On Distributional Properties of Perpetuities

Gerold Alsmeyer · Alex Iksanov · Uwe Rösler

Abstract We study probability distributions of convergent random series of a special structure, called perpetuities. By giving a new argument, we prove that such distributions are of pure type: degenerate, absolutely continuous, or continuously singular. We further provide necessary and sufficient criteria for the finiteness of p-moments, $p > 0$, as well as exponential moments. In particular, a formula for the abscissa of convergence of the moment generating function is provided. The results are illustrated with a number of examples at the end of the article.

Keywords Perpetuity · Continuity of distribution · Atom · Moment · Exponential moment · Abscissa of convergence

Mathematics Subject Classification (2000) Primary 60E99 · Secondary 60G50

G. Alsmeyer
Institut für Mathematische Statistik, Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany
e-mail: gerolda@math.uni-muenster.de

A. Iksanov (✉)
Faculty of Cybernetics, National T. Shevchenko University, 01033 Kiev, Ukraine
e-mail: iksan@unicyb.kiev.ua

U. Rösler
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, 24098 Kiel, Germany
e-mail: roesler@math.uni-kiel.de

1 Introduction and Results

Let $\{(M_n, Q_n) : n = 1, 2, \ldots\}$ be a sequence of i.i.d. $\mathbb{R}^2$-valued random vectors with generic copy (M, Q). Put

$$\Pi_0 \stackrel{\text{def}}{=} 1 \quad \text{and} \quad \Pi_n \stackrel{\text{def}}{=} M_1 M_2 \cdots M_n, \quad n = 1, 2, \ldots.$$

Under conditions ensuring the almost sure convergence of the sequence

$$Z_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} \Pi_{k-1} Q_k, \quad n = 1, 2, \ldots \tag{1}$$

the limiting random variable

$$Z_\infty \stackrel{\text{def}}{=} \sum_{k \geq 1} \Pi_{k-1} Q_k \tag{2}$$

is often called perpetuity due to its occurrence in the realm of insurance and finance as a sum of discounted payment streams. It also arises naturally from shot-noise processes with exponentially decaying after-effect. In the study of the (forward) iterated function system

$$\Phi_n \stackrel{\text{def}}{=} \Psi_n(\Phi_{n-1}) = \Psi_n \circ \cdots \circ \Psi_1(\Phi_0), \quad n = 1, 2, \ldots, \tag{3}$$

where $\Psi_n(t) \stackrel{\text{def}}{=} Q_n + M_n t$ for $n = 1, 2, \ldots$ and Φ_0 is independent of $\{(M_n, Q_n) : n = 1, 2, \ldots\}$, the law of Z_∞ forms a stationary distribution for this recursive Markov chain and thus a distributional fixed point of the equation

$$\Phi \stackrel{d}{=} Q + M\Phi \tag{4}$$

where as usual the variable Φ is assumed to be independent of (M, Q). Note that Z_∞ is obtained as the a.s. limit of the backward system associated with (3) when started at $\Phi_0 \equiv 0$, i.e.

$$Z_\infty = \lim_{n \to \infty} \Psi_1 \circ \cdots \circ \Psi_n(0). \tag{5}$$

This is in contrast to the forward sequence $\Psi_n \circ \cdots \circ \Psi_1(0)$ which converges to Z_∞ in distribution only.

When focussing on (4), Vervaat [23] already showed that the law of Z_∞ forms the only possible solution, unless $Q + Mc = c$ a.s. for some $c \in \mathbb{R}$. Under the latter degeneracy condition, the solutions to (4) are either all distributions on $\mathbb{R}$, or those symmetric about c, or just the Dirac measure at c. This explains that subsequent work has primarily dealt with a further study of the random variable Z_∞ and conditions that ensure its existence. As to the last problem, Goldie and Maller [10] gave the following complete characterization of the a.s. convergence of the series in (2). For $x > 0$, define

$$A(x) \stackrel{\text{def}}{=} \int_0^x \mathbb{P}\{-\log|M| > y\} dy = \mathbb{E} \min(\log^- |M|, x). \tag{6}$$

Proposition 1.1 ([10], Theorem 2.1) *Suppose*

$$\mathbb{P}\{M = 0\} = 0 \quad and \quad \mathbb{P}\{Q = 0\} < 1. \tag{7}$$

Then

$$\lim_{n \to \infty} \Pi_n = 0 \quad a.s. \quad and \quad I \stackrel{\text{def}}{=} \int_{(1,\infty)} \frac{\log x}{A(\log x)} \mathbb{P}\{|Q| \in dx\} < \infty, \tag{8}$$

and

$$Z_\infty^* \stackrel{\text{def}}{=} \sum_{n \geq 1} |\Pi_{n-1} Q_n| < \infty \quad a.s. \tag{9}$$

are equivalent conditions, which imply

$$\lim_{n \to \infty} Z_n = Z_\infty \quad a.s. \quad and \quad |Z_\infty| < \infty \quad a.s.$$

Moreover, if

$$\mathbb{P}\{Q + Mc = c\} < 1 \quad for \ all \ c \in \mathbb{R}, \tag{10}$$

and if at least one of the conditions in (8) *fails to hold, then* $\lim_{n \to \infty} |Z_n| = \infty$ *in probability.*

Remark 1.2 As to condition (7) note that, if $\mathbb{P}\{Q = 0\} = 1$, Z_∞ trivially exists and equals zero a.s. If $\mathbb{P}\{M = 0\} > 0$, then

$$N \stackrel{\text{def}}{=} \min\{n \geq 1 : M_n = 0\},$$

is a.s. finite and

$$Z_\infty = Z_n = \sum_{k=1}^{N} \Pi_{k-1} Q_k$$

for all $n \geq N$. Hence, in this case no condition on the distribution of Q is needed to ensure the existence of Z_∞.

Our first result states that the distribution of Z_∞ is pure.

Theorem 1.3 *If* $\mathbb{P}\{M = 0\} = 0$ *and* $|Z_\infty| < \infty$ *a.s., then the distribution of* Z_∞ *is either degenerate, absolutely continuous, or singular and continuous.*

To our knowledge this result is new in the given generality. It was obtained by Grinčevičius in [11] under the additional condition $\mathbb{E}\log |M| \in (-\infty, 0)$. However, for the conclusion that Z_∞ is continuous if nondegenerate his analytic argument is quite different from ours. This latter conclusion may also be derived from his Theorem 1 in [12], as has been done in Lemma 2.1 in [2]. Let us further point out that $Z_\infty = c$ a.s. for some $c \in \mathbb{R}$ implies $\mathbb{P}\{Q + Mc = c\} = 1$, as following from the fact

that Z_∞ satisfies (4). Hence the law of Z_∞ is continuous whenever (7) and (10) are assumed.

If M and Q are independent, Pakes [20] provided sufficient conditions for the absolute continuity of the distribution of Z_∞. His proof relies heavily upon studying the behavior of corresponding characteristic functions and used moment assumptions as an indispensable ingredient. Without such assumptions it is not clear how absolute continuity of the law of Z_∞ may be derived via an analytic approach.

Theorem 1.4 *Assuming* (7) *and* (10), *the following assertions are equivalent for any* $p > 0$:

$$\mathbb{E}|M|^p < 1 \quad and \quad \mathbb{E}|Q|^p < \infty, \tag{11}$$

$$\mathbb{E}|Z_\infty|^p < \infty, \tag{12}$$

$$\mathbb{E}Z_\infty^{*p} < \infty, \tag{13}$$

where Z_∞^* *is defined in* (9).

Theorem 1.4 seems to be new in the stated generality but was given as Proposition 10.1 in [18] for the case that $M, Q \geq 0$ (in which (12) and (13) are clearly identical). If $p > 1$, Vervaat [23] proved that (11) implies (12).

Remark 1.5 It is not difficult to see that further conditions equivalent to those in the previous theorem are given by

$$\mathbb{E}\sup_{n \geq 1}|\Pi_{n-1}Q_n|^p < \infty, \tag{14}$$

$$\mathbb{E}\sup_{n \geq 1}|Z_n|^p < \infty, \tag{15}$$

$$\mathbb{E}\left(\sum_{n \geq 1}\Pi_{n-1}^2 Q_n^2\right)^{p/2} < \infty, \tag{16}$$

where Z_n is defined in (1). Some comments regarding the proof can be found in Remark 2.1 after the proof of Theorem 1.4.

Given any real-valued random variable Z, let us define

$$r(Z) \stackrel{\text{def}}{=} \sup\{r > 0 : \mathbb{E}e^{r|Z|} < \infty\},$$

called the abscissa of convergence of the moment generating function of $|Z|$. Note that $\mathbb{E}e^{r(Z)|Z|}$ may be finite or infinite.

Our next two results provide complete information on how $r(Z_\infty)$ relates to $r(Q)$. For convenience we distinguish the cases where $\mathbb{P}\{|M| = 1\} = 0$ and $\mathbb{P}\{|M| = 1\} \in (0, 1)$. Recall that if conditions (7) and (10) hold then the law of Z_∞ is nondegenerate if $|Z_\infty| < \infty$ a.s.

Theorem 1.6 *Suppose* (7), (10) *and* $\mathbb{P}\{|M| = 1\} = 0$, *and let* $s > 0$. *Then* $\mathbb{E}e^{s|Z_\infty|} < \infty$ *holds if, and only if,*

$$\mathbb{P}\{|M| < 1\} = 1 \quad and \quad \mathbb{E}e^{s|Q|} < \infty. \tag{17}$$

In particular, if $\mathbb{P}\{|M| < 1\} = 1$, *then* $r(Z_\infty) = r(Q)$.

Theorem 1.7 *Suppose* (7), (10) *and* $\mathbb{P}\{|M| = 1\} \in (0, 1)$, *and let* $s > 0$. *Then* $\mathbb{E}e^{s|Z_\infty|} < \infty$ *holds if, and only if,*

$$\mathbb{P}\{|M| \le 1\} = 1, \quad \mathbb{E}e^{s|Q|} < \infty \quad and \tag{18}$$

$$b_- b_+ < (1 - a_-)(1 - a_+), \tag{19}$$

where $a_\pm = a_\pm(s) \stackrel{\text{def}}{=} \mathbb{E}e^{\pm sQ}\mathbf{1}_{\{M=1\}}$ *and* $b_\pm = b_\pm(s) \stackrel{\text{def}}{=} \mathbb{E}e^{\pm sQ}\mathbf{1}_{\{M=-1\}}$. *In particular, if* $\mathbb{P}\{|M| \le 1\} = 1$ *and* $\mathbb{P}\{|M| = 1\} \in (0, 1)$, *then* $r(Z_\infty) = \min(r(Q), r^*(M, Q))$, *where*

$$r^*(M, Q) \stackrel{\text{def}}{=} \sup\{r > 0 : b_-(r)b_+(r) < (1 - a_-(r))(1 - a_+(r))\}.$$

The reader should notice $\max(a_- + b_-, a_+ + b_+) < 1$ provides a sufficient condition for (19).

We are aware of two papers that deal with the existence of exponential moments of $|Z_\infty|$. By using the contraction principle, Goldie and Grübel [9] proved that, if $|M| \le 1$ a.s. and $\mathbb{E}e^{r|Q|} < \infty$ for some $r > 0$, then $\mathbb{E}e^{t|Z_\infty|} < \infty$ for $0 \le t < \sup\{\theta : \mathbb{E}e^{\theta|Q|}|M| < 1\}$ (see their Theorem 2.1). For the case of nonnegative M and Q, a stronger result was obtained by Kellerer [18], namely that $\mathbb{E}e^{rZ_\infty} < \infty$ for some $r > 0$ iff $M \le 1$ a.s. and $\mathbb{E}e^{tQ} < \infty$ for some $t > 0$ (see his Proposition 10.2).

The rest of the paper is organized as follows. All proofs are given in Sect. 2. Section 3 collects a number of examples which illustrate our main results, followed by a number of concluding remarks in Sect. 4.

2 Proofs

Let us start by pointing out that, if $|Z_\infty| < \infty$ a.s.,

$$Z_\infty = Q_1 + M_1 Z_\infty^{(1)} = Q^{(m)} + \Pi_m Z_\infty^{(m)}, \tag{20}$$

holds true for each $m \ge 1$, where (setting $\Pi_{k:l} \stackrel{\text{def}}{=} M_k \cdots M_l$)

$$Q^{(m)} \stackrel{\text{def}}{=} \sum_{k=1}^{m} \Pi_{k-1} Q_k \quad and \quad Z_\infty^{(m)} \stackrel{\text{def}}{=} Q_{m+1} + \sum_{k \ge m+2} \Pi_{m+1:k-1} Q_k. \tag{21}$$

The latter variable is a copy of Z_∞ and independent of $(M_1, Q_1), \ldots, (M_m, Q_m)$. We thus see that Z_∞ may be viewed as the perpetuity generated by i.i.d. copies of $(\Pi_m, Q^{(m)})$ for any fixed $m \ge 1$.

Proof of Theorem 1.3 Suppose immediately that $\mathbb{P}\{Q = 0\} < 1$ and (10) hold true, for otherwise the law of Z_∞ is clearly degenerate. By Proposition 1.1 we thus also have that $\Pi_n \to 0$ a.s.

We first show that if the law of Z_∞ is having atoms then again it must be degenerate. Let $b_1, \ldots, b_d$ denote the atoms with maximal probability ϱ, say. Notice that $d \leq \varrho^{-1}$. In view of (20) we have

$$\mathbb{P}\{Z_\infty = b_i\} = \sum_{a \in A} \mathbb{P}\{Q^{(m)} + \Pi_m a = b_i\}\mathbb{P}\{Z_\infty = a\}, \quad i = 1, \ldots, d \qquad (22)$$

for each $m = 1, 2, \ldots$, where A is the set of all atoms of the distribution of Z_∞. Since $\mathbb{P}\{M = 0\} = 0$, we have $\sum_{a \in A} \mathbb{P}\{Q^{(m)} + \Pi_m a = b_i\} \leq 1$. Now use $\mathbb{P}\{Z_\infty = a\} \leq \mathbb{P}\{Z_\infty = b_i\}$ to conclude that (22) can only hold if the summation extends only over b_j, $j = 1, \ldots, d$, and so

$$\sum_{j=1}^d \mathbb{P}\{Q^{(m)} + \Pi_m b_j = b_i\} = 1, \quad i = 1, \ldots, d, \qquad (23)$$

for each $m = 1, 2, \ldots$. By letting m tend to infinity and using $(\Pi_m, Q^{(m)}) \to (0, Z_\infty)$ a.s. in (23), we arrive at

$$\mathbb{P}\{Z_\infty = b_i\} = d^{-1}, \quad i = 1, \ldots, d. \qquad (24)$$

In order to see that this already yields degeneracy of Z_∞, suppose $d \geq 2$ and let U, V be independent copies of Z_∞ which are also independent of $\{(M_n, Q_n) : n = 1, 2, \ldots\}$. Put $Z_\infty^{(s)} \stackrel{\text{def}}{=} U - V$, clearly a symmetrization of Z_∞ with support given as $\Gamma \stackrel{\text{def}}{=} \{b_i - b_j : i, j = 1, \ldots, d\}$. Since $Q^{(m)} + \Pi_m U \stackrel{d}{=} Q^{(m)} + \Pi_m V \stackrel{d}{=} Z_\infty$ for each $m = 1, 2, \ldots$, we see that

$$D_m \stackrel{\text{def}}{=} \left(Q^{(m)} + \Pi_m U\right) - \left(Q^{(m)} + \Pi_m V\right) = \Pi_m Z_\infty^{(s)}$$

has a support Γ_m contained in Γ. Put $\gamma_* \stackrel{\text{def}}{=} \min(\Gamma \cap (0, \infty))$ and $\gamma^* \stackrel{\text{def}}{=} \max \Gamma$. Using the independence of Π_m and $Z_\infty^{(s)}$ in combination with $\mathbb{P}\{M = 0\} = 0$, we now infer

$$0 = \mathbb{P}\{|D_m| \in (0, \gamma_*)\} = \mathbb{P}\{|\Pi_m Z_\infty^{(s)}| \in (0, \gamma_*)\}$$

$$\geq \mathbb{P}\{|\Pi_m| < \gamma_*/\gamma^*\}\mathbb{P}\{|Z_\infty^{(s)}| \in (0, \gamma^*]\}$$

and therefore $\mathbb{P}\{|\Pi_m| < \gamma_*/\gamma^*\} = 0$ because $\mathbb{P}\{|Z_\infty^{(s)}| \in (0, \gamma^*]\} = 1 - d\varrho^2 > 0$. But this contradicts $\Pi_m \to 0$ a.s. and so $d = 1$, i.e. $Z_\infty = b_1$ a.s. by (24).

It remains to verify that a continuously distributed Z_∞ is of pure type. Apart from minor modifications, the following argument is due to Grincevičius [11] and stated here for completeness.

Let $\phi(t)$ be the characteristic function (ch.f.) of Z_∞. By Lebesgue's decomposition theorem $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$, where $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and ϕ_1, ϕ_2 are the ch.f. of the absolutely continuous and the continuously singular components of the law of Z_∞, respectively. Suppose $\alpha_1 > 0$ so that $\phi = \phi_1$ must be verified.

Since the law of Z_∞ satisfies the stochastic fixed point equation (4), we infer in terms of its ch.f.

$$\phi(t) = \mathbb{E}e^{itQ}\phi(Mt) \tag{25}$$

and thus

$$\alpha_1\phi_1(t) + \alpha_2\phi_2(t) = \alpha_1\mathbb{E}e^{itQ}\phi_1(Mt) + \alpha_2\mathbb{E}e^{itQ}\phi_2(Mt). \tag{26}$$

It is easily seen that $\mathbb{E}e^{itQ}\phi_1(Mt)$ is the ch.f. of an absolutely continuous distribution, because this is true for ϕ_1 and $\mathbb{P}\{M = 0\} = 0$. If

$$\alpha_2\mathbb{E}e^{itQ}\phi_2(Mt) = \alpha_3\phi_3(t) + \alpha_4\phi_4(t),$$

where $\alpha_3, \alpha_4 \geq 0$, $\alpha_3 + \alpha_4 = 1$, and ϕ_3, ϕ_4 are the absolutely continuous and the continuously singular components, respectively, then the uniqueness of the Lebesgue decomposition renders in (26) that $\alpha_1\phi_1(t) = \alpha_1\mathbb{E}e^{itQ}\phi_1(Mt) + \alpha_2\alpha_3\phi_3(t)$ and thus upon setting $t = 0$ that $\alpha_2\alpha_3 = 0$. Consequently, $\phi_1(t) = \mathbb{E}e^{itQ}\phi_1(Mt)$ which means that ϕ_1 is also a solution to the functional equation (25). By considering the bounded continuous function $\phi - \phi_1$ and utilizing $\phi(0) - \phi_1(0) = 0$ in combination with $\Pi_n \to 0$ a.s., we infer upon iterating (25) for $\phi - \phi_1$ and an appeal to the dominated convergence theorem that

$$|\phi(t) - \phi_1(t)| \leq \lim_{n \to \infty} \mathbb{E}|\phi(\Pi_n t) - \phi_1(\Pi_n t)| = 0$$

for all $t \neq 0$ and so $\phi = \phi_1$. $\qquad\square$

Proof of Theorem 1.4 Since $|Z_\infty| \leq Z_\infty^*$ it suffices to prove "(11) $\Rightarrow$ (13)" and "(12) $\Rightarrow$ (11)". A proof of the first implication, though known from [18], is easy and thus stated for completeness here.

"(11) $\Rightarrow$ (13)": If $0 < p \leq 1$, just use the subadditivity of $x \mapsto x^p$ on $[0, \infty)$ in combination with the independence of Π_{k-1} and Q_k for each k to infer

$$\mathbb{E}Z_\infty^{*p} \leq \sum_{k \geq 1} \mathbb{E}|\Pi_{k-1}|^p \mathbb{E}|Q_k|^p$$
$$= \sum_{k \geq 1} (\mathbb{E}|M|^p)^{k-1}\mathbb{E}|Q|^p = \frac{\mathbb{E}|Q|^p}{1 - \mathbb{E}|M|^p} < \infty.$$

If $p > 1$, a similar inequality holds for $\|Z_\infty\|_p$, where $\|\cdot\|_p$ denotes the usual L_p-norm. Namely, by Minkowski's inequality,

$$\|Z_\infty^*\|_p \leq \sum_{k \geq 1} \|\Pi_{k-1}Q_k\|_p = \sum_{k \geq 1} \|M\|_p^{k-1}\|Q\|_p = \frac{\|Q\|_p}{1 - \|M\|_p} < \infty.$$

"(12) $\Rightarrow$ (11)" Let us start by pointing out that (11) is equivalent to

$$\mathbb{E}|Q_1 + M_1Q_2|^p < \infty \quad \text{and} \quad \mathbb{E}|M_1M_2|^p < 1, \tag{27}$$

which, in the notation introduced at the beginning of this section, is nothing but condition (11) for the pair $(\Pi_2, Q^{(2)})$. We only remark concerning "(27) $\Rightarrow$ (11)" that in the case $p \geq 1$, by Minkowski's inequality,

$$\|Q_1 \mathbf{1}_{\{|Q_1| \leq b, |Q_2| \leq c\}}\|_p \leq \|(Q_1 + M_1 Q_2)\mathbf{1}_{\{|Q_1| \leq b, |Q_2| \leq c\}}\|_p$$
$$+ \|M_1 \mathbf{1}_{\{|Q_1| \leq b\}}\|_p \|Q_2 \mathbf{1}_{\{|Q_2| \leq c\}}\|_p$$

for all $b, c > 0$ and therefore (upon letting b tend to ∞ and picking c large enough)

$$\|Q\|_p \leq \frac{\|Q_1 + M_1 Q_2\|_p + c\|M\|_p}{\mathbb{P}\{|Q| \leq c\}^{1/p}} < \infty.$$

If $0 < p < 1$ a similar argument using the subadditivity of $0 \leq x \mapsto x^p$ yields the conclusion. Next, we note that the conditional law of $Q_1 + M_1 Q_2$ given Π_2 cannot be degenerate, for otherwise either $Q + cM = c$ or $(M_1, Q_1) = (1, c)$ a.s. for some $c \in \mathbb{R}$ by Proposition 1 in [13]. But both alternatives are here impossible, the first by our assumption (10), the second by $|Z_\infty| < \infty$ a.s. Let us also mention that $|Z_\infty| < \infty$ a.s. in combination with (10) ensures $\Pi_n \to 0$ a.s. by Theorem 1.1.

The following argument based upon conditional symmetrization may be viewed as a streamlined version of a similar one given in the proof of Proposition 3 in [17]. Put $Q_n^{(2)} \stackrel{\text{def}}{=} Q_{2n-1} + M_{2n-1} Q_{2n}$ for $n = 1, 2, \ldots$ and note that $\{(M_{2n-1} M_{2n}, Q_n^{(2)}) : n = 1, 2, \ldots\}$ is a family of independent copies of $(\Pi_2, Q^{(2)})$. Let $\overline{Q}_n^{(2)}$ be a conditional symmetrization of $Q_n^{(2)}$ given $M_{2n-1} M_{2n}$ such that $(M_{2n-1} M_{2n}, \overline{Q}_n^{(2)})$, $n = 1, 2, \ldots$ are also i.i.d. More precisely, $\overline{Q}_n^{(2)} = Q_n^{(2)} - \widehat{Q}_n^{(2)}$, where $\{(M_{2n-1} M_{2n}, Q_n^{(2)}, \widehat{Q}_n^{(2)}) : n = 1, 2, \ldots\}$ consists of i.i.d. random variables and $Q_n^{(2)}, \widehat{Q}_n^{(2)}$ are conditionally i.i.d. given $M_{2n-1} M_{2n}$. By what has been pointed out above, the law of $\overline{Q}_n^{(2)}$, and thus also of $Q_n^{(2)}$, is nondegenerate. Putting $\mathcal{B}_n \stackrel{\text{def}}{=} \sigma(M_1, \ldots, M_n)$ for $n = 1, 2, \ldots$, we now infer with the help of Lévy's symmetrization inequality (see [3], Corollary 5 on p. 72)

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |\Pi_{2k-2} \overline{Q}_k^{(2)}| > x \big| \mathcal{B}_{2n}\right) \leq 2\mathbb{P}\left(\left|\sum_{k=1}^n \Pi_{2k-2} \overline{Q}_k^{(2)}\right| > x \big| \mathcal{B}_{2n}\right)$$
$$\leq 4\mathbb{P}\left(\left|\sum_{k=1}^n \Pi_{2k-2} Q_k^{(2)}\right| > \frac{x}{2} \big| \mathcal{B}_{2n}\right)$$
$$= 4\mathbb{P}(|Z_{2n}| > x/2 | \mathcal{B}_{2n}) \quad \text{a.s.}$$

for all $x > 0$ and thus (recalling that the law of Z_∞ is continuous in the present situation as pointed out right after Theorem 1.3)

$$\mathbb{P}\left\{\sup_{k \geq 1} |\Pi_{2k-2} \overline{Q}_k^{(2)}| > x\right\} \leq 4\mathbb{P}\{|Z_\infty| > x/2\}. \tag{28}$$

As a consequence of this in combination with $\mathbb{E}|Z_\infty|^p < \infty$ we conclude

$$\mathbb{E} \sup_{k \geq 1} |\Pi_{2k-2} \overline{Q}_k^{(2)}|^p \leq 2^{p+2} \mathbb{E}|Z_\infty|^p < \infty.$$

Now put $S_0 \overset{\text{def}}{=} 0$ and

$$S_n \overset{\text{def}}{=} \log |\Pi_{2n}| = \sum_{k=1}^n \log |M_{2k-1} M_{2k}| \quad \text{and} \quad Y_n \overset{\text{def}}{=} \mathbf{1}_{\{\overline{Q}_n^{(2)} \neq 0\}} \log |\overline{Q}_n^{(2)}|$$

for $n = 1, 2, \ldots$. Then $\{S_n : n = 0, 1, \ldots\}$ forms an ordinary zero-delayed random walk with $S_n \to -\infty$ a.s. (recall $\Pi_n \to 0$ a.s. from above), and

$$\mathbb{P}\{Y_n = 0 \text{ i.o.}\} = \mathbb{P}\{\overline{Q}_n^{(2)} \in \{0, 1\} \text{ i.o.}\} = 0$$

by the nondegeneracy and symmetry of the $\overline{Q}_n^{(2)}$. With this we see that

$$\mathbb{E} \sup_{k \geq 1} |\Pi_{2k-2} \overline{Q}_k^{(2)}|^p = \mathbb{E} \exp \left(p \sup_{n \geq 0} (S_n + Y_{n+1}) \right) < \infty.$$

Since the pairs $(\log |M_{2n-1} M_{2n}|, Y_n)$, $n = 1, 2, \ldots$, are i.i.d., an application of Lemma 2.2 stated below yields

$$\mathbb{E} \sup_{k \geq 0} |\Pi_{2k}|^p = \mathbb{E} \exp \left(p \sup_{k \geq 0} S_k \right) < \infty.$$

But we further have that $W \overset{\text{def}}{=} \sup_{k \geq 0} |\Pi_{2k}|^p$ and its copy $W' \overset{\text{def}}{=} \sup_{k \geq 1} |\Pi_{3:2k}|^p$ (setting $\Pi_{3:2} \overset{\text{def}}{=} 1$) satisfy

$$W = \max(1, |\Pi_2|^p W') \geq |\Pi_2|^p W' \quad \text{and} \quad \mathbb{E}W \geq \mathbb{E}|\Pi_2|^p \mathbb{E}W' \tag{29}$$

whence $\mathbb{E}|\Pi_2|^p = \mathbb{E}|M_1 M_2|^p \leq 1$. In order to conclude strict inequality note first that $\mathbb{E}|\Pi_2|^p = 1$ in (29) would give $|\Pi_2|^p W' = W \geq 1$ a.s. But since $p \overset{\text{def}}{=} \mathbb{P}\{W = 1\} \geq \mathbb{P}\{\sup_{n \geq 1} |\Pi_{2n}| < 1\} > 0$ as argued below, the independence of W' and Π_2 would further imply

$$\mathbb{P}\{|\Pi_2|^p W' < 1\} \geq \mathbb{P}\{|\Pi_2| < 1, W' = 1\} = p \mathbb{P}\{|\Pi_2| < 1\} > 0$$

which is a contradiction. Therefore $\mathbb{E}|\Pi_2|^p < 1$, that is the second half of (27) holds true.

In order to show $\mathbb{P}\{\sup_{n \geq 1} |\Pi_{2n}| < 1\} > 0$, let us recall that $S_n = -\log |\Pi_{2n}|$, $n = 0, 1, \ldots$, forms an ordinary random walk converging a.s. to $-\infty$ (as $\Pi_n \to 0$ a.s.). Consequently, the associated first strictly descending ladder epoch $\tau_- \overset{\text{def}}{=} \inf\{n : S_n < 0\}$ has finite mean (see Corollary 1 on p. 153 in [3]) and with its dual $\tau_+ \overset{\text{def}}{=} \inf\{n : S_n \geq 0\}$ it satisfies the relation

$$\mathbb{P}\{\tau_+ = \infty\} = 1/\mathbb{E}\tau_- > 0,$$

see Theorem 2 on p. 151 in [3]. But $\{\tau_+ = \infty\} = \{\sup_{n \geq 1} |\Pi_{2n}| < 1\}$.

Left with the first half of (27), namely $\|Q^{(2)}\|_p < \infty$, use (20) with $m = 2$ rendering $|Q^{(2)}| \leq |Z_\infty| + |\Pi_2 Z_\infty^{(2)}|$ and therefore

$$\|Q^{(2)}\|_p \leq \|Z_\infty\|_p (1 + \|\Pi_2\|_p) < \infty$$

in the case $p \geq 1$. The case $0 < p < 1$ is handled similarly. This completes the proof of the theorem. $\qquad\square$

Remark 2.1 Here are a few comments on how to obtain the additional equivalences stated in Remark 1.5:

"(14) $\Rightarrow$ (11)" is contained in the above proof of Theorem 1.4.

"(11) $\Rightarrow$ (14)": Use "(11) $\Rightarrow$ (13)" and $\sup_{n \geq 1} |\Pi_{n-1} Q_n| \leq Z_\infty^*$.

"(11) $\Rightarrow$ (15)": Use "(11) $\Rightarrow$ (13)" and $\sup_{n \geq 1} |Z_n| \leq Z_\infty^*$.

"(15) $\Rightarrow$ (11)": Use $|Z_\infty| \leq \sup_{n \geq 0} |Z_n|$ and then "(12) $\Rightarrow$ (11)".

"(16) $\Rightarrow$ (11)": Use $\sup_{n \geq 1} |\Pi_{n-1} Q_n| \leq (\sum_{n \geq 1} \Pi_{n-1}^2 Q_n^2)^{1/2}$ and "(14) $\Rightarrow$ (11)".

"(11) $\Rightarrow$ (16)": Use "(11) $\Rightarrow$ (13)" and $(\sum_{n \geq 1} \Pi_{n-1}^2 Q_n^2)^{1/2} \leq Z_\infty^*$.

We continue with the lemma used at the end of the previous proof. It contains a tail inequality first given in [8]. A similar result was stated as Lemma 2 in [17], but that result is correct only for the case of independent M and Q.

Lemma 2.2 *Let $\{(X_k, Y_k) : k = 1, 2, \ldots\}$ be a family of i.i.d. $\mathbb{R}^2$-valued random vectors. Put $S_n \stackrel{\text{def}}{=} X_1 + \cdots + X_n$ for $n = 0, 1, \ldots, \xi \stackrel{\text{def}}{=} \sup_{n \geq 0} S_n$ and $\zeta \stackrel{\text{def}}{=} \sup_{n \geq 0}(S_n + Y_{n+1})$. Then*

$$\mathbb{P}\{\zeta > x\} \geq \mathbb{P}\{Y_1 > y\}\mathbb{P}\{\xi > x - y\} \tag{30}$$

for all $x, y \in \mathbb{R}$. Furthermore, if $\Phi : [0, \infty) \to [0, \infty)$ is any nondecreasing, differentiable function, then

$$\mathbb{E}\Phi(\xi) \leq \Phi(0) + c\mathbb{E}\Phi(c + \zeta^+) \tag{31}$$

for a constant $c \in (1, \infty)$ that does not depend on Φ.

Proof For any fixed $x, y \in \mathbb{R}$, put $\tau \stackrel{\text{def}}{=} \inf\{k \geq 0 : S_k > x - y\}$ with the usual convention $\inf \emptyset \stackrel{\text{def}}{=} \infty$. Note that $\{\xi > x - y\} = \{\tau < \infty\}$ and $\{\zeta > x\} \supset \{\tau < \infty, Y_{\tau+1} > y\}$. Inequality (30) now follows from

$$\mathbb{P}\{\zeta > x\} \geq \mathbb{P}\{\tau < \infty, Y_{\tau+1} > y\} = \sum_{n \geq 0} \mathbb{P}\{\tau = n, Y_{n+1} > y\}$$

$$= \mathbb{P}\{Y_1 > y\} \sum_{n \geq 0} \mathbb{P}\{\tau = n\} = \mathbb{P}\{Y_1 > y\}\mathbb{P}\{\tau < \infty\}$$

$$= \mathbb{P}\{Y_1 > y\}\mathbb{P}\{\xi > x - y\}.$$

In order to get (31) fix any $c > 1$ such that $\mathbb{P}\{Y_1 > -c\} \geq 1/c$. Then (30) with $y = -c$ provides us with

$$\mathbb{P}\{\zeta + c > x\} \geq \mathbb{P}\{\xi > x\}/c$$

for $x \in \mathbb{R}$ which in combination with $\mathbb{E}\Phi(\xi) - \Phi(0) = \int_0^\infty \Phi'(x)\mathbb{P}\{\xi > x\}dx$ finally shows (31). □

Proof of Theorem 1.6 Let us define $\psi(t) \overset{\text{def}}{=} \mathbb{E}e^{tZ_\infty}$, $\widehat{\psi}(t) \overset{\text{def}}{=} \mathbb{E}e^{t|Z_\infty|}$, $\varphi(t) \overset{\text{def}}{=} \mathbb{E}e^{tQ}$ and $\widehat{\varphi}(t) \overset{\text{def}}{=} \mathbb{E}e^{t|Q|}$. Note that $\psi(t) \leq \widehat{\psi}(s)$ for all $t \in [-s, s]$ and $s > 0$, and that $\max(\psi(-t), \psi(t)) \leq \widehat{\psi}(t) \leq \psi(t) + \psi(-t)$ for all $t \in \mathbb{R}$. From the fixed point equation $Z_\infty \overset{d}{=} Q + MZ_\infty$, we infer $\psi(t) = \mathbb{E}e^{tQ}\psi(Mt)$ for all $t \in \mathbb{R}$. These facts will be used in several places hereafter.

(a) Sufficiency: Suppose that (17) is valid. The almost sure finiteness of Z_∞ follows from Proposition 1.1. We have to check that $r(Z_\infty) \geq r(Q)$. To this end, we fix an arbitrary $s \in (0, r(Q))$ and divide the subsequent proof into two steps.

Step (a1). Assume first that $|M| \leq \beta < 1$ a.s. for some $\beta > 0$. Since the function $\widehat{\varphi}$ is convex and differentiable on $[0, \beta s]$, its derivative is nondecreasing on that interval. Therefore, for each $k = 2, 3, \ldots$, there exists $\theta_k \in [0, \beta^{k-1}s]$ such that

$$0 \leq \widehat{\varphi}(\beta^{k-1}s) - 1 = \widehat{\varphi}'(\theta_k)\beta^{k-1}s \leq \widehat{\varphi}'(\beta s)\beta^{k-1}s. \tag{32}$$

Now $r(Z_\infty) \geq r(Q)$ follows from

$$\widehat{\psi}(s) \leq \mathbb{E}\exp\left(s\sum_{k\geq 1}|\Pi_{k-1}Q_k|\right)$$

$$\leq \mathbb{E}\exp\left(s\sum_{k\geq 1}\beta^{k-1}|Q_k|\right) = \prod_{k\geq 1}\widehat{\varphi}(\beta^{k-1}s)$$

$$\leq \widehat{\varphi}(s)\exp\left(\sum_{k\geq 2}(\widehat{\varphi}(\beta^{k-1}s) - 1)\right)$$

$$\leq \widehat{\varphi}(s)\exp\left(\widehat{\varphi}'(\beta s)\beta s(1 - \beta)^{-1}\right) < \infty \quad \text{(by (32))}.$$

Step (a2). Consider now the general case. Since $\mathbb{P}\{|M| = 1\} = 0$, we can choose $\beta \in (0, 1)$ such that

$$\mathbb{P}\{|M| > \beta\} < 1 \quad \text{and} \quad \gamma \overset{\text{def}}{=} \mathbb{E}e^{s|Q|}\mathbf{1}_{\{|M|>\beta\}} < 1.$$

Define the a.s. finite stopping times

$$T_0 \overset{\text{def}}{=} 0, \qquad T_k \overset{\text{def}}{=} \inf\{n > T_{k-1} : |M_n| \leq \beta\}, \quad k = 1, 2, \ldots.$$

We have $Z_\infty = Q_1^* + \sum_{k \geq 1} M_1^* \cdots M_{k-1}^* Q_k^*$, where for $k = 1, 2, \ldots$

$$M_k^* \stackrel{\text{def}}{=} M_{T_{k-1}+1} \cdots M_{T_k} = \Pi_{T_{k-1}+1:T_k} \quad \text{and} \tag{33}$$

$$Q_k^* \stackrel{\text{def}}{=} Q_{T_{k-1}+1} + M_{T_{k-1}+1} Q_{T_{k-1}+2} + \cdots + M_{T_{k-1}+1} \cdots M_{T_k-1} Q_{T_k}, \tag{34}$$

so that (M_k^*, Q_k^*) are independent copies of

$$(M^*, Q^*) \stackrel{\text{def}}{=} \left(\Pi_{T_1}, Q_1 + \sum_{k=1}^{T_1} \Pi_{k-1} Q_k \right).$$

Since $|M^*| \leq \beta$ a.s., Step (a1) of the proof provides the desired conclusion if we still verify that $\widehat{\varphi}(s) < \infty$ implies $\mathbb{E}e^{s|Q^*|} < \infty$. This is checked as follows:

$$\mathbb{E}e^{s|Q^*|} \leq \mathbb{E}e^{s(|Q_1| + \cdots + |Q_{T_1}|)} = \sum_{n \geq 1} \mathbb{E}e^{s(|Q_1| + \cdots + |Q_n|)} \mathbf{1}_{\{T_1 = n\}}$$

$$= \sum_{n \geq 1} \mathbb{E}\left[e^{s|Q_n|} \mathbf{1}_{\{|M_n| \leq \beta\}} \prod_{k=1}^{n-1} e^{s|Q_k|} \mathbf{1}_{\{|M_k| > \beta\}} \right]$$

$$= \mathbb{E}e^{s|Q|} \mathbf{1}_{\{|M| \leq \beta\}} \sum_{n \geq 1} \gamma^{n-1}$$

$$\leq \widehat{\varphi}(s)(1 - \gamma)^{-1} < \infty.$$

(b) Necessity: If $\mathbb{E}e^{s|Z_\infty|} < \infty$, we have $\mathbb{E}|Z_\infty|^p < \infty$ and therefore, by Theorem 1.4, $\mathbb{E}|M|^p < 1$ for all $p > 0$. The latter in combination with $\mathbb{P}\{|M| = 1\} = 0$ implies $|M| < 1$ a.s. Finally, if $\widehat{\psi}(s) < \infty$ and $c \stackrel{\text{def}}{=} \min_{|t| \leq s} \psi(t)$ (clearly > 0), then

$$\infty > \psi(t) = \mathbb{E}e^{tQ} \psi(Mt) \geq c\varphi(t), \quad t \in \{-s, s\}, \tag{35}$$

and thus $\widehat{\varphi}(s) \leq \varphi(s) + \varphi(-s) < \infty$. This shows $r(Z_\infty) \leq r(Q)$. $\qquad \square$

Proof of Theorem 1.7 Recall that $a_\pm \stackrel{\text{def}}{=} \mathbb{E}e^{\pm sQ} \mathbf{1}_{\{M=1\}}$, $b_\pm \stackrel{\text{def}}{=} \mathbb{E}e^{\pm sQ} \mathbf{1}_{\{M=-1\}}$.

(a) Necessity: By the same argument as in part (b) of the proof of Theorem 1.6, we infer $\mathbb{E}|M|^p < 1$ for all $p > 0$ and thereby $|M| \leq 1$ a.s. Moreover, as $\widehat{\psi}(s) < \infty$, inequality (35) holds here as well and gives $\widehat{\varphi}(s) < \infty$. This shows (18) and leaves us with the proof of (19), for which we will proceed in two steps.

Step (a1). Suppose first that $\mathbb{P}\{M = -1\} = 0$ in which case $b_\pm = 0$ and thus (19) reduces to $a_\pm < 1$. We have

$$\psi(s) = \mathbb{E}e^{sQ} \psi(Ms) \mathbf{1}_{\{|M| < 1\}} + \psi(s) \mathbb{E}e^{sQ} \mathbf{1}_{\{M=1\}} \quad \text{and}$$

$$\psi(-s) = \mathbb{E}e^{-sQ} \psi(-Ms) \mathbf{1}_{\{|M| < 1\}} + \psi(-s) \mathbb{E}e^{-sQ} \mathbf{1}_{\{M=1\}},$$

which together with $\mathbb{E}e^{sQ} \psi(\pm Ms) \mathbf{1}_{\{|M| < 1\}} > 0$ (as $\mathbb{P}\{|M| < 1\} > 0$) implies

$$\mathbb{E}e^{\pm sQ} \mathbf{1}_{\{M=1\}} = a_\pm < 1$$

as required.

Step (a2). Assuming now $\mathbb{P}\{M = -1\} > 0$, let $\{(M_k^*, Q_k^*) : k = 1, 2, \ldots\}$ be defined as in (33), (34), but with

$$T_0 \stackrel{\text{def}}{=} 0, \qquad T_k \stackrel{\text{def}}{=} \inf\{n > T_{k-1} : \Pi_{T_{k-1}+1:n} > -1\}, \quad k = 1, 2, \ldots.$$

Then $\mathbb{P}\{M^* = -1\} = 0$, and we infer from Step (a1) that $\mathbb{E}e^{\pm s Q^*}\mathbf{1}_{\{M^*=1\}} < 1$. But

$$e^{\pm s Q_1^*}\mathbf{1}_{\{M_1^*=1\}} = e^{\pm s Q_1}\mathbf{1}_{\{M_1=1\}} + e^{\pm s(Q_1-Q_2)}\mathbf{1}_{\{M_1=M_2=-1\}}$$

$$+ \sum_{n\geq 3} e^{\pm s(Q_1-Q_2-\cdots-Q_n)}\mathbf{1}_{\{M_1=-1, M_2=\cdots=M_{n-1}=1, M_n=-1\}}$$

implies

$$1 > \mathbb{E}e^{\pm s Q^*}\mathbf{1}_{\{M^*=1\}} = a_\pm + \sum_{n\geq 0} b_\pm a_\mp^n b_\mp = a_\pm + \frac{b_\pm b_\mp}{1 - a_\mp}$$

and thus (19).

(b) Sufficiency: Let $\{(M_k^*, Q_k^*) : k = 1, 2, \ldots\}$ be as defined in Step (a2). Assuming (19) we thus have

$$a_\pm^* \stackrel{\text{def}}{=} \mathbb{E}e^{\pm s Q^*}\mathbf{1}_{\{M^*=1\}} = a_\pm + \frac{b_\pm b_\mp}{1 - a_\mp} < 1.$$

Note that (19) particularly implies $a_\pm, b_\pm \in [0, 1)$. Using this and

$$e^{\pm s Q_1^*} = e^{\pm s Q_1}\mathbf{1}_{\{M_1>-1\}} + e^{\pm s(Q_1-Q_2)}\mathbf{1}_{\{M_1=-1, M_2>-1\}}$$

$$+ \sum_{n\geq 3} e^{\pm s(Q_1-Q_2-\cdots-Q_n)}\mathbf{1}_{\{M_1=-1, M_2=\cdots=M_{n-1}=1, M_n>-1\}}$$

we further obtain that

$$\mathbb{E}e^{\pm s Q^*} = \mathbb{E}e^{\pm s Q}\mathbf{1}_{\{M>-1\}} + \mathbb{E}e^{\mp s Q}\mathbf{1}_{\{M>-1\}}\frac{b_\pm}{1 - a_\mp} < \infty.$$

Now let

$$\widehat{T}_0 \stackrel{\text{def}}{=} 0, \qquad \widehat{T}_k \stackrel{\text{def}}{=} \inf\{n > T_{k-1} : M_n^* < 1\}, \quad k = 1, 2, \ldots$$

and then $\{(\widehat{M}_k, \widehat{Q}_k) : k = 1, 2, \ldots\}$ in accordance with (33), (34) for these stopping times. We claim that $\mathbb{E}e^{\pm s \widehat{Q}} < \infty$ and thus $\mathbb{E}e^{s|\widehat{Q}|} < \infty$. Indeed,

$$\mathbb{E}e^{\pm s \widehat{Q}} = \sum_{n\geq 1} \mathbb{E}e^{\pm s(Q_1^* + M_1^* Q_2^* + \cdots + M_1^* \cdots M_{n-1}^* Q_n^*)}\mathbf{1}_{\{M_1^*=\cdots=M_{n-1}^*=1, M_n^*<1\}}$$

$$= \mathbb{E}e^{\pm s Q^*}\mathbf{1}_{\{M^*<1\}} \sum_{n\geq 1} a_\pm^{*n-1}$$

$$\leq \frac{\mathbb{E}e^{\pm s Q^*}}{1 - a_\pm} < \infty.$$

So we have $\mathbb{P}\{|\widehat{M}| = 1\} = 0$ and $\mathbb{E}e^{s|\widehat{Q}|} < \infty$ and may thus invoke Theorem 1.6 to finally conclude $\mathbb{E}e^{s|Z_\infty|} < \infty$ because Z_∞ is also the perpetuity generated by $(\widehat{M}, \widehat{Q})$. $\qquad\square$

3 Examples

We begin with an example that shows that condition $\mathbb{P}\{M = 0\} = 0$ in Theorem 1.3 is indispensable.

Example 3.1 If $\mathbb{P}\{M = 0\} = p = 1 - \mathbb{P}\{M = 1\}$ for some $p \in (0, 1)$ and $Q = 1$ a.s., the distribution of Z_∞ is geometric with parameter p. This can be seen from Remark 1.2, as the random variable N defined there has a geometric distribution with parameter $\mathbb{P}\{M = 0\}$.

The next examples illustrate that the distribution of Z_∞ can indeed be continuously singular as well as absolutely continuous. Denote by $\mathcal{L}(X)$ the distribution of a random variable X.

Example 3.2 (Deterministic M) Consider the situation where M is a.s. equal to a constant $c \in (0, 1)$, so

$$Z_\infty \overset{d}{=} cZ_\infty + Q.$$

(a) If $c = 1/2$ and $\mathcal{L}(Q)$ is a Poisson distribution with parameter $\lambda > 0$, then $\mathcal{L}(Z_\infty)$ is singularly continuous according to Example 4.3 by Watanabe [24].

(b) If $c = 1/n$ for some fixed positive integer n and $\mathcal{L}(Q)$ is the discrete uniform distribution on $\{0, \ldots, n - 1\}$, then it can be easily verified that Z_∞ has the uniform distribution on $(0, 1)$ and is thus absolutely continuous. This example is a special case of one due to Letac, see Example A2 in [19].

(c) A particularly well-studied class of special cases is when $\mathbb{P}\{Q = 1\} = \mathbb{P}\{Q = -1\} = 1/2$. A short survey can be found in [4]. If $c = 1/2$, then Z_∞ is uniformly distributed on $[-2, 2]$, while Z_∞ is continuously singular if $0 < c < 1/2$. One would expect Z_∞ to be absolutely continuous whenever $c \in (1/2, 1)$. However, this is not true as there are values of c between $1/2$ and 1 giving a singular $\mathcal{L}(Z_\infty)$, for example, if $c = (\sqrt{5} - 1)/2 = 0.618\ldots$, see [5, 6]. Meanwhile it has been proved by Solomyak [22] that, on the other hand, $\mathcal{L}(Z_\infty)$ is indeed absolutely continuous for almost all values of $c \in (1/2, 1)$.

Example 3.3 Assume that M and Q are independent.

(a) If M has a beta distribution with parameters 1 and $\alpha > 0$, i.e.

$$\mathbb{P}\{M \in dx\} = \alpha(1 - x)^{\alpha - 1}\mathbf{1}_{(0,1)}(x)dx,$$

and if Q has a $\Gamma(\alpha, \alpha)$-distribution, i.e.

$$\mathbb{P}\{Q \in dx\} = \frac{\alpha^\alpha}{\Gamma(\alpha)}x^{\alpha - 1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(x)dx,$$

then $\mathcal{L}(Z_\infty) = \Gamma(\alpha + 1, \alpha)$ as one can easily verify by direct calculation.

(b) If M has a Weibull distribution with parameter $1/2$, i.e.

$$\mathbb{P}\{M \in dx\} = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}\mathbf{1}_{(0,\infty)}(x)dx,$$

and Q is nonnegative with Laplace transform $\mathbb{E}e^{-sQ} = (1 + b\sqrt{s})e^{-b\sqrt{s}}$, $s \geq 0$, $b > 0$, then

$$\int_0^\infty e^{-sx}\mathbb{P}\{Z_\infty \in dx\} = e^{-b\sqrt{s}}, \quad s \geq 0$$

i.e. $\mathcal{L}(Z_\infty)$ is the positive stable law with index $1/2$, as was found independently in [7] and [16].

(c) If $\mathbb{P}\{M \in dx\} = (x^{-1/2} - 1)\mathbf{1}_{(0,1)}(x)dx$ and Q has Laplace transform

$$\varphi(s) \stackrel{\text{def}}{=} \left(\frac{\sqrt{2s}}{\sinh\sqrt{2s}}\right)^2, \quad s \geq 0,$$

then the Laplace transform of Z_∞ takes the form

$$\mathbb{E}e^{-sZ_\infty} = \frac{3(\sinh\sqrt{2s} - \sqrt{2s}\cosh\sqrt{2s})}{\sinh^3\sqrt{2s}} = -\frac{\varphi'(s)}{\mathbb{E}Q}, \quad s \geq 0.$$

This result was obtained in [21].

Absolute continuity of $\mathcal{L}(Z_\infty)$ in (a) and (b) is obvious. In (c) it follows from the fact that the corresponding $\mathcal{L}(M)$ is absolutely continuous.

Recall that the *size-biased distribution* $\overline{\mu}$ pertaining to a probability distribution μ on $[0, \infty)$ with finite mean $m > 0$ is defined as

$$\overline{\mu}(dx) \stackrel{\text{def}}{=} m^{-1}x\mu(dx).$$

In all three previous examples $\mathcal{L}(Z_\infty)$ is the size-biased distribution pertaining to $\mathcal{L}(Q)$. The study of distributions solving the fixed point equation (4) and having this additional property was initiated by Pitman and Yor [21] and then continued in [15] and [16].

Our last example provides an illustration of Theorem 1.7.

Example 3.4 Let Q be an exponential random variable with parameter $a > 0$ and M be independent of Q with $\mathbb{P}\{0 \leq M \leq 1\} = 1$ and $\mathbb{E}M < 1$. Then Z_∞ is a.s. finite and it can be checked directly or by using the fact that $Z_\infty \stackrel{d}{=} \int_0^\infty e^{-Y_t}dt$ for an appropriate compound Poisson process $\{Y_t : t \geq 0\}$ starting at zero, that

$$\mathbb{E}Z_\infty^n = \frac{n!}{a^n(1 - \mathbb{E}M)(1 - \mathbb{E}M^2)\cdots(1 - \mathbb{E}M^n)}.$$

Put $a_n \stackrel{\text{def}}{=} \mathbb{E}Z_\infty^n/n!$ and note that $\lim_{n\to\infty} a_{n+1}^{-1}a_n = a\mathbb{P}\{M < 1\}$. Hence, by the Cauchy-Hadamard formula, $r(Z_\infty) = a\mathbb{P}\{M < 1\}$ which is in full accordance with

Theorem 1.7 according to which $r(Z_\infty)$ is the positive solution to the equation $\frac{a}{a-s}\mathbb{P}\{M=1\}=1$ and thus indeed equal to $a\mathbb{P}\{M<1\}$.

4 Concluding Remarks

Although settling a number of open questions about perpetuities, this work gives also rise for further research. For example, it is natural to ask for conditions under which Z_∞ is of each of the three possible types. As already pointed out after Theorem 1.3, the law of Z_∞ can only be degenerate if $\mathbb{P}\{Q+cM=c\}=1$ for some $c \in \mathbb{R}$. However, the discussion in Example 3.2(c) indicates that a similar characterization for singularity and absolute continuity of the law of Z_∞ remains open even in the special case where M is deterministic and Q very simple.

Another natural problem that arises when regarding our Theorems 1.6 and 1.7 is to determine $r_*(Z_\infty) \leq 0$ and $r^*(Z_\infty) \geq 0$, given by

$$r_*(Z_\infty) \stackrel{\text{def}}{=} \inf\{r \leq 0 : \mathbb{E}e^{sZ_\infty} < \infty \text{ for all } r \leq s \leq 0\}$$

and

$$r^*(Z_\infty) \stackrel{\text{def}}{=} \sup\{r \geq 0 : \mathbb{E}e^{sZ_\infty} < \infty \text{ for all } 0 \leq s \leq r\}.$$

While this would provide information about when Z_∞ has exponential left and right tails, one may also aim at conditions that ensure existence of log-type moments of Z_∞ of the form $\mathbb{E}(\log^+ |Z_\infty|)^\beta$ for $\beta > 0$. The latter is studied in recent work by the first two authors [1] and is of additional interest due to the connection of perpetuities with certain intrinsic martingales in the supercritical branching random walk. This connection was first observed in [14] and further exploited in [17].

References

1. Alsmeyer, G., Iksanov, A.: A log-type moment result for perpetuities and its application to martingales in supercritical branching random walks. Preprint (2008)
2. Bertoin, J., Lindner, A., Maller, R.: On continuity properties of integrals of Lévy processes. Preprint available at http://www-m4.ma.tum.de/pers/lindner/ (2006)
3. Chow, Y.S., Teicher, H.: Probability Theory: Independence, Interchangeability, Martingales, 3rd edn. Springer, New York (1997)
4. Diaconis, P., Freedman, D.: Iterated random functions. SIAM Rev. **41**, 45–76 (1999)
5. Erdös, P.: On a family of symmetric Bernoulli convolutions. Am. J. Math. **61**, 974–975 (1939)
6. Erdös, P.: On the smoothness properties of Bernoulli convolutions. Am. J. Math. **62**, 180–186 (1940)
7. Donati-Martin, C., Ghomrasni, R., Yor, M.: Affine random equations and the stable ($\frac{1}{2}$) distribution. Stud. Sci. Math. Hung. **36**, 387–405 (2000)
8. Goldie, C.M.: Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. **1**, 126–166 (1991)

9. Goldie, C.M., Grübel, R.: Perpetuities with thin tails. Adv. Appl. Probab. **28**, 463–480 (1996)

10. Goldie, C.M., Maller, R.A.: Stability of perpetuities. Ann. Probab. **28**, 1195–1218 (2000)

11. Grincevičius, A.K.: On the continuity of the distribution of a sum of dependent variables connected with independent walks on lines. Theory Probab. Appl. **19**, 163–168 (1974)

12. Grincevičius, A.K.: Products of random affine transformations. Lith. Math. J. **20**, 279–282 (1980)

13. Grincevičius, A.K.: A random difference equation. Lith. Math. J. **21**, 302–306 (1981)

14. Iksanov, A.M.: Elementary fixed points of the BRW smoothing transforms with infinite number of summands. Stoch. Process. Their Appl. **114**, 27–50 (2004)

15. Iksanov, A.M., Kim, C.S.: On a Pitman-Yor problem. Stat. Probab. Lett. **68**, 61–72 (2004)

16. Iksanov, A.M., Kim, C.S.: New explicit examples of fixed points of Poisson shot noise transforms. Aust. N. Z. J. Stat. **46**, 313–321 (2004)

17. Iksanov, A.M., Rösler, U.: Some moment results about the limit of a martingale related to the super-critical branching random walk and perpetuities. Ukr. Math. J. **58**, 505–528 (2006)

18. Kellerer, H.G.: Ergodic behaviour of affine recursions III: positive recurrence and null recurrence. Technical reports, Math. Inst. Univ. München, Theresienstrasse 39, 8000 München, Germany (1992)

19. Letac, G.: A contraction principle for certain Markov chains and its applications. In: Random Matrices and Their Applications (Brunswick, Maine, 1984). Contemp. Math., vol. 50, pp. 263–273. Am. Math. Soc., Providence (1986)

20. Pakes, A.G.: Some properties of a random linear difference equation. Aust. J. Stat. **25**, 345–357 (1983)

21. Pitman, J., Yor, M.: Infinitely divisible laws associated with hyperbolic functions. Can. J. Math. **55**, 292–330 (2003)

22. Solomyak, B.: On the random series $\pm\lambda^i$ (an Erdös problem). Ann. Math. **242**, 611–625 (1995)

23. Vervaat, W.: On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. Adv. Appl. Probab. **11**, 750–783 (1979)

24. Watanabe, T.: Absolute continuity of some semi-selfdecomposable distributions and self-similar measures. Probab. Theory Relat. Fields **117**, 387–405 (2000)