the survival event $\{|\mathbb{T}| = \infty\}$, $\widehat{Z}_n, \widehat{Z}_{n,\beta}$ and $\widehat{M}_n$ have the same asymptotic behavior as their "nonhatted" counterparts (in an appropriate sense).

(3) Then use the results of Section 1 for the "hatted" variables to draw the same conclusions for the "nonhatted" ones on $\{|\mathbb{T}|=\infty\}$.

The key tool that facilitates Steps 2 and 3 is the following simple lemma.

Lemma 2.1. *Assume* (C1–4) *and let k be so large that $q \le 1-\frac{1}{k}$. Then, for each $n \ge 0$ and $\beta \ge 1$,*

$$\widehat{Z}_{n,\beta} \;\le\; Z_{n,\beta} \;\le\; \sum_{j=1}^{k} \widehat{Z}_{n,\beta}^{(j)} \quad \textit{and} \quad \min_{1\le j\le k} \widehat{M}_n^{(j)} \;\le\; M_n \;\le\; \widehat{M}_n,$$

where the $(\widehat{Z}_{n,\beta}^{(j)}, \widehat{M}_n^{(j)})$, $j \ge 1$, are identically distributed with $(\widehat{Z}_{n,\beta}^{(1)}, \widehat{M}_n^{(1)}) \stackrel{\text{def}}{=} (\widehat{Z}_{n,\beta}, \widehat{M}_n)$.

We emphasize that we do neither claim the whole sequences $(\widehat{Z}_{n,\beta}^{(j)}, \widehat{M}_n^{(j)})_{n\ge 0}$ to form copies of $(\widehat{Z}_n, \widehat{M}_n)_{n\ge 0}$, nor the independence of the $(\widehat{Z}_{n,\beta}^{(j)}, \widehat{M}_n^{(j)})$, $1 \le j \le n$, for n fixed. This will become clear from the proof below. To define the $(\widehat{Z}_{n,\beta}^{(j)}, \widehat{M}_n^{(j)})$ it may be necessary to enlarge the underlying probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Proof. Putting $I(v) \stackrel{\text{def}}{=} \mathbf{1}_{\{|\mathbb{T}(v)|=\infty\}}$ for $v \in \mathbb{T}$, we have $I(v) \stackrel{d}{=} \text{Bernoulli}(1-q)$ and

$$\widehat{Z}_{n,\beta} \;=\; \sum_{v\in\mathbb{T}_n} e^{-\beta S(v)} I(v)$$

for each $n \ge 0$ and $\beta \ge 1$. Let $\{(I_1(v), ..., I_k(v)) : v \in \mathbb{T}\}$ be a family of i.i.d. k-variate exchangeable random vectors independent of the $(S(v))_{v\in\mathbb{T}}$, taking values in $\{0,1\}^k$ and with generic copy $(I_1, ..., I_k)$ such that

$$\mathbb{P}(I_1 + ... + I_k = 1) \;=\; \gamma \;=\; 1 - \mathbb{P}(I_1 + ... + I_k = k), \quad \gamma \;\stackrel{\text{def}}{=}\; q\left(1 + \frac{1}{k-1}\right).$$

This distribution is obtained as the convex combination with weights γ and $1-\gamma$ of the conditional distributions of a k-variate vector of independent Bernoulli$(1-q)$ variables conditioned upon its sum being 1 and k, respectively. By exchangeability, $\mathbb{P}(I_j = 1) = \frac{\gamma}{k} + 1 - \gamma = 1 - q$ and thus indeed $I_j \stackrel{d}{=} \text{Bernoulli}(1-q)$ for each $j = 1, ..., k$.

The assertions of the lemma are now easily obtained when defining $\widehat{\mathbb{T}}_n^{(j)} \stackrel{\text{def}}{=} \{v \in \mathbb{T}_n : I_j(v) = 1\}$,

$$\widehat{Z}_{n,\beta}^{(j)} \;\stackrel{\text{def}}{=}\; \sum_{v\in\widehat{\mathbb{T}}_n^{(j)}} e^{-\beta S(v)} \;=\; \sum_{v\in\mathbb{T}_n} e^{-\beta S(v)} I_j(v) \quad \text{and} \quad \widehat{M}_n^{(j)} \;\stackrel{\text{def}}{=}\; \min_{v\in\widehat{\mathbb{T}}_n^{(j)}} S(v). \qquad \diamondsuit$$

The following proposition shows for some parts of the results in Section 1 how, once proved under (C5), they carry over to the general supercritical case, especially (2.2) will allow us to prove Theorem 1.1 for the nonextinctive situation only.

Proposition 2.2. *Assume* (C1–4) *and the results of Section 1 be true for the hatted variables* $\widehat{Z}_n, \widehat{Z}_{n,\beta}$ *and* $\widehat{M}_n$. *Then, as* $n \to \infty$,

$$\frac{\widehat{Z}_n}{Z_n} \stackrel{\mathbf{P}}{\to} 1-q, \tag{2.2}$$

$$\frac{\log Z_{n,\beta}}{\log n} - \frac{\log \widehat{Z}_{n,\beta}}{\log n} \stackrel{\mathbf{P}}{\to} 0 \tag{2.3}$$

for each $\beta \geq 1$, *and*

$$\frac{M_n}{\log n} - \frac{\widehat{M}_n}{\log n} \stackrel{\mathbf{P}}{\to} 0. \tag{2.4}$$

Proof. By (1.11) for $\widehat{Z}_{n,\beta}$, we have

$$\lim_{n\to\infty} \mathbf{P}(n^{-\varepsilon} \leq n^{3\beta/2}\widehat{Z}_{n,\beta} \leq n^{\varepsilon}) \;=\; 1$$

for all $\varepsilon > 0$, and this implies

$$\lim_{n\to\infty} \mathbf{P}\left(n^{-\varepsilon} \leq n^{3\beta/2} \sum_{j=1}^{k} \widehat{Z}_{n,\beta}^{(j)} \leq n^{\varepsilon} \right) \;=\; 1$$

for all $\varepsilon > 0$. Now take logarithms in (2.1) and divide by $\log n$ to see that (2.3) holds true for any $\beta \geq 1$.

In order to infer (2.4), it suffices to note that (1.6) for $\widehat{M}_n$ implies the very same for $\min_{1\leq j\leq k} \widehat{M}_n^{(j)}$ in Lemma 2.1, so that the assertion is a direct consequence of that lemma.

Turning to (2.2), write

$$\frac{\widehat{Z}_n}{Z_n} - (1-q) \;=\; \sum_{v\in\mathbb{T}_n} \frac{e^{-S(v)}}{Z_n}\Big(I(v) - (1-q)\Big).$$

By conditioning upon $\mathcal{F}_n$, we infer

$$\begin{aligned}
\mathbf{E}\left(\frac{\widehat{Z}_n}{Z_n} - (1-q)\right)^2 \;&\leq\; \frac{1}{1-q}\,\mathbb{E}\left(\mathbf{1}_{\{Z_n>0\}} \sum_{v\in\mathbb{T}_n} \frac{e^{-2S(v)}}{Z_n^2}\mathbb{E}\Big(I(v)-(1-q)\Big)^2\right) \\
&=\; q\,\mathbb{E}\left(\mathbf{1}_{\{Z_n>0\}} \sum_{v\in\mathbb{T}_n} \frac{e^{-2S(v)}}{Z_n^2}\right) \\
&=\; q\,\mathbb{E}\left(\mathbf{1}_{\{Z_n>0\}} \frac{Z_{n,2}}{Z_n^2}\right).
\end{aligned} \tag{2.5}$$

Now use (2.3) in combination with (1.9) and (1.11) for $\widehat{Z}_n$ and $\widehat{Z}_{n,2}$, respectively, to infer

$$\frac{\log \frac{Z_{n,2}}{Z_n^2}}{\log n} \;=\; \frac{\log \widehat{Z}_{n,2}}{\log n} - \frac{2\log \widehat{Z}_n}{\log n} + \frac{\log \frac{Z_{n,2}}{Z_n^2} - \log \frac{\widehat{Z}_{n,2}}{\widehat{Z}_n^2}}{\log n} \stackrel{\mathbf{P}}{\to} -\frac{3}{2} + 1 + 0 \;=\; -\frac{1}{2}$$

and thereupon in (2.5) by dominated convergence (as $\frac{Z_{n,2}}{Z_n^2} \leq 1$) that

$$\lim_{n\to\infty} \mathbb{E}\left(\mathbf{1}_{\{Z_n>0\}}\frac{Z_{n,2}}{Z_n^2}\right) \;=\; 0.$$

Consequently, $\frac{\widehat{Z}_n}{Z_n}$ converges to $1-q$ in L_2 and particularly in probability (under $\mathbf{P}$). $\diamondsuit$

2.2. Size-biasing and spinal trees

In the following, we will briefly present some required material on size-biasing and spinal trees in connection with BRW. Generally speaking, size-biasing has proved to be a very effective tool from harmonic analysis in the study of various branching models. Here we restrict ourselves to a rather informal description of those facts that are needed in this article.

As $(Z_n)_{n\geq 0}$ constitutes a positive martingale, we can uniquely define a new probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_\infty)$, $\mathcal{F}_\infty \stackrel{\text{def}}{=} \sigma(\cup_{n\geq 0}\mathcal{F}_n)$, such that

$$\widehat{\mathbb{P}}_{|\mathcal{F}_n} \;=\; Z_n\,\mathbb{P}_{|\mathcal{F}_n}$$

for all $n \geq 0$. Fix n and define a random variable $V_n = V_n^{(n)}$ taking values in $\mathbb{T}_n$ such that

$$\widehat{\mathbb{P}}(V_n = v|\mathcal{F}_\infty) \;=\; \frac{e^{-S(v)}}{Z_n}$$

for each $v \in \mathbb{T}_n$. Hence V_n picks a node $\in \mathbb{T}_n$ in accordance with the size-biased distribution obtained from $\{e^{-S(v)} : v \in \mathbb{T}_n\}$. Let $(V_0, ..., V_n)$ denote the unique random path connecting the root $V_0 = \emptyset$ with V_n. It is not difficult to verify that, conditioned upon $\mathcal{F}_\infty$, this random vector constitutes a Markov chain on the random subtree $\mathbb{T}_{\leq n} \stackrel{\text{def}}{=} \{v \in \mathbb{T} : |v| \leq n\}$ with one-step transition probabilities

$$P(v, vi) \;\stackrel{\text{def}}{=}\; \frac{e^{-(S(vi)-S(v))}Z_{n-k-1}(vi)}{Z_{n-k}(v)}, \quad v \in \mathbb{T}_k,\ vi \in \mathbb{T}_{k+1}.$$

Though suppressed in the notation, it should be noticed that $P(\cdot,\cdot)$ depends on n. The thus obtained random line of individuals $(V_0, ..., V_n)$ in $\mathbb{T}_{\leq n}$ is called its *spine*, and the main observation stated in Proposition 2.3 below is that these individuals produce offspring and pick a position in a different way than the other population members.

For an individual $v \in \mathbb{T}\backslash\{\emptyset\}$ let $\hat{v}$ denote its mother and put

$$\Delta S(v) \;\stackrel{\text{def}}{=}\; S(v) - S(\hat{v}).$$

For $1 \leq k \leq n$, define

$$\mathcal{I}_k^{(n)} \;\stackrel{\text{def}}{=}\; \big\{v \in \mathbb{T}_k : \hat{v} = V_{k-1} \text{ and } v \neq V_k\big\}$$

to be the random set of nonspinal offspring in generation k stemming from the spinal mother V_{k-1}. Notice that $\mathcal{I}_k^{(n)}$ may be empty. Finally, put

$$\mathcal{G}_n \;\stackrel{\text{def}}{=}\; \sigma\left((V_k, S(V_k), \mathcal{I}_k^{(n)})_{1\le k\le n}, \sum_{v\in\mathcal{I}_k^{(n)}} \delta_{\Delta S(v)}\right),$$

and for later purposes

$$n^* \;\stackrel{\text{def}}{=}\; \min\left\{0\le k<n : S(V_k) = \min_{0\le j<n} S(V_j)\right\} \quad\text{and}\quad V_n^* \;\stackrel{\text{def}}{=}\; V_{n^*}.$$

Plainly, n^*, V_n^* and $S(V_n^*)$ are $\mathcal{G}_n$-measurable. Recall that $\mathbb{S} = \{(v, S(v)) : v\in\mathbb{T}\}$ and put $\mathbb{S}_{\le n} \stackrel{\text{def}}{=} \{(v, S(v)) : v\in\mathbb{T}_{\le n}\}$. Following our usual convention, we let $\mathbb{S}_{\le n}(v)$ denote the shifted counterpart of $\mathbb{S}_{\le n} = \mathbb{S}_{\le n}(\emptyset)$ rooted at v, more precisely

$$\mathbb{S}_{\le n}(v) \;\stackrel{\text{def}}{=}\; \{(vw, S(vw)-S(v)) : w\in\mathbb{T}_{\le n}(v)\}.$$

The following proposition provides all relevant information on the distribution of $\mathbb{S}_{\le n}$ and the spine under $\widehat{\mathbb{P}}$.

Proposition 2.3. *Assume (C4). Then the following assertions hold true under the probability measure $\widehat{\mathbb{P}}$ for any fixed $n\ge 1$:*

(a) The random variables $\big(\sum_{v\in\mathcal{I}_k^{(n)}}\delta_{\Delta S(v)}, \Delta S(V_k)\big)$, $1\le k\le n$, are i.i.d. with the same distribution as $\big(\sum_{v\in\mathcal{I}_1^{(n)}}\delta_{\Delta S(v)}, \Delta S(V_1)\big)$.

(b) Conditioned upon $\mathcal{G}_n$, the shifted weighted subtrees $\mathbb{S}_{\le n-|v|}(v)$, $v\in\cup_{k=1}^n\mathcal{I}_k^{(n)}$, are independent, and $\widehat{\mathbb{P}}(\mathbb{S}_{\le n-|v|}(v)\in\cdot|\mathcal{G}_n) = \mathbb{P}(\mathbb{S}_{n-|v|}\in\cdot)$.

(c) The random vector $\big(S(V_k), |\mathcal{I}_k^{(n)}|\big)_{0\le k\le n}$, where $|\mathcal{I}_0^{(n)}| = 0$, has i.i.d. increments and $\widehat{\mathbb{E}}S(V_1) = 0$.

(d) For any nonnegative measurable $G:\mathbb{R}\to[0,\infty)$ and $1\le k\le n$

$$\widehat{\mathbb{E}}G(S(V_k)) \;=\; \mathbb{E}\left(\sum_{|v|=k} e^{-S(v)}G(S(v))\right). \tag{2.6}$$

We omit the proof of this result which follows along similar arguments as those provided for supercritical Galton-Watson trees by Lyons et al. [25]. Equation (2.6) may also be found in Biggins and Kyprianou [10].

3. Preliminary tail and moment results for Z_n

In this section, we prove some auxiliary results that provide information on the law of Z_n near 0. For simplicity, almost certain survival of the given BRW (condition (C5)) will be a standing assumption throughout this section.

Proposition 3.1. *For any $\varepsilon > 0$, there exists $\kappa > 0$ such that*

$$\mathbb{P}\Big(Z_n \le n^{-(1/2)-\varepsilon}\Big) \;\le\; \exp(-n^{\kappa})$$

for all sufficiently large n.

Proof. Recall from the end of Subsection 1.1 that (under (C1-5)), the derivative martingale $W_n = \sum_{|v|=n} S(v)e^{-S(v)}$, $n \ge 0$, converges a.s. to a positive random variable W the law of which forms the unique solution up to scaling of the stochastic fixed point equation (1.3).

Let φ denote the Laplace transform of W. Then (1.3) in terms of Laplace transforms becomes

$$\varphi(t) \;=\; \mathbb{E}\Bigg(\prod_{|v|=n} \varphi(e^{-S(v)}t)\Bigg), \quad t \ge 0, \tag{3.1}$$

for each $n \ge 1$. The process

$$\Phi_n(t) \;\stackrel{\text{def}}{=}\; \prod_{|v|=n} \varphi(e^{-S(v)}t), \quad n \ge 0 \tag{3.2}$$

is also a martingale for any $t \in [0,\infty)$ with a.s. limit $\Phi_\infty(t)$, say. We recall from Liu [24] that

$$-\log\varphi(t) \;\simeq\; -t\log t, \quad \text{as } t \to 0, \tag{3.3}$$

and this implies

$$-\log\Phi_\infty(t) \;=\; tW \quad \text{a.s.}$$

for all $t \ge 0$. We put $\Phi_n \stackrel{\text{def}}{=} \Phi_n(1)$, hence $W = -\log\Phi_\infty$ a.s.

Lemma 3.2. *There exists $\kappa > 0$ and $a_0 \ge 1$ such that*

$$\mathbb{E}\Phi_n^a \;\le\; \mathbb{E}\Phi_\infty^a \;\le\; \exp(-a^{\kappa}) \tag{3.4}$$

for all $a \ge a_0$ and $n \ge 1$.

Proof. The first inequality is clear because $(\Phi_n^a)_{n\ge 0}$ forms a submartingale for any $a \ge 1$. For the second one, notice that $\mathbb{E}\Phi_\infty^a = \mathbb{E}e^{-aW} = \varphi(a)$. Put $\psi(t) \stackrel{\text{def}}{=} -\log\varphi(t)$ for $t \ge 0$. Then ψ is a decreasing function, and we infer with help of (3.1)

$$\begin{aligned}
\psi(t) \;&=\; -\log\mathbb{E}\Bigg(\prod_{|v|=n} \varphi(e^{-S(v)}t)\Bigg) \\
&\ge\; -\log\mathbb{E}\varphi(e^{-n\overline{x}}t)^{N_n} \\
&=\; -\log f_n\varphi(e^{-n\overline{x}}t) \\
&=\; -\log f_n(\exp[-\psi(e^{-n\overline{x}}t)])
\end{aligned}$$

for all $t \geq 0$ and $n \geq 1$, where f_n denotes the generating function of N_n (and thus the n-fold iteration of f). Now use that, for any fixed $s \in [0,1)$, we have $f_n(s) \leq C_s\gamma^n$ for some $C > 0$, $\gamma \in (0,1)$ and all $n \geq 1$ (see Section I.11 in [6] and recall the almost certain survival of the BRW) to obtain

$$\psi(e^{n\overline{x}}) \;\geq\; -\log f_n(\psi(1)) \;\geq\; -\log(C_{\psi(1)}\gamma^n)$$

or, equivalently,

$$\varphi(e^{n\overline{x}}) \;\leq\; C_{\psi(1)}\gamma^n$$

for all $n \geq 1$. Obviously, this gives the second inequality in (3.4) for some $\kappa > 0$. $\diamondsuit$

Proof of Proposition 3.1. In the following, C will always denote a suitable generic constant which may differ from line to line. Pick $c > 0$ such that $\mathbb{P}(W \leq c) \geq e^{-1}$. Then $\varphi(t) = \mathbb{E}e^{-tW} \geq e^{-ct}\mathbb{P}(W \leq c) \geq e^{-ct-1}$ and thus $\psi(t) \leq ct + 1$ for all $t \geq 0$. With the help of (3.2) and (3.3), we next infer for any $\lambda \geq 1$

$$\begin{aligned} -\log\Phi_n &= -\sum_{|v|=n}\psi(e^{-S(v)}) \\ &\leq C\sum_{|v|=n}\mathbf{1}_{\{S(v)\geq 1\}}S(v)e^{-S(v)} \;+\; \sum_{|v|=n}\mathbf{1}_{\{S(v)<1\}}\big(ce^{-S(v)}+1\big) \\ &\leq C\sum_{|v|=n}\mathbf{1}_{\{S(v)\geq\lambda\}}S(v)e^{-S(v)} \;+\; (e^{-1}+c+1)\sum_{|v|=n}e^{-S(v)} \\ &\leq C\left(\sum_{|v|=n}\mathbf{1}_{\{S(v)\geq\lambda\}}S(v)e^{-S(v)} + \lambda Z_n\right) \end{aligned} \tag{3.5}$$

where C does not depend on λ. Consequently,

$$\begin{aligned} \mathbb{P}(\lambda Z_n \leq z) &\leq \mathbb{P}\left(-\frac{1}{C}\log\Phi_n - \sum_{|v|=n}\mathbf{1}_{\{S(v)\geq\lambda\}}S(v)e^{-S(v)} \leq z\right) \\ &\leq \mathbb{P}(-\log\Phi_n \leq 2Cz) \;+\; \mathbb{P}\left(\sum_{|v|=n}\mathbf{1}_{\{S(v)\geq\lambda\}}S(v)e^{-S(v)} \geq z\right) \end{aligned} \tag{3.6}$$

for $0 < z \leq 1$.

Using Markov's inequality in combination with Lemma 3.2, we obtain

$$\mathbb{P}(-\log\Phi_n \leq 2Cz) \;=\; \mathbb{P}(\Phi_n \geq e^{-2Cz}) \;\leq\; e^{2C}\mathbb{E}\Phi_n^{1/z} \;\leq\; C\exp(-z^{-\kappa}) \tag{3.7}$$

for all $n \geq 1$ and $z \in (0, a_0^{-1}]$.

As for the second probability in (3.6), use $S(v) \leq n\overline{x}$ and the large deviations estimate

$\mathbb{P}(S_n \ge \lambda) \le e^{-\lambda^2/nC}$ for all $n \ge 1$ to infer

$$\begin{aligned}\mathbb{P}\left(\sum_{|v|=n} \mathbf{1}_{\{S(v)\ge\lambda\}} S(v) e^{-S(v)} \ge z\right)\\ \le\ \frac{n\overline{x}}{z}\mathbb{E}\left(\sum_{|v|=n} \mathbf{1}_{\{S(v)\ge\lambda\}} e^{-S(v)}\right)\ =\ \frac{n\overline{x}}{z}\mathbb{P}(S_n \ge \lambda)\ \le\ \frac{n\overline{x}}{z} e^{-\lambda^2/nC}\end{aligned} \tag{3.8}$$

for all $n \ge 1$, $z \in (0,1]$ and $\lambda \ge 1$.

Now a combination of (3.6), (3.7) and (3.8) provides us with

$$\mathbb{P}(\lambda Z_n \le z)\ \le\ C\exp(-z^{-\kappa})\ +\ \frac{n\overline{x}}{z} e^{-\lambda^2/nC} \tag{3.9}$$

for all $n \ge 1$, $z \in (0, a_0^{-1}]$, $\lambda \ge 1$ and a sufficiently large $C > 0$. Choosing $\lambda = n^{(1+\varepsilon)/2}$ and $z = n^{-\varepsilon/2}$ for any fixed $\varepsilon \in (0,1)$, we arrive at

$$\mathbb{P}(Z_n \le n^{-\varepsilon})\ =\ \mathbb{P}(n^{(1+\varepsilon)/2} Z_n \le n^{-\varepsilon/2})\ \le\ C\exp(-n^{\kappa\varepsilon/2}) + \overline{x} n^{1+\varepsilon/2}\exp(-n^{\varepsilon}/C)$$

for all n sufficiently large, and this clearly yields Proposition 3.1. ◊

Choosing $z = x^{2/(\kappa+2)}$ and $\lambda = n^{1/2} x^{-\kappa/(\kappa+2)}$ in (3.9), we get

$$\mathbb{P}(n^{1/2} Z_n \le x)\ \le\ C\Big(\exp(-x^{-2\kappa/(\kappa+2)}) + n\overline{x} x^{-2/(\kappa+2)}\exp(-x^{-2\kappa/(\kappa+2)}/C)\Big) \tag{3.10}$$

for all $n \ge 1$, $x \in (0, 1/a_0^{(\kappa+2)/2}]$ and some $C > 0$. In fact, since $x^{-2/(\kappa+2)} \ge 1$, the bound in (3.10) may be replaced with 1 if $C\overline{x}n\exp(-x^{-2\kappa/(\kappa+2)}/C) \ge 1$ and thus for $x \ge a_n \stackrel{\text{def}}{=} (C\log(C\overline{x}n))^{-(\kappa+2)/2\kappa}$. For $0 < x \le a_n$ and any $\varepsilon \in (0,1)$ we further infer

$$C\overline{x}n x^{-2/(\kappa+2)}\exp(-x^{-2\kappa/(\kappa+2)}/C)\ \le\ n^{\varepsilon} h_{\varepsilon}(x),$$

where

$$h_{\varepsilon}(x)\ \stackrel{\text{def}}{=}\ (C\overline{x})^{\varepsilon} x^{-2/(\kappa+2)}\exp(-\varepsilon x^{-2\kappa/(\kappa+2)}/C).$$

Plainly, $x^{-s}h_{\varepsilon}(x)$ remains bounded on $(0,\infty)$ for all $s > 0$. With these estimates we can now easily prove the following useful lemma.

Lemma 3.3. *There exists $s_0 > 0$ such that*

$$\mathbb{E}Z_n^{-s}\ \le\ n^{(s/2)+o(1)}, \quad n \to \infty$$

for all $0 < s < s_0$.

PROOF. Fix $s>0$ and any $\varepsilon\in(0,1)$. Put $b\stackrel{\rm def}{=}1/a_0^{(\kappa+2)/2}$ and $\widehat{Z}_n\stackrel{\rm def}{=}n^{1/2}Z_n$. Then

$$\begin{aligned}
\mathbb{E}\widehat{Z}_n^{-s} &\le \mathbb{E}(\widehat{Z}_n\wedge b)^{-s} \\
&= b^{-s} + \int_{b^{-s}}^{\infty}\mathbb{P}(\widehat{Z}_n^{-s}\ge x)\,dx \\
&= b^{-s} + \int_0^b sx^{-(s+1)}\,\mathbb{P}(\widehat{Z}_n\le x)\,dx \\
&\le b^{-s}(1+C) + a_n^{-s} + n^{\varepsilon}\int_0^{a_n} sx^{-(s+1)}\,h_\varepsilon(x)\,dx \\
&\le C\Big(b^{-s} + a_n^{-s} + n^{\varepsilon}a_n\Big)
\end{aligned}$$

for all $n\ge 2$, and this gives the assertion because $\varepsilon>0$ was arbitrarily chosen. ◊

Lemma 3.4. *Let $W_{2,n}\stackrel{\rm def}{=}\sum_{|v|=n}(n^{1/2}\vee S(v))e^{-S(v)}$ for $n\ge 0$. Then*

$$\sup_{n\ge 0}\mathbb{E}W_{2,n}^{-s} < \infty$$

for all $s>0$.

PROOF. It follows from (3.5) with $\lambda=1$ that

$$W_{2,n} \ge \frac{1}{2}\left(\sum_{|v|=n}\mathbf{1}_{\{S(v)\ge 1\}}S(v)e^{-S(v)} + Z_n\right) \ge -C\log\Phi_n$$

for all $n\ge 0$. Therefore

$$\mathbb{P}(W_{2,n}\le x) \le \mathbb{P}(-\log\Phi_n\le x/C) \le e^{-C}\mathbb{E}\Phi_n^{1/x},$$

which, by Lemma 3.2, is bounded by e^{-1/x^κ} for $0<x\le a_0^{-1}$ and suitbaly chosen $\kappa>0, a_0\ge 1$. Consequently,

$$\begin{aligned}
\mathbb{E}W_{2,n}^{-s} &= \int_0^\infty sx^{-(s+1)}\,\mathbb{P}(W_{2,n}\le x)\,dx \\
&\le e^{-C}\int_0^{a_0} sx^{-(s+1)}\,e^{-1/x^\kappa}\,dx + a_0^{-s}
\end{aligned}$$

which proves the assertion as $\int_0^{a_0}sx^{-(s+1)}\,e^{-1/x^\kappa}\,dx<\infty$ for any $s>0$. ◊

4. The almost sure behavior of M_n: upper bounds

In this section, we will prove one half of (1.5) of Theorem 1.2, namely that $\frac{1}{2}$ and $\frac{3}{2}$ are almost sure upper bounds for the liminf, respectively limsup of $M_n/\log n$ as $n\to\infty$. Throughout the remainder of this article and as before, C will always denote a positive finite constant that may differ from line to line.

Proposition 4.1. *Assume* (C1–4). *Then*

$$\liminf_{n\to\infty}\frac{M_n}{\log n} \;\le\; \frac{1}{2} \quad \textit{and} \quad \limsup_{n\to\infty}\frac{M_n}{\log n} \;\le\; \frac{3}{2} \quad \mathbf{P}\text{-}a.s. \tag{4.1}$$

PROOF. It suffices to consider the nonextinctive case ($q = 0$) because, in the notation of Subsection 2.1, $M_n \le \widehat{M}_n$. So we further assume (C5) hereafter.

Fix any $a > \frac{1}{2}$ and $b > 0$ (greater than the lattice span d of S_1 under $\widehat{\mathbb{P}}$ if $d > 0$), and put $n_j \stackrel{\text{def}}{=} 2^j$ for $j \ge 0$. For $j \ge 0$, $n \in \{n_j, ..., n_{j+1} - 1\}$ and $x \in \mathbb{R}$ let $\mathbb{A}_n^x$ be the random set of individuals $v \in \mathbb{T}_n$ such that

$$x + S(v|k) \ge 0 \quad \text{for } k = 0, ..., n \quad \text{and} \quad x + S(v) \in [a \log n, a \log n + b].$$

Following our usual convention, let $\mathbb{A}_n^x(w)$ denote the corresponding set when considering the BRW associated with the Galton-Watson subtree $\mathbb{T}(w)$ of $\mathbb{T}$ rooted at w. For convenience put also $\mathbb{A}_n^{-\infty} \stackrel{\text{def}}{=} \emptyset$.

We start by estimating $A_n(x) \stackrel{\text{def}}{=} \mathbb{E}|\mathbb{A}_n^x| = \mathbb{E}[\sum_{|v|=n} \mathbf{1}_{\mathbb{A}_n^x}(v)]$ for $x \ge 0$. By the usual change of measure argument and a subsequent appeal to Theorem A.4, we infer

$$\begin{aligned} A_n(x) \;&=\; \widehat{\mathbb{E}}\Big(e^{S_n}\mathbf{1}_{\{S_k \ge -x,\, 0 \le k \le n, S_n - a\log n_j \in [-x, b-x]\}}\Big) \\ &\ge\; n^a e^{-x}\,\widehat{\mathbb{P}}\big(S_k \ge 0,\, 0 \le k \le n, S_n - a\log n \in [-x, b-x]\big) \\ &\ge\; C_1 n^{a-3/2}\log n\, e^{-x}\mathbf{1}_{[0,n^\varepsilon]}(x) \end{aligned} \tag{4.2}$$

for any $0 < \varepsilon < \frac{1}{2}$, $x \ge 0$ and some $C_1 \in (0, \infty)$ (that may depend on ε). A similar estimation shows

$$A_n(x) \;\le\; C_2 n^{a-3/2}\log n\, e^{b-x}, \tag{4.3}$$

for all $x \ge 0$ and some $C_2 \in (0, \infty)$.

Next define

$$\mathbb{L}_n^x \;\stackrel{\text{def}}{=}\; \{x + S_k > g_k(n),\, 0 \le k \le n,\, x + S_n \in [a\log n, a\log n + b]\},$$

where $g_n(x) = g_{n,\varepsilon}(x) \stackrel{\text{def}}{=} \varepsilon\min(x^{1/3}, n^\varepsilon, (n-x)^{1/3})$ for $\varepsilon \in (0, a - \frac{1}{2})$. Similar to (4.2), we get

$$L_n(0) \;\stackrel{\text{def}}{=}\; \mathbb{E}\mathbb{L}_n^0 \;\ge\; n^a\,\widehat{\mathbb{P}}\big(S_k \ge g_n(k),\, 0 \le k \le n, S_n - a\log n \in [0, b]\big),$$

and by Lemma A.5 we can choose $\varepsilon > 0$ so small that the occurring probability is bounded from below by $\widehat{\mathbb{P}}\big(S_k \ge 0,\, 0 \le k \le n, S_n - a\log n \in [0, b]\big)/2$ which in turn is lower bounded by $C_1 n^{-3/2}$, thus

$$L_n(0) \;\ge\; C_1 n^{a-3/2}\log n. \tag{4.4}$$

Now consider

$$\widehat{\mathbb{L}}_j \;\stackrel{\text{def}}{=}\; \sum_{n=n_j}^{n_{j+1}} \mathbb{L}_n^0.$$

Our goal is to provide a lower estimate for $\mathbb{P}(\widehat{\mathbb{L}}_j \neq \emptyset)$ with the help of the Paley-Zygmund inequality

$$\mathbb{P}(\widehat{\mathbb{L}}_j \neq \emptyset) \;\geq\; \frac{1}{4}\,\frac{[\mathbb{E}|\widehat{\mathbb{L}}_j|]^2}{\mathbb{E}|\widehat{\mathbb{L}}_j|^2}. \tag{4.5}$$

As for $\mathbb{E}|\widehat{\mathbb{L}}_j| = \sum_{n=n_j}^{n_{j+1}} \mathbb{E}[\sum_{|v|=n} \mathbf{1}_{\mathbb{L}_n^0}(v)]$, (4.4) provides us with

$$\mathbb{E}|\widehat{\mathbb{L}}_j| \;=\; \sum_{n=n_j}^{n_{j+1}-1} \mathbb{E}|\mathbb{L}_n^0| \;\geq\; C(n_{j+1}-n_j)n_j^{a-3/2}\log n_j \;=\; Cn_j^{a-1/2}\log n_j. \tag{4.6}$$

An upper estimate of $\mathbb{E}|\widehat{\mathbb{L}}_j|^2$ requires more work. By definition,

$$\mathbb{E}|\widehat{\mathbb{L}}_j|^2 \;=\; 2\sum_{n=n_j}^{n_{j+1}}\sum_{m=n}^{n_{j+1}} \mathbb{E}\Bigg[\sum_{|v|=n}\sum_{|w|=m} \mathbf{1}_{\mathbb{L}_n^0}(v)\mathbf{1}_{\mathbb{L}_m^0}(w)\Bigg].$$

Let u be the most recent common ancestor of v and w in this sum and observe that $\mathbf{1}_{\mathbb{L}_m^0}(uw') \leq \mathbf{1}_{\mathbb{A}_{m-k}^{S(u)}(u)}(w')$ for all $m \geq 1$ and $u, w' \in \mathbb{T}$ with $|u| = k$, $|w'| = m-k$. Then

$$\begin{aligned}
\mathbb{E}|\widehat{\mathbb{L}}_j|^2 \;&=\; 2\sum_{n=n_j}^{n_{j+1}-1}\sum_{m=n}^{n_{j+1}-1}\sum_{k=0}^{n-1} \mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k}\sum_{|w|=m-k} \mathbf{1}_{\mathbb{L}_n^0}(uv)\mathbf{1}_{\mathbb{L}_m^0}(uw)\Bigg] \\
&\leq\; 2\sum_{n=n_j}^{n_{j+1}-1}\sum_{m=n}^{n_{j+1}-1}\sum_{k=0}^{n-1} \mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k}\sum_{|w|=m-k} \mathbf{1}_{\mathbb{L}_n^0}(uv)\mathbf{1}_{\mathbb{A}_{m-k}^{S(u)}(u)}(w)\Bigg].
\end{aligned} \tag{4.7}$$

Condition the expectation in the previous line, denoted as $\mathcal{S}(n,m,k)$ hereafter, with respect to the history up to time k, use the branching structure, (4.3) and $S(u) > g_n(|u|)$ if $uv \in \mathbb{L}_n^0$ to infer that

$$\begin{aligned}
\mathcal{S}(n,m,k) \;&=\; \mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k} \mathbf{1}_{\mathbb{L}_n^0}(uv)A_{m-k}(S(u))\Bigg] \\
&\leq\; C_2(m-k)^{a-3/2}\,\mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k} e^{b-S(u)}\mathbf{1}_{\mathbb{L}_n^0}(uv)\Bigg] \\
&\leq\; C_2(m-k)^{a-3/2}\log(m-k)\,e^{b-g_n(k)}\,\mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k} \mathbf{1}_{\mathbb{L}_n^0}(uv)\Bigg].
\end{aligned} \tag{4.8}$$

We now split up the summation over k in (4.7) into $0 \leq k \leq n^{8\varepsilon}$, $n^{8\varepsilon} < k \leq n - n^{8\varepsilon}$, and $n - n^{8\varepsilon} < k < n$:

CASE 1. If $0 \leq k \leq n^{8\varepsilon}$, then $g_n(k) = k^{1/3}$ and $(m-k)^{-3/2} = O(n^{-3/2})$ for $m \geq n$, whence (4.8) implies

$$\sum_{n=n_j}^{n_{j+1}-1}\sum_{m=n}^{n_{j+1}-1}\sum_{0\leq k\leq n^{8\varepsilon}} \mathcal{S}(n,m,k) \;\leq\; Cn_j^{a-1/2}\log n_j \sum_{n=n_j}^{n_{j+1}-1} \mathbb{E}\Bigg[\sum_{|u|=k}\sum_{|v|=n-k} \mathbf{1}_{\mathbb{L}_n^0}(uv)\Bigg]$$

$$= \; C n_j^{a-1/2} \log n_j \, \mathbb{E}\widehat{\mathbb{L}}_j.$$

CASE 2. If $n^{8\varepsilon} < k \le n - n^{8\varepsilon}$, then $g_n(k) = n^{\varepsilon}$ and $(m-k)^{-3/2} = O(n^{-9\varepsilon/2})$ for $m \ge n$, whence in this case

$$\sum_{n=n_j}^{n_{j+1}-1} \sum_{m=n}^{n_{j+1}-1} \sum_{n^{8\varepsilon} < k \le n - n^{8\varepsilon}} \mathcal{S}(n,m,k) \; \le \; C n_j^{1-9\varepsilon/2} \log n_j \, e^{b - n_j^{\varepsilon}} \, \mathbb{E}\widehat{\mathbb{L}}_j.$$

CASE 3. Finally, if $n - n^{8\varepsilon} < k < n$, then $g_n(k) = (n-k)^{1/3}$, and we obtain

$$\begin{aligned}
&\sum_{n=n_j}^{n_{j+1}-1} \sum_{m=n}^{n_{j+1}-1} \sum_{n - n^{8\varepsilon} < k < n} \mathcal{S}(n,m,k) \\
&\qquad \le \; C \sum_{n=n_j}^{n_{j+1}-1} \sum_{k=0}^{n-1} e^{-(n-k)^{1/3}} \sum_{m=n}^{n_{j+1}-1} \frac{\log(m-k)}{(m-k)^{3/2}} \, \mathbb{E}\Bigg[\sum_{|u|=k} \sum_{|v|=n-k} \mathbf{1}_{\mathbb{L}_n^0}(uv) \Bigg] \\
&\qquad \le \; C \sum_{m \ge 1} \frac{\log m}{m^{3/2}} \sum_{k \ge 1} e^{-k^{1/3}} \, \mathbb{E}|\widehat{\mathbb{L}}_j| \\
&\qquad \le \; C \, \mathbb{E}|\widehat{\mathbb{L}}_j|.
\end{aligned}$$

By combining the three cases, we see that

$$\mathbb{E}|\widehat{\mathbb{L}}_j|^2 \; \le \; C n_j^{a-1/2} \log n_j \, \mathbb{E}|\widehat{\mathbb{L}}_j| \tag{4.9}$$

for a suitable $C \in (0,\infty)$. Now use (4.5), (4.6) and (4.9) to arrive at the conclusion

$$\inf_{j \ge 0} \mathbb{P}(\widehat{\mathbb{L}}_j \ne \emptyset) \; > \; 0,$$

a fortiori

$$\varrho \; \stackrel{\text{def}}{=} \; \inf_{j \ge 0} \mathbb{P}\Big(\min_{n_j \le n < n_{j+1}} M_n \le a \log n_j + b \Big) \; > \; 0.$$

Defining $\tau_j \stackrel{\text{def}}{=} \inf\{k : N_k \ge j\}$, we now infer

$$\begin{aligned}
&\mathbb{P}\Big(\min_{\tau_j + n_j \le n < \tau_j + n_{j+1}} M_n \ge \overline{x}\tau_j + a \log n_j + b \Big) \\
&\qquad\qquad = \; \mathbb{P}\Big(\min_{v \in \mathbb{T}_{\tau_j}} \min_{n_j \le n < n_{j+1}} (S(v) + M_n(v)) \ge \overline{x}\tau_j + a \log n_j + b \Big) \\
&\qquad\qquad \le \; \mathbb{P}\Big(\min_{n_j \le n < n_{j+1}} M_n \ge a \log n_j + b \Big)^j \\
&\qquad\qquad = \; (1-\varrho)^j
\end{aligned}$$

and thus, by the Borel-Cantelli lemma,

$$\mathbb{P}\Big(\min_{\tau_j + n_j \le n < \tau_j + n_{j+1}} M_n \le \overline{x}\tau_j + a \log n_j + b \text{ i.o.} \Big) \; = \; 1.$$

Finally, since $\tau_j = o(\log n_j)$ a.s. and $a > 1/2$ was arbitrarily chosen, the desired conclusion $\liminf_{n\to\infty} \frac{M_n}{\log n} \le \frac{1}{2}$ a.s. follows.

In order to prove the second assertion of (4.1), we now fix any $a > \frac{3}{2}$ and show that

$$\inf_{n\ge 1} \mathbb{P}(\mathbb{A}_n^0 \ne \emptyset) \;\ge\; \frac{1}{4} \inf_{n\ge 1} \frac{[\mathbb{E}|\mathbb{A}_n^0|]^2}{\mathbb{E}|\mathbb{A}_n^0|^2} \;>\; 0. \tag{4.10}$$

Since $\mathbb{E}|\mathbb{A}_n^0| = A_n(0) \ge C_1 n^{a-3/2}\log n$ by (4.3), only an upper bound for

$$\mathbb{E}|A_n^0|^2 \;=\; 2\,\mathbb{E}\left[\sum_{|v|=n}\sum_{|w|=n} \mathbf{1}_{\mathbb{A}_n^0}(v)\mathbf{1}_{\mathbb{A}_n^0}(w)\right]$$

is still to be found. As in the first part of the proof, let u be the most recent common ancestor of v and w in this sum and note that, for all u, v', w' with $|u| = k$ and $|v'| = |w'| = n-k$,

$$\mathbf{1}_{\mathbb{A}_n^0}(uv')\mathbf{1}_{\mathbb{A}_n^0}(uw') \;\le\; \mathbf{1}_{\{S(u|j)>0, 1\le j\le k\}}\mathbf{1}_{\mathbb{A}_{n-k}^{S(u)}}(v')\mathbf{1}_{\mathbb{A}_{n-k}^{S(u)}(u)}(w').$$

Along similar lines as for (4.7) and (4.8), we now infer with the help of (4.3) and Lemma A.6

$$\begin{aligned}
\mathbb{E}|\mathbb{A}_n^0|^2 \;&=\; \sum_{k=0}^{n-1}\mathbb{E}\left[\sum_{|u|=k}\mathbf{1}_{\{S(u|j)>0,1\le j\le k\}}\sum_{|v|=n-k}\sum_{|w|=n-k}\mathbf{1}_{\mathbb{A}_{n-k}^{S(u)}}(v)\mathbf{1}_{\mathbb{A}_{n-k}^{S(u)}(u)}(w)\right]\\
&=\; \sum_{k=0}^{n-1}\mathbb{E}\left[\sum_{|u|=k}\mathbf{1}_{\{S(u|j)>0,1\le j\le k\}}A_{n-k}^2(S(u))\right]\\
&\le\; C\sum_{k=0}^{n-1}(n-k)^{2a-3}\log^2(n-k)\,\mathbb{E}\left[\sum_{|u|=k}\mathbf{1}_{\{S(u|j)>0,1\le j\le k\}}e^{-2S(u)}\right]\\
&=\; C\sum_{k=0}^{n-1}(n-k)^{2a-3}\log^2(n-k)\,\widehat{\mathbb{E}}\Big(e^{-S_k}\mathbf{1}_{\{\sigma^->k\}}\Big)\\
&\le\; C\sum_{k=0}^{n-1}(n-k)^{2a-3}\log^2(n-k)k^{-3/2}\log k\\
&\le\; C\,n^{2a-3}\log^2 n
\end{aligned}$$

and thereupon (4.10). In particular,

$$\chi_a \;\stackrel{\text{def}}{=}\; \inf_{n\ge 1}\mathbb{P}(M_n \le a\log n_j + b) \;>\; 0$$

for any $a > \frac{3}{2}$ and $b > 0$ sufficiently large. The remaining Borel-Cantelli argument is very similar to that for the liminf case: Defining $\tau(n) \stackrel{\text{def}}{=} \inf\{k \ge 0 : \log N_k \ge \varepsilon \log m \log n\}$ for any fixed $\varepsilon > 0$, we have $\frac{\tau(n)}{\log n} \to \varepsilon$ a.s. and thus $\mathbb{P}(\tau(n) > k(n) \text{ i.o.}) = 0$, where $k(n) \stackrel{\text{def}}{=} \lfloor 2\varepsilon \log n\rfloor$. Then

$$\mathbb{P}(M_n > (a + 2\varepsilon\overline{x})\log n, \tau(n) \le k(n) + b)$$

$$
\begin{aligned}
&\le \mathbb{P}\Big(N_{k(n)} \ge n^{\varepsilon \log m}, \min_{v \in \mathbb{T}_{k(n)}} \big(S(v) + M_{n-k(n)}(v)\big) > (a + 2\varepsilon\overline{x}) \log n + b\Big) \\
&\le \mathbb{P}(M_{n-k(n)} > a \log n + b)^{n^{\varepsilon \log m}} \\
&\le (1 - \chi_a)^{n^{\varepsilon \log m}}
\end{aligned}
$$

in combination with the Borel-Cantelli lemma implies

$$\mathbb{P}(M_n > (a + 2\varepsilon\overline{x}) \log n \text{ i.o.}) \ = \mathbb{P}(\tau(n) \le k(n), M_n > (a + 2\varepsilon\overline{x}) \log n \text{ i.o.}) \ = \ 0,$$

and since $a > \frac{3}{2}$ and $\varepsilon > 0$ were chosen arbitrarily, the proof is complete. $\diamondsuit$

5. Two useful moment results

In this section, we will prove two moment results which, besides being of independent interest, provide the key to the proofs of our main results.

Theorem 5.1. *Assuming* (C1–4), *for each* $r \in [0, 1)$,

$$0 \ < \ \liminf_{n\to\infty} n^{r/2}\mathbb{E}Z_n^r \ < \ \limsup_{n\to\infty} n^{r/2}\mathbb{E}Z_n^r \ < \ \infty. \tag{5.1}$$

Proof. In view of Lemma 2.1 and the subadditivity of $x \mapsto x^r$ it is enough to prove the result under the additional condition (C5).

Note that $W_{1,n} \le n^{1/2} Z_n \le W_{2,n}$ for each $n \ge 0$, where

$$W_{1,n} \stackrel{\text{def}}{=} \sum_{|v|=n} (n^{1/2} \wedge S(v)^+) e^{-S(v)} \quad \text{and} \quad W_{2,n} \stackrel{\text{def}}{=} \sum_{|v|=n} (n^{1/2} \vee S(v)) e^{-S(v)}. \tag{5.2}$$

Upper bound. We first prove the upper bound in (5.1) and start by observing that, putting $s = 1 - r$ and recalling (5.17),

$$n^{r/2}\mathbb{E}Z_n^r \ \le \ \mathbb{E}W_{2,n}^r \ = \ \widehat{\mathbb{E}}\Big((n^{1/2} \vee S(V_n)) W_{2,n}^{-s}\Big). \tag{5.3}$$

In order for finding an upper bound for $W_{2,n}^{-s}$, let $\tau(n) \stackrel{\text{def}}{=} \min\{j > n^* : \mathcal{I}_j^{(n)} \ne \emptyset \text{ or } j = n+1\}$ and observe that $\widehat{\mathbb{P}}(\tau(n) > j | \mathcal{G}_n) = (1-\gamma)^{j-n^*}$ for $n^* < j \le n$, where $\gamma \stackrel{\text{def}}{=} \widehat{\mathbb{P}}(\mathcal{I}_1^{(1)} \ne \emptyset)$. Thus, $\tau(n) - n^*$ given $\mathcal{G}_n$ has a truncated geometric distribution. Now we further estimate

$$
\begin{aligned}
W_{2,n} \ &= \ \sum_{j=1}^{n} \sum_{v \in \mathcal{I}_j^{(n)}} \sum_{|w|=n-j} (n^{1/2} \vee S(vw)) e^{-S(vw)} \\
&\ge \ \mathbf{1}_{\{\tau(n) \le n\}} \sum_{v \in \mathcal{I}_{\tau(n)}^{(n)}} \sum_{|w|=n-\tau(n)} (n^{1/2} \vee S(vw)) e^{-S(vw)} \\
&\quad + \ \mathbf{1}_{\{\tau(n)=n+1\}} (n^{1/2} \vee S(V_n)) e^{-S(V_n)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mathbf{1}_{\{\tau(n)\leq n\}} e^{-S(V_{\tau(n)})-\overline{x}} \sum_{v\in\mathcal{I}^{(n)}_{\tau(n)}} \sum_{|w|=n-\tau(n)} (n^{1/2}\vee S(vw)) e^{-(S(vw)-S(v))} \\
&\quad + \mathbf{1}_{\{\tau(n)=n+1\}} (n^{1/2}\vee S(V_n)) e^{-S(V_n^*)-\overline{x}(\tau(n)-n^*)}.
\end{aligned}
\tag{5.4}
$$

If $\tau(n)\leq n$ and $S(V_{\tau(n)})\geq -\frac{1}{s}\log n+\overline{x}$, then

$$
\begin{aligned}
n^{1/2}\vee S(vw) &\geq \max[n^{1/2}, S(V_{\tau(n)})-\overline{x}+(S(vw)-S(v))] \\
&\geq \max[n^{1/2}, S(vw)-S(v)]/2
\end{aligned}
$$

for sufficiently large n (this is clear if $S(vw)-S(v)\leq n^{1/2}$ and follows from $S(vw)\geq -\frac{1}{s}\log n+S(vw)-S(v)\geq \frac{1}{2}(S(vw)-S(v))$, otherwise). Hence in this case we further estimate in (5.4)

$$
\begin{aligned}
W_{2,n} &\geq \frac{1}{2} e^{-S(V_{\tau(n)})-\overline{x}} \sum_{v\in\mathcal{I}^{(n)}_{\tau(n)}} \sum_{|w|=n-\tau(n)} \left[n^{1/2}\vee (S(vw)-S(v))\right] \\
&\geq \frac{1}{2} e^{-S(V_n^*)-(\tau(n)-n^*)\overline{x}}\, W_{2,n-\tau(n)}(v^*)
\end{aligned}
\tag{5.5}
$$

on $\{\tau(n)\leq n\}$ for all large n, where v^* is an arbitrary element from $\mathcal{I}^{(n)}_{\tau(n)}$ and $W_{2,j}(v)$ is as usual defined as $W_{2,j}$, but for the subtree rooted at v.

If $\tau(n)\leq n$ and $S(V_{\tau(n)})\leq -\frac{1}{s}\log n+\overline{x}$, then we get in (5.4)

$$
W_{2,n} \;\geq\; n^{(1/2)+(1/s)} Z_{n-\tau(n)}(v^*) \tag{5.6}
$$

on $\{\tau(n)\leq n\}$, where v^* is again an arbitrary element from $\mathcal{I}^{(n)}_{\tau(n)}$.

By combining (5.3–6) and using the subadditivity of $x\mapsto x^s$, we see that

$$
\begin{aligned}
n^{r/2}\,\mathbb{E}Z_n^r &\leq C\,\widehat{\mathbb{E}}\Big((n^{1/2}\vee S(V_n)) e^{s[S(V_n^*)+\overline{x}(\tau(n)-n^*)]} W^{-s}_{2,n-\tau(n)}(v^*)\mathbf{1}_{\{\tau(n)\leq n\}}\Big) \\
&\quad + n^{-1-s/2}\,\widehat{\mathbb{E}}\Big((n^{1/2}\vee S(V_n)) Z^{-s}_{n-\tau(n)}(v^*)\mathbf{1}_{\{\tau(n)\leq n\}}\Big) \\
&\quad + \widehat{\mathbb{E}}\Big((n^{1/2}\vee S(V_n))^{1-s} e^{s[S(V_n^*)+\overline{x}(\tau(n)-n^*)]}\mathbf{1}_{\{\tau(n)=n+1\}}\Big).
\end{aligned}
\tag{5.7}
$$

In order to conclude that the three terms on the right-hand side of (5.7) remain bounded as $n\to\infty$, we need the following moment estimates, all valid for any $s\in(0,1)$:

$$
\mathbb{E}Z_n^{-s} \;\leq\; n^{s/2+o(1)}, \quad n\to\infty, \tag{5.8}
$$

$$
\mu(s) \;\stackrel{\text{def}}{=}\; \sup_{n\geq 1}\mathbb{E}W_{2,n}^{-s} \;<\; \infty \tag{5.9}
$$

and

$$
\nu(s) \;\stackrel{\text{def}}{=}\; \sup_{n\geq 1}\widehat{\mathbb{E}}(e^{s\overline{x}(\tau(n)-n^*)}) \;<\; \infty. \tag{5.10}
$$

The latter follows because, as mentioned above, $\tau(n)-n^*$ conditioned upon $\mathcal{G}_n$ has a truncated geometric distribution with parameter $\gamma=\widehat{\mathbb{P}}(\mathcal{I}_1^{(1)}\neq\emptyset)$ not depending on n. (5.8) and (5.9) are proved as Lemmata 3.3 and 3.4, respectively. We will also need below that

$$\sup_{n\geq 1}\widehat{\mathbb{E}}\Big[(n^{1/2}\vee S_n)\exp\Big(s\min_{0\leq j<n}S_j\Big)\Big] \;<\; \infty \tag{5.11}$$

which follows from Lemma A.3 in our Appendix.

Now observe that, given $\mathcal{G}_n$ and $\tau(n)=k\leq n$, the conditional distribution under $\widehat{\mathbb{P}}$ of $(Z_{n-\tau(n)}(v^*), W_{2,n-\tau(n)}(v^*))$ equals the distribution of $(Z_{n-k}, W_{2,n-k})$ under $\mathbb{P}$. Using this and the above estimates (5.8-10), we finally conclude from (5.7)

$$\begin{aligned} n^{r/2}\,\mathbb{E}Z_n^r \;&\leq\; C\,\nu(s)\,\mu(s)\,\widehat{\mathbb{E}}\Big[(n^{1/2}\vee S_n)\exp\Big(s\min_{0\leq j<n}S_j\Big)\Big] \\ &\quad+\; n^{-1-(s/2)}\,o(n^{s/2+o(1)})\,\widehat{\mathbb{E}}(n^{1/2}\vee S_n) \\ &\quad+\; \nu(s)\,\widehat{\mathbb{E}}\Big[(n^{1/2}\vee S_n)^{1-s}\exp\Big(s\min_{0\leq j<n}S_j\Big)\Big] \;=\; O(1), \quad n\to\infty \end{aligned}$$

for any s and thus r between 0 and 1. This completes the proof of the upper bound in (5.1).

Lower bound. We turn to the lower bound in (5.1) and start with the inequality

$$\begin{aligned} n^{r/2}\mathbb{E}Z_n^r \;&\geq\; \mathbb{E}W_{1,n}^r \;=\; \widehat{\mathbb{E}}(n^{1/2}\wedge S(V_n)^+)W_{1,n}^{-s} \\ &\geq\; \widehat{\mathbb{E}}\left((n^{1/2}\wedge S(V_n)^+)\left[\sum_{j=1}^n\sum_{v\in\mathcal{I}_j^{(n)}}\sum_{|w|=n-j}S(vw)^+e^{-S(vw)}\right]^{-s}\right) \end{aligned} \tag{5.12}$$

Since $S(vw)^+\leq S(V_j)^+ + \overline{x} + (S(vw)-S(v))^+$ for each $v\in\mathcal{I}_j^{(n)}$, we infer

$$\begin{aligned} \sum_{j=1}^n\sum_{v\in\mathcal{I}_j^{(n)}}\sum_{|w|=n-j}S(vw)^+e^{-S(vw)} \;&\leq\; e^{\overline{x}}\sum_{j=1}^n[S(V_j)^+ + \overline{x}]e^{-S(V_j)}\sum_{v\in\mathcal{I}_j^{(n)}}Z_{n-j}(v) \\ &\quad+\; e^{\overline{x}}\sum_{j=1}^n e^{-S(V_j)}\sum_{v\in\mathcal{I}_j^{(n)}}W_{n-j}^{(+)}(v), \end{aligned} \tag{5.13}$$

where $W_n^{(+)}\stackrel{\text{def}}{=}\sum_{|w|=n}S(w)^+e^{-S(w)}$. Note that, by (3.3),

$$W_n^{(+)} \;\leq\; C\sum_{|w|=n}-\log\varphi(e^{-S(w)}) \;=\; C\,\Phi_n$$

for all $n\geq 0$ and some $C>0$. Therefore, by Lemma 3.2,

$$\sup_{n\geq 0}\mathbb{E}W_n^{(+)s} \;\leq\; \sup_{n\geq 0}[\mathbb{E}W_n^{(+)}]^s \;\leq\; C\,[\mathbb{E}\Phi_\infty]^s \;<\; \infty,$$

recalling that $(\Phi_n)_{n\geq 0}$ is a submartingale. The same arguments as for (5.25) then imply

$$\widehat{\mathbb{E}}\left(\sum_{v\in\mathcal{I}_j^{(n)}}W_{n-j}^{(+)s}(v)\middle|\mathcal{G}_n\right) \;\leq\; \widehat{\mathbb{E}}|\mathcal{I}_1^{(1)}|\,\mathbb{E}W_{n-j}^{(+)s} \;\leq\; C\,\mathbb{E}N_1^2 \tag{5.14}$$

for all $0 \le j \le n$. A combination of (5.13), (5.25) and (5.14) provides us now with the estimate

$$\begin{aligned}
&\widehat{\mathbb{E}}\left(\left[\sum_{j=1}^{n}\sum_{v\in\mathcal{I}_j^{(n)}}\sum_{|w|=n-j} S(vw)^{+}e^{-S(vw)}\right]^{-s}\middle|\mathcal{G}_n\right)\\
&\ge\ e^{-s\overline{x}}\left[\sum_{j=1}^{n}[S(V_j)^{+}+\overline{x}]^{s}e^{-sS(V_j)}\widehat{\mathbb{E}}\left(\sum_{v\in\mathcal{I}_j^{(n)}} Z_{n-j}^{s}(v)\middle|\mathcal{G}_n\right)\right.\\
&\qquad\qquad\left.+\ \sum_{j=1}^{n} e^{-sS(V_j)}\widehat{\mathbb{E}}\left(\sum_{v\in\mathcal{I}_j^{(n)}} W_{n-j}^{(+)s}(v)\middle|\mathcal{G}_n\right)\right]^{-1}\\
&\ge\ C\left[\sum_{j=1}^{n} e^{-sS(V_j)}[(S(V_j)^{+})^{s}+1]\right]^{-1}
\end{aligned}$$

and then upon returning to (5.12)

$$\begin{aligned}
n^{r/2}\mathbb{E}Z_n^r\ &\ge\ C\,\widehat{\mathbb{E}}\left((n^{1/2}\wedge S_n^{+})\left[\sum_{j=1}^{n} e^{-sS_j}[(S_j^{+})^{s}+1]\right]^{-1}\right)\\
&\ge\ C\,\widehat{\mathbb{E}}\left((n^{1/2}\wedge S_n)\left[\sum_{j=0}^{n} e^{-sS_j/2}\right]^{-1}\mathbf{1}_{\{\sigma^{-}>n\}}\right)\\
&\ge\ C\,\widehat{\mathbb{E}}\left((n^{1/2}\wedge S_n)\left[\sum_{j=0}^{n} e^{-sS_j/2}\right]^{-1}\mathbf{1}_{\{\sigma^{-}>2n\}}\right),
\end{aligned}\tag{5.15}$$

where $\sigma^{-} \stackrel{\text{def}}{=} \inf\{n \ge 1 : S_n \le 0\}$ and $(x+1)e^{-sx} \le Ce^{-sx/2}$ for $x \ge 0$ has been utilized for the penultimate inequality. It is a well-known result from fluctuation theory of random walks that

$$\widehat{\mathbb{P}}(\sigma^{-} > n)\ =\ \widehat{\mathbb{P}}\Big(\min_{1\le j\le n} S_n > 0\Big)\ \sim\ cn^{-1/2}$$

for some (known) $c > 0$. Therefore we can further estimate in (5.15)

$$\begin{aligned}
&\ge\ C\varepsilon n^{1/2}\,\widehat{\mathbb{E}}\left(\left[\sum_{j=0}^{n} e^{-sS_j/2}\right]^{-1}\mathbf{1}_{\{\sigma^{-}>n,S_n>\varepsilon n^{1/2}\}}\right)\\
&\ge\ C\varepsilon n^{1/2}\left(\widehat{\mathbb{E}}\left(\left[\sum_{j=0}^{n} e^{-sS_j/2}\right]^{-1}\mathbf{1}_{\{\sigma^{-}>2n\}}\right)-\widehat{\mathbb{P}}(S_n\le\varepsilon n^{1/2},\sigma^{-}>2n)\right)
\end{aligned}$$

$$\geq\ C\left(\left[\widehat{\mathbb{E}}\left(\sum_{j=0}^{n} e^{-sS_j/2}\middle|\sigma^- > 2n\right)\right]^{-1} - \widehat{\mathbb{P}}(S_n \leq \varepsilon n^{1/2}|\sigma^- > 2n)\right)$$

for any $\varepsilon > 0$. Now Lemma A.2 ensures that, for sufficiently small ε, the final line stays bounded away from 0 as $n \to \infty$ and thus $\liminf_{n\to\infty} n^{r/2}\,\mathbb{E}Z_n^r > 0$. ♢

Theorem 5.2. *Assuming* (C1–4), *for each* $\beta > 1$ *and* $0 < r < 1/\beta$,

$$\mathbb{E}Z_{n,\beta}^r \;=\; n^{-(3/2)\beta r+o(1)}, \quad n \to \infty. \tag{5.16}$$

PROOF. Again, it suffices to prove (5.16) under the additonal assumption (C5) because then, in the notation of Subsection 2.1, $\mathbb{E}\widehat{Z}_{n,\beta}^r = (1-q)\mathbf{E}\widehat{Z}_{n,\beta}^r = n^{-(3/2)\beta r+o(1)}$ holds true which in combination with Lemma 2.1 readily shows (5.16) under (C1–4). So assume (C1–5) hereafter.

UPPER BOUND. For any $\mathcal{F}_n$-measurable Y, we have

$$\begin{aligned}\mathbb{E}Z_{n,\beta}Y \;&=\; \widehat{\mathbb{E}}\left(\sum_{|v|=n}\frac{e^{-\beta S(v)}}{Z_n}Y\right)\\ &=\; \widehat{\mathbb{E}}\left(\sum_{|v|=n}\mathbf{1}_{\{V_n=v\}}e^{-(\beta-1)S(v)}Y\right) \;=\; \widehat{\mathbb{E}}\Big(e^{-(\beta-1)S(V_n)}Y\Big).\end{aligned} \tag{5.17}$$

Fix $s \in (\frac{\beta-1}{\beta}, 1)$ and $\lambda > 0$ (later to be chosen as $\frac{3}{2}$). Then

$$\begin{aligned}\mathbb{E}Z_{n,\beta}^{1-s} \;&\leq\; n^{-(1-s)\beta\lambda} \;+\; \mathbb{E}\Big(Z_{n,\beta}^{1-s}\mathbf{1}_{\{Z_{n,\beta}>n^{-\beta\lambda}\}}\Big)\\ &=\; n^{-(1-s)\beta\lambda} \;+\; \widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{\{Z_{n,\beta}>n^{-\beta\lambda}\}}\right)\end{aligned} \tag{5.18}$$

which leaves us with a further estimation of the final expectation in (5.18).

Let us fix three positive constants a, b and ϱ such that

$$(\beta-1)a - s\beta\lambda \;>\; 3/2, \quad [\beta s - (\beta-1)]b \;>\; 3/2$$

and $\varrho > b$ (the choice of ϱ will be further specified a little later). We note that $a > \lambda$, for $(\beta-1)(a-\lambda) > (\beta-1)a - s\beta\lambda > 0$. Consider the events

$$\begin{aligned}E_{1,n} \;&\stackrel{\text{def}}{=}\; \{S(V_n) \notin [-b\log n, a\log n]\},\\ E_{2,n} \;&\stackrel{\text{def}}{=}\; \{S(V_n^*) < -\varrho\log n,\ S(V_n) \in [-b\log n, a\log n]\},\\ E_{3,n} \;&\stackrel{\text{def}}{=}\; \{S(V_n^*) \geq -\varrho\log n,\ S(V_n) \in [-b\log n, \lambda\log n]\},\\ E_{4,n} \;&\stackrel{\text{def}}{=}\; \{S(V_n^*) \geq -\varrho\log n,\ S(V_n) \in (\lambda\log n, a\log n]\},\end{aligned}$$

which clearly cover the whole space. We split up the final expected value in (5.18) into the four integrals over the $E_{i,n}$, $i = 1, ..., 4$, and estimate these integrals separately.

On $E_{1,n} \cap \{Z_{1,\beta} > n^{-\beta\lambda}\}$ either $S(V_n) > a \log n$, in which case

$$\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s} \ \leq\ n^{s\beta\lambda-(\beta-1)a} \ \leq\ n^{-3/2},$$

or $S(V_n) < -b \log n$, in which case $Z_{n,\beta} \geq e^{-\beta S(V_n)}$ provides us with

$$\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s} \ \leq\ e^{[\beta s-(\beta-1)]S(V_n)} \ \leq\ n^{-[\beta s-(\beta-1)]b} \ \leq\ n^{-3/2}.$$

Consequently,

$$\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{E_{1,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}}\right) \ \leq\ n^{-3/2}. \tag{5.19}$$

Turning to the integral on $E_{2,n} \cap \{Z_{n,\beta} > n^{-\beta\lambda}\}$, define

$$I_n \ \stackrel{\text{def}}{=}\ \bigcap_{k=1}^{n-\gamma\log n}\ \bigcup_{j=k}^{k+\gamma\log n} \left\{\mathcal{I}_j^{(n)} \neq \emptyset \ \text{ and } \ \max_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v) > n^{-\beta\kappa}\right\} \tag{5.20}$$

for $\gamma, \kappa > 0$ with κ so large that

$$\xi \ \stackrel{\text{def}}{=}\ \sup_{n\geq 2}\ \sup_{1\leq j\leq n} \widehat{\mathbb{P}}\Big(\mathcal{I}_j^{(n)} = \emptyset \ \text{ or } \ \max_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v) \leq n^{-\beta\kappa}\Big) \ \in (0,1).$$

The existence of such a κ is shown in Lemma 5.3 at the end of this section. It follows that

$$\widehat{\mathbb{P}}(I_n^c) \ \leq\ n\,\xi^{\gamma\log n} \ =\ n^{1+\gamma\log\xi}$$

for all $n \geq 2$. By combining this with the crude bound $e^{-(\beta-1)S(V_n)}/Z_{n,\beta}^s \leq n^{\beta\lambda+(\beta-1)sb}$ on $E_{2,n}$ and fixing γ sufficiently large, we infer

$$\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{E_{2,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}\cap I_n^c}\right) \ \leq\ n^{\beta\lambda+(\beta-1)sb+1+\gamma\log\xi} \ \leq\ n^{-2}.$$

To estimate the integral on $E_{2,n} \cap \{Z_{n,\beta} > n^{-\beta\lambda}\} \cap I_n$, we pick any $p \in (0, s)$ to obtain

$$\begin{aligned}\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s} \ &=\ \frac{e^{\beta pS(V_n^*)-(\beta-1)S(V_n)}}{Z_{n,\beta}^{s-p}}\,\frac{e^{-\beta pS(V_n^*)}}{Z_{n,\beta}^p}\\ &\leq\ n^{-\varrho p\beta+(\beta-1)b+\beta\lambda(s-p)}\,\frac{e^{-\beta pS(V_n^*)}}{Z_{n,\beta}^p}\end{aligned}$$

on that event. We will show in Lemma 5.4 that

$$\widehat{\mathbb{E}}\left(\frac{e^{-\beta pS(V_n^*)}}{Z_{n,\beta}^p}\mathbf{1}_{I_n}\right) \ \leq\ n^{\theta}$$

for some $\theta > 0$ and all n sufficiently large. Consequently, by fixing $\varrho > b$ sufficiently large, we infer

$$\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{E_{2,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}}\right) \ \le\ n^{-\varrho p\beta+(\beta-1)b+\beta\lambda(s-p)+\theta}+n^{-2} \ \le\ n^{-3/2} \tag{5.21}$$

for all n sufficiently large.

On $E_{3,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}$ we use $e^{-\beta S(V_n)}\le Z_{n,\beta}$ in combination with $\beta s-(\beta-1)>0$ to infer

$$\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s} \ \le\ e^{[\beta s-(\beta-1)]S(V_n)} \ \le\ n^{[\beta s-(\beta-1)]\lambda}. \tag{5.22}$$

Moreover, since $(S(V_n))_{n\ge 0}$ is a centered random walk under $\widehat{\mathbb{P}}$, we have, as $n\to\infty$,

$$\widehat{\mathbb{P}}(E_{3,n}) \ =\ \widehat{\mathbb{P}}\left(\min_{0\le k\le n} S(V_n)\ge-\varrho\log n, -b\log n\le S(V_n)\le\lambda\log n\right) \ =\ n^{-3/2+o(1)}.$$

Therefore, as $n\to\infty$

$$\begin{aligned}\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{E_{3,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}}\right) \ &\le\ n^{[\beta s-(\beta-1)]\lambda}\widehat{\mathbb{P}}(E_{3,n})\\ &=\ n^{[\beta s-(\beta-1)]\lambda-3/2+o(1)}\\ &=\ n^{-(1-s)\beta\lambda+(\lambda-3/2)+o(1)}.\end{aligned} \tag{5.23}$$

Finally turning to the integral on $E_{4,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}$, a combination of $e^{-(\beta-1)S(V_n)}\le n^{-(\beta-1)\lambda}$ and $Z_{n,\beta}>n^{-\beta\lambda}$ once again implies

$$\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s} \ \le\ n^{[\beta s-(\beta-1)]\lambda}.$$

Since, furthermore,

$$\widehat{\mathbb{P}}(E_{4,n}) \ =\ \widehat{\mathbb{P}}\left(\min_{0\le k\le n} S(V_n)\ge-\varrho\log n, \lambda\log n< S(V_n)\le a\log n\right) \ =\ n^{-3/2+o(1)},$$

as $n\to\infty$, we arrive at a similar estimate as in (5.23), namely

$$\begin{aligned}\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{Z_{n,\beta}^s}\mathbf{1}_{E_{4,n}\cap\{Z_{n,\beta}>n^{-\beta\lambda}\}}\right) \ &\le\ n^{[\beta s-(\beta-1)]\lambda}\widehat{\mathbb{P}}(E_{4,n})\\ &=\ n^{-(1-s)\beta\lambda+(\lambda-3/2)+o(1)},\end{aligned} \tag{5.24}$$

as $n\to\infty$. Now put $\lambda=\frac{3}{2}$, $r=1-s$ and combine (5.18–6), (5.23) and (5.24) to conclude

$$\mathbb{E}Z_{n,\beta}^r \ \le\ n^{-(3/2)\beta r+o(1)},$$

which is the desired upper bound. $\diamondsuit$

LOWER BOUND. As before, let $\beta > 1$ and $s \in (\frac{\beta-1}{\beta}, 1)$. Using the subadditivity of $x \mapsto x^{1-s}$ on $[0,\infty)$, we have

$$Z_{n,\beta}^{1-s} = \left(\sum_{j=1}^{n}\sum_{v\in\mathcal{I}_j^{(n)}} e^{-\beta S(v)} Z_{n-j,\beta}(v)\right)^{1-s} \le \sum_{j=1}^{n} e^{-(1-s)\beta(S(V_j)-\overline{x})}\Lambda_{n,j,\beta}^{1-s},$$

where

$$\Lambda_{n,j,\beta} \stackrel{\text{def}}{=} \sum_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v).$$

Under $\widehat{\mathbb{P}}$ and given $\mathcal{G}_n = \sigma((V_j, S(V_j)_{0\le j\le n})$, the $(\mathcal{I}_j^{(n)}, \Lambda_{n,j,\beta})$, $1 \le j \le n$, are conditionally independent with

$$\widehat{\mathbb{P}}((\mathcal{I}_j^{(n)}, \Lambda_{n,j,\beta}) \in \cdot|\mathcal{G}_n) = \widehat{\mathbb{P}}\left(\left(\mathcal{I}_1^{(1)}, \sum_{v\in\mathcal{I}_1^{(1)}} Z_{n-j,\beta}(v)\right) \in \cdot\right).$$

Furthermore, the $Z_{n-j,\beta}(v)$ are independent of $\mathcal{I}_j^{(n)}$ and the $S(v)$, $v \in \mathcal{I}_j^{(n)}$, and they have the same distribution as $Z_{n-j,\beta}$ under $\mathbb{P}$. Therefore,

$$\widehat{\mathbb{E}}(\Lambda_{n,j,\beta}^{1-s}|\mathcal{G}_n) \le \widehat{\mathbb{E}}\left(\sum_{v\in\mathcal{I}_1^{(1)}} Z_{n-j,\beta}^{1-s}(v)\right) \le \widehat{\mathbb{E}}|\mathcal{I}_1^{(1)}|\,\mathbb{E}Z_{n-j,\beta}^{1-s}. \tag{5.25}$$

Fix any small $\varepsilon > 0$ such that $\lambda \stackrel{\text{def}}{=} \frac{3}{2}(1-s)\beta - \varepsilon > 0$. Using $\widehat{\mathbb{E}}|\mathcal{I}_1^{(1)}| \le \widehat{\mathbb{E}}N_1 = \mathbb{E}N_1^2 < \infty$ and the already proved upper bound for $\mathbb{E}Z_{n,\beta}^{1-s}$, we now infer

$$\begin{aligned}\widehat{\mathbb{E}}(Z_{n,\beta}^{1-s}|\mathcal{G}_n) &\le e^{(1-s)\beta\overline{x}}\,\mathbb{E}N_1^2\sum_{j=1}^{n} e^{-(1-s)\beta S(V_j)}\mathbb{E}Z_{n-j,\beta}^{1-s} \\ &\le C\sum_{j=1}^{n} e^{-(1-s)\beta S(V_j)}(n-j+1)^{-\lambda}\end{aligned}$$

for some $C > 0$. Since $\mathbb{E}Z_{n,\beta}^{1-s} = \widehat{\mathbb{E}}(e^{-(\beta-1)S(V_n)}Z_{n,\beta}^{-s})$ (by (5.17)), it follows by an appeal to Jensen's inequality,

$$\begin{aligned}\mathbb{E}Z_{n,\beta}^{1-s} &= \widehat{\mathbb{E}}\Big(e^{-(\beta-1)S(V_n)}\widehat{\mathbb{E}}(Z_{n,\beta}^{-s}|\mathcal{G}_n)\Big) \\ &\ge \widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{\widehat{\mathbb{E}}(Z_{n,\beta}^{1-s}|\mathcal{G}_n)^{s/(1-s)}}\right) \\ &\ge C\,\widehat{\mathbb{E}}\left(\frac{e^{-(\beta-1)S(V_n)}}{\left[\sum_{j=1}^{n} e^{-(1-s)\beta S(V_j)}(n-j+1)^{-\lambda}\right]^{s/(1-s)}}\right).\end{aligned}$$

So we have produced a lower bound for $\mathbb{E}Z_{n,\beta}^{1-s}$ which is just a functional of the centered random walk $S_n \stackrel{\text{def}}{=} S(V_n)$, $n \ge 0$ (under $\widehat{\mathbb{P}}$). Since $(S_k)_{0\le k\le n} \stackrel{d}{=} (S_n - S_{n-k})_{0\le k\le n}$, the last expectation further equals

$$
\begin{aligned}
&= \widehat{\mathbb{E}}\left(\frac{e^{[\beta s-(\beta-1)]S_n}}{\big[\sum_{j=1}^n e^{(1-s)\beta(S_n-S_j)}(n-j+1)^{-\lambda}\big]^{s/(1-s)}}\right)\\
&= \widehat{\mathbb{E}}\left(\frac{e^{[\beta s-(\beta-1)]S_n}}{\big[\sum_{j=1}^n e^{(1-s)\beta S_{n-j}}(n-j+1)^{-\lambda}\big]^{s/(1-s)}}\right)\\
&\geq \widehat{\mathbb{E}}\left(\frac{e^{[\beta s-(\beta-1)]S_n}}{\big[\sum_{j=0}^n e^{(1-s)\beta S_j}(j+1)^{-\lambda}\big]^{s/(1-s)}}\right).
\end{aligned}
$$

Therefore it remains to show that the last expectation is of the order at least $n^{-(3/2)(1-s)\beta-\eta}$ for some $\eta = \eta(\varepsilon)$ converging to 0 as $\varepsilon \to 0$. This will be accomplished by finding a suitable event $A_n = A_n(\varepsilon)$ such that

$$
\widehat{\mathbb{E}}\left(\frac{e^{[\beta s-(\beta-1)]S_n}}{\big[\sum_{j=0}^n e^{(1-s)\beta S_j}(j+1)^{-\lambda}\big]^{s/(1-s)}}\right)\mathbf{1}_{A_n} \tag{5.26}
$$

is at least of this order for $n \to \infty$.

Fix any small ε, and put $k(n) \stackrel{\text{def}}{=} \lfloor n^\varepsilon \rfloor$ and $l(n) \stackrel{\text{def}}{=} n - \lfloor n^\varepsilon \rfloor$. Define

$$
A_n \stackrel{\text{def}}{=} A_{n,1} \cap A_{n,2} \cap A_{n,3},
$$

where

$$
\begin{aligned}
A_{n,1} &\stackrel{\text{def}}{=} \Big\{ \max_{1\leq k\leq k(n)} S_k \leq 0,\ S_{k(n)} \in [-n^{\varepsilon/2}, -n^{\varepsilon/2}+\overline{x}] \Big\},\\
A_{n,2} &\stackrel{\text{def}}{=} \Big\{ \max_{k(n)<k\leq l(n)} S_k \leq -n^{\varepsilon},\ S_{l(n)} \in [-n^{\varepsilon/2}, -n^{\varepsilon/2}+\overline{x}] \Big\},\\
A_{n,3} &\stackrel{\text{def}}{=} \Big\{ \max_{l(n)<k\leq n} S_k \leq \tfrac{3}{2}\log n,\ S_n \in [\tfrac{3}{2}\log n, \tfrac{3+\varepsilon}{2}\log n] \Big\}.
\end{aligned}
$$

By Theorem A.4, we infer upon setting $I_n \stackrel{\text{def}}{=} [-n^{\varepsilon/2}, -n^{\varepsilon/2}+\overline{x}]$ and $J_n \stackrel{\text{def}}{=} [\frac{3}{2}\log n, \frac{3+\varepsilon}{2}\log n]$

$$
\begin{aligned}
\widehat{\mathbb{P}}(A_{n,1}) &\geq C\,n^{-3\varepsilon/2},\\
\widehat{\mathbb{P}}(A_{n,1}\cap A_{n,2}) &= \int_{I_n} \widehat{\mathbb{P}}(A_{n,2}|S_{k(n)}=x)\,\widehat{\mathbb{P}}(S_{k(n)}\in dx; A_{n,1})\\
&\geq \int_{I_n} \widehat{\mathbb{P}}\Big(\max_{1\leq k\leq l(n)-k(n)} S_k \leq -n^\varepsilon - x, S_{l(n)-k(n)} \in I_n - x\Big)\\
&\qquad\qquad \times \widehat{\mathbb{P}}(S_{k(n)}\in dx; A_{n,1})\\
&\geq \int_{I_n} C\,n^{-3/2}\,\widehat{\mathbb{P}}(S_{k(n)}\in dx; A_{n,1})\\
&\geq C\,n^{-(3/2)(1+\varepsilon)},
\end{aligned}
$$

and finally (note $n - l(n) = k(n)$)

$$
\widehat{\mathbb{P}}(A_n) = \int_{J_n} \widehat{\mathbb{P}}(A_{n,3}|S_{l(n)}=x)\,\widehat{\mathbb{P}}(S_{k(n)}\in dx; A_{n,1}\cap A_{n,2})
$$

$$\geq \int_{J_n} \widehat{\mathbb{P}}\Big(\max_{1\leq k\leq k(n)} S_k \leq \tfrac{3}{2}\log n - x, S_{k(n)} \in J_n - x\Big)$$
$$\times\, \widehat{\mathbb{P}}(S_{l(n)} \in dx; A_{n,1}\cap A_{n,2})$$
$$\geq \int_{J_n} C\, n^{-3\varepsilon/2}\, \widehat{\mathbb{P}}(S_{k(n)} \in dx; A_{n,1}\cap A_{n,2})$$
$$\geq C\, n^{-(3/2)(1+2\varepsilon)} \tag{5.27}$$

for a suitable $C>0$ and n sufficiently large.

It remains to estimate the integrand in (5.26) on A_n. Plainly,

$$e^{[\beta s-(\beta-1)]S_n} \;\geq\; n^{(3/2)[\beta s-(\beta-1)]} \;=\; n^{-(3/2)(1-s)\beta} n^{3/2}. \tag{5.28}$$

As for the denominator, we split up the sum $\sum_{j=0}^n e^{(1-s)\beta S_j}(j+1)^{-\lambda}$ into $\sum_{j=0}^{k(n)}$, $\sum_{j=k(n)+1}^{l(n)}$ and $\sum_{j=l(n)+1}^{n}$. On A_n, the first sum is clearly bounded by $k(n)$, the second by a finite constant C independent of n, and the third one by $k(n)l(n)^{-\lambda}\, n^{(3/2)(1-s)\beta(1+\varepsilon)}$. This gives

$$\left[\sum_{j=0}^n e^{(1-s)\beta S_j}(j+1)^{-\lambda}\right]^{s/(1-s)} \;\leq\; C\, n^{(3\varepsilon/2)s\beta+3\varepsilon s/(1-s)} \tag{5.29}$$

for some $C>0$ and all n sufficiently. By combining (5.26–29), we finally conclude

$$\begin{aligned}
\widehat{\mathbb{E}} Z_{n,\beta} &\geq \widehat{\mathbb{E}}\left(\frac{e^{[\beta s-(\beta-1)]S_n}}{\left[\sum_{j=0}^n e^{(1-s)\beta S_j}(j+1)^{-\lambda}\right]^{s/(1-s)}}\right)\mathbf{1}_{A_n}\\
&\geq C\, n^{-(3/2)(1-s)\beta} n^{3/2} n^{-(3\varepsilon/2)s\beta-3\varepsilon s/(1-s)}\widehat{\mathbb{P}}(A_n)\\
&\geq C\, n^{-(3/2)(1-s)\beta} n^{-(3\varepsilon/2)s\beta-3\varepsilon s/(1-s)} n^{-3\varepsilon}\\
&= C\, n^{-(3/2)(1-s)\beta-\eta}
\end{aligned}$$

with a suitable $\eta=\eta(\varepsilon)$ converging to 0 as $\varepsilon\to 0$. This completes the proof of the lower bound. ♢

Lemma 5.3. *Suppose* (C1–5). *Then*

$$\xi \;\stackrel{\text{def}}{=}\; \sup_{n\geq 2}\sup_{1\leq j\leq n} \widehat{\mathbb{P}}\Big(\mathcal{I}_j^{(n)}=\emptyset \;\; or \;\; \max_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v)\leq n^{-\beta\kappa}\Big) \;\in (0,1)$$

for $\kappa>0$ sufficiently large.

Proof. The assertion follows from

$$\begin{aligned}
&\widehat{\mathbb{P}}\Big(\mathcal{I}_j^{(n)}=\emptyset \text{ or } \max_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v)\leq n^{-\beta\kappa}\Big)\\
&\qquad\leq \widehat{\mathbb{P}}\Big(\mathcal{I}_j^{(n)}=\emptyset \text{ or } \max_{v\in\mathcal{I}_j^{(n)}} M_{n-j}(v)\geq \kappa\log n\Big)\\
&\qquad\leq \widehat{\mathbb{P}}(I_1^{(1)}=\emptyset) \;+\; \mathbb{P}\left(\sup_{j\geq 0}\frac{M_j}{\log(j\vee n)}\geq\kappa\right)
\end{aligned}$$

for all $n \geq 2$ and $1 \leq j \leq n$ in combination with the facts that $\widehat{\mathbb{P}}(I_1^{(1)} = \emptyset) \in (0,1)$ and $\limsup_{j\to\infty} \frac{M_j}{\log j} \leq \frac{3}{2}$ a.s. (Proposition 4.1). $\diamondsuit$

Lemma 5.4. *Suppose* (C1–5). *Let $p \in (0,1)$ and I_n be defined as in (5.20) for any $\gamma, \kappa > 0$. Then*

$$\widehat{\mathbb{E}}\left(\frac{e^{-\beta p S(V_n^*)}}{Z_{n,\beta}^p}\mathbf{1}_{I_n}\right) \;\leq\; n^\theta$$

for some $\theta > 0$ and all n sufficiently large.

Proof. Fix any $n \geq e^2$, recall that $n^* = \min\{0 \leq k < n : S(V_k) = \min_{0\leq j<n} S(V_j)\}$ and put

$$\mathcal{L} \stackrel{\text{def}}{=} \{j \in \{1,...,n\} : |j - n^*| \leq \gamma \log n,\ \mathcal{I}_j^{(n)} \neq \emptyset \text{ and } \max_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v) > n^{-\beta\kappa}\}.$$

It follows that $I_n \subset \{\mathcal{L} \neq \emptyset\}$. Furthermore, $|S(v) - S(V_j)| \leq 2\overline{x} \leq \overline{x}\log n$ and $|S(V_j) - S(V_n^*)| \leq \overline{x}\gamma\log n$ for $j \in \mathcal{L}$ and $v \in \mathcal{I}_j^{(n)}$ whence

$$S(v) \;\leq\; S(V_n^*) + \overline{x}(1+\gamma)\log n$$

for all such j, v. Consequently, on the event $\{\mathcal{L} \neq \emptyset\}$,

$$Z_{n,\beta} \;\geq\; \sum_{j\in\mathcal{L}}\sum_{v\in\mathcal{I}_j^{(n)}} e^{-\beta S(v)} Z_{n-j,\beta}(v) \;\geq\; \frac{e^{-\beta S(V_n^*)}}{n^{\overline{x}(1+\gamma)}}\sum_{j\in\mathcal{L}}\sum_{v\in\mathcal{I}_j^{(n)}} Z_{n-j,\beta}(v) \;\geq\; \frac{e^{-\beta S(V_n^*)}}{n^{\overline{x}(1+\gamma)+\beta\kappa}}$$

and therefore

$$\frac{e^{-\beta S(V_n^*)}}{Z_{n,\beta}} \;\leq\; n^{\overline{x}(1+\gamma)+\beta\kappa}.$$

This proves the assertion with $\theta \stackrel{\text{def}}{=} \overline{x}(1+\gamma) + \beta\kappa$. $\diamondsuit$

6. Proof of the main results

Proof of Theorem 1.3. We start with the proof of $\limsup_{n\to\infty} \frac{\log Z_n}{\log n} \leq -\frac{1}{2}$ $\mathbb{P}$-a.s. and consider the nonnegative supermartingale $(Z_n^r)_{n\geq 0}$ for any fixed $r \in (0,1)$. By Doob's maximal inequality and Theorem 5.1, we have for all $\lambda > 0$ and $n \leq m$

$$\mathbb{P}\Big(\max_{n\leq j\leq m} Z_j^r \geq \lambda\Big) \;\leq\; \frac{\mathbb{E}Z_n^r}{\lambda} \;\leq\; \frac{C}{\lambda}\, n^{-r/2},$$

where C does not depend on n, m, λ. Pick any $\varepsilon \in (0, \frac{1}{2})$ and put $n_k \stackrel{\text{def}}{=} \lfloor k^{2/\varepsilon r}\rfloor$ for $k \geq 0$. Then

$$\begin{aligned}\sum_{k\geq 1}\mathbb{P}\Big(\max_{n_k\leq j\leq n_{k+1}} j^{1/2-\varepsilon} Z_j \geq 1\Big) \;&\leq\; \sum_{k\geq 1}\mathbb{P}\Big(\max_{n_k\leq j\leq n_{k+1}} Z_j \geq n_{k+1}^{-1/2+\varepsilon}\Big)\\ &\leq\; C\sum_{k\geq 1} n_{k+1}^{-\varepsilon r} \;<\; \infty,\end{aligned}$$

whence $\limsup_{n\to\infty} n^{1/2-\varepsilon} Z_n \le 1$ $\mathbb{P}$-a.s. and, a fortiori, $\limsup_{n\to\infty} \frac{\log Z_n}{\log n} \le -\frac{1}{2} + \varepsilon$ $\mathbb{P}$-a.s. follows by the Borel-Cantelli lemma.

For the proof of $\liminf_{n\to\infty} \frac{\log Z_n}{\log n} \ge -\frac{1}{2}$ $\mathbf{P}$-a.s. it suffices to consider the nonextinctive case (thus $\mathbf{P} = \mathbb{P}$), for Lemma 2.1 gives $\liminf_{n\to\infty} \frac{\log Z_n}{\log n} \ge \liminf_{n\to\infty} \frac{\log \widehat{Z}_n}{\log n}$ in the general situation. Fix an arbitrary $\varepsilon \in (0, \frac{1}{2})$. The Paley-Zygmund inequality in combination with (5.1) yields, for each $r \in (0, \frac{1}{2})$, suitable $\rho > 0$ and sufficiently large n

$$\mathbb{P}\big(Z_n^r > \tfrac{1}{2}\,\mathbb{E}Z_n^r\big) \ \ge\ \frac{1}{4}\,\frac{[\mathbb{E}Z_n^r]^2}{\mathbb{E}Z_n^{2r}} \ \ge\ \rho, \tag{6.1}$$

so that

$$\mathbb{P}(Z_n \le n^{-1/2-\varepsilon}) \ \le\ \mathbb{P}\big(Z_n^r \le \tfrac{1}{2}\,\mathbb{E}Z_n^r\big) \ \le\ 1-\rho \tag{6.2}$$

for all n sufficiently large.

Put $\tau(n) \stackrel{\text{def}}{=} \inf\{k \ge 1 : \log N_k \ge \frac{\varepsilon}{2\overline{x}} \log m \log n\}$ and note that

$$\frac{\tau(n)}{\log n} \ \to\ \frac{\varepsilon}{2\overline{x}} \quad \text{a.s.},$$

for $(N_k)_{k\ge0}$ is an ordinary supercritical nonextinctive Galton-Watson process with offspring mean m. As a consequence, $\mathbf{1}_{\{\tau(n)\le k(n)\}} \to 1$ a.s. for $k(n) \stackrel{\text{def}}{=} \lfloor \frac{\varepsilon}{\overline{x}} \log n \rfloor$. Since, with $l(n) \stackrel{\text{def}}{=} n - k(n)$,

$$Z_n \ =\ \sum_{|v|=k(n)} e^{-S(v)} Z_{l(n)}(v) \ \ge\ e^{-\overline{x}k(n)} \sum_{|v|=k(n)} Z_{l(n)}(v) \ \ge\ n^{-\varepsilon} \sum_{|v|=k(n)} Z_{l(n)}(v),$$

and the $(Z_{l(n)}(v))_{|v|=k(n)}$ are mutually independent and independent of $\mathcal{F}_{k(n)}$ with the same distribution as $Z_{l(n)}$, we infer with the help of (6.2)

$$\begin{aligned}
&\sum_{n\ge1} \mathbb{P}(Z_n \le l(n)^{1/2-2\varepsilon}, \tau(n) \le k(n)) \\
&\qquad \le\ \sum_{n\ge1} \mathbb{E}\Big(\mathbb{P}(Z_{l(n)} \le l(n)^{1/2-\varepsilon})^{N_{k(n)}} \mathbf{1}_{\{N_{k(n)} \ge n^{\varepsilon \log m/\overline{x}}\}}\Big) \\
&\qquad \le\ C \sum_{n\ge1} (1-\rho)^{n^{\varepsilon \log m/\overline{x}}} \ <\ \infty
\end{aligned} \tag{6.3}$$

for sufficiently large n and then the desired conclusion by another appeal to the Borel-Cantelli lemma (notice also $l(n) \simeq n$ and recall $\mathbf{1}_{\{\tau(n)\le k(n)\}} \to 1$ a.s.)

Proof of Theorem 1.4. Use $e^{-\beta M_n} \le Z_{n,\beta} \le Z_n^\beta$ in combination with Proposition 4.1 and Theorem 1.3 to infer

$$-\frac{\beta}{2} \ \le\ -\beta \liminf_{n\to\infty} \frac{M_n}{\log n} \ \le\ \limsup_{n\to\infty} \frac{\log Z_{n,\beta}}{\log n} \le\ \limsup_{n\to\infty} \frac{\beta \log Z_n}{\log n} \ =\ -\frac{\beta}{2} \quad \mathbf{P}\text{-a.s.}$$

This leaves us with the proof of (1.11) and the lower bound in (1.10). The following arguments are very similar to those in the previous proof and therefore presented in abridged form.

By an application of Markov's inequality and (5.16) of Theorem 5.2, we infer

$$\mathbb{P}(n^{(3/2)\beta} Z_{n,\beta} > n^{\varepsilon}) \;\le\; \frac{\mathbb{E}Z_{n,\beta}^{r}}{n^{-(3/2)\beta r+\varepsilon}} \;=\; n^{o(1)-\varepsilon} \;\to\; 0, \quad n\to\infty, \tag{6.4}$$

for each $\varepsilon > 0$, in particular $\liminf_{n\to\infty} \frac{\log Z_{n,\beta}}{\log n} \le -\frac{3\beta}{2}$ $\mathbb{P}$-a.s.

The subsequent Borel-Cantelli-type argument, for which it is again no loss of generality to consider the nonextinctive case only, will show $\liminf_{n\to\infty} n^{(3/2)\beta+\varepsilon} Z_{n,\beta} \ge 0$ $\mathbf{P}$-a.s. for each $\varepsilon > 0$ which in combination with (6.4) implies (1.11) as well as the lower bound in (1.10).

Fix an arbitrary $\varepsilon \in (0, \frac{1}{2})$. The Paley-Zygmund inequality in combination with (5.16) yields, for each $r \in (0, \frac{1}{2\beta})$ and suitable $\varepsilon_n \to 0$,

$$\mathbb{P}\big(Z_{n,\beta}^{r} > \tfrac{1}{2}\,\mathbb{E}Z_{n,\beta}^{r}\big) \;\ge\; \frac{1}{4}\,\frac{[\mathbb{E}Z_{n,\beta}^{r}]^2}{\mathbb{E}Z_{n,\beta}^{2r}} \;=\; n^{-\varepsilon_n},$$

so that

$$\mathbb{P}(Z_{n,\beta} \le n^{-(3/2)\beta-\varepsilon}) \;\le\; \mathbb{P}\big(Z_{n,\beta}^{r} \le \tfrac{1}{2}\,\mathbb{E}Z_{n,\beta}^{r}\big) \;\le\; 1 - n^{\varepsilon_n} \;\le\; \exp(-n^{-\varepsilon_n}) \tag{6.5}$$

for all n sufficiently large. Similar to the proof of Theorem 1.3, put $\tau(n) \stackrel{\text{def}}{=} \inf\{k \ge 1 : \log N_k \ge \frac{\varepsilon}{2\beta\overline{x}} \log m \log n\}$, $k(n) \stackrel{\text{def}}{=} \lfloor \frac{\varepsilon}{\beta\overline{x}} \log n \rfloor$ and $l(n) \stackrel{\text{def}}{=} n - k(n)$ to obtain

$$Z_{n,\beta} \;\ge\; e^{-\beta\overline{x}k(n)} \sum_{|v|=k(n)} Z_{l(n),\beta}(v) \;\ge\; n^{-\varepsilon} \sum_{|v|=k(n)} Z_{l(n),\beta}(v).$$

and thereupon with the help of (6.5) (compare (6.3))

$$\sum_{n\ge 1} \mathbb{P}(Z_{n,\beta} \le e^{-\beta\overline{x}k} n^{(3/2)\beta-2\varepsilon}, \tau(n) \le k(n)) \;\le\; C \sum_{n\ge 1} \exp(-n^{-\varepsilon_n+\varepsilon\log m/\beta\overline{x}}) \;<\; \infty$$

for all n sufficiently large. We arrive at the desired conclusion by invoking the Borel-Cantelli lemma and using $\mathbf{1}_{\{\tau(n)\le k(n)\}} \to 1$ a.s. (as $\frac{\tau(n)}{\log n} \to \frac{\varepsilon}{2\beta\overline{x}}$ a.s.) $\diamondsuit$

We are now ready to prove Theorem 1.2 for which, in view of Proposition 4.1, it remains to show (1.6) and the reverse inequalities in (4.1).

Proof of Theorem 1.2. Using $e^{-\beta M_n} \le Z_{n,\beta}$ for each $n \ge 0$ and $\beta \ge 1$ in combination with Theorem 1.3 and 1.4, we infer with $\beta = 1$

$$\liminf_{n\to\infty} \frac{M_n}{\log n} \;\ge\; -\limsup_{n\to\infty} \frac{\log Z_n}{\log n} \;=\; \frac{1}{2} \quad \mathbf{P}\text{-a.s.}$$

and with any $\beta > 1$

$$\limsup_{n\to\infty} \frac{M_n}{\log n} \;\ge\; -\limsup_{n\to\infty} \frac{\log Z_{n,\beta}}{\beta \log n} \;=\; \frac{3}{2} \quad \mathbf{P}\text{-a.s.}$$

This together with (4.1) proves (1.5).

As to (1.6), note that, for all $1 < \beta < \gamma$ and $n \geq 0$,

$$-\beta^{-1} \log Z_{n,\beta} \;\leq\; M_n \;\leq\; (\gamma - \beta)^{-1}(\log Z_{n,\beta} - \log Z_{n,\gamma})$$

where the upper bound follows from $Z_{n,\gamma} \exp\big((\gamma-\beta)M_n\big) \leq\; Z_{n,\beta}$. Then the assertion follows by taking logarithms and invoking Theorem 1.4. $\diamondsuit$

Finally, we must prove Theorem 1.1 and need the following lemma as a prerequisite.

Lemma 6.1. *Suppose (C1–5). For any* $\gamma \in (0, 1)$,

$$\mathbb{P}\left(\left|\frac{Z_{n+1}}{Z_n} - 1\right| \geq n^{-\gamma}\right) \;\leq\; n^{-(1-\gamma)/2+o(1)}, \quad n \to \infty.$$

Proof. For any $n \geq 1$, write

$$\frac{Z_{n+1}}{Z_n} - 1 \;=\; \sum_{|v|=n} \frac{e^{-S(v)}}{Z_n}\left(\sum_{j\geq 1} e^{-X_j(v)} - 1\right)$$

and notice that this expression conditioned upon $\mathcal{F}_n$, may be viewed as a weighted sum of i.i.d. centered random variables (having the distribution of $Z_1 - 1$) and thus as a martingale limit. By applying the Topchii-Vatutin inequality [5], we infer for any $1 < \beta \leq 2$

$$\mathbb{E}\left(\left|\frac{Z_{n+1}}{Z_n} - 1\right|^{\beta}\middle|\mathcal{F}_n\right) \;\leq\; 2\sum_{|v|=n} \frac{e^{-\beta S(v)}}{Z_n^{\beta}}\, \mathbb{E}|Z_1 - 1|^{\beta} \;=\; c\,\frac{Z_{n,\beta}}{Z_n^{\beta}} \quad \text{a.s.}$$

where $c \stackrel{\text{def}}{=} 2\,\mathbb{E}|Z_1 - 1|^{\beta}$.

Next fix $\gamma \in (0, 1)$, $\varepsilon, \eta > 0$ and $r \in (0, \frac{1}{\beta})$. Define the event $A_n \in \mathcal{F}_n$ by

$$A_n \;\stackrel{\text{def}}{=}\; \{Z_n \geq n^{-(1/2)-\varepsilon}, Z_{n,\beta} \leq n^{-(3/2)\beta+\eta}\}$$

on which we obviously have

$$\frac{Z_{n,\beta}}{Z_n^{\beta}} \;\leq\; n^{-(1-\varepsilon)\beta+\eta}.$$

By Proposition 3.1, there exists a $\kappa > 0$ such that

$$\mathbb{P}(Z_n \leq n^{-(1/2)-\varepsilon}) \;\leq\; \exp(-n^{\kappa}),$$

while Theorem 5.2 implies

$$\mathbb{P}(Z_{n,\beta} \geq n^{-(3/2)\beta+\eta}) \;\leq\; n^{(3/2)\beta r-\eta r}\mathbb{E}Z_{n,\beta}^r \;=\; n^{-\eta r+o(1)}, \quad n \to \infty.$$

Consequently, $\mathbb{P}(A_n^c) \le n^{-\eta r + o(1)}$, as $n \to \infty$. Putting the facts together, we arrive at

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{Z_{n+1}}{Z_n} - 1\right| \ge n^{-\gamma}\right) \; &\le \; \mathbb{P}(A_n^c) \; + \; n^{\gamma\beta} \int_{A_n} \mathbb{E}\left(\left|\frac{Z_{n+1}}{Z_n} - 1\right|^{\beta} \middle| \mathcal{F}_n\right) d\mathbb{P} \\
&\le \; \mathbb{P}(A_n^c) \; + \; c\,\mathbb{E}\left(\mathbf{1}_{A_n} \frac{Z_{n,\beta}}{Z_n^{\beta}}\right) \\
&\le \; n^{-\eta r + \varepsilon} \; + \; n^{\gamma\beta - (1-\varepsilon)\beta + \eta}.
\end{aligned}$$

for all sufficiently large n. Choosing $\eta = \frac{1-\gamma}{2}$ and by sending $\varepsilon \to 0, \beta \to 1$ and finally $r \to 1$, the assertion follows. ♢

PROOF OF THEOREM 1.1. Owing to (2.2), it obviously suffices to prove the assertion for the nonextinctive case only, for otherwise we may consider $c_n \widehat{Z}_n$ under $\mathbf{P}$ instead of $c_n Z_n$ under $\mathbb{P}$.

Let c_n be defined by $\mathbb{E}(c_n Z_n)^{1/2} = 1$, so $c_n \stackrel{\text{def}}{=} (\mathbb{E} Z_n^{1/2})^{-2}$. By Theorem 5.1, these constants satisfy (1.2), and the sequence $(c_n Z_n)_{n \ge 1}$ is tight. Let Y be any weak limit of a subsequence $(c_{n(k)} Z_{n(k)})_{k \ge 1}$. We will prove uniqueness of Y. By another appeal to Theorem 5.1 and dominated convergence, we infer $\mathbb{E} Y^{1/2} = 1$. Moreover, Lemma 6.1 ensures $Z_{n+1}/Z_n \to 1$ in probability and thus the weak convergence of $c_{n(k)} Z_{n(k)+1}$ to Y. Now

$$c_{n(k)} Z_{n+1} \;=\; \sum_{|v|=1} e^{-S(v)} c_{n(k)} Z_{n(k)}(v)$$

and since conditioned upon $\mathcal{F}_1$ the $Z_{n(k)}(v)$ are independent copies of $Z_{n(k)}$, we infer, as $k \to \infty$, that

$$Y \;\stackrel{d}{=}\; \sum_{|v|=1} e^{-S(v)} Y(v)$$

with $Y(v)$ being independent copies of Y. Hence Y is a nondegenerate solution to the fixed point equation (1.3) and thus, by essential uniqueness, a constant multiple of W, the a.s. limit of the derivative martingale $(W_n)_{n \ge 0}$. This completes the proof of the theorem. ♢

Appendix

Lemma A.1. *Let f be the generating function of a probability distribution $(p_n)_{n \ge 0}$ on the set $\mathbb{N}_0$. Then $g(t) = -\log f(e^{-t})$, $t \ge 0$, is an increasing smooth convex function with $g(0) = 0$.*

PROOF. Just note that g is the composition of three smooth convex functions. ♢

Lemma A.2. *Let $(S_n)_{n \ge 0}$ be a zero-delayed centered random walk with i.i.d. increments $X_1, X_2, \ldots$ having finite variance. Put $\sigma^- \stackrel{\text{def}}{=} \inf\{n \ge 1 : S_n \le 0\}$. Then*

$$\sup_{n \ge 1} \mathbb{E}\left(\sum_{k=0}^{n} e^{-S_k} \middle| \sigma^- > 2n\right) \;<\; \infty \qquad (A1)$$

as well as

$$\limsup_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}(S_n\le\varepsilon n^{1/2}|\sigma^->2n) \;=\; 0. \tag{A2}$$

Jensen's inequality in combination with (A1) implies

$$\liminf_{n\to\infty}\mathbb{E}\left(\Big[\textstyle\sum_{k=0}^n e^{-S_k}\Big]^{-1}\Big|\sigma^->2n\right) \;>\; 0.$$

The same statement with conditioning event $\{\sigma^->n\}$ was given by Kozlov in [22, eq. (34)]. However, his proof appears to be flawed and we have been unable to fix it.

Proof. We start by gathering some required facts about certain quantities associated with the random walk $(S_n)_{n\ge0}$ which are all known from classical fluctuation theory. As usual we will use C for a generic positive and finite constant that may differ from line to line.

Let $(\sigma_k^+)_{k\ge1}$ be the sequence of strictly ascending ladder epochs associated with $(S_n)_{n\ge0}$, i.e. $\sigma_0^+\stackrel{\text{def}}{=}0$ and $\sigma_n^+\stackrel{\text{def}}{=}\inf\{k>\sigma_{n-1}^+ : S_k>\widehat{S}_{n-1}\}$ for $n\ge1$, where $\widehat{S}_n\stackrel{\text{def}}{=}S_{\sigma_n^+}$. Define $Q_n\stackrel{\text{def}}{=}\mathbb{P}(S_n\in\cdot,\sigma^->n)$ for $n\ge0$ and note that

$$Q_n \;=\; \sum_{k\ge0}\mathbb{P}(S_n\in\cdot,\sigma_k^+=n) \tag{A3}$$

as well as

$$\sum_{n\ge0}Q_n \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma^--1}\mathbf{1}_{\{S_n\in\cdot\}}\right) \;=\; U^+$$

where U^+ denotes the renewal measure associated with $(\widehat{S}_n)_{n\ge0}$. Since $\sup_{t\ge0}U^+[t,t+1]<\infty$, we infer

$$\mathbb{E}\left(\sum_{n=0}^{\sigma^--1}e^{-sS_j}\right) \;=\; \int_{(0,\infty)}e^{-sx}\;U^+(dx) \;<\; \infty \tag{A4}$$

for any $s>0$.

Finally, putting $p_n(x)\stackrel{\text{def}}{=}\mathbb{P}(\min_{0\le k\le n}S_k>-x)$ for $x\ge0$ (hence $p_n(0)=\mathbb{P}(\sigma^->n)$), Kozlov [22, Theorem A] has shown that

$$p_n(x) \;\le\; C(x+1)n^{-1/2} \tag{A5}$$

for all $x\in[0,\infty)$ and $n\ge1$.

Turning to the first assertion of the lemma, we use $\mathbb{P}(\sigma^->n)\ge Cn^{-1/2}$ for n sufficiently large and (A5) to infer

$$\begin{aligned}\mathbb{E}\left(\sum_{k=0}^n e^{-S_k}\Big|\sigma^->2n\right) \;&\le\; Cn^{1/2}\sum_{k=0}^n\mathbb{E}\Big(e^{-S_k}\mathbf{1}_{\{\sigma^->k,\min_{k\le j\le 2n}S_j-S_k>-S_k\}}\Big)\\ &=\; Cn^{1/2}\sum_{k=0}^n\int_{(0,\infty)}e^{-x}p_{2n-k}(x)\;Q_k(dx)\end{aligned}$$

$$\le C \sum_{k\ge 0} \int_{(0,\infty)} e^{-x}(x+1)\; Q_k(dx)$$
$$= C\,\mathbb{E}\Bigg(\sum_{k=0}^{\sigma^- -1} e^{-S_k}(S_k+1)\Bigg).$$

But the last expectation is clearly bounded by a constant times the expectation in (A4) for any $s \in (0,1)$ and is thus finite. This proves (A1).

Turning to (A2), we have

$$\begin{aligned}
\mathbb{P}(S_n \le \varepsilon n^{1/2} | \sigma^- > 2n) &\le Cn^{1/2}\,\mathbb{P}(S_n \le \varepsilon n^{1/2}, \sigma^- > n, \min_{n\le j\le 2n} S_j - S_n > -S_n)\\
&= Cn^{1/2} \int_{(0,\varepsilon n^{1/2}]} p_n(x)\; Q_n(dx)\\
&\le C \int_{(0,\varepsilon n^{1/2}]} (x+1)\; Q_n(dx)\\
&\le C\varepsilon n^{1/2}\,\mathbb{P}(\sigma^- > n)
\end{aligned}$$

which yields the desired conclusion because $n^{1/2}\mathbb{P}(\sigma^- > n)$ stays bounded as $n \to \infty$. ♢

Lemma A.3. *Under the assumptions of the previous lemma, put* $M_n \stackrel{\text{def}}{=} \min_{0\le k\le n} S_k$ *for* $n \ge 0$. *Then*

$$\sup_{n\ge 1} n^{1/2}\,\mathbb{E}e^{sM_n} \;<\; \infty \tag{A6}$$

as well as

$$\sup_{n\ge 1} \mathbb{E}S_n^+ e^{sM_n} \;<\; \infty \tag{A7}$$

for each $s > 0$.

Proof. It suffices to consider $s = 1$. Using Kozlov's estimate (A5), we infer

$$\begin{aligned}
\mathbb{E}e^{M_n} &\le \sum_{k=0}^{n} e^{-k} p_n(k+1) \;+\; \mathbb{E}e^{M_n}\mathbf{1}_{\{M_n<-n\}}\\
&\le Cn^{-1/2}\sum_{k\ge 0}(k+2)e^{-k} \;+\; e^{-n}
\end{aligned}$$

for all $n \ge 1$ and thereby (A6).

Note that

$$\mathbb{E}e^{M_n} \;=\; \sum_{k=0}^{n} \mathbb{E}e^{S_k}\mathbf{1}_{\{\sigma^+>k\}}\,\mathbb{P}(\sigma^- > n-k).$$

and furthermore that, again by (A6),

$$\mathbb{E}|M_n|e^{M_n} \;\le\; C\,\mathbb{E}e^{M_n/2} \;=\; O(n^{-1/2}), \quad n \to \infty.$$

Since $\mathbb{E}S_n^+ e^{M_n} \le \mathbb{E}|M_n|e^{M_n} + \mathbb{E}(S_n - M_n)e^{M_n}$, it remains to verify $\mathbb{E}(S_n - M_n)e^{M_n} = O(1)$, as $n \to \infty$. But

$$\begin{aligned}
\mathbb{E}(S_n - M_n)e^{M_n} &= \sum_{k=0}^{n} \mathbb{E}e^{S_k}\mathbf{1}_{\{\sigma^+ > k\}}\, \mathbb{E}S_{n-k}\mathbf{1}_{\{\sigma^- > n-k\}} \\
&\le n^{1/2} \sup_{k\ge 1} \mathbb{E}\left(\frac{S_k}{k^{1/2}}\Big|\sigma^- > k\right) \sum_{k=0}^{n} \mathbb{E}e^{S_k}\mathbf{1}_{\{\sigma^+ > k\}}\, \mathbb{P}(\sigma^- > n-k) \\
&= \sup_{k\ge 1} \mathbb{E}\left(\frac{S_k}{k^{1/2}}\Big|\sigma^- > k\right) n^{1/2}\, \mathbb{E}e^{M_n},
\end{aligned}$$

which gives the desired conclusion if we still verify that the supremum in the last line is finite or, equivalently (as $k^{1/2}\mathbb{P}(\sigma^- > k)$ converges to a positive limit),

$$\sup_{n\ge 1} \mathbb{E}S_n\mathbf{1}_{\{\sigma^- > n\}} < \infty. \tag{A8}$$

As in the proof of Lemma A.2, let $(\widehat{S}_n)_{n\ge 0}$ be the renewal process of strictly ascending ladder heights. We will use the following useful formula discovered by Alili and Doney [4].

$$\mathbb{P}(\sigma_k^+ = n, \widehat{S}_k \in \cdot) = \frac{k}{n}\,\mathbb{P}(\widehat{S}_{k-1} < S_n \le \widehat{S}_k, S_n \in \cdot)$$

for all $k, n \ge 1$. Summing over k and recalling (A3), we then infer

$$Q_n = \sum_{k\ge 1} \frac{k}{n}\,\mathbb{P}(\widehat{S}_{k-1} < S_n \le \widehat{S}_k, S_n \in \cdot)$$

and therefore particularly

$$\mathbb{E}S_n\mathbf{1}_{\{\sigma^- > n\}} = \sum_{k\ge 1} \frac{k}{n}\,\mathbb{E}S_n\mathbf{1}_{\{\widehat{S}_{k-1} < S_n \le \widehat{S}_k\}} \tag{A9}$$

for all $n \ge 1$. Note that $\mathbb{E}S_1^2 < \infty$ implies $\mathbb{E}\widehat{S}_1 < \infty$, and we can therefore fix $\varepsilon > 0$ so small that, with $\varphi(\lambda) \overset{\text{def}}{=} \mathbb{E}e^{-\lambda \widehat{S}_1}$, we have $\eta \overset{\text{def}}{=} e^{\varepsilon}\varphi(\varepsilon) < 1$ implying

$$\mathbb{P}(\widehat{S}_{k-1} \le \varepsilon k) \le e^{\varepsilon k}\varphi(\varepsilon)^{k-1} = \varepsilon\eta^{k-1}$$

for all $k \ge 1$. Finally, for such ε and all $k, n \ge 1$

$$k\,\mathbb{E}S_n\mathbf{1}_{\{\widehat{S}_{k-1} < S_n \le \widehat{S}_k\}} \le \varepsilon^{-1}\mathbb{E}S_n^2\mathbf{1}_{\{\widehat{S}_{k-1} < S_n \le \widehat{S}_k\}} + k^2\,\mathbb{P}(\widehat{S}_{k-1} \le \varepsilon k)$$

so that in (A9)

$$\begin{aligned}
\mathbb{E}S_n\mathbf{1}_{\{\sigma^- > n\}} &\le \frac{1}{n}\sum_{k\ge 1}\left(\varepsilon^{-1}\mathbb{E}S_n^2\mathbf{1}_{\{\widehat{S}_{k-1} < S_n \le \widehat{S}_k\}} + \varepsilon k^2\eta^{k-1}\right) \\
&\le \frac{\mathbb{E}S_n^2}{\varepsilon n} + \frac{C}{n} = \frac{\mathbb{E}S_1^2}{\varepsilon} + \frac{C}{n}
\end{aligned}$$

for all $n \geq 1$. $\diamondsuit$

Theorem A.4. (Addario-Berry & Reed [2]) *Let $(S_n)_{n\geq 0}$ be a zero-delayed centered random walk with i.i.d. increments $X_1, X_2, \ldots$ having a finite moment of order $2+\delta$ for some $\delta > 0$ and lattice span d. Suppose $m_n \geq 0$ and $m_n = O(n^{1/2})$. Then for any fixed $a > 0$, there are constants $C_1, C_2 \in (0,\infty)$, such that*

$$C_1 n^{-3/2} \ \leq \ \frac{\mathbb{P}(k \leq S_n \leq k+a, S_j > -m, 1 \leq j < n)}{(m+1)(k+m+1)} \ \leq \ C_2 n^{-3/2}$$

for all $m \in [0, m_n]$, $k \in [-m, m]$ (both in $d\mathbb{Z}$, if $d > 0$) and all sufficiently large n. The upper bound even holds true for all $m \geq 0$, $k \geq -m$ and $n \in \mathbb{N}$.

With the help of this theorem, we can now prove the following

Lemma A.5. *Let $(S_n)_{n\geq 0}$ be a zero-delayed centered random walk with i.i.d. increments $X_1, X_2, \ldots$ having $\mathbb{E}|X_1|^3 < \infty$. For $\varepsilon, a > 0$, $n \in \mathbb{N}$ and $x \in [0,n]$, define*

$$g_n(x) \ = \ g_{n,\varepsilon,a}(x) \ \stackrel{\text{def}}{=} \ \varepsilon \min\Big(x^{1/8}, n^{\varepsilon}, (n-x)^{1/8}\Big).$$

Then

$$\lim_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \frac{\mathbb{P}(S_j > g_n(j),\, 1 \leq j < n,\, S_n \in [a\log n, a\log n + b])}{\mathbb{P}(\sigma^- > n, S_n \in [a\log n, a\log n + b])} \ = \ 1 \qquad (A10)$$

for all $b > 0$.

Proof. Fix any $a, b > 0$ and note that

$$g_n(x) \ = \ \begin{cases} \varepsilon x^{1/8}, & \text{if } x \in [0, n^{8\varepsilon}], \\ \varepsilon n^{\varepsilon}, & \text{if } x \in (n^{8\varepsilon}, n - n^{8\varepsilon}), \\ \varepsilon (n-x)^{1/8}, & \text{if } x \in [n - n^{8\varepsilon}, n]. \end{cases}$$

Put

$$p_n(x,y,z) \ \stackrel{\text{def}}{=} \ \mathbb{P}(x + S_j > 0,\, 1 \leq j \leq n,\, x + S_n \in [y, y+z])$$

and note further that

$$p_n(0, a\log n, b) \ = \ \mathbb{P}(\sigma^- > n, S_n \in [a\log n, a\log n + b]).$$

Instead of (A10), we will prove the equivalent assertion

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \frac{\mathbb{P}(\tau_n < n < \sigma^-,\, S_n \in [a\log n, a\log n + b])}{p_n(0, a\log n, b)} \ = \ 0, \qquad (A11)$$

where $\tau_n = \tau_{n,\varepsilon} \stackrel{\text{def}}{=} \inf\{k \geq 1 : 0 < S_k \leq g_n(k) \text{ or } k = n\}$. We will split up the event in the enumerator into the subevents where $\tau_n \leq n^{8\varepsilon}$, $\tau_n > n - n^{8\varepsilon}$, and $n^{8\varepsilon} < \tau_n \leq n - n^{8\varepsilon}$.

By Theorem A.4, $\sup_{0\le x\le n^\varepsilon} \frac{p_n(x,a\log n,b)}{p_n(0,a\log n,b)} \le C(x+1)$ for all n as well as

$$p_n(0,a\log n,b) \;=\; \Theta(n^{-3/2}\log^2 n), \quad n\to\infty.$$

This will be utilized hereafter at several places without further notice.

Using the Berry-Esséen theorem, it follows that, for any fixed m,

$$\mathbb{P}(\tau_n \le m) \;\le\; \sum_{k=1}^{m} \mathbb{P}(0 < S_k \le \varepsilon k^{1/8}) \;\le\; O(\varepsilon), \quad \varepsilon \downarrow 0. \tag{A12}$$

Furthermore, by another appeal to Theorem A.4,

$$\begin{aligned}
\sum_{k=m+1}^{n^{8\varepsilon}} k^{1/8}\mathbb{P}(\sigma^- > k, \tau_n = k) \;&\le\; \sum_{k=m+1}^{n^{8\varepsilon}} k^{1/8} p_k(0,0,\varepsilon k^{1/8}) \\
&\le\; \sum_{k=m+1}^{n^{8\varepsilon}} k^{1/8} \sum_{j=0}^{\varepsilon k^{1/8}} p_k(0,j,1) \\
&\le\; \sum_{k>m} k^{1/8} \sum_{j=0}^{\varepsilon k^{1/8}} \frac{j+1}{k^{3/2}} \\
&\le\; C\varepsilon^2 \sum_{k>m} k^{-9/8}
\end{aligned} \tag{A13}$$

for all n sufficiently large. A combination of (A12) and (A13) gives

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \sum_{k=1}^{n^{8\varepsilon}} k^{1/8}\mathbb{P}(\sigma^- > k, \tau_n = k) \;=\; 0$$

and thus

$$\begin{aligned}
&\frac{\mathbb{P}(\sigma^- > n, \tau_n \le n^{8\varepsilon},\, S_n \in [a\log n, a\log n + b])}{p_n(0,a\log n,b)} \\
&= \sum_{k=1}^{n^{8\varepsilon}} \int_{(0,g_n(k)]} \frac{p_{n-k}(x,a\log n,b)}{p_n(0,a\log n,b)}\, \mathbb{P}(\sigma^- > k, \tau_n = k, S_k \in dx) \\
&\le C \sum_{k=1}^{n^{8\varepsilon}} \int_{(0,g_n(k)]} \frac{p_n(x,a\log n,b)}{p_n(0,a\log n,b)}\, \mathbb{P}(\sigma^- > k, \tau_n = k, S_k \in dx) \\
&\le C \sum_{k=1}^{n^{8\varepsilon}} k^{1/8}\mathbb{P}(\sigma^- > k, \tau_n = k) \;\to\; 0,
\end{aligned} \tag{A14}$$

as $n\to\infty$ and $\varepsilon\downarrow 0$ (in this order). In a similar manner, we obtain

$$\begin{aligned}
&\frac{\mathbb{P}(\sigma^- > n,\, \tau_n > n - n^{8\varepsilon},\, S_n \in [a\log n, a\log n + b])}{p_n(0,a\log n,b)} \\
&= \sum_{n-n^{8\varepsilon}<k<n} \int_{(0,g_n(k)]} \frac{p_{n-k}(x,a\log n,b)}{p_n(0,a\log n,b)}\, \mathbb{P}(\sigma^- > k, \tau_n = k, S_k \in dx)
\end{aligned}$$

$$
\begin{aligned}
&\leq\ Cn^{3/2} \sum_{n-n^{8\varepsilon}<k<n} (n-k)^{-3/2} g_n(k) p_k(0,0,g_n(k)) \\
&\leq\ Cn^{3/2} \sum_{n-n^{8\varepsilon}<k<n} (n-k)^{-11/8} \sum_{j=0}^{\varepsilon(n-k)^{1/8}} p_k(0,j,1) \\
&\leq\ Cn^{3/2} \sum_{n-n^{8\varepsilon}<k<n} (n-k)^{-11/8} \sum_{j=0}^{\varepsilon(n-k)^{1/8}} (j+1)k^{-3/2} \\
&\leq\ Cn^{3/2} \sum_{n-n^{8\varepsilon}<k<n} (n-k)^{-11/8} \varepsilon^2 (n-k)^{1/4} k^{-3/2} \\
&\leq\ C\varepsilon^2 \sum_{1\leq k<n^{8\varepsilon}} k^{-9/8},
\end{aligned}
$$

so that

$$
\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \frac{\mathbb{P}(\sigma^- > n, n-n^{8\varepsilon} < \tau_n < n,\ S_n \in [a\log n, a\log n + b])}{p_n(0, a\log n, b)} \ =\ 0. \tag{A15}
$$

Finally,

$$
\begin{aligned}
&\frac{\mathbb{P}(\sigma^- > n,\ n^{8\varepsilon} < \tau_n \leq n-n^{8\varepsilon},\ S_n \in [a\log n, a\log n + b])}{p_n(0, a\log n, b)} \\
&=\ \sum_{n^{8\varepsilon}<k\leq n-n^{8\varepsilon}} \int_{(0,\varepsilon n^{\varepsilon}]} \frac{p_{n-k}(x, a\log n, b)}{p_n(0, a\log n, b)}\ \mathbb{P}(\sigma^- > k, \tau_n = k, S_k \in dx) \\
&\leq\ Cn^{3/2} \sum_{n^{8\varepsilon}<k\leq n-n^{8\varepsilon}} (n-k)^{-3/2} n^{\varepsilon} p_k(0,0,\varepsilon n^{\varepsilon}) \\
&\leq\ Cn^{3/2+\varepsilon} \sum_{n^{8\varepsilon}<k\leq n-n^{8\varepsilon}} (n-k)^{-3/2} \sum_{j=0}^{\varepsilon n^{\varepsilon}} p_k(0,j,1) \\
&\leq\ C\varepsilon^2 n^{3/2+3\varepsilon} \sum_{n^{8\varepsilon}<k\leq n-n^{8\varepsilon}} (n-k)^{-3/2} k^{-3/2} \\
&\leq\ C\varepsilon^2 n^{3\varepsilon} \sum_{n^{8\varepsilon}<k\leq n/2} k^{-3/2} \\
&\leq C\varepsilon^2 n^{-\varepsilon}
\end{aligned}
$$

which clearly goes to 0 as $n\to\infty$ and $\varepsilon\downarrow 0$. By combining this with (A14) and (A15), we arrive at the desired conclusion (A11). ◊

Lemma A.6. *Let $(S_n)_{n\geq 0}$ be a zero-delayed centered random walk with i.i.d. increments $X_1, X_2, \ldots$ having a finite moment of order $2+\delta$ for some $\delta > 0$. Then*

$$
\mathbb{E}\Big(\mathbf{1}_{\{\sigma^- > n\}} e^{-S_n}\Big) \ =\ O(n^{-3/2}\log n), \quad n\to\infty.
$$

Proof. The assertion follows directly with the help of Theorem A.4 when using the obvious inequality

$$\mathbb{E}\Big(\mathbf{1}_{\{\sigma^->n\}}e^{-S_n}\Big) \;\leq\; n^{-c}+\mathbb{P}(\sigma^- > n, S_n\in(0,c]),$$

valid for all $c>0$. $\diamondsuit$

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