Dirk Kuhlbusch

Moment conditions for weighted branching processes

– 2004 –

Angewandte Mathematik

Moment conditions for weighted branching processes

Inaugural-Dissertation zur
Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
durch den Fachbereich Mathematik und Informatik
der Westfälischen Wilhelms-Universität Münster

vorgelegt von

Dirk Kuhlbusch
aus Warendorf

– 2004 –

Dekan:	Prof. Dr. K. Hinrichs
Erster Gutachter:	Prof. Dr. G. Alsmeyer
Zweiter Gutachter:	Prof. Dr. U. Rösler (Kiel)
Tag der mündlichen Prüfung:	
Tag der Promotion:	

Contents

Introduction **1**

1 The underlying models **5**

1.1 Ordinary weighted branching processes 5

1.1.1 The model 5

1.1.2 Weighted branching processes vs. branching random walks 10

1.1.3 Weighted branching processes and stochastic fixed point equations . 12

1.1.4 A useful relation of tail probabilities 14

1.2 Weighted branching processes in random environment 16

1.2.1 Introduction 16

1.2.2 A suitable σ-algebra on $\mathbb{M}$ 18

1.2.3 Examples and some further notation 19

1.2.4 The intrinsic martingale 21

2 Limit theorems for normalized WBPRE **25**

2.1 The problem 25

2.2 The method of proof 27

2.2.1 Weighted trees 28

2.2.2 Size-biased weighted trees 29

2.3 Some auxiliary results 36

2.4 Main results 43

2.4.1 The general case 43

2.4.2 Stationary and m-dependent random environment 53

2.4.3 The case of i.i.d. environment 54

2.4.4 Applications to ordinary WBP 59

2.4.5 A counterexample and additional remarks 63

3 A characterization of $\mathfrak{L}_\alpha$-convergence for WBP **67**
3.1 Motivation . . . 67
3.2 The double martingale structure . . . 69
3.3 Preparatory moment results . . . 72
3.4 Results . . . 77
3.5 Some remarks on the functions $\mu(\cdot)$ and $g(\cdot)$ . . . 90
3.6 An a posteriori justification of the assumptions . . . 92

4 On the existence of ϕ-moments of the martingale limit of a WBP **97**
4.1 Introduction . . . 97
4.2 Functions of regular variation . . . 99
4.2.1 Regular variation and convexity . . . 101
4.2.2 Regular variation and submultiplicativity . . . 104
4.3 The related multiplicative random walk . . . 108
4.4 Results concerning $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ . . . 117
4.5 Results when the weights are decreasing along each line of descent . . . 124
4.6 Examples . . . 132

5 Some applications to tail probabilities **137**
5.1 On the tail probabilities of $W - W_n$ . . . 137
5.2 Some relations between the tail probabilities of Z_1, W and W^* . . . 143

A Two asymptotic results for stationary ergodic sequences of random variables **149**

B Two convex function inequalities for martingales **151**

List of abbreviations **153**

Bibliography **155**

Introduction

Branching processes form one of the classical fields of applied probability theory. In the second half of the last century, the classical *Galton-Watson process (GWP)* has been one of the most popular branching models. It may be seen as the prototype of a branching process and has been studied in numerous publications. A survey of relevant references can be found in the books by Asmussen and Hering [7], Athreya and Ney [12] and Jagers [59]. The underlying (discrete-time) branching mechanism is based on the assumptions that individuals reproduce independently of one another and according to the same probability distribution, and that each individual has a fixed life length of one unit of time and can generate offspring only at its death. In most cases, it is assumed that the population consists of exactly one individual when the process starts.

This branching model restricts itself to counting the number of individuals which are alive at different points of time and focuses on the inherent branching mechanism without considering exogenous effects like immigration or environmental changes. The most important facts concerning the extinction probability, martingale results and conditional limit theorems are collected in the aforementioned books.

During the last 35 years, many efforts have been made to generalize the GWP model and to consider branching models which are not only interesting from the mathematical point of view, but also accessible for applications. For instance, GWP allowing *immigration* have been considered in a series of articles. Again, we refer to the above books for a list of references.

Moreover, allowing environmental changes over time has led to *GWP in varying environment* and *GWP in random environment*. In terms of the GWP model, this means that the assumption of i.i.d. reproduction is relaxed. In the case of varying environments, individuals still reproduce independently, but the offspring distribution depends on the generation, while random environmental changes result in the fact that the independence assumption is replaced by a certain *conditional* independence. A list of references on GWP in random environment will be given in Section 1.2.

As a further way of generalization, we exemplarily mention (continuous-time) *age-dependent* branching processes (cf. [12], Chapter IV), where individuals live for a random length of time. Often, these life lengths are assumed to be i.i.d. with an exponential distribution which leads to *continuous time Markov branching processes* (cf. [12], Chapter III).

This thesis deals with *weighted branching processes (WBP)* which have been introduced by Rösler [94, 95] who also studied them together with Topchii and Vatutin [97, 98]. However, special types of WBP have also been used by Graf et al. [51] and by Mauldin and Williams [86] for the study of random Cantor sets and their Hausdorff dimensions. We will prove limit theorems for *ordinary WBP* as well as for the more general *weighted branching processes in random environment (WBPRE)*. Loosely speaking, an ordinary WBP can be seen as a GWP where individuals are allowed to have countably many children, and where a nonnegative random weight or size is assigned to each individual. The weight of an individual is given by the product of the weight of its mother and a random factor which is independent of the mother's weight. We mention that WBP are closely related to *branching random walks* as will be explained in detail in Subsection 1.1.2.

Correspondingly, WBPRE can be interpreted as generalizations of Galton-Watson processes in random environment. Again, each cell or individual carries a random weight which is assumed to be nonnegative and is defined in a multiplicative way. Throughout this thesis, we suppose that the environmental changes under consideration are modelled by a stationary ergodic random sequence $(U_n)_{n\geq 0}$ of probability measures on $([0,\infty)^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}_{[0,\infty)})$.

The further organization of this thesis is as follows. Chapter 1 rigorously introduces both branching models to be considered in this thesis and relates these models to well-studied special cases. Moreover, we describe the connection to *stochastic fixed point equations* in Subsection 1.1.3.

The second chapter is devoted to the question of $\mathfrak{L}_1$-convergence of the martingale which is obtained when normalizing a weighted branching process in random environment with its conditional mean (given the environment). Adapting the *spinal tree method* which was pioneered by Lyons et al. [83] for GWP, and by Lyons [82] for branching random walks, we find necessary and sufficient conditions for $\mathfrak{L}_1$-convergence when $(U_n)_{n\geq 0}$ is stationary ergodic. It will turn out that in contrast to supercritical Galton-Watson processes (where the so-called *$Z\log Z$-condition* is crucial), a pair of conditions has to be imposed. One of these conditions can be seen as a *generalized $Z\log Z$-condition*. However, the (strict) $Z\log Z$-condition "$\mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right] < \infty$", where Z_1 is the total size of the first generation and $\mu(U_0)$ denotes its conditional mean, is not necessary for $\mathfrak{L}_1$-convergence in this general setting. Moreover, we see that in the special case of stationary m-dependent environment ($m \geq 0$), the conditions can be substantially simplified. As an application, we consider ordinary WBP which may be viewed as WBPRE with deterministic and nonvarying environment. In this way, we are able to confirm a limit theorem due to Lyons [82].

From Chapter 3 on, we focus on ordinary weighted branching processes. For each $\alpha \in (1, \infty)$, Chapter 3 gives a complete characterization of $\mathfrak{L}_\alpha$-convergence of normalized weighted branching processes. Denoting by W the limit of the martingale obtained by normalization, $\mathfrak{L}_\alpha$-convergence holds if and only if W has a finite and positive αth moment. We find necessary and sufficient conditions for $0 < \mathbb{E}W^\alpha < \infty$ by exploiting an important inherent probabilistic feature of the underlying branching model which expresses itself in a *double martingale structure*. This justifies the repeated application of certain convex function inequalities for martingales, an approach that has first been used by Alsmeyer and Rösler [6] for supercritical GWP and a more general class of convex functions.

In Chapter 4, we analyze when the martingale limit W has a finite and positive ϕ-moment, where ϕ is a regularly varying function of index $\alpha \geq 1$. This will be carried out by using similar methods as in Chapter 3. Under an additional condition on ϕ (actually guaranteeing that ϕ is asymptotically equivalent to some submultiplicative regularly varying function), we solve this problem completely. On the other hand, if the random weights fulfill a certain boundedness property which implies that the weights are decreasing along each fixed line of descent, no serious restriction on ϕ is required.

The final chapter uses the previous results to give bounds for the rate of convergence of the intrinsic martingale and collects some further results on tail probabilities.

Acknowledgements. I would like to express my appreciation to my supervisor Prof. Dr. Gerold Alsmeyer for his encouragement and many interesting and fruitful discussions throughout this project. Moreover, my sincere thanks go to Prof. Dr. Uwe Rösler and Dr. Amke Caliebe (both Kiel) for their interest and some valuable hints. I would also like to thank all members of the *Institut für Mathematische Statistik* at the *WWU Münster* for a good working environment, in particular Markus Jaeger who had a careful look at a draft version of this thesis.

Last but not least, I would like to thank my girlfriend Kirsten Mersch for her continuous moral support during the past two years.

Chapter 1

The underlying models

This chapter introduces the branching models to be investigated throughout this thesis.

Hereafter, let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a fixed probability space which is supposed to be large enough to carry all random variables encountered.

1.1 Ordinary weighted branching processes

1.1.1 The model

As mentioned in the Introduction, this thesis provides limit theorems for normalized weighted branching processes and normalized weighted branching processes in random environment. Weighted branching processes have been introduced by Rösler [94, 95] and have also been treated by Rösler, Topchii and Vatutin in [97] and [98]. Bibliographical remarks on related but more specific models will be given at the end of this subsection.

In order to give an informal description of the underlying model, first without random environment, we consider a population of cells all stemming from one ancestor cell, denoted $(\emptyset)$. As usual, we label each cell or individual of a subsequent generation by an element $v = (v_1, ..., v_n)$ of the *Ulam-Harris tree* $\mathcal{N} := \{(\emptyset)\} \cup \bigcup_{n\geq 1} \mathbb{N}^n$ which encodes its line of descent. Suppose that each cell $v = (v_1, ..., v_n)$ has a lifespan of one unit of time and carries a random weight (or size) $L(v)$ obtained as the product of the mother's weight $L(v_1, ..., v_{n-1})$ and a random factor $T_{v_n}(v_1, ..., v_{n-1})$. This reflects that every cell grows or shrinks during its life period and splits into a random number of offspring cells at its death. By normalization, let the ancestor have weight one, i.e. $L(\emptyset) = 1$, and let the infinite vectors $T(v) = (T_i(v))_{i\geq 1}$, $v \in \mathcal{N}$ be i.i.d.. The latter expresses the standard assumption in many branching models that all population members exhibit the same biological performance (growth, reproduction) and that all cells develop independently of one another. Note that every cell can produce infinitely many daughter cells, but that only cells with positive weight are alive. Now, if

$$Z_n = \sum_{|v|=n} L(v) \tag{1.1.1}$$

denotes the total weight of all cells of the nth generation, then $(Z_n)_{n\geq 0}$ is called a *weighted branching process (WBP)*. Here and in the following, $|v|$ denotes the *length* or *generation* of v, in particular $|(\emptyset)| = 0$.

The following picture shows the first generations (denoted by $\mathbf{G}_0, ..., \mathbf{G}_3$) of the corresponding family tree. It only contains living cells, i.e. those cells with a positive weight.

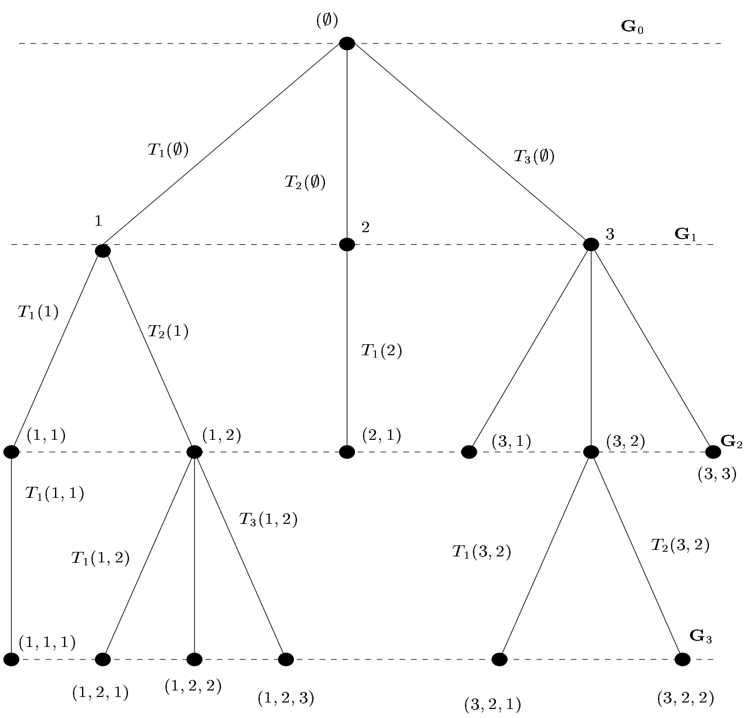

Fig. 1.1: A typical realization of a weighted branching process

Hereafter, we are interested in the total weight or size of the nth generation (given by Z_n), and its behaviour as $n \to \infty$ (when appropriately normalized).

We emphasize that it is a crucial property of WBP that the underlying random vectors

$$T(v) = (T_i(v))_{i\geq 1} : \Omega \to [0,\infty)^{\mathbb{N}}, \quad v \in \mathcal{N},$$

are i.i.d.. Notice that Rösler et al. [97, 98] also allowed negative weights. A modification of the above concept as in Section 1.2 leads to *weighted branching processes in random environment*. However, even in the present i.i.d. setting, we allow arbitrary dependencies of the random variables $T_i(v)$, $i \geq 1$ when $v \in \mathcal{N}$ is fixed.

Given $v, w \in \mathcal{N}$, we will say that w *stems from* v if $v = (\emptyset)$ or if $w = (v, v')$ for some $v' \in \mathcal{N}$, where given $v = (v_1, ..., v_r)$ and $v' = (v_1', ..., v_s') \in \mathcal{N}$, (v, v') is to be understood as $(v_1, ..., v_r, v_1', ..., v_s')$, in particular $(v, \emptyset) := (\emptyset, v) := v$ for any $v \in \mathcal{N}$. v is called *alive* if its weight is positive. In addition, we abbreviate

$$T := (T_i)_{i\geq 1} := T(\emptyset).$$

By assumption, this random sequence determines the distribution of the entire process $(Z_n)_{n\geq 0}$. Therefore, the sequence $T_1, T_2, ...$ is often called *generic weight sequence* or *sequence of generic weights*. As indicated above, the family of weights $L(v)$, $v \in \mathcal{N}$ satisfies $L(\emptyset) = 1$ and

$$L(i_1, ..., i_n) = T_{i_1} \cdot T_{i_2}(i_1) \cdot ... \cdot T_{i_n}(i_1, ..., i_{n-1}), \qquad (i_1, ..., i_n) \in \mathbb{N}^n \quad (n \geq 1).$$

In combination with (1.1.1), this multiplicative structure leads to the frequently used *forward equation*

$$Z_{n+1} = \sum_{|v|=n} L(v) \cdot \sum_{i\geq 1} T_i(v) = \sum_{|v|=n} \sum_{i\geq 1} L(v) T_i(v), \qquad n \geq 0, \tag{1.1.2}$$

while a reversed view in time explains the *backward equation*

$$Z_{n+1} = \sum_{i\geq 1} T_i Z_{n,i}, \qquad n \geq 0. \tag{1.1.3}$$

Here, the sequences $(Z_{n,i})_{n\geq 0}$, $i \geq 1$, are i.i.d. with the same distribution as $(Z_n)_{n\geq 0}$ and independent of $T = T(\emptyset)$. This representation reflects the immanent self-similar structure of the underlying branching model. Notice that on the event $\{T_j > 0\}$, $Z_{n,j}$ can be written as $Z_{n,j} = \sum_{|v|=n} \frac{L(j,v)}{T_j}$. Hence, $Z_{n,j}$ can be viewed as the normalized size of the subfamily founded by individual j at time $n+1$. While the backward equation is closely related to iterations of distributions and thus to stochastic fixed point equations (cf. Subsection 1.1.3), the forward equation discloses the martingale structure associated with (Z_n). If the *reproduction mean* $\mu := \mathbb{E}Z_1$ is positive and finite, it is a well-known fact (cf. [94], [95] or [97]) and also follows from the more general Lemma 1.2.3 below that the normalized sequence

$$W_n := Z_n/\mu^n, \qquad n \geq 0,$$

is a nonnegative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$, defined by

$$\mathcal{F}_0 := \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_{n+1} := \sigma(T(v) : |v| \leq n), \quad n \geq 0.$$

Plainly, this implies $\mathbb{E}Z_n = \mu^n$ for $n \geq 0$, and $\mathcal{F}_n$ contains the σ-algebra generated by $(L(v))_{|v|\leq n}$, $n \geq 0$. By the martingale convergence theorem and Fatou's lemma , W_n converges a.s. to some nonnegative and finite random variable W with $\mathbb{E}W \leq \mathbb{E}W_0 = 1$. It seems worth noting that $(W_n)_{n\geq 0}$ can itself be viewed as a WBP with generic weights $\mu^{-1}T_1, \mu^{-1}T_2, ...$, and is naturally normalized. Therefore, we may (and will) often assume without loss of generality that $\mu = 1$, hence $Z_n = W_n$ for all $n \geq 0$. If $L(v) \in \{0,1\}$ a.s. for all $v \in \mathcal{N}$, and each cell has only a finite number of living daughters, $(Z_n)_{n\geq 0}$ is a Galton-Watson process as will be explained below. Note that the simplifying assumption of unit mean is a proper restriction in the context of Galton-Watson processes where the reproduction mean is a crucial parameter.

The weighted branching process unites a branching mechanism and a multiplicative structure. To give (extreme) examples, it is natural to suppress one of these components and focus on the other. Removing the multiplicative structure leads to the well-known Galton-Watson process:

Example 1.1.1. (Galton-Watson processes)
Suppose that $T_i \in \{0,1\}$ a.s. for all $i \geq 1$. By induction, it follows that any individual can only have weight 0 or 1. Thus, $(Z_n)_{n\geq 0}$ merely counts the number of living individuals without distinguishing their weights or sizes. Put

$$p_k := \mathbb{P}\left(\sum_{i\geq 1} T_i = k\right), \qquad k \in \overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}.$$

If Z_1 is a.s. finite (i.e. $p_\infty = 0$), then $(Z_n)_{n\geq 0}$ is a *Galton-Watson process (GWP)* with offspring distribution $(p_k)_{k\geq 0}$, whereas otherwise, $(Z_n)_{n\geq 0}$ may be viewed as a Galton-Watson process with state space $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$. In case $p_\infty = 0$, the reproduction mean $\mu = \mathbb{E}Z_1 = \sum_{k\geq 1} kp_k$ is an important parameter. In contrast to the general WBP, the

GWP forms a (homogeneous) Markov chain and is a well studied object. One of the main mathematical tools utilized for their analysis is the *generating function* $f(s) := \mathbb{E}s^{Z_1} = \sum_{k\geq 0} p_k s^k$ for $s \in [0,1]$ together with its iterates and functional equations connected to them. For more background material on GWP, we refer to the books [7], [12] and [59].

A general WBP also considers the random sizes of different individuals. This contrast to Galton-Watson processes is emphasized by introducing the corresponding GWP with state space $\overline{\mathbb{N}}_0$, given by

$$Z'_n := \sum_{|v|=n} \mathbf{1}_{\{L(v)>0\}}, \qquad n \geq 0,$$

which counts the number of living cells in generation n. We shall frequently use the notation

$$N := Z'_1 = \sum_{i\geq 1} \mathbf{1}_{\{T_i>0\}}, \tag{1.1.4}$$

giving the number of cells with positive weight in the first generation.

On the other hand, if there is no branching at all, we arrive at a multiplicative random walk:

Example 1.1.2. (Multiplicative random walks)
Assume that each individual has at most one successor, that is $\mathbb{P}(N \leq 1) = 1$ which means that the infinite sum $\sum_{i\geq 1} T_i$ consists of at most one strictly positive term. Without loss of generality, we may suppose that $T_2 = T_3 = ... = 0$ almost surely. Then it is obvious that Z_n is the product of independent copies of $Z_1 = T_1$. In other words, $(\log Z_n)_{n\geq 0}$ forms a random walk taking values in $\mathbb{R} \cup \{-\infty\}$. If $\varrho = \mathbb{E}\log Z_1$ exists in $\overline{\mathbb{R}} = [-\infty, \infty]$, Z_n converges to 0 (∞) iff ϱ is strictly negative (strictly positive) by the strong law of large numbers. If $\varrho = 0$ and $Z_1 \neq 1$ with positive probability, the Chung-Fuchs theorem implies that Z_n oscillates between 0 and ∞.

Motivated by Example 1.1.1, we define

$$c := \mathbb{P}(T_i \in \{0, \mu\} \text{ for all } i \geq 1) = \mathbb{P}(T_i/\mu \in \{0,1\} \text{ for all } i \geq 1) \tag{1.1.5}$$

for a given WBP $(Z_n)_{n\geq 0}$ with generic weights $T_1, T_2, ...$ and reproduction mean $\mu = \mathbb{E}Z_1 \in (0,\infty)$. Then $c = 1$ if and only if the (normalized) WBP $(W_n)_{n\geq 0}$ is a critical Galton-Watson process. This situation is often excluded in the remainder of this thesis.

Weighted branching processes have been discussed in a variety of contexts. In a less general form, they were introduced by Mandelbrot [84] who assumed that $\|N\|_\infty := \text{ess sup}\, N = r < \infty$ and that $T_1, ..., T_r$ are i.i.d., a classical model which has thereafter also been considered by Kahane and Peyrière [62], Guivarc'h [52] and Barral [14]. Holley and Liggett [56] considered the case when $\|N\|_\infty = r < \infty$ and $T_1, ..., T_r$ are fixed multiples of a positive random variable, whereas Durrett and Liggett [44] allowed $(T_1, ..., T_r)$ to have arbitrary joint distribution. A similar model was also investigated by Waymire and Williams [108].

Graf et al. [51] and Mauldin and Williams [86] used weighted branching processes for the study of random Cantor sets and their Hausdorff dimensions. In this context, the weights $L(v)$, $|v| = n$, denote the volumes of the remaining sets after n steps in a so-called *random recursive construction* (without loss of generality assuming that the starting set $J_{(\emptyset)}$ is normalized to have volume 1). Thus, the weights $T_i, i \geq 1$ are a.s. bounded by 1, and with $\mathcal{C}$ denoting the resulting Cantor set, then under mild conditions, $\mathcal{C} \neq \emptyset$ with positive probability, and the Hausdorff dimension of $\mathcal{C}$ is a.s. constant on the event $\{\mathcal{C} \neq \emptyset\}$ and given by the smallest $\alpha \in (0, \infty)$ such that $\sum_{i\geq 1} \mathbb{E}T_i^\alpha \leq 1$. We remark that for the results in [51], the assumption $\|N\|_\infty < \infty$ is needed. In this framework, it means that there is a (non-random) bound for the number of remaining sets after the first step of the random construction.

Weighted branching processes are also closely related to the Laplace functional of *branching random walks* and to so-called *multiplicative cascades.* These stochastic processes actually correspond to the situation where N is a.s. finite, but not necessarily $\|N\|_\infty < \infty$, i.e. N is not a.s. bounded. The following subsection is devoted to these connections.

1.1.2 Weighted branching processes vs. branching random walks

The objective of this subsection is to stress that weighted branching processes as just introduced are closely related to the *Laplace functional* of a *branching random walk (BRW).* This kind of process has first been considered by Kingman [65] and extensively studied by Biggins in a series of articles [17, 18, 19, 20], to mention but a few references. In the following, we briefly outline the construction of a branching random walk. An initial ancestor, forming the 0th generation, is created at the origin. His children form the first generation and their positions on the real line are described by a point process Υ^1 on

$\mathbb{R}$. The individuals in the nth generation give birth independently of one another and of the preceding generations to form the $(n+1)$th generation. The point process modelling the displacements of the children of an individual x from the position of x has the same distribution as Υ^1.

Let $(X_r^n)_r$ be an enumeration of the positions of the people in the nth generation, define

$$m(\theta) := \mathbb{E}\left[\sum_r \exp(-\theta X_r^1)\right] = \mathbb{E}\left[\int_{\mathbb{R}} e^{-\theta t}\,\Upsilon^1(dt)\right]$$

and assume that $m(0) \in (1, \infty)$ and $m(\theta) < \infty$ for some $\theta > 0$. These assumptions imply that the generation sizes $\Upsilon^n(-\infty, \infty)$ in the branching random walk form a supercritical Galton-Watson process with reproduction mean $m(0)$. Particularly, individuals can have only finitely many children. Then for any θ with $m(\theta) < \infty$, the *Laplace functional*

$$\Xi^n(\theta) := m(\theta)^{-n} \sum_r \exp(-\theta X_r^n), \qquad n \geq 0$$

of the BRW is a nonnegative (and therefore a.s. convergent) martingale with respect to $(\mathcal{E}_n)_{n\geq 0}$, where $\mathcal{E}_n$ is the σ-field generated by the births in the first n generations. Evidently, $\Xi^n(0) = \Upsilon^n(-\infty, \infty)$ is the underlying Galton-Watson process.

Moreover, it is a trivial observation that when θ is fixed and $(Z_n)_{n\geq 0}$ denotes the WBP with generic weight sequence $T_i := \exp(-\theta X_i^1)\mathbf{1}_{\{\Upsilon^1(\mathbb{R})\geq i\}}$, $i \geq 1$, then $\Xi^n(\theta) = m(\theta)^{-n} Z_n$ for $n \geq 0$. In other words, the BRW is a special WBP, taking an additive point of view instead of a multiplicative one. However, the concept of WBP is slightly more general because it gives up the assumption of finite generation sizes. This is worth noting since many results on branching random walks heavily rely on the analysis of Laplace transforms and generating functions, exploiting well-known results for the underlying Galton-Watson process. As an example, we mention the Seneta-Heyde type result proved by Biggins and Kyprianou in [23] and [24] which we shall quote in Subsection 2.4.4. A further amenity of the WBP model consists of the fact that it can be easily extended to the case of real-valued (instead of nonnegative) weights, even though that case seems to be much more difficult to analyse (cf. [97], [98]). However, Lyons [82] and Iksanov [58] recently also considered BRW with not necessarily finite offspring numbers.

In Section 4.5 we will study in detail moment conditions for WBP satisfying the condition $\mathbb{P}\left(\sup_{i\geq 1} T_i \leq 1\right) = 1$ which means that the weights are a.s. decreasing along each fixed line of descent. In the BRW model, this regime corresponds to the case where individuals can only be born to the right of their mother, i.e. the BRW can only take positive steps. The latter condition can also be expressed as $\Upsilon^1(-\infty,0) = 0$ a.s., a restriction imposed by Kingman in his pioneering article [65].

Liu [78, 79] has studied a multiplicative variant of the BRW called *multiplicative cascade* which, however, also requires the condition of finite offspring numbers. In addition, he treated extensively the case of *homogeneous branching random walks* in [77, 79, 80] which corresponds to a WBP where with probability 1, all positive (generic) weights T_i are equal. Some of his results on multiplicative cascades will be quoted in this thesis for the sake of completeness.

Finally, we mention that multiplicative cascades taking values in a Banach algebra (instead of being $[0,\infty)$-valued) have been introduced by Barral [15].

1.1.3 Weighted branching processes and stochastic fixed point equations

Weighted branching processes have been discussed in a variety of contexts. For instance, they have appeared in the analysis of certain stochastic fixed point equations. The connection to this popular object of modern probability theory is exemplified by the following two results, the first of which can also be found in [35], [94], [95], [96] and [97].

Theorem 1.1.3. *The martingale limit W of a weighted branching process $(Z_n)_{n\geq 0}$ with reproduction mean $\mu = \mathbb{E}Z_1 \in (0,\infty)$ can be written in the form*

$$W = \sum_{i\geq 1} \frac{T_i}{\mu} \cdot W_{(i)} \quad a.s., \tag{1.1.6}$$

where $W_{(1)}, W_{(2)}, \ldots$ are independent copies of W and also independent of $\mathcal{F}_1 = \sigma(T_1, T_2, \ldots)$.

Proof. Without loss of generality, suppose that $\mu = 1$ and let $n \geq 0$. Then using the backward equation (1.1.3), $W_{n+1} = Z_{n+1}$ can be rewritten as

$$W_{n+1} = \sum_{i\geq 1} T_i W_{n,i} \text{ a.s.},$$

where the random variables $W_{n,i}$, $i \geq 1$ are independent copies of W_n and also independent of $\mathcal{F}_1 = \sigma(T_1, T_2, \ldots)$. Hence, each $W_{n,i}$ has an a.s. limit $W_{(i)}$ distributed as W and independent of $\mathcal{F}_1$. Furthermore, $W_{(1)}, W_{(2)}, ...$ are independent random variables. Now by Fatou's lemma,

$$W = \liminf_{n\to\infty} W_{n+1} \geq \sum_{i\geq 1} T_i \liminf_{n\to\infty} W_{n,i} = \sum_{i\geq 1} T_i W_{(i)} \text{ a.s..}$$

In addition, both W and the expression on the right-hand side have the same finite mean and are therefore equal with probability 1. □

In particular,

$$W \stackrel{d}{=} \sum_{i\geq 1} T_i W_{(i)} \tag{1.1.7}$$

if $\mu = 1$, where "$\stackrel{d}{=}$" means equality in distribution. In other words, if $\mathcal{D}$ ($\overline{\mathcal{D}}$) denotes the space of distributions on $[0,\infty)$ ($[0,\infty]$), then the distribution of W is a fixed point of the so-called *smoothing transformation* $\mathcal{K} : \mathcal{D} \to \overline{\mathcal{D}}$,

$$\mathcal{K}(\nu) := \mathbb{P}\left(\sum_{i\geq 1} T_i X_i \in \cdot\right),$$

where $X_1, X_2, ...$ are independent random variables with distribution ν such that $(T_i)_{i\geq 1}$ and $(X_i)_{i\geq 1}$ are independent. With this notation, the backward equation (1.1.3) reads

$$\mathbb{P}(Z_{n+1} \in \cdot) = \mathcal{K}(\mathbb{P}(Z_n \in \cdot)), \qquad n \geq 0$$

and thus ($Z_0 = 1$ a.s.)

$$\mathbb{P}(Z_n \in \cdot) = \mathcal{K}^n(\delta_1),$$

where δ_1 is the Dirac measure in 1 and $\mathcal{K}^n$ is the n-fold iteration of $\mathcal{K}$. In other words, the essence of the backward equation is that the distribution of Z_n evolves as an iteration of distributions.

In terms of the characteristic function Φ_W of W, (1.1.7) can be expressed by the functional equation

$$\Phi_W(s) = \mathbb{E}\left[\prod_{j\geq 1} \Phi_W(sT_j)\right], \qquad s \in \mathbb{R}.$$

By the same argument as in Theorem 1.1.3, we obtain the following representation of the martingale limit:

Theorem 1.1.4. *Fix $n \geq 2$. Then W has the representation*

$$W = \sum_{|v|=n} \frac{L(v)}{\mu^n} \cdot W_{(v)} \quad a.s., \tag{1.1.8}$$

where $W_{(v)}$, $v \in \mathbb{N}^n$ are independent copies of W and also independent of $(T(w))_{|w|\leq n-1}$.

Remark 1.1.5. Rösler [92, 93] considered the slightly more general distributional fixed point equation

$$X \stackrel{d}{=} \sum_{j\geq 1} T_j X_j + C \tag{1.1.9}$$

for i.i.d. real-valued random variables $X, X_1, X_2, ...$ which are independent of $(C, T_1, T_2, ...)$, where C and $T = (T_j)_{j\geq 1}$ are in general neither independent nor nonnegative. Fixed points of the latter equation naturally arise as limits of recursive equations of the form

$$X^{(n+1)} \stackrel{d}{=} \sum_{j\geq 1} T_j^{(n)} X_j^{(n)} + C^{(n)}, \qquad n \geq 0.$$

Here $(T^{(n)}, C^{(n)}), X_1^{(n)}, X_2^{(n)}, ...$ are independent for each n, and the joint distribution of $T^{(n)}$ and $C^{(n)}$ is known and converges to that of (C, T). A recursive system like this has been dealt with in [92] and [93] in the context of the sorting algorithm QUICKSORT. Although the method provided in these articles can be successfully applied to more general divide and conquer algorithms, QUICKSORT still constitutes the most prominent example. The method relies on the contraction property of the operator $\mathcal{K}_C$ associated with (1.1.9) which is defined on the set of all probability measures with finite (absolute) pth moment. This set is endowed with the so-called *Mallows metric*. Further information on the method employed, including applications and generalizations, can be found in [91] and [96].

For a comprehensive account of stochastic fixed point equations, the reader may also consult [5], [13], [27], [33], [34], [35], [44], [55], [58], [68], [75], [76], [79] and [80].

1.1.4 A useful relation of tail probabilities

If W' denotes the nondegenerate limit of a normalized supercritical Galton-Watson process $(Z'_n)_{n\geq 0}$ fulfilling $\mathbb{E}Z'_1 \log^+ Z'_1 < \infty$, it is well-known (see for example Lemma II.2.6 in [7]) that

$$\mathbb{E}f(W') < \infty \quad \Leftrightarrow \quad \mathbb{E}f\left(\sup_{n\geq 0} Z'_n/\mathbb{E}Z'_n\right) < \infty$$

whenever $f : [0, \infty) \to [0, \infty)$ is nondecreasing and continuous on $[0, \infty)$, continuously differentiable[1] on $(0, \infty)$ and satisfies $f(0) = 0$ and $f(2x) \leq cf(x)$ for some $c > 0$ and

[1] Although the result can be extended to a larger class of functions, we focus on smooth functions in this subsection.

all $x \geq 0$. The following result, due to Biggins [19] in the situation of branching random walks, shows that this equivalence also holds in our context. Put $W^* := \sup_{n\geq 0} W_n$.

Theorem 1.1.6. *Suppose that* $\mathbb{E}W = 1$. *For any* $a \in (0,1)$, *there is a constant* $B = B(a) \in (0,\infty)$ *such that*

$$\mathbb{P}(W > at) \geq B\mathbb{P}(W^* > t), \quad t > 1 \tag{1.1.10}$$

(cf. Lemma II.2.6 in [7] for the Galton-Watson case). Moreover, if $f : [0,\infty) \to [0,\infty)$ *is nondecreasing and continuous on* $[0,\infty)$, *continuously differentiable on* $(0,\infty)$ *and satisfies* $f(0) = 0$ *and* $f(2x) \leq cf(x)$ *for some* $c > 0$ *and all* $x \geq 0$, *we have the equivalence*

$$\mathbb{E}f(W) < \infty \quad \Leftrightarrow \quad \mathbb{E}f\left(\sup_{n\geq 0} W_n\right) < \infty. \tag{1.1.11}$$

The following proof has been given by Biggins for branching random walks (Lemma 2 in [19]) and only requires slight modifications in our scenery:

Proof. Without loss of generality, suppose that $\mu = \mathbb{E}Z_1 = 1$. For the proof of the tail estimate, fix $t > 1$ and put $E_n := \{W_n > t, \max_{0\leq k<n} W_k \leq t\}$ for $n \geq 1$. Then

$$\mathbb{P}(W > at) \geq \sum_{n:\mathbb{P}(E_n)>0} \mathbb{P}(W > at|E_n)\mathbb{P}(E_n). \tag{1.1.12}$$

For any $n \geq 1$, Theorem 1.1.4 gives the representation

$$W = \sum_{|v|=n} L(v)W_{(v)} \quad \text{a.s.,} \tag{1.1.13}$$

where $W_{(v)}$, $v \in \mathbb{N}^n$ are independent copies of W and also independent of $(T(w))_{|w|\leq n-1}$. Consequently, for all n with $\mathbb{P}(E_n) > 0$,

$$\begin{aligned}
\mathbb{P}(W > at|E_n) \;&\geq\; \mathbb{P}\left(\sum_{|v|=n} L(v)W_{(v)} > at \middle| E_n\right) \\
&=\; \mathbb{P}\left(\sum_{|v|=n} \frac{L(v)}{W_n}(W_{(v)} - 1) > \frac{at}{W_n} - 1 \middle| E_n\right) \\
&\geq\; \mathbb{P}\left(\sum_{|v|=n} \frac{L(v)}{W_n}(W_{(v)} - 1) > a - 1 \middle| E_n\right) \\
&=\; \mathbb{P}(E_n)^{-1} \int_{E_n} \mathbb{P}\left(\sum_{|v|=n} \frac{L(v)}{W_n}(W_{(v)} - 1) > a - 1 \middle| \mathcal{F}_n\right) d\mathbb{P}
\end{aligned}$$

because $E_n \in \mathcal{F}_n$ and $W_n > t$ on E_n. Since the random variables $W_{(v)} - 1$, $|v| = n$ are independent of each other and of $\mathcal{F}_n$ with zero mean, Lemma 1 in [19] shows that for some $B = B(a) > 0$,

$$\int_{E_n} \mathbb{P}\left(\sum_{|v|=n} \frac{L(v)}{W_n}(W_{(v)} - 1) > a - 1 \,\middle|\, \mathcal{F}_n\right) d\mathbb{P} \geq B\mathbb{P}(E_n),$$

whence by (1.1.12),

$$\mathbb{P}(W > at) \geq B \sum_{n \geq 1} \mathbb{P}(E_n) = B\mathbb{P}(W^* > t).$$

Now if f is as described, the moment assertion follows easily from (1.1.10) and the facts that

$$\mathbb{E}f(X) = \int_{(0,\infty)} f'(t)\mathbb{P}(X > t)\,\boldsymbol{\lambda}(dt)$$

and

$$\mathbb{E}f(rX) < \infty \quad \text{iff} \quad \mathbb{E}f(X) < \infty$$

which are valid for any $r > 0$ and any nonnegative random variable X. □

1.2 Weighted branching processes in random environment

1.2.1 Introduction

In many biological contexts the assumption of independent and identical biological performance of all cells seems doubtful because various exogenous environmental factors like temperature, food supply, competition etc. may cause a variation of cell behaviour over generations. If this variation shows a recurrent (seasonal) pattern the following generalization of the above model (which can also be found in the prepublication [66]) may be viewed as a reasonable alternative: Suppose that, given a stationary ergodic sequence $\mathbf{U} = (U_n)_{n \geq 0}$, the weight factors $T(v)$, $v \in \mathcal{N}$, are conditionally independent with a conditional distribution depending only on U_n if v belongs to the nth generation. This means that the U_n are the random parameters reflecting the environmental fluctuations over time. A rigorous description is given further below. The sequence $(Z_n)_{n \geq 0}$ thus obtained is our main object of interest in Chapter 2 and called a *weighted branching process in random environment (WBPRE)*.

A rigorous description of the setting now follows. As indicated, environmental changes over time shall be modelled by a sequence of random probability measures on the set $([0,\infty)^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}_{[0,\infty)})$ of nonnegative sequences in which the weight sequences $T(v)$, $v \in \mathcal{N}$ take their values. More precisely, denote by Σ the measurable mapping

$$\Sigma : [0,\infty)^{\mathbb{N}} \to [0,\infty], \qquad \Sigma((x_i)_{i\geq 1}) := \sum_{i\geq 1} x_i,$$

and let $\mathbb{M}$ denote the collection of probability measures Q on $([0,\infty)^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}_{[0,\infty)})$ fulfilling

$$0 < \mu(Q) := \int_{[0,\infty)^{\mathbb{N}}} \Sigma \, dQ = \int_{[0,\infty]} y \; Q^{\Sigma}(dy) < \infty.$$

In Subsection 1.2.2 further below, we will endow $\mathbb{M}$ with an appropriate σ-algebra $\mathfrak{M}$. Moreover, suppose that $\mathbf{U} = (U_n)_{n\geq 0}$ is a stationary ergodic sequence of random variables taking values in $(\mathbb{M}, \mathfrak{M})$, i.e. satisfies (cf. [31], Chapter 6)

- *Stationarity:* $\mathcal{S}\mathbf{U} \stackrel{d}{=} \mathbf{U}$, where the shift operator $\mathcal{S} : (\mathbb{M}^{\mathbb{N}_0}, \mathfrak{M}^{\mathbb{N}_0}) \to (\mathbb{M}^{\mathbb{N}_0}, \mathfrak{M}^{\mathbb{N}_0})$ is defined by $\mathcal{S}(\zeta_0, \zeta_1, ...) = (\zeta_1, \zeta_2, ...)$ (and is obviously measurable). Note that this implies $\mathcal{S}^n\mathbf{U} \stackrel{d}{=} \mathbf{U}$ for any $n \geq 1$, where $\mathcal{S}^n(\zeta_0, \zeta_1, ...) = (\zeta_n, \zeta_{n+1}, ...)$.

- *Ergodicity:* The σ-algebra $\mathfrak{I} := \{A \in \mathbb{B}^{\mathbb{N}}_{[0,\infty)} : \mathbf{1}_A(\mathbf{U}) = \mathbf{1}_A(\mathcal{S}\mathbf{U}) \ \mathbb{P}\text{-a.s.}\}$ of a.s. shift-invariant events is $\mathbb{P}^{\mathbf{U}}$-trivial in the sense that $\mathbb{P}(\mathbf{U} \in A) \in \{0,1\}$ for all $A \in \mathfrak{I}$.

In the new model the assumption of i.i.d. reproduction which is characteristic for the situation of ordinary WBP is replaced by the assumption that the random variables $T(v), v \in \mathcal{N}$ are conditionally independent given $\mathbf{U}$ with conditional distributions determined by

$$\mathbb{P}(T(v) \in \cdot|\mathbf{U}) = U_{|v|} \quad \text{a.s. for all } v \in \mathcal{N}.$$

Obviously, this ensures that all individuals still show identical biological performance since for each $v \in \mathcal{N}$ and all measurable $A \subset [0,\infty)^{\mathbb{N}}$,

$$\mathbb{P}(T(v) \in A) = \int \mathbb{P}(T(v) \in A|\mathbf{U}) \, d\mathbb{P} = \mathbb{E}U_{|v|}(A) = \mathbb{E}U_0(A)$$

by the stationarity of $(U_n)_{n\geq 0}$, but in general, the assumption of independent reproduction is not fulfilled. However, once it is known how the environmental conditions change in the course of time, individuals reproduce independently of one another, but in general not in an identical manner. Nevertheless, taking this conditional point of view, individuals within one generation also exhibit the same reproduction mechanism. In other words, after picking the random distributions $(U_n)_{n\geq 0}$, $(Z_n)_{n\geq 0}$ can be viewed as a *weighted branching process in varying environments*. Obviously, the forward equation (1.1.2) can be carried over without changes to the situation of WBPRE. However, to give an analogue of the backward equation (1.1.3), write the conditional distribution of $(Z_n)_{n\geq 0}$ in the form $\mathbb{P}((Z_n)_{n\geq 0} \in \cdot|\mathbf{U}) = \mathbb{L}(\mathbf{U}, \cdot)$ a.s. for some suitable stochastic kernel $\mathbb{L}$. Then (1.1.3) remains true if $(Z_{n,i})_{n\geq 0}$, $i \geq 1$ and $T(\emptyset)$ are supposed to be conditionally independent given $\mathbf{U}$ with $\mathbb{P}((Z_{n,i})_{n\geq 0} \in \cdot|\mathbf{U}) = \mathbb{L}(\mathcal{S}\mathbf{U}, \cdot)$ a.s. for all $i \geq 1$.

By Kolmogorov' zero-one law, any sequence consisting of i.i.d. components is stationary ergodic. In this case, individuals in different generations reproduce independently of each other as we will see in Subsection 2.4.3.

Before giving examples, we turn to the definition of an appropriate σ-field $\mathfrak{M}$ on $\mathbb{M}$.

1.2.2 A suitable σ-algebra on $\mathbb{M}$

We now endow $\mathbb{M}$ with a suitable σ-algebra $\mathfrak{M}$. For this purpose, let $\mathbb{M}'$ be the collection of distributions on $(\mathbb{R}, \mathbb{B})$ and endow $\mathbb{M}'$ with the σ-algebra $\mathfrak{M}'$ generated by the total variation norm $d_{\mathbb{M}'}$ on $\mathbb{M}'$, i.e. by all subsets which are open with respect to the metric

$$d_{\mathbb{M}'}(Q_1', Q_2') := \sup\{|Q_1'(A) - Q_2'(A)| : A \in \mathbb{B}\}.$$

Concerning an appropriate σ-field $\mathfrak{M}$ on $\mathbb{M}$, we then stipulate the following:

M.1 For any $A \in \mathbb{B}^{\mathbb{N}}_{[0,\infty)}$, the *projection* $Q \mapsto Q(A) = \int \mathbf{1}_A \, dQ$ is $\mathfrak{M}$-$\mathbb{B}$-measurable. Note that by the standard extension principle for measurable functions, this implies measurability of the mapping $\mathcal{I}_f : \mathbb{M} \to [0, \infty]$, defined by

$$\mathcal{I}_f(Q) := \int f \, dQ,$$

whenever f is nonnegative and $\mathbb{B}^{\mathbb{N}}_{[0,\infty)}$-$\overline{\mathbb{B}}$-measurable. Using this for $f = \Sigma$ gives measurability of the mapping $Q \mapsto \mu(Q)$.

M.2 The mapping $Q \mapsto \tilde{Q}$ is $\mathfrak{M}$-$\mathfrak{M}'$-measurable, where $\tilde{Q}(\cdot) \in \mathbb{M}'$ is defined by

$$\tilde{Q}(B) := \mu(Q)^{-1} \int_{\{\Sigma \in B\}} \Sigma \, dQ, \quad B \in \mathbb{B}.$$

M.3 The same is true for the mapping $Q \mapsto Q^*$, where $Q^*(\cdot) \in \mathbb{M}'$ is defined by

$$Q^*(B) := \mu(Q)^{-1} \int \left(\sum_{i\geq 1} \pi_i \mathbf{1}_{\{\pi_i \in B\}} \right) dQ, \quad B \in \mathbb{B}.$$

Throughout this thesis, $\pi_i : [0,\infty)^{\mathbb{N}} \to [0,\infty)$ denotes the projection to the ith component, i.e. $\pi_i((x_j)_{j\geq 1}) = x_i$ for $i \geq 1$. In particular, $\Sigma = \sum_{i\geq 1} \pi_i$.

M.1 is obviously a natural condition, whereas the relevance of **M.2** and **M.3** (which seem rather artificial at first glance) will turn out in Section 2.3 where they are used to ensure the measurability of certain auxiliary random variables.

1.2.3 Examples and some further notation

We now briefly describe two special cases of WBPRE. First, we explain the connection to Galton-Watson processes in random environment.

Example 1.2.1. (Galton-Watson processes in random environment)
Suppose that $\mathbb{P}(T_i \in \{0,1\}$ for all $i \geq 1) = 1$, or equivalently, $U_0(\{0,1\}^{\mathbb{N}}) = 1$ a.s.. Since $T(v)$, $v \in \mathcal{N}$ are identically distributed, this yields

$$\mathbb{P}(L(v) \in \{0,1\}\, \forall v \in \mathcal{N}) = 1,$$

and $(Z_n)_{n\geq 0}$ is a *Galton-Watson process in random environment (GWPRE)*. The latter process has been introduced by Smith and Wilkinson [101] in the special situation of i.i.d. random environment and later by Athreya and Karlin [10, 11] in the more general setting of stationary ergodic random environment. Although Galton-Watson processes in random environments are well-studied objects (see for instance [1], [10], [11], [38], [39], [49], [50], [73], [83], [101], [103]–[106]), there seems to be no work on weighted branching processes in random environment, apart from a recent article by Biggins and Kyprianou [26] dealing with branching processes with a general type space. As an application of their investigations, Biggins and Kyprianou sketch a result on *branching random walks in random environment* which confirms the results we obtain in Chapter 2. In this context, it should be mentioned that their article was written independently of and contemporaneously with the article [66] which is actually a prepublication of Chapter 2.

The precise connection of our results to similar results on Galton-Watson processes in random environment will be discussed in detail in Remark 2.4.23 in Subsection 2.4.5.

Obviously, the concept of WBPRE also generalizes that of ordinary WBP as introduced in Section 1.1.

Example 1.2.2. (Ordinary weighted branching processes)
If the sequence $(U_n)_{n\geq 0}$ is deterministic and nonvarying, i.e. $\mathbb{P}(U_0 \in \cdot) = \mathbb{P}(U_1 \in \cdot) = ... = \delta_\Gamma$ for some $\Gamma \in \mathbb{M}$, it is easily checked that $T(v)$, $v \in \mathcal{N}$ are even independent. Hence, $(Z_n)_{n\geq 0}$ forms an ordinary weighted branching process with $\mu = \mathbb{E}Z_1 = \mu(\Gamma) \in (0,\infty)$.

If the conditions of the previous examples hold simultaneously, $(Z_n)_{n\geq 0}$ is a Galton-Watson process with reproduction mean $\mu \in (0,\infty)$ (cf. Example 1.1.1). If there is no branching (as in Example 1.1.2), $(Z_n)_{n\geq 0}$ can plainly be considered as a *multiplicative random walk with stationary ergodic increments.*

We close this subsection with some additional notation and assumptions. Given $\mathbf{Q} = (Q_n)_{n\geq 0} \in \mathbb{M}^{\mathbb{N}_0}$, we abbreviate $\mu_0(\mathbf{Q}) := 1$ and

$$\mu_n(\mathbf{Q}) := \prod_{i=0}^{n-1} \mu(Q_i) \in (0,\infty), \qquad n \geq 1.$$

Generalizing (1.1.5), we put

$$\begin{aligned} c &:= \mathbb{P}\left(T_i = 0 \text{ or } T_i = \mu(U_0) \text{ for all } i \geq 1\right) \\ &= \mathbb{P}\left(T_i/\mu(U_0) \in \{0,1\} \text{ for all } i \geq 1\right). \end{aligned}$$

In addition, we assume that

$$\beta := \mathbb{E}[\log \mu(U_0)] \in (-\infty,\infty) \tag{1.2.1}$$

and that

$$\gamma := \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log \frac{T_i}{\mu(U_0)}\right] \text{ exists in } \overline{\mathbb{R}}.$$

In case

$$\gamma_+ := \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{T_i}{\mu(U_0)}\right] \qquad \text{or} \qquad \gamma_- := \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^- \frac{T_i}{\mu(U_0)}\right]$$

is finite, we have the decomposition $\gamma = \gamma_+ - \gamma_-$. Finally, note that by **M.1** and Proposition 6.31 in [31], the sequence $(\mu(U_n))_{n\geq 0}$ is stationary ergodic as well. Moreover, $\mu(U_n) \in (0,\infty)$ and hence $\mu_n(\mathbf{U}) \in (0,\infty)$ a.s. for each $n \geq 0$ by (1.2.1).

1.2.4 The intrinsic martingale

Concerning WBPRE as just introduced, our main object of interest is the sequence $(W_n)_{n\geq 0}$, defined by

$$W_n := Z_n/\mu_n(\mathbf{U}), \qquad n \geq 0.$$

We will see in Lemma 1.2.3 below that $(W_n)_{n\geq 0}$ forms a nonnegative martingale with respect to an appropriate filtration $(\mathcal{G}_n)_{n\geq 0}$ of $(\Omega, \mathfrak{A}, \mathbb{P})$ and therefore converges a.s. to a nonnegative random variable W satisfying $\mathbb{E}W \leq 1$.
Informally speaking, it is natural to ask for necessary and sufficient conditions for

$$Z_n \approx \mu_n(\mathbf{U}) \text{ as } n \to \infty.$$

More precisely, our aim is to determine when $\mu_n(\mathbf{U})$, the conditional mean of Z_n given $\mathbf{U}$, appropriately describes the growth of Z_n as $n \to \infty$ in the sense that $W = \lim_{n\to\infty} W_n$ is nondegenerate, i.e. satisfies $\mathbb{P}(W > 0) > 0$, or even fulfills $\mathbb{E}W = 1$. This problem will be treated in Chapter 2 which is based on the recent article [66]. A more explicit motivation and relevant references can be found in Section 2.1.

Introducing the filtration $(\mathcal{G}_n)_{n\geq 0}$, given by $\mathcal{G}_0 := \sigma(\mathbf{U})$ and $\mathcal{G}_n := \sigma(\mathbf{U}, T(v), |v| \leq n-1)$ for $n \geq 1$, we can easily establish the a.s. convergence of $(W_n)_{n\geq 0}$. As in Section 1.1, it is easily checked that $\mathcal{G}_n = \sigma(\mathbf{U}, \mathcal{F}_n)$ satisfies $\mathcal{G}_n \supset \sigma(\mathbf{U}, (L(v))_{|v|\leq n})$.

Lemma 1.2.3. *(a) Given any $w \in \mathcal{N}$, $\mathbb{E}\left(\sum_{i\geq 1} T_i(w)|\mathbf{U}\right) = \mu(U_{|w|})$ a.s..*

(b) $\mathbb{E}(Z_n|\mathbf{U}) = \mu_n(\mathbf{U})$ a.s. for all $n \geq 0$.

(c) The sequence $(W_n)_{n\geq 0}$ forms a nonnegative martingale with respect to $(\mathcal{G}_n)_{n\geq 0}$ with $\mathbb{E}W_0 = 1$ and therefore converges a.s. to a nonnegative random variable W satisfying $\mathbb{E}W \leq 1$.

Proof. (a) By construction, we have that

$$\begin{aligned} \mathbb{E}\left(\sum_{i\geq 1} T_i(w)\middle|\mathbf{U}\right) &= \mathbb{E}(\Sigma \circ T(w)|\mathbf{U}) \\ &= \int_{[0,\infty)^{\mathbb{N}}} \Sigma \, d\mathbb{P}^{T(w)|\mathbf{U}} = \int_{[0,\infty)^{\mathbb{N}}} \Sigma \, dU_{|w|} = \mu(U_{|w|}) \quad \text{a.s..} \end{aligned}$$

(b) We prove the claim by induction. As $Z_0 = \mu_0(\mathbf{U}) = 1$ a.s., we may suppose that $\mathbb{E}(Z_n|\mathbf{U}) = \mu_n(\mathbf{U})$ a.s. for some $n \geq 0$. Then using the forward equation

$$Z_{n+1} = \sum_{|v|=n+1} L(v) = \sum_{|w|=n} L(w) \sum_{i\geq 1} T_i(w) = \sum_{|w|=n} L(w) \cdot \Sigma \circ T(w),$$

the conditional independence of $L(w)$ and $\Sigma \circ T(w)$, the monotone convergence theorem and part (a), we infer that

$$\begin{aligned}
\mathbb{E}(Z_{n+1}|\mathbf{U}) &= \sum_{|w|=n} \mathbb{E}(L(w) \cdot \Sigma \circ T(w)|\mathbf{U}) \\
&= \sum_{|w|=n} \mathbb{E}(L(w)|\mathbf{U}) \cdot \mathbb{E}(\Sigma \circ T(w)|\mathbf{U}) \\
&\overset{(a)}{=} \mu(U_n) \cdot \sum_{|w|=n} \mathbb{E}(L(w)|\mathbf{U}) \\
&= \mu(U_n) \cdot \mathbb{E}(Z_n|\mathbf{U}) \\
&= \mu(U_n) \cdot \mu_n(\mathbf{U}) = \mu_{n+1}(\mathbf{U}) \quad \text{a.s.},
\end{aligned}$$

as demanded.

(c) Fix $n \geq 0$. Obviously, W_n is measurable with respect to $\mathcal{G}_n$. Since $L(w)$ is $\mathcal{G}_n$-measurable if $|w| = n$, it follows that

$$\begin{aligned}
\mathbb{E}(W_{n+1}|\mathcal{G}_n) &= \mu_{n+1}(\mathbf{U})^{-1}\mathbb{E}\left(\sum_{|v|=n+1} L(v)\middle|\mathcal{G}_n\right) \\
&= \mu_{n+1}(\mathbf{U})^{-1} \sum_{|w|=n} \mathbb{E}\left(L(w) \cdot \Sigma \circ T(w)|\mathcal{G}_n\right) \\
&= \mu_{n+1}(\mathbf{U})^{-1} \sum_{|w|=n} L(w)\mathbb{E}\left(\Sigma \circ T(w)|\mathcal{G}_n\right) \\
&\overset{(\star)}{=} \mu_{n+1}(\mathbf{U})^{-1}\mu(U_n) \sum_{|w|=n} L(w) \\
&= \mu_n(\mathbf{U})^{-1} Z_n = W_n \quad \text{a.s.}.
\end{aligned}$$

To justify $(\star)$, observe that for all $A \in \mathfrak{M}^{\mathbb{N}_0}$ and $B \in \bigotimes_{v\in\mathcal{N},\,|v|\leq n-1} \mathbb{B}^{\mathbb{N}}$,

$$\begin{aligned}
&\int_{\{\mathbf{U}\in A,(T(v))_{|v|\leq n-1}\in B\}} \Sigma \circ T(w)\, d\mathbb{P} \\
&= \int_{\{\mathbf{U}\in A\}} \mathbb{E}\left(\mathbf{1}_{\{(T(v))_{|v|\leq n-1}\in B\}} \cdot \Sigma \circ T(w)\middle|\mathbf{U}\right) d\mathbb{P} \\
&= \int_{\{\mathbf{U}\in A\}} \mu(U_n) \cdot \mathbb{P}\left((T(v))_{|v|\leq n-1} \in B\middle|\mathbf{U}\right) d\mathbb{P}
\end{aligned}$$

$$= \int_{\{\mathbf{U}\in A\}} \mathbb{E}\left(\mu(U_n)\mathbf{1}_{\{(T(v))_{|v|\le n-1}\in B\}}\middle|\mathbf{U}\right)\,d\mathbb{P}$$
$$= \int_{\{\mathbf{U}\in A,(T(v))_{|v|\le n-1}\in B\}} \mu(U_n)\,d\mathbb{P},$$

for the random variables $\Sigma \circ T(w)$ and $(T(v))_{|v|\le n-1}$ are conditionally independent with $\mathbb{E}(\Sigma \circ T(w)|\mathbf{U}) = \mu(U_n)$ a.s. as seen in (a). Since $W_0 = 1$ we can finish the proof by applying the martingale convergence theorem and Fatou's lemma.

□

In particular, the normalization $W_n = Z_n/\mu^n$, $n \ge 0$ arising in ordinary weighted branching processes with reproduction mean $\mu = \mathbb{E}Z_1 \in (0,\infty)$ leads in fact to a martingale which converges a.s. to some limit W satisfying $\mathbb{E}W \le 1$. To be rigorous, we put $W = \limsup_{n\to\infty} W_n$ whenever W has to be defined on the entire probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Chapter 2

Limit theorems for normalized weighted branching processes in random environment

This chapter comprises the contents of the article [66] and gives an almost comprehensive treatment of the problem indicated at the end of the previous chapter. As far as we know (and as already mentioned in Example 1.2.1), the only further result on weighted branching processes in random environment is stated by Biggins and Kyprianou in their article [26], confirming our results (under some additional assumptions).

Unless stated otherwise, suppose that $(Z_n)_{n\geq 0}$ forms a WBPRE with environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$ satisfying $\beta = \mathbb{E}\log\mu(U_0) \in (-\infty, \infty)$.

2.1 The problem

Given a supercritical Galton-Watson process $(Z_n')_{n\geq 0}$ with reproduction mean $\mu' \in (1, \infty)$ and a.s. martingale limit W', a well-known result going back to Kesten and Stigum (cf. Theorem II.2.1 in [7] or Theorem I.10.1 in [12]) says that the martingale $(Z_n'/\mu'^n)_{n\geq 0}$ is uniformly integrable if and only if the so-called $Z\log Z$*-condition* is fulfilled, i.e. if $\mathbb{E}Z_1' \log^+ Z_1' < \infty$. In other words, $\mathbb{E}W' = 1$ iff $\mathbb{E}Z_1' \log^+ Z_1' < \infty$, and $\mathbb{E}W = 0$ otherwise. Even in essentially more general branching models, similar conditions have turned out to be crucial for the nondegeneracy of the corresponding martingale limit as long as individuals behave independently of one another and follow the same reproduction mechanism. For instance, we mention results by Doney [41, 42] on *Bellman-Harris age-dependent* *processes* and *Crump-Mode processes* and *age-dependent birth and death processes*, and by Jagers [60] and Olofsson [90] on so-called *general branching processes*. The $Z\log Z$-condition is also of great importance for the articles [87] by Nerman and [61] by Jagers and Nerman on general *Crump-Mode-Jagers processes*.

If in the framework of Galton-Watson processes in random environment, the hypothesis of independent reproduction is given up and replaced by conditional independence given the environmental sequence, the $Z\log Z$-condition still guarantees uniform integrability of the intrinsic martingale if the environmental sequence is stationary ergodic. This has been shown by Tanny [106] when an additional assumption holds, and by Lyons et al. [83] in full generality. However, Tanny has given an example which proves that in this case, the $Z\log Z$-condition is not *necessary*, i.e. is not implied by uniform integrability. However, if the environmental sequence is assumed to consist of independent components, it is also due to Tanny [106] that the $Z\log Z$-condition does in fact characterize uniform integrability.
Turning to WBPRE, these circumstances motivate the following since they suggest to search for necessary and sufficient conditions for the martingale $W_n = Z_n/\mu_n(\mathbf{U})$, $n \geq 0$ from Chapter 1 to be uniformly integrable. As is well-known from probability theory, this is tantamount with analysing when the a.s. martingale limit W has mean 1, i.e. when W_n converges to W in $\mathfrak{L}_1$. In terms of the underlying branching model, the condition $\mathbb{E}W = 1$ (or, more generally the condition $\mathbb{P}(W > 0) > 0$) means that the sequence $\mu_n(\mathbf{U}) = \mathbb{E}(Z_n|\mathbf{U})$ of conditional means is a proper (random) normalization of the branching process $(Z_n)_{n\geq 0}$, at least on a set of positive probability. In other words, on the event $\{W > 0\}$, the rate of growth of the considered population is roughly $\mu_n(\mathbf{U})$, whereas on $\{W = 0\}$, Z_n grows essentially slower (or shrinks faster) than $\mu_n(\mathbf{U})$, or even dies out.

It will turn out in Section 2.4 that in contrast to Galton-Watson processes in random environment, there are essentially two assumptions which have to be imposed to ensure uniform integrability. The first condition can be viewed as a *generalized* $Z\log Z$*-condition* and is expressed in terms of a certain random series which actually depends on the underlying process only via the sum $\sum_{i\geq 1} T_i(v)/\mu(U_{|v|})$ of weights related to each individual $v \in \mathcal{N}$. When dealing with Galton-Watson processes in random environment, this condition has been shown to be necessary and sufficient for uniform integrability (cf. [83] and [106]). But in the much more general context we consider, a second assumption which depends on the entire sequence $(T_i(\emptyset)/\mu(U_0))_{i\geq 1}$ is also needed. This condition is automatically satisfied and therefore superfluous in the case of Galton-Watson processes in

random environment. Moreover, we can show that the first condition is implied by the $Z \log Z$-condition (which reads $\mathbb{E}[\mu(U_0)^{-1} Z_1 \log^+ Z_1] < \infty$ in this case), thus justifying the notion *generalized* $Z \log Z$*-condition*. If the environmental sequence $\mathbf{U}$ has independent components, both conditions are equivalent, whereas in the general case of stationary ergodic environment, the $Z \log Z$-condition is a strictly stronger assumption.
The further organization of the chapter is as follows. The subsequent section presents the method of proof which forms the fundament of the rest of this chapter. The approach we use has been heavily inspired by the articles by Lyons et al. [83] and Lyons [82] who gave probabilistic proofs of well-known limit theorems on Galton-Watson processes and branching random walks, replacing the purely analytic approaches which had been preferred in the past. Section 2.3 provides some additional lemmata which furnish the proofs of our main results. These are stated and proved in Section 2.4 which also gives an instructive counterexample showing that in stationary ergodic random environment, the $Z \log Z$-condition is in general not necessary for $\mathfrak{L}_1$-convergence. Furthermore, we compare our results to former results in the literature on Galton-Watson processes and Galton-Watson processes in random environment.

2.2 The method of proof

The approach we follow is based on viewing a weighted branching process as the sequence of generation sizes of an appropriate weighted random family tree and involves modern techniques utilizing the construction of size-biased measures on the space of marked trees generated by the underlying process. In other words, the basic idea expresses itself in a change of measure on the space of marked trees which makes it possible to restrict to purely probabilistic arguments instead of analytic considerations. This concept is called *size-biased method* or *spinal tree method* which aims at the fact that in the construction of the size-biased tree a randomly chosen line of descent called *spine* occurs. It goes back to Lyons et al. who used it to reprove a variety of classical limit theorems for Galton-Watson processes (not only restricting to supercritical ones) in their article [83]. As an important tool, they apply limit theorems for GWP allowing *immigration* which can be established by standard techniques of general probability theory. However, Lyons et al. assert that a related construction is due to Waymire and Williams [108] who studied a somewhat different scenery.

The application of this method to more general classes of branching processes was

pioneered by Lyons in his article [82] which contains a new probabilistic proof of Biggins' martingale convergence theorem for branching random walks, improving on the analytic version given in [17]. It should be emphasized that the construction used in this chapter is an adaption of that sketched by Lyons in [82] to the more general situation where environmental changes are allowed. Thus, our results generalize [82], and his main result will be obtained as a by-product in Subsection 2.4.4.
Modifications of the size-biased tree method have been developed and applied by Kurtz et al. [67] for multi-type Galton-Watson branching processes, and by Olofsson [90] for so-called *general branching processes* who also used corresponding processes allowing immigration as a key tool. These results have been extended by Athreya [9] to a Kesten-Stigum-type result in a Markov chain context which made the method applicable to supercritical measure-valued branching processes on a general type space, particularly comprising multi-type Galton-Watson processes and single-type Bellman-Harris processes. In addition, the spinal tree method enabled Kyprianou and Rahimzadeh Sani [69] to prove a generalization of Biggins' martingale convergence theorem for multi-type branching random walks. The approach has also been fruitfully used by Biggins and Kyprianou [25] in the context of stopped branching random walks by establishing a relationship between stopping lines in the corresponding family tree and certain stopping times which also requires a change of measure argument for the space of marked trees. The same authors indicate further refinements and applications of the underlying method in the article [26] which is closely related to Athreya's article [9]. Recently, the change of measure arguments pioneered by Lyons et al. have been utilized by Iksanov [58] in the study of fixed points of the so-called *branching random walk transform* when the number of children is not necessarily finite. Moreover, the robustness of the method is stressed by another application outlined by Kyprianou [68] who studied travelling-wave solutions of the so-called *Kolmogorov-Petrovskii-Piskounov equation.*

Finally, let us mention that a somewhat different construction has been used by Geiger [48] to give probabilistic proofs of well-known conditional limit theorems for critical and subcritical Galton-Watson processes.

2.2.1 Weighted trees

We have already mentioned that our method of proof is based on analysing the weighted family tree associated with $(Z_n)_{n\geq 0}$. For this reason, we formally introduce the space of weighted trees in this section and endow it with an appropriate σ-algebra, following

Chauvin's and Neveu's approaches (see [36], [88]).

Denote by $\mathbb{T}$ the set of all nonnegative mappings defined on $\mathcal{N}$, i.e.

$$\mathbb{T} := \{t | \, t : \mathcal{N} \to [0, \infty)\}.$$

Any $t \in \mathbb{T}$ is called a *weighted tree* or *marked tree* and will be identified with the family $(t(v))_{v \in \mathcal{N}}$. Additionally, given any $v \in \mathcal{N}$, define the projection $\ell_v : \mathbb{T} \to [0, \infty)$, $\ell_v(t) := t(v)$. The filtration $(\mathcal{T}_n)_{n \geq 0}$, given by

$$\mathcal{T}_n := \sigma(\ell_v, |v| \leq n) \qquad (n \geq 0),$$

will play an important role in our analysis. Finally, endow $\mathbb{T}$ with the σ-algebra

$$\mathcal{T} := \sigma(\cup_{n \geq 0} \mathcal{T}_n) = \sigma(\ell_v, v \in \mathcal{N}).$$

To get the connection to the weighted branching process $(Z_n)_{n \geq 0}$ introduced in Chapter 1, we consider the random tree

$$L = (L(v))_{v \in \mathcal{N}}$$

which is obviously $\mathcal{T}$-measurable because each $L(v)$ is a random variable.

2.2.2 Size-biased weighted trees

We now explain in detail the aforementioned change of measure on the space of marked trees under consideration. The proofs of our main results exploit a fundamental relation between the random weighted tree $L = (L(v))_{v \in \mathcal{N}}$ and the so-called *size-biased tree* $\hat{L} = (\hat{L}(v))_{v \in \mathcal{N}}$ to be defined hereafter. Lemma 2.2.2 compares the distributions of L and $\hat{L}$, showing that on $\mathcal{T}_n$, the σ-algebra generated by the first n levels of trees, the distribution of $\hat{L}$ is dominated by the probability measure $\mathbb{E} W_n \mathbf{1}_{\{L \in \cdot\}}$. As in the articles [82] and [83], this allows us to apply a measure-theoretic result due to Durrett [43] which provides an essential dichotomy for the martingale limit W.

However, we start with the formal construction. Let

$$((\hat{T}(n), C_n))_{n \geq 0} = (((\hat{T}_i(n))_{i \geq 1}, C_n))_{n \geq 0}$$

be a sequence of random variables which are defined on $(\Omega, \mathfrak{A}, \mathbb{P})$, take values in $[0, \infty)^{\mathbb{N}} \times \mathbb{N}$ and meet the following conditions:

- Conditionally upon $\mathbf{U}$, $(T(v))_{v \in \mathcal{N}}$ and $((\hat{T}(n), C_n))_{n \geq 0}$ are independent.

- Conditionally upon $\mathbf{U}$, the random variables $(\hat{T}(n), C_n), n \geq 0$ are independent with conditional distributions determined by

$$\begin{aligned}
&\mathbb{P}(\hat{T}(n) \in A, C_n = i|\mathbf{U}) \\
&= \mu(U_n)^{-1}\mathbb{E}\left(\mathbf{1}_{\{T(\mathbf{v}_n)\in A\}}T_i(\mathbf{v}_n)|\mathbf{U}\right) \qquad (2.2.1)\\
&= \mu(U_n)^{-1}\int_A \pi_i \, dU_n \quad \text{a.s.} \qquad (A \in \mathbb{B}^{\mathbb{N}}, i \geq 1),
\end{aligned}$$

where $\mathbf{v}_n$ is an arbitrary element of $\mathbb{N}^n$ and $\pi_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ denotes the projection to the ith coordinate. Note that this implies

$$\begin{aligned}
\mathbb{P}(\hat{T}(n) \in A|\mathbf{U}) &= \mu(U_n)^{-1}\mathbb{E}\left(\mathbf{1}_{\{T(\mathbf{v}_n)\in A\}} \cdot \Sigma \circ T(\mathbf{v}_n)|\mathbf{U}\right) \qquad (2.2.2)\\
&= \mu(U_n)^{-1}\int_A \Sigma \, dU_n \quad \text{a.s.} \qquad (A \in \mathbb{B}^{\mathbb{N}}, |\mathbf{v}_n| = n),
\end{aligned}$$

$\mathbb{P}(C_n = i|\mathbf{U}) = \mu(U_n)^{-1}\mathbb{E}(T_i(\mathbf{v}_n)|\mathbf{U})$ a.s. $(i \geq 1, |\mathbf{v}_n| = n)$, and

$$\mathbb{P}\left(0 < \sum_{i\geq 1} \hat{T}_i(n) < \infty\right) = 1 \qquad (2.2.3)$$

because $\mathbb{P}(\hat{T}(n) \in \Sigma^{-1}((0,\infty))|\mathbf{U}) = \mu(U_n)^{-1}\int_{\Sigma^{-1}((0,\infty))} \Sigma \, dU_n = 1$ a.s..

Now define the random variables $V_0 := (\emptyset)$, $\hat{L}(\emptyset) = 1$, $V_n := (C_0, ..., C_{n-1}) \in \mathbb{N}^n$ for $n \geq 1$, and

$$\hat{L}(w,i) := \begin{cases} \hat{L}(V_n) \cdot \hat{T}_i(n), & w = V_n \text{ for some } n \geq 0, \\ \hat{L}(w) \cdot T_i(w), & \text{otherwise} \end{cases}$$

for all $w \in \mathcal{N}$ and $i \geq 1$. The sequence $(V_n)_{n\geq 0}$ is a randomly chosen line of descent in the size-biased tree and called *spine*. After these preliminaries, we note that the size-biased random tree $\hat{L} = (\hat{L}(v))_{v\in\mathcal{N}}$ is $\mathcal{T}$-measurable as well. Figure 2.1 below shows a typical realization of the first generations (denoted $\hat{\mathbf{G}}_0, ..., \hat{\mathbf{G}}_3$) and gives a graphical explanation of the construction. Individuals are enumerated in the same way as in the original tree. To explain the notation, observe that since the original random weighted tree $L = (L(v))_{v\in\mathcal{N}}$ takes values in the polish space $\mathbb{R}^{\mathcal{N}}$, its conditional distribution given $\mathbf{U}$ can be written in the form $\mathbb{P}(L \in \cdot|\mathbf{U}) = \mathbb{K}(\mathbf{U}, \cdot)$ a.s. for some stochastic kernel $\mathbb{K}$. Then for any $v \in \mathcal{N}$ off the spine, $L^{(v)}$ is a random weighted tree with conditional distribution $\mathbb{K}(\mathcal{S}^{|v|}\mathbf{U}, \cdot)$ which is conditionally independent of the reproduction of any individual not stemming from v. Recall that $\mathcal{S}$ is the shift operator defined in Chapter 1. The nodes V_n belonging to the spine are indicated by the symbol ■, whereas the symbol • stands for individuals not belonging to the spine. The interpretation of the sequence $(V_n)_{n\geq 0}$ as a spine or backbone is indicated by the broken lines. Here, $C_0 = 2, C_1 = 3$ and $C_2 = 1$.

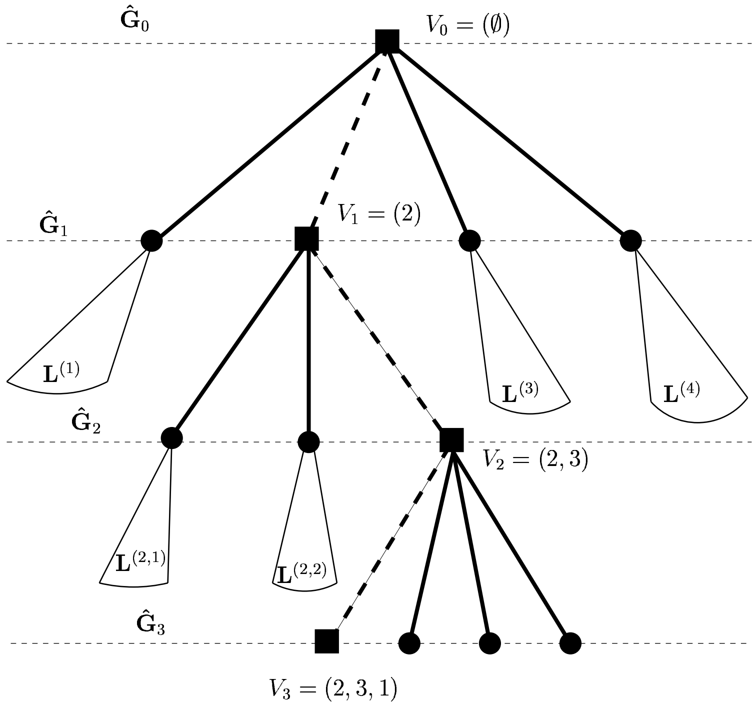

Fig. 2.1: A typical realization of the size-biased tree

An informal illustration of the construction just given is the following:

— Again, the population consists of merely one individual $(\emptyset)$ at the beginning of the process, and this element is the ancestor of the spinal sequence $(V_n)_{n\geq 0}$, i.e. $V_0 = (\emptyset)$ with weight 1.

— The first generation of the size-biased tree is made up of individuals with weights $\hat{T}_1(0), \hat{T}_2(0), \ldots$. Among these individuals, choose $V_1 = (C_0)$ according to the distribution given above. In particular, $\hat{L}(V_1) = \hat{T}_{C_0}(0)$.

— From now on, each individual V_n $(n \geq 1)$ on the spine generates offspring with weights $\hat{L}(V_n) \cdot \hat{T}_1(n), \hat{L}(V_n) \cdot \hat{T}_2(n), \ldots$, and among these children, V_{n+1} is picked via $V_{n+1} = (V_n, C_n)$ and has weight $\hat{L}(V_{n+1}) = \prod_{j=0}^{n} \hat{T}_{C_n}(n)$.

— Any individual $v = (v_1, \ldots, v_m)$ off the spine has a descendant tree $\mathbf{L}^{(v)}$ which has the conditional distribution $\mathbb{K}(\mathcal{S}^m\mathbf{U}, \cdot)$ defined before. This means that any individual $\tilde{v} = (v_1, \ldots, v_m, v_1', \ldots, v_r')$ in the size-biased tree which stems from v has weight

$\hat{L}(\tilde{v}) = \hat{L}(v) \cdot T_{v_1'}(v) \cdot ... \cdot T_{v_r'}(v_1, ..., v_m, v_1', ..., v_{r-1}')$. On the event $\{L(v) > 0\}$, $\hat{L}(\tilde{v})$ can be rewritten as $\hat{L}(\tilde{v}) = \hat{L}(v) \cdot L(\tilde{v})/L(v)$.

— We emphasize that conditionally upon $\mathbf{U}$, all individuals reproduce independently of one another. Particularly, the family $(\mathbf{L}^{(v)})_{v\in\mathcal{N}}$ is conditionally independent of $(\hat{T}(n), C_n)_{n\geq 0}$, and if v, w do not belong to the same line of descent (i.e. neither v stems from w nor vice versa), then $\mathbf{L}^{(v)}$ and $\mathbf{L}^{(w)}$ are also conditionally independent.

Remark 2.2.1. Given the situation of ordinary weighted branching processes with reproduction mean μ (where $U_0 = U_1 = ... = \Gamma$ a.s. for some $\Gamma \in \mathbb{M}$), it follows that the random variables $(\hat{T}(n), C_n)$, $n \geq 0$ are i.i.d. with distribution

$$\mathbb{P}(\hat{T}(0) \in \cdot, C_0 = i) = \mu^{-1}\mathbb{E}(\mathbf{1}_{\{T\in\cdot\}}T_i), \qquad i \geq 1$$

(cf. [82]), in particular $\mathbb{P}(\hat{T}(0) \in \cdot) = \mu^{-1}\mathbb{E}(\mathbf{1}_{\{T\in\cdot\}}Z_1)$, i.e. $\frac{d\mathbb{P}^{\hat{T}(0)}}{d\mathbb{P}^T} = \frac{\Sigma}{\mu}$. This implies that for all $i \geq 1$,

$$\mathbb{P}\left(C_0 = i \middle| \hat{T}(0)\right) = \frac{\hat{T}_i(0)}{\sum_{j\geq 1}\hat{T}_j(0)} =: \hat{D}_i \quad \text{a.s.}$$

because for any $A \in \mathbb{B}^{\mathbb{N}}$,

$$\int_{\{\hat{T}(0)\in A\}} \hat{D}_i \, d\mathbb{P} = \int_A \frac{\pi_i}{\Sigma} \, d\mathbb{P}^{\hat{T}(0)} = \int_A \frac{\pi_i}{\Sigma} \cdot \frac{\Sigma}{\mu} \, d\mathbb{P}^T = \int_{\{T\in A\}} \frac{T_i}{\mu} \, d\mathbb{P} = \mathbb{P}(\hat{T}(0) \in A, C_0 = i).$$

In terms of the underlying construction, this actually says that once the first generation of the size-biased tree is born, the random spinal element is chosen according to the (conditional) probability distribution $(\hat{D}_1, \hat{D}_2, ...)$ on the set of positive integers. $\hat{D}_i$ gives nothing but the weight contributed by individual i relative to the size of the entire first generation.

In the remainder of this chapter we shall make frequent use of the relation between

$$\hat{\mathcal{Q}} := \mathbb{P}((\hat{L}, \mathbf{U}) \in \cdot) \quad \text{and} \quad \mathcal{Q} := \mathbb{P}((L, \mathbf{U}) \in \cdot)$$

described in Lemma 2.2.2 below. For this purpose, put

$$z_n(t) := \sum_{|v|=n} t(v) \qquad (t \in \mathbb{T}, n \geq 0),$$
$$w_n(t, u) := z_n(t)/\mu_n(u) \qquad (t \in \mathbb{T}, u \in \mathbb{M}^{\mathbb{N}_0}, n \geq 0),$$

$$\begin{aligned} w &:= \limsup_{n\to\infty} w_n, \\ \hat{W}_n &:= w_n \circ (\hat{L}, \mathbf{U}), \quad \hat{Z}_n := z_n \circ \hat{L}, \\ \mathcal{T}_n' &:= \ \mathcal{T}_n \otimes \mathfrak{M}^{\mathbb{N}_0} \end{aligned}$$

and note that w_n is $\mathcal{T}_n'$-measurable ($n \geq 0$). In addition, we have the representations

$$W = w \circ (L, \mathbf{U}) \text{ and } W_n = w_n \circ (L, \mathbf{U}).$$

The subsequent lemma reveals that size-biasing along the spine has the effect that for each $n \geq 0$, the new probability measure $\hat{\mathcal{Q}}_{|\mathcal{T}_n'}$ is dominated by the original measure $\mathcal{Q}_{|\mathcal{T}_n'}$ and has Radon-Nikodym derivative w_n. Consequently, the problem of $\mathfrak{L}_1$-convergence of $(W_n)_{n\geq 0}$ is transformed to studying the asymptotic behaviour of the sequence $(w_n)_{n\geq 0}$ under the new measure because w_n is measurable with respect to $\mathcal{T}_n'$. The final part of the lemma applies a measure-theoretic result by Durrett [43] to obtain the crucial dichotomy on the asymptotic behaviour of $(W_n)_{n\geq 0}$.

Lemma 2.2.2. *Let $n \geq 0$.*

(a) For any $A \in \mathcal{T}_n$, we have the identity

$$\mathbb{P}(\hat{L} \in A|\mathbf{U}) = \mathbb{E}(W_n \cdot \mathbf{1}_{\{L\in A\}}|\mathbf{U}) \quad a.s..$$

(b) For any $B \in \mathcal{T}_n'$,

$$\hat{\mathcal{Q}}(B) = \mathbb{E}[W_n \cdot \mathbf{1}_{\{(L,\mathbf{U})\in B\}}] = \int_B w_n(t,u)\, \mathcal{Q}(dt, du),$$

i.e. $\hat{\mathcal{Q}}_{|\mathcal{T}_n'} \ll \mathcal{Q}_{|\mathcal{T}_n'}$ with $\frac{d\hat{\mathcal{Q}}_{|\mathcal{T}_n'}}{d\mathcal{Q}_{|\mathcal{T}_n'}}(t,u) = w_n(t,u)$.

(c) We have the dichotomy

(i) $\hat{\mathcal{Q}}(w < \infty) = 1 \Longleftrightarrow \mathbb{E}W = 1$ and

(ii) $\hat{\mathcal{Q}}(w = \infty) = 1 \Longleftrightarrow \mathcal{Q}(w = 0) = 1$.

Proof. (a) Obviously, it suffices to show that for all $n \geq 0$ and $A_v \in \mathbb{B}$ ($|v| \leq n$),

$$\mathbb{P}(\hat{L}(v) \in A_v \ \forall |v| \leq n|\mathbf{U}) = \mathbb{E}\left(\mathbf{1}_{\{\hat{L}(v)\in A_v \ \forall |v|\leq n\}} W_n|\mathbf{U}\right) \quad \text{a.s..}$$

For this purpose, we introduce the auxiliary random variables $\tilde{T}(v)$, $v \in \mathcal{N}$, defined by

$$\tilde{T}(v) := \begin{cases} \hat{T}(n), & v = V_n \text{ for some } n \geq 0, \\ T(v), & \text{otherwise.} \end{cases}$$

Since $\hat{L}(\emptyset) = L(\emptyset) = 1$, we may suppose $n \geq 1$. Considering the representations

$$(\hat{L}(v))_{|v| \leq n} = \Psi \circ (\tilde{T}(v))_{|v| \leq n-1}$$

and

$$(L(v))_{|v| \leq n} = \Psi \circ (T(v))_{|v| \leq n-1}$$

for some appropriate measurable mapping $\Psi : \times_{v \in \mathcal{N}, |v| \leq n-1} \mathbb{R}^{\mathbb{N}} \to \times_{v \in \mathcal{N}, |v| \leq n} \mathbb{R}$, it is enough to prove that for all $B_v \in \mathbb{B}^{\mathbb{N}}$ $(|v| \leq n-1)$,

$$\mathbb{P}(\tilde{T}(v) \in B_v \ \forall |v| \leq n-1 | \mathbf{U}) = \mathbb{E}\left(\mathbf{1}_{\{T(v) \in B_v \ \forall |v| \leq n-1\}} W_n | \mathbf{U}\right) \quad \text{a.s..} \tag{2.2.4}$$

Choose $\sigma \in \mathbb{N}^n$. We claim (and prove by induction) that

$$\begin{aligned} & \mathbb{P}(\tilde{T}(v) \in B_v \ \forall |v| \leq n-1, \, V_n = \sigma | \mathbf{U}) \\ = \ & \mu_n(\mathbf{U})^{-1} \mathbb{E}\left(\mathbf{1}_{\{T(v) \in B_v \ \forall |v| \leq n-1\}} L(\sigma) | \mathbf{U}\right) \quad \text{a.s..} \end{aligned} \tag{2.2.5}$$

Once this identity is verified, summation over all $\sigma \in \mathbb{N}^n$ yields (2.2.4).

First note that the case $n = 1$ in (2.2.5) is obviously true because

$$\begin{aligned} \mathbb{P}(\tilde{T}(\emptyset) \in B, \, V_1 = \sigma | \mathbf{U}) &= \mathbb{P}(\hat{T}(0) \in B, C_0 = \sigma | \mathbf{U}) \\ &= \mu(U_0)^{-1} \mathbb{E}\left[T_\sigma \cdot \mathbf{1}_{\{T \in B\}} | \mathbf{U}\right] \quad \text{a.s.} \end{aligned}$$

for any $B \in \mathbb{B}$ and $\sigma \geq 1$. Then fix n and suppose that (2.2.5) is proved for all $B_v \in \mathbb{B}^{\mathbb{N}}$ $(|v| \leq n-1)$ and all $\sigma \in \mathbb{N}^n$. Let $\tau = (i_0, ..., i_n) \in \mathbb{N}^{n+1}$, put $\sigma := (i_0, ..., i_{n-1})$ and observe that $L(\tau) = L(\sigma) \cdot T_{i_n}(\sigma)$, and $V_{n+1} = \tau$ iff $V_n = \sigma$ and $C_n = i_n$. Then it follows by hypothesis and construction that

$$\begin{aligned} & \mathbb{P}(\tilde{T}(v) \in B_v \ \forall |v| \leq n, \, V_{n+1} = \tau | \mathbf{U}) \\ = \ & \mathbb{P}(\tilde{T}(v) \in B_v \ \forall |v| \leq n-1, \, V_n = \sigma | \mathbf{U}) \\ & \quad \cdot \mathbb{P}(T(v) \in B_v \ \forall |v| = n, v \neq \sigma | \mathbf{U}) \cdot \mathbb{P}(\hat{T}(n) \in B_\sigma, C_n = i_n | \mathbf{U}) \\ = \ & \mu_n(\mathbf{U})^{-1} \mathbb{E}\left[L(\sigma) \cdot \mathbf{1}_{\{T(v) \in B_v \ \forall |v| \leq n-1\}} | \mathbf{U}\right] \cdot \mathbb{E}\left[\mathbf{1}_{\{T(v) \in B_v \ \forall |v| = n, v \neq \sigma\}} | \mathbf{U}\right] \\ & \quad \cdot \mu(U_n)^{-1} \mathbb{E}\left[T_{i_n}(\sigma) \mathbf{1}_{\{T(\sigma) \in B_\sigma\}} | \mathbf{U}\right] \end{aligned}$$

$$= \mu_{n+1}(\mathbf{U})^{-1}\mathbb{E}\left[\mathbf{1}_{\{T(v)\in B_v\ \forall|v|\le n\}}\cdot L(\sigma)T_{i_n}(\sigma)|\mathbf{U}\right]$$
$$= \mu_{n+1}(\mathbf{U})^{-1}\mathbb{E}\left[\mathbf{1}_{\{T(v)\in B_v\ \forall|v|\le n\}}\cdot L(\tau)|\mathbf{U}\right] \quad \text{a.s.},$$

where the first identity holds in view of the conditional independence of the random variables $((\tilde{T}(v))_{|v|\le n-1}, V_n)$, $(T(v))_{|v|=n,v\ne\sigma}$ and $(\hat{T}(n), C_n)$. The inductive hypothesis and (2.2.1) have been used in the second equation, while the third identity is once more due to conditional independence.

The proof of (a) is now complete.

(b) Without loss of generality, we may suppose that B is of the form $B = C \times D$ for some $C \in \mathcal{T}_n$ and $D \in \mathfrak{M}^{\mathbb{N}_0}$. Now (a) gives

$$\begin{aligned}\hat{\mathcal{Q}}(B) &= \int_{\{\mathbf{U}\in D\}} \mathbb{P}(\hat{L}\in C|\mathbf{U})\,d\mathbb{P}\\ &= \int_{\{\mathbf{U}\in D\}} \mathbb{E}(W_n\cdot \mathbf{1}_{\{L\in C\}}|\mathbf{U})\,d\mathbb{P}\\ &= \mathbb{E}[W_n\cdot \mathbf{1}_{\{(L,\mathbf{U})\in C\times D\}}]\\ &= \int_B w_n(t,u)\,\mathcal{Q}(dt,du).\end{aligned}$$

(c) Since $(\mathcal{T}_n')_{n\ge 0}$ is a filtration of $\mathbb{T}\times\mathbb{M}^{\mathbb{N}_0}$ satisfying $\mathcal{T}\otimes\mathfrak{M}^{\mathbb{N}_0} = \sigma(\cup_{n\ge 0}\mathcal{T}_n')$, part (b) and Theorem (4.3.4) in [43] imply that for all $C\in\mathcal{T}\otimes\mathfrak{M}^{\mathbb{N}_0}$,

$$\hat{\mathcal{Q}}(C) = \int_C w\,d\mathcal{Q} + \hat{\mathcal{Q}}(C\cap\{w=\infty\}),$$

in particular

$$\mathbb{E}W = \int_{\mathbb{T}\times\mathbb{M}^{\mathbb{N}_0}} w\,d\mathcal{Q} = 1 - \hat{\mathcal{Q}}(w=\infty),$$

ensuring (i) and (ii). □

Remark 2.2.3. In the situation of Lemma 2.2.2(b), let $n\ge 0$, $A\in\mathbb{B}$ and put $B := \{w_n\in A\}\in\mathcal{T}_n'$. Then

$$\mathbb{P}(\hat{W}_n\in A) = \hat{\mathcal{Q}}(w_n\in A) = \int_B w_n\,d\mathcal{Q} = \frac{\mathbb{E}W_n\mathbf{1}_{\{W_n\in A\}}}{\mathbb{E}W_n},$$

i.e. $\hat{W}_n$ can be obtained by *size-biasing* W_n which explains the notion of *size-biased trees.* Note that this implies $\mathbb{P}(\hat{W}_n\in\cdot)\ll\mathbb{P}(W_n\in\cdot)$ with $\frac{d\mathbb{P}(\hat{W}_n\in\cdot)}{d\mathbb{P}(W_n\in\cdot)}(x) = x$, hence

$$\mathbb{E}h(\hat{W}_n) = \mathbb{E}W_n h(W_n)$$

for any measurable and nonnegative or bounded function h. For some more background information (and further references) on the concept of size-biasing, we mention the article [83] by Lyons et al.. If z_n is understood as a mapping defined on $\mathbb{T} \times \mathbb{M}^{\mathbb{N}_0}$ (by letting $z_n(t,u) = z_n(t)$ for all (t,u)), z_n is also $\mathcal{T}_n'$-measurable, and we come to

$$\mathbb{P}(\hat{Z}_n \in A) = \mathbb{E}W_n \mathbf{1}_{\{Z_n \in A\}} = \mathbb{E}\left[\frac{Z_n \mathbf{1}_{\{Z_n \in A\}}}{\mu_n(\mathbf{U})}\right],$$

which can be interpreted as *conditional size-biasing* Z_n.

2.3 Some auxiliary results

This section is devoted to some preparatory results furnishing the proofs of the main results of this chapter which will be stated in Section 2.4.

The first of these auxiliary results asserts that the probabilistic structure of the sequence $\mathbf{U} = (U_n)_{n\geq 0}$ is passed on to certain sequences of random variables encountered in the construction from Section 2.2.2 in the sense that these sequences are also stationary and ergodic. Moreover, they exhibit the same independence structure as the environmental sequence $(U_n)_{n\geq 0}$. This circumstance will help us to prove the main results of this chapter because it permits us to apply well-known limit theorems for stationary ergodic or even i.i.d. random variables, such as Birkhoff's ergodic theorem or the Chung-Fuchs theorem.

Recall from Chapter 1 that $\mathbb{M}'$, the set of all probability measures on $(\mathbb{R}, \mathbb{B})$, is endowed with the σ-algebra $\mathfrak{M}'$ generated by the total variation norm $d_{\mathbb{M}'}$ on $\mathbb{M}'$. Moreover, put $\Phi : \mathbb{R} \times \mathbb{M}' \to \mathbb{R}$,

$$\Phi(t, Q') := Q'(t) := Q'((-\infty, t]) \qquad (t \in \mathbb{R}, Q' \in \mathbb{M}').$$

Furthermore, let $m \geq 0$ and recall that a stationary sequence of random variables $(Y_n)_{n\geq 0}$ is called *m-dependent* if for each $k \geq 0$, $(X_0, ..., X_k)$ and $(X_j)_{j\geq m+k+1}$ are independent, i.e. if

$$\mathbb{P}^{(X_0,...,X_k,X_{k+m+1},X_{k+m+2},...)} = \mathbb{P}^{(X_0,...,X_k)} \otimes \mathbb{P}^{(X_j)_{j\geq m+k+1}}.$$

By stationarity, this entails independence of $(X_l, ..., X_{l+k})$ and $(X_j)_{j\geq m+k+l+1}$ whenever $l, k \geq 0$. Obviously, 0-dependence of a stationary sequence means nothing but independence.

Finally, we note that it is a well-known fact (and can also be checked by modifying the proof of Kolmogorov's zero-one law) that any stationary and m-dependent sequence of random variables is also ergodic.

Lemma 2.3.1. *(a) The mapping* $\Phi^{-1} : (0,1) \times \mathbb{M}' \to \mathbb{R}$,

$$\Phi^{-1}(t, Q') := \inf\{x \in \mathbb{R} : \Phi(x, Q') \geq t\} \qquad (0 < t < 1, Q' \in \mathbb{M}'),$$

is $\mathbb{B} \otimes \mathfrak{M}'$-$\mathbb{B}$-*measurable.*

(b) The sequence $(X_n)_{n\geq 0}$*, defined by* $X_n := \Sigma \circ \hat{T}(n) = \sum_{i\geq 1} \hat{T}_i(n)$ *for* $n \geq 0$*, is stationary ergodic.*

(c) The same holds true for the sequence $(\tilde{X}_n)_{n\geq 0}$*, where* $\tilde{X}_n := \frac{\hat{T}_{C_n}(n)}{\mu(U_n)}$ $(n \geq 0)$.

(d) If $m \geq 0$ *and* $\mathbf{U} = (U_n)_{n\geq 0}$ *is a stationary* m*-dependent sequence, the same is true for the sequences* $(X_n)_{n\geq 0}$ *and* $(\tilde{X}_n)_{n\geq 0}$*, respectively.*

Proof. (a) Fix any $\alpha \in \mathbb{R}$. Then we have that

$$\begin{aligned} \{(t, Q') : \Phi^{-1}(t, Q') \leq \alpha\} &= \{(t, Q') : \inf\{x \in \mathbb{R} : Q'(x) \geq t\} \leq \alpha\} \\ &= \{(t, Q') : Q'(\alpha) \geq t\} \\ &= \{(t, Q') : t - Q'(\alpha) \leq 0\}. \end{aligned}$$

Since the estimate $|Q_1'(\alpha) - Q_2'(\alpha)| \leq d_{\mathbb{M}'}(Q_1', Q_2')$ trivially holds for any fixed α and for arbitrary $Q_1', Q_2' \in \mathbb{M}'$, the mapping $Q' \mapsto Q'(\alpha)$ is continuous (and therefore measurable), and the claim follows.

(b) Let $\mathbf{Y} = (Y_n)_{n\geq 0}$ be a sequence of i.i.d. random variables on $(\Omega, \mathfrak{A}, \mathbb{P})$ which are uniformly distributed in the interval $(0,1)$ such that $\mathbf{Y}$ and $\mathbf{U} = (U_n)_{n\geq 0}$ are independent. First note that by (2.2.3), $\mathbb{P}(0 < X_n < \infty) = 1$. Then (2.2.2) says that for any $A \in \mathbb{B}$,

$$\mathbb{P}(X_n \in A | \mathbf{U}) = \mu(U_n)^{-1} \int_{\{\Sigma \in A\}} \Sigma \, dU_n =: \tilde{U}_n(A) \quad \text{a.s.,} \tag{2.3.1}$$

and from the measurability assumption **M.2** of Subsection 1.2.2, we infer that $\tilde{U}_n$ is an $\mathfrak{A}$-$\mathfrak{M}'$-measurable random variable. Since $\mathbf{U}$ is stationary ergodic, the same is true for $\tilde{\mathbf{U}} = (\tilde{U}_n)_{n\geq 0}$. Now put $X_n' := \Phi^{-1}(Y_n, \tilde{U}_n), n \geq 0$. Then $(X_n')_{n\geq 0}$ is stationary ergodic as well (see Proposition I.4.1.6 in [32] and Proposition 6.31 in [31]), and it is readily checked that the random variables $X'_n, n \geq 0$ are conditionally independent given $\mathbf{U}$ with $\mathbb{P}(X'_n \in \cdot|\mathbf{U}) = \tilde{U}_n$ a.s. for $n \geq 0$. Thus, $(X_n)_{n\geq 0} \stackrel{d}{=} (X'_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ is stationary ergodic, as asserted.

(c) The conditional independence of $(\hat{T}(n), C_n), n \geq 0$ yields that for any sequence $(A_n)_{n\geq 0}$ in $\mathbb{B}$,

$$\mathbb{P}(\hat{T}_{C_n}(n) \in A_n \ \forall n \geq 0|\mathbf{U}) = \prod_{n\geq 0} U_n^*(A_n) \quad \text{a.s.},$$

where for $n \geq 0$, (2.2.1) shows that

$$\begin{aligned} U_n^*(A_n) &:= \mathbb{P}(\hat{T}_{C_n}(n) \in A_n|\mathbf{U}) \\ &= \sum_{i\geq 1} \mathbb{P}(\hat{T}_i(n) \in A_n, C_n = i|\mathbf{U}) \\ &= \mu(U_n)^{-1} \sum_{i\geq 1} \int_{\{\pi_i \in A_n\}} \pi_i \, dU_n \quad \text{a.s..} \end{aligned}$$

As in (b), we obtain that in view of assumption **M.3** from Chapter 1, U_n^* is an $\mathfrak{A}$-$\mathfrak{M}'$-measurable random variable. Then define $X''_n := \Phi^{-1}(Y_n, U_n^*)$ and $X'''_n := X''_n/\mu(U_n), n \geq 0$. Then it is again easy to see that the random variables $(X''_n)_{n\geq 0}$ are conditionally independent given $\mathbf{U}$ having conditional distribution $\mathbb{P}(X''_n \in \cdot|\mathbf{U}) = U_n^*$ a.s. for all $n \geq 0$. Thus, $(X'''_n)_{n\geq 0}$ forms a copy of $(\tilde{X}_n)_{n\geq 0}$, and since $(X'''_n)_{n\geq 0}$ is stationary ergodic (see Proposition I.4.1.6 in [32] and Proposition 6.31 in [31]), the proof is complete.

(d) Since the case $m \geq 1$ can be proved in a similar manner, we restrict ourselves to the proof of the case $m = 0$ which means that $U_0, U_1, ...$ (and hence $\tilde{U}_0, \tilde{U}_1, ...$) are independent and identically distributed. As the random variables $X_n, n \geq 0$ are conditionally independent given $\mathbf{U}$ with $\mathbb{P}(X_n \in \cdot|\mathbf{U}) = \tilde{U}_n$ a.s., the independence of $\tilde{U}_n, n \geq 0$ ensures that for any sequence $(A_n)_{n\geq 0}$ in $\mathbb{B}$,

$$\begin{aligned} \mathbb{P}(X_n \in A_n \ \forall n \geq 0) &= \mathbb{E}\left[\mathbb{P}(X_n \in A_n \ \forall n \geq 0|\mathbf{U})\right] \\ &= \mathbb{E}\left[\prod_{n\geq 0} \tilde{U}_n(A_n)\right] \\ &= \prod_{n\geq 0} \mathbb{E}\tilde{U}_n(A_n) \\ &= \prod_{n\geq 0} \mathbb{P}(X_n \in A_n), \end{aligned}$$

i.e. the independence of $(X_n)_{n\geq 0}$. The assertion on $(\tilde{X}_n)_{n\geq 0}$ follows by an analogous argument. □

Remark 2.3.2. In the recent proof, we have seen that for each n, the nonnegative random variable $\hat{T}_{C_n}(n)$ has conditional distribution U_n^*. If $(Z_n)_{n\geq 0}$ forms an ordinary WBP, this distribution is an unconditional one and independent of n. If we assume without loss of generality that $\mu = \mathbb{E}Z_1 = 1$, U_n^* a.s. takes the form

$$U_n^* = \sum_{i\geq 1} \int_{\{\pi_i \in \cdot\}} \pi_i \, d\mathbb{P}^{T(\emptyset)} = \sum_{i\geq 1} \mathbb{E}T_i \mathbf{1}_{\{T_i \in \cdot\}} =: \xi.$$

In Chapter 4 this probability measure ξ will be of great importance which expresses itself in the investigation of a multiplicative random walk with incremental distribution ξ.

The following lemma consists of some moment calculations which clarify the relation between the quantities γ and $\mathbb{E}[Z_1 \log^+ Z_1/\mu(U_0)]$ on the one hand and the random variables X_n, $\tilde{X}_n$ on the other hand. As we will see in the forthcoming section, these expressions are crucial for the statement of our results.

The final part of the lemma will be used for an application of the Chung-Fuchs theorem in the proof of Theorem 2.4.12.

Recall that $\gamma_\pm = \mathbb{E}\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^\pm \frac{T_k}{\mu(U_0)}\right]$ and $c = \mathbb{P}\left(\bigcap_{i\geq 1}\{T_i = 0 \text{ or } T_i = \mu(U_0)\}\right)$.

Lemma 2.3.3. *(a) Suppose that γ_+ or γ_- is finite. Then*

$$\mathbb{E}\log \tilde{X}_0 = \mathbb{E}\left[\log \frac{\hat{T}_{C_0}(0)}{\mu(U_0)}\right] = \gamma_+ - \gamma_- = \gamma \in \overline{\mathbb{R}}.$$

(b) $\mathbb{E}\log^+ X_0 = \mathbb{E}\left[\log^+ \sum_{i\geq 1} \hat{T}_i(0)\right] = \mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]$.

(c) $\tilde{X}_0 = 1$ *a.s.* $\iff c = 1$.

Proof. (a) By construction, it follows that for all $A \in \mathbb{B}^{\mathbb{N}}, i \geq 1$ and $B \in \mathfrak{M}^{\mathbb{N}_0}$,

$$\begin{aligned}
\mathbb{P}(\hat{T}(0) \in A, C_0 = i, \mathbf{U} \in B) &= \int_{\{\mathbf{U}\in B\}} \mathbb{P}(\hat{T}(0) \in A, C_0 = i | \mathbf{U}) \, d\mathbb{P} \\
&= \int_{\{\mathbf{U}\in B\}} \mathbb{E}\left[\mu(U_0)^{-1} \mathbf{1}_{\{T\in A\}} T_i | \mathbf{U}\right] \, d\mathbb{P} \qquad (2.3.2) \\
&= \mathbb{E}\left[\mathbf{1}_{\{T\in A, \mathbf{U}\in B\}} \frac{T_i}{\mu(U_0)}\right],
\end{aligned}$$

proving that

$$\mathbb{P}((\hat{T}(0),\mathbf{U})\in C, C_0=i)=\mathbb{E}\left[\frac{T_i}{\mu(U_0)}\mathbf{1}_{\{(T,\mathbf{U})\in C\}}\right]$$

for all $C\in\mathbb{B}^{\mathbb{N}}\otimes\mathfrak{M}^{\mathbb{N}_0}$ and $i\ge 1$. Consequently,

$$\mathbb{P}\left(\frac{\hat{T}_i(0)}{\mu(U_0)}\in D, C_0=i\right)=\int_{\left\{\frac{T_i}{\mu(U_0)}\in D\right\}}\frac{T_i}{\mu(U_0)}\,d\mathbb{P} \tag{2.3.3}$$

for all $D\in\mathbb{B}$ and $i\ge 1$. Now first turning to the positive part of $\log\frac{\hat{T}_{C_0}(0)}{\mu(U_0)}$, we obtain the decomposition

$$\log^+\frac{\hat{T}_{C_0}(0)}{\mu(U_0)}=\sum_{k\ge 1}\Upsilon_k,$$

where

$$\Upsilon_k:=\mathbf{1}_{\{C_0=k\}}\log^+\frac{\hat{T}_k(0)}{\mu(U_0)}=\mathbf{1}_{\{C_0=k,\hat{T}_k(0)\ge\mu(U_0)\}}\log\frac{\hat{T}_k(0)}{\mu(U_0)},\quad k\ge 1.$$

On the basis of (2.3.3), we infer that for arbitrary $t>0$ and $k\ge 1$,

$$\mathbb{P}(\Upsilon_k>t)=\mathbb{P}\left(C_0=k,\log\frac{\hat{T}_k(0)}{\mu(U_0)}>t\right)=\int_{\left\{\log\frac{T_k}{\mu(U_0)}>t\right\}}\frac{T_k}{\mu(U_0)}\,d\mathbb{P}$$

and therefore by Fubini's theorem

$$\begin{aligned}\mathbb{E}\Upsilon_k &= \int_{(0,\infty)}\mathbb{P}(\Upsilon_k>t)\,\boldsymbol{\lambda}(dt)\\ &= \mathbb{E}\left[\frac{T_k}{\mu(U_0)}\int_0^\infty\mathbf{1}_{\left\{t<\log\frac{T_k}{\mu(U_0)}\right\}}\,\boldsymbol{\lambda}(dt)\right]\\ &= \mathbb{E}\left[\frac{T_k}{\mu(U_0)}\log^+\frac{T_k}{\mu(U_0)}\right].\end{aligned}$$

Thus, by monotone convergence, we have

$$\mathbb{E}\left[\log^+\frac{\hat{T}_{C_0}(0)}{\mu(U_0)}\right]=\sum_{k\ge 1}\mathbb{E}\Upsilon_k=\sum_{k\ge 1}\mathbb{E}\left[\frac{T_k}{\mu(U_0)}\log^+\frac{T_k}{\mu(U_0)}\right]=\gamma_+.$$

Since an analogous calculation gives

$$\mathbb{E}\left[\log^-\frac{\hat{T}_{C_0}(0)}{\mu(U_0)}\right]=\gamma_-$$

and one of the terms γ_+,γ_- is finite by assumption, it follows that

$$\mathbb{E}\left[\log\frac{\hat{T}_{C_0}(0)}{\mu(U_0)}\right]=\gamma_+-\gamma_-=\gamma=\mathbb{E}\left[\sum_{k\ge 1}\frac{T_k}{\mu(U_0)}\log\frac{T_k}{\mu(U_0)}\right]\in\overline{\mathbb{R}},$$

as demanded.

(b) As seen in the proof of Lemma 2.3.1(b),

$$\begin{aligned}\mathbb{P}(\log^+ X_0 > t) &= \mathbb{P}(X_0 > e^t) \\ &= \mathbb{E}\tilde{U}_0((e^t,\infty]) \\ &= \mathbb{E}\left[\mu(U_0)^{-1} Z_1 \mathbf{1}_{\{Z_1>e^t\}}\right] \\ &= \mathbb{E}\left[\mu(U_0)^{-1} Z_1 \mathbf{1}_{\{\log^+ Z_1>t\}}\right] \quad \text{for all } t>0\end{aligned}$$

because $Z_1 = \sum_{i\geq 1} T_i$. This implies by Fubini's theorem that

$$\begin{aligned}\mathbb{E}\log^+ X_0 &= \int_{(0,\infty)} \mathbb{P}(\log^+ X_0 > t)\,\boldsymbol{\lambda}(dt) \\ &= \mathbb{E}\left[\mu(U_0)^{-1} Z_1 \int_{(0,\infty)} \mathbf{1}_{\{\log^+ Z_1>t\}}\,\boldsymbol{\lambda}(dt)\right] \\ &= \mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right].\end{aligned}$$

(c) In analogy to (2.3.2), we have that for all $A, B \in \mathbb{B}$,

$$\begin{aligned}\mathbb{P}(\hat{T}_{C_0}(0)\in A, \mu(U_0)\in B|\mathbf{U}) &= \mathbf{1}_{\{\mu(U_0)\in B\}}\sum_{i\geq 1}\mu(U_0)^{-1}\mathbb{E}(\mathbf{1}_{\{T_i\in A\}}T_i|\mathbf{U}) \\ &= \sum_{i\geq 1}\mu(U_0)^{-1}\mathbb{E}(\mathbf{1}_{\{T_i\in A,\mu(U_0)\in B\}}T_i|\mathbf{U}) \quad \text{a.s.,}\end{aligned}$$

hence

$$\begin{aligned}\mathbb{P}(\tilde{X}_0 = 1|\mathbf{U}) &= \mathbb{P}(\hat{T}_{C_0}(0) = \mu(U_0) > 0|\mathbf{U}) \\ &= \sum_{i\geq 1}\mathbb{E}(\mu(U_0)^{-1}T_i\mathbf{1}_{\{T_i=\mu(U_0)\}}|\mathbf{U}) \\ &= \sum_{i\geq 1}\mathbb{P}(T_i = \mu(U_0)|\mathbf{U}) \quad \text{a.s.}\end{aligned}$$

Then the claim follows from the decomposition

$$\begin{aligned}\mu(U_0) = \mathbb{E}(Z_1|\mathbf{U}) &= \sum_{i\geq 1}\mathbb{E}(T_i\mathbf{1}_{\{T_i=\mu(U_0)\}}|\mathbf{U}) + \sum_{i\geq 1}\mathbb{E}(T_i\mathbf{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \\ &= \mu(U_0)\sum_{i\geq 1}\mathbb{P}(T_i=\mu(U_0)|\mathbf{U}) + \sum_{i\geq 1}\mathbb{E}(T_i\mathbf{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \\ &= \mu(U_0)\mathbb{P}(\tilde{X}_0=1|\mathbf{U}) + \sum_{i\geq 1}\mathbb{E}(T_i\mathbf{1}_{\{T_i\neq\mu(U_0)\}}|\mathbf{U}) \quad \text{a.s.}\end{aligned}$$

because $\mathbb{P}(0<\mu(U_0)<\infty) = 1$. □

For any $a > 1$, define the random variable

$$\mathbb{G}(\mathbf{U}, a) := \sum_{n\geq 0} \mu(U_n)^{-1} \int_{\{\Sigma > a^n\}} \Sigma \, dU_n = \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x \, U_n^{\Sigma}(dx) \quad \text{a.s..}$$

This random variable will turn out to be very important for the characterization of the uniform integrability and hence $\mathfrak{L}_1$-convergence of the martingale $(W_n)_{n\geq 0}$. On the one hand, the following result on the family of random variables $(\mathbb{G}(\mathbf{U}, a))_{a>1}$ connects it to the sequence $(X_n)_{n\geq 0}$, on the other hand it shows that this family has the property that either $\mathbb{G}(\mathbf{U}, a) = \infty$ a.s. for all $a > 1$, or $\mathbb{G}(\mathbf{U}, a) < \infty$ a.s. for all $a > 1$.

Observe that if $|\mathbf{v}_n| = n$ for all $n \geq 0$, $S(\mathbf{v}_n) := \sum_{i\geq 1} T_i(\mathbf{v}_n)$ and $a > 1$, then

$$\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x \, \mathbb{P}^{S(\mathbf{v}_n)|\mathbf{U}}(dx) = \sum_{n\geq 0} \mu(U_n)^{-1} \mathbb{E}\left[S(\mathbf{v}_n) \mathbf{1}_{\{S(\mathbf{v}_n) > a^n\}} \middle| \mathbf{U} \right]$$

almost surely. Here, the path $(\emptyset) = \mathbf{v}_0 \to \mathbf{v}_1 \to \mathbf{v}_2 \to \ldots$ may be interpreted as an arbitrary non-random line of descent, and for fixed $v \in \mathcal{N}$, $S(v)$ can be seen as a *growth factor* since on the event $\{L(v) > 0\}$, $S(v) = \frac{\sum_{i\geq 1} L(v,i)}{L(v)}$, where $\sum_{i\geq 1} L(v, i)$ gives the weight of v before splitting.

Lemma 2.3.4. *(a) For $a > 1$, $\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} \mathbb{P}(X_n > a^n | \mathbf{U})$ a.s..*

(b) If $\mathbb{G}(\mathbf{U}, a) < \infty$ with positive probability for some $a > 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \log^+ X_n = 0 \quad a.s.$$

(c) If $\mathbb{G}(\mathbf{U}, a) = \infty$ with positive probability for some $a > 1$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = \infty \quad a.s..$$

(d) $\mathbb{G}(\mathbf{U}, a)$ is finite with positive probability for some $a > 1$ if and only if it is finite a.s. for all $a > 1$.

Proof. (a) Fix $n \geq 0$ and $a > 1$. Then (2.2.3) and (2.3.1) yield

$$\mathbb{P}(X_n > a^n | \mathbf{U}) = \tilde{U}_n((a^n, \infty)) = \mu(U_n)^{-1} \int_{\{\Sigma > a^n\}} \Sigma \, dU_n \quad \text{a.s.}$$

and therefore

$$\sum_{n\geq 0} \mathbb{P}(X_n > a^n | \mathbf{U}) = \mathbb{G}(\mathbf{U}, a) \quad \text{a.s..}$$

(b) By Theorem 1 in [102], Lemma 4 in [89] or Lemma 7.2 in [83], $\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n = 0$ a.s. or $\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n = \infty$ a.s., for the sequence $(X_n)_{n\geq 0}$ is stationary ergodic (Lemma 2.3.1). Now if $\mathbb{G}(\mathbf{U}, a)$ is finite with positive probability for some $a > 1$, the Borel-Cantelli Lemma implies that

$$\mathbb{P}(X_n > a^n \text{ i.o.}|\mathbf{U}) = \mathbb{P}\left(\frac{1}{n}\log X_n > \log a \text{ i.o.}\middle|\mathbf{U}\right) = 0 \quad \text{w.p.p.},$$

i.e. $\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n < \infty$ a.s.. Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\log^+ X_n = 0 \quad \text{a.s.}$$

by what has been mentioned above.

(c) Considering the conditional independence of $X_n, n \geq 0$, a similar argument gives

$$\mathbb{P}(X_n > a^n \text{ i.o.}|\mathbf{U}) = \mathbb{P}\left(\frac{1}{n}\log X_n > \log a \text{ i.o.}\middle|\mathbf{U}\right) = 1 \quad \text{w.p.p.},$$

i.e.

$$\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n \geq \log a > 0 \quad \text{a.s.}.$$

Since $\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n \in \{0, \infty\}$ a.s., this implies

$$\limsup_{n\to\infty} \frac{1}{n}\log^+ X_n = \infty \quad \text{a.s.}$$

(d) is an immediate consequence of (b) and (c). □

2.4 Main results

This section contains the main results of this chapter. As mentioned at the beginning of this chapter, similar but weaker results are given in [26].

2.4.1 The general case

In this subsection, we consider the general case of stationary ergodic random environment. The following two theorems show that under mild additional assumptions, uniform integrability is tantamount with the pair of conditions

$$\gamma < 0 \quad \text{and} \quad \mathbb{G}(\mathbf{U}, a) < \infty \text{ w.p.p. for some } a > 1.$$

Recall the definitions

$$\gamma_{\pm} = \mathbb{E}\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log^{\pm} \frac{T_k}{\mu(U_0)}\right]$$

and the assumption that

$$\gamma = \mathbb{E}\left[\sum_{k\geq 1} \frac{T_k}{\mu(U_0)} \log \frac{T_k}{\mu(U_0)}\right] \quad \text{exists in } \overline{\mathbb{R}}.$$

Theorem 2.4.1. *Suppose that $\gamma_+ < \infty$, $-\infty \leq \gamma < 0$ and that there is some $a > 1$ such that*

$$\begin{aligned} \mathbb{G}(\mathbf{U}, a) &= \sum_{n\geq 0} \mu(U_n)^{-1} \int_{\{\Sigma > a^n\}} \Sigma \; dU_n \\ &= \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x \; U_n^{\Sigma}(dx) < \infty \quad \textit{with positive probability.} \end{aligned}$$

Then $W = \lim_{n\to\infty} W_n$ satisfies $\mathbb{E}W = 1$.

Proof. Define $\mathcal{Y} := \sigma((\hat{T}_k, C_k)_{k\geq 0}, \mathbf{U})$ and fix $n \geq 0$. We start by calculating the conditional expectation of $\hat{W}_n$ given $\mathcal{Y}$. For this purpose, we decompose $\hat{Z}_n = z_n \circ \hat{L}$, the sum of all weights in the nth generation of $\hat{L}$ in the following way: Given any $k \in \{0, ..., n-1\}$, let $\hat{R}_k$ be the sum of weights of all cells in generation n stemming from V_k, but not from V_{k+1}. Then we have the representation

$$\hat{Z}_n = \hat{L}(V_n) + \sum_{k=0}^{n-1} \hat{R}_k, \tag{2.4.1}$$

or more explicitly

$$\hat{Z}_n = \hat{L}(V_n) + \sum_{|\sigma|=n} \mathbf{1}_{\{V_n=\sigma\}} \sum_{k=0}^{n-1} \hat{L}(V_k) D_{k,\sigma},$$

where

$$D_{k,\sigma} := \sum_{j\geq 1, j\neq \sigma_{k+1}} \hat{T}_j(k) \cdot \Xi_{\sigma,k,j},$$

$\Xi_{\sigma,n-1,j} := 1$ and

$$\Xi_{\sigma,k,j} := \sum_{|v|=n-k-1} T_{v_1}(\sigma_1, ..., \sigma_k, j) \cdot ... \cdot T_{v_{n-k-1}}(\sigma_1, ..., \sigma_k, j, v_1, ..., v_{n-k-2}) \quad \text{if } k < n-1,$$

writing $\sigma = (\sigma_1, ..., \sigma_n)$ and $v = (v_1, ..., v_{n-k-1})$. We claim that

$$\mathbb{E}(\Xi_{\sigma,k,j}|\mathcal{Y}) = \mathbb{E}(\Xi_{\sigma,k,j}|\mathbf{U}) = \prod_{l=k+1}^{n-1} \mu(U_l) = \frac{\mu_n(\mathbf{U})}{\mu_{k+1}(\mathbf{U})} \quad \text{a.s..} \tag{2.4.2}$$

To establish (2.4.2), let $B \in \bigotimes_{n\geq 0}(\mathbb{B}^{\mathbb{N}} \otimes \mathbb{B})$, $C \in \mathfrak{M}^{\mathbb{N}_0}$ and

$$A := \{(\hat{T}(n), C_n)_{n\geq 0} \in B, \mathbf{U} \in C\} \in \mathcal{Y}.$$

Then the conditional independence of $\Xi_{\sigma,k,j}$ and $(\hat{T}(n), C_n)_{n\geq 0}$ (given $\mathbf{U}$) implies

$$\begin{aligned}
\int_A \Xi_{\sigma,k,j}\, d\mathbb{P} &= \int_{\{\mathbf{U}\in C\}} \mathbb{E}\left(\Xi_{\sigma,k,j} \cdot \mathbf{1}_{\{(\hat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) d\mathbb{P} \\
&= \int_{\{\mathbf{U}\in C\}} \mathbb{E}\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \cdot \mathbb{E}\left(\mathbf{1}_{\{(\hat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) d\mathbb{P} \\
&= \int_{\{\mathbf{U}\in C\}} \mathbb{E}\left(\mathbb{E}\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \cdot \mathbf{1}_{\{(\hat{T}(n),C_n)_{n\geq 0}\in B\}}\middle|\mathbf{U}\right) d\mathbb{P} \\
&= \int_A \mathbb{E}\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) d\mathbb{P} \quad \text{a.s.,}
\end{aligned}$$

i.e.

$$\mathbb{E}\left(\Xi_{\sigma,k,j}|\mathcal{Y}\right) = \mathbb{E}\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \quad \text{a.s.}$$

because $\sigma(\mathbf{U}) \subset \mathcal{Y}$. Moreover, it follows by construction that for all $k \geq 0$,

$$\begin{aligned}
\prod_{l=k+1}^{n-1} \mu(U_l) &= \prod_{l=k+1}^{n-1} \mathbb{E}\left[\Sigma \circ T(\mathbf{v}_l)|\mathbf{U}\right] \\
&= \mathbb{E}\left[\prod_{l=k+1}^{n-1} \Sigma \circ T(\mathbf{v}_l)\middle|\mathbf{U}\right] \\
&= \mathbb{E}\left[\prod_{l=k+1}^{n-1} \sum_{m_l\geq 1} T_{m_l}(\mathbf{v}_l)\middle|\mathbf{U}\right] \\
&= \mathbb{E}\left(\Xi_{\sigma,k,j}|\mathbf{U}\right) \quad \text{a.s.,}
\end{aligned}$$

where for $l \geq 1$, $\mathbf{v}_l$ is an arbitrary element of $\mathbb{N}^l$. Thus, (2.4.2) is proved, and consequently,

$$\mathbb{E}(D_{k,\sigma}|\mathcal{Y}) = \sum_{j\geq 1, j\neq\sigma_{k+1}} \hat{T}_j(k) \cdot \prod_{l=k+1}^{n-1} \mu(U_l) = \sum_{j\geq 1, j\neq\sigma_{k+1}} \hat{T}_j(k) \cdot \frac{\mu_n(\mathbf{U})}{\mu_{k+1}(\mathbf{U})} \quad \text{a.s..}$$

Now invoking the $\mathcal{Y}$-measurability of $V_0, ..., V_n$,

$$\begin{aligned}\mathbb{E}(\hat{W}_n|\mathcal{Y}) &= \frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{|\sigma|=n} \mathbf{1}_{\{V_n=\sigma\}} \sum_{k=0}^{n-1} \frac{\hat{L}(V_k)}{\mu_n(\mathbf{U})} \cdot \mathbb{E}(D_{k,\sigma}|\mathcal{Y}) \\ &= \frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{|\sigma|=n} \mathbf{1}_{\{V_n=\sigma\}} \sum_{k=0}^{n-1} \frac{\hat{L}(V_k)}{\mu_{k+1}(\mathbf{U})} \cdot \sum_{j\geq 1, j\neq\sigma_{k+1}} \hat{T}_j(k) \\ &\leq \frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})} + \sum_{k=0}^{n-1} \frac{1}{\mu(U_k)} \cdot \frac{\hat{L}(V_k)}{\mu_k(\mathbf{U})} \cdot \exp\left(\log^+ X_k\right) \quad \text{a.s..}\end{aligned}$$

Our aim is to see that the last expression converges a.s. as $n \to \infty$. Then using the obvious product representation

$$\hat{L}(V_n) = \prod_{k=0}^{n-1} \hat{T}_{C_k}(k),$$

a straight forward extension of the ergodic theorem (see Lemma A.2) and Lemma 2.3.3(a) show that

$$\begin{aligned}\left[\frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})}\right]^{1/n} &= \left(\prod_{k=0}^{n-1} \frac{\hat{T}_{C_k}(k)}{\mu(U_k)}\right)^{1/n} \\ &= \exp\left[\frac{1}{n}\sum_{k=0}^{n-1} \log \tilde{X}_k\right] \qquad (2.4.3) \\ &\underset{n\to\infty}{\longrightarrow} \exp(\mathbb{E}\log \tilde{X}_0) = e^{\gamma} \in [0,1) \quad \text{a.s.,}\end{aligned}$$

i.e.

$$\lim_{n\to\infty} \frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})} = 0 \quad \text{a.s..}$$

It remains to verify that

$$\sum_{k\geq 0} \frac{1}{\mu(U_k)} \cdot \frac{\hat{L}(V_k)}{\mu_k(\mathbf{U})} \cdot \exp\left(\log^+ X_k\right) \qquad (2.4.4)$$

converges a.s.. For this purpose, note that the ergodic theorem and (1.2.1) give

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \mu(U_k) = \beta = \mathbb{E}[\log \mu(U_0)] \in (-\infty, \infty) \quad \text{a.s.}$$

and therefore

$$\lim_{n\to\infty} (\mu(U_n))^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{1}{n} \log \mu(U_n)\right) = 1 \quad \text{a.s..} \qquad (2.4.5)$$

Hence

$$\lim_{n\to\infty}\left[\frac{1}{\mu(U_n)}\cdot\frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})}\right]^{1/n} = e^{\gamma} < 1 \quad \text{a.s.} \tag{2.4.6}$$

by (2.4.3). Referring to Lemma 2.3.4, the assumption $\mathbb{G}(\mathbf{U},a) < \infty$ with positive probability for some $a > 1$ ensures

$$\lim_{n\to\infty}\frac{1}{n}\log^+ X_n = 0 \quad \text{a.s.,} \tag{2.4.7}$$

and together with (2.4.6), this guarantees that there is some $\delta \in (e^{\gamma},1)$ such that for almost every $\omega \in \Omega$, fixed $\eta \in (1,\delta^{-1})$ and all sufficiently large k,

$$\frac{\hat{L}(V_k)(\omega)}{\mu_k(\mathbf{U})(\omega)\cdot\mu(U_k)(\omega)} \leq \delta^k$$

as well as

$$\exp\left(\log^+ X_k\right)(\omega) \leq \eta^k.$$

Now putting these estimates together, the a.s. convergence of the series (2.4.4) is evident since $\delta\eta < 1$. Therefore, it follows by Fatou's lemma that

$$\mathbb{E}(\liminf_{n\to\infty}\hat{W}_n|\mathcal{Y}) \leq \liminf_{n\to\infty}\mathbb{E}(\hat{W}_n|\mathcal{Y}) < \infty \quad \text{a.s.,}$$

i.e. $\liminf_{n\to\infty}\hat{W}_n < \infty$ a.s., in other words

$$\hat{\mathcal{Q}}(\liminf_{n\to\infty} w_n < \infty) = 1. \tag{2.4.8}$$

In addition, note that $\hat{\mathcal{Q}}(w_n > 0) = 1$ for all n since by Lemma 2.2.2(b),

$$\hat{\mathcal{Q}}(w_n = 0) = \int_{\{w_n=0\}} w_n \, d\mathcal{Q} = 0.$$

Obviously, w_n^{-1} is $\mathcal{T}_n'$-measurable for all $n \geq 0$. In the next step we show that the sequence $((w_n^{-1},\mathcal{T}_n'))_{n\geq 0}$ is a nonnegative supermartingale w.r.t. $\hat{\mathcal{Q}}$. In fact, letting $n \geq 0$, $A \in \mathcal{T}_n' \subset \mathcal{T}_{n+1}'$ and recalling Lemma 2.2.2(b), the computation

$$\begin{aligned}
\int_A w_{n+1}^{-1}\, d\hat{\mathcal{Q}} &= \int_{A\cap\{w_{n+1}>0\}} w_{n+1}^{-1}\, d\hat{\mathcal{Q}}_{|\mathcal{T}_{n+1}'} \\
&= \int_{A\cap\{w_{n+1}>0\}} w_{n+1}^{-1}\cdot w_{n+1}\, d\mathcal{Q}_{|\mathcal{T}_{n+1}'} \\
&= \mathcal{Q}(A) - \mathcal{Q}(A\cap\{w_{n+1}=0\}) \\
&= \mathcal{Q}_{|\mathcal{T}_n'}(A\cap\{w_n>0\}) + \mathcal{Q}(A\cap\{w_n=0\}) - \mathcal{Q}_{|\mathcal{T}_{n+1}'}(A\cap\{w_{n+1}=0\})
\end{aligned}$$

$$= \int_{A\cap\{w_n>0\}} w_n^{-1}\cdot w_n\, d\mathcal{Q}_{|\mathcal{T}_n'} + \underbrace{\mathcal{Q}(A\cap\{w_n=0\}) - \mathcal{Q}(A\cap\{w_{n+1}=0\})}_{\le 0}$$
$$\le \int_A w_n^{-1}\, d\hat{\mathcal{Q}}$$

proves that $\mathbb{E}_{\hat{\mathcal{Q}}}(w_{n+1}^{-1}|\mathcal{T}_n') \le w_n^{-1}$ $\hat{\mathcal{Q}}$-a.s. as well as the integrability of w_n^{-1} with respect to $\hat{\mathcal{Q}}$. To justify the last inequality, consider that $\{W_n = 0\} \subset \{W_{n+1} = 0\}$ for $n \ge 0$. Therefore, $(w_n^{-1})_{n\ge 0}$ converges $\hat{\mathcal{Q}}$-a.s., in particular

$$\hat{\mathcal{Q}}(\liminf_{n\to\infty} w_n = w) = 1.$$

Consequently, (2.4.8) gives

$$\hat{\mathcal{Q}}(w < \infty) = 1,$$

which in turn implies

$$\mathbb{E}W = \int_{\mathbb{T}} w\, d\mathcal{Q} = 1$$

by another appeal to Lemma 2.2.2. This is the claim. □

Remark 2.4.2. The preceding proof particularly contains the following result:

$$\gamma_+ < \infty,\ -\infty \le \gamma < 0,\ \limsup_{n\to\infty} \frac{1}{n}\log^+ X_n = 0 \quad \text{a.s.} \implies \mathbb{E}W = 1. \tag{2.4.9}$$

This implication is stated here explicitly because it will be used in the proof of Theorem 2.4.10.

Theorem 2.4.3. *Suppose that $\gamma_- < \infty$. If $0 < \gamma \le \infty$, or $\mathbb{G}(\mathbf{U}, a)$ is infinite with positive probability for some $a > 1$, then $\mathbb{P}(W = 0) = 1$.*

Proof. First suppose that $\gamma = \gamma_+ - \gamma_- = \mathbb{E}\left[\log \frac{\hat{T}_{C_0}(0)}{\mu(U_0)}\right] \in (0,\infty]$. Then (2.4.1), the extended ergodic theorem (Lemma A.2) and Lemma 2.3.3(a) show that analogously to (2.4.3),

$$\begin{aligned} \hat{W}_n^{1/n} = \left[\frac{\hat{Z}_n}{\mu_n(\mathbf{U})}\right]^{1/n} &\ge \left[\frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})}\right]^{1/n} \\ &= \exp\left[\frac{1}{n}\sum_{k=0}^{n-1} \log \tilde{X}_k\right] \\ &\underset{n\to\infty}{\longrightarrow} \exp(\mathbb{E}\log \tilde{X}_0) = e^{\gamma} \in (1,\infty] \quad \text{a.s.} \end{aligned} \tag{2.4.10}$$

Thus,

$$\hat{\mathcal{Q}}(w=\infty)=\mathbb{P}(\limsup_{n\to\infty}\hat{W}_n=\infty)=1$$

and

$$\mathbb{P}(W=0)=\mathcal{Q}(w=0)=1$$

by Lemma 2.2.2(c).

If on the other hand, $\mathbb{G}(\mathbf{U},a)$ is infinite with positive probability for some $a>1$, then Lemma 2.3.4 shows that

$$\limsup_{n\to\infty}\frac{1}{n}\log X_n=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i\geq 1}\hat{T}_i(n)=\infty\quad\text{a.s..}\tag{2.4.11}$$

Furthermore, we have the estimate

$$\hat{Z}_{n+1}\geq\hat{L}(V_n)\cdot X_n=\hat{L}(V_n)\cdot\sum_{i\geq 1}\hat{T}_i(n)$$

and therefore

$$\begin{aligned}\left(\hat{W}_{n+1}\right)^{1/n}&\geq\left[\frac{1}{\mu(U_{n+1})}\cdot\frac{\hat{L}(V_n)}{\mu_n(\mathbf{U})}\cdot X_n\right]^{1/n}\\&=\exp\left(-\frac{1}{n}\log\mu(U_{n+1})\right)\cdot\exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\log\tilde{X}_k\right)\cdot\exp\left(\frac{1}{n}\log X_n\right).\end{aligned}\tag{2.4.12}$$

By (2.4.5),

$$\lim_{n\to\infty}\exp\left(-\frac{1}{n}\log\mu(U_{n+1})\right)=1\quad\text{a.s.,}$$

and considering the fact that $\gamma_-<\infty$ implies $\gamma>-\infty$ and hence

$$\lim_{n\to\infty}\exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\log\tilde{X}_k\right)=e^{\gamma}>0\quad\text{a.s.,}$$

(2.4.11) and (2.4.12) yield

$$\limsup_{n\to\infty}\hat{W}_n=\infty\quad\text{a.s.,}$$

i.e. once more $W=0$ a.s. by Lemma 2.2.2(c). □

Remark 2.4.4. We have particularly proved the implication

$$\gamma_-<\infty,\ \limsup_{n\to\infty}\frac{1}{n}\log X_n=\infty\quad\text{a.s.}\implies\mathbb{P}(W=0)=1\tag{2.4.13}$$

which will be used in the proof of Theorem 2.4.10.

The subsequent corollary follows immediately from Theorem 2.4.1 and Theorem 2.4.3.

Corollary 2.4.5. *Suppose that* $\gamma_+ \vee \gamma_- < \infty$ *and* $\gamma \neq 0$.

(a) The following are equivalent:

(i) $\mathbb{E}W = 1$,

(ii) $\gamma < 0$ *and* $\mathbb{G}(\mathbf{U}, a) < \infty$ *w.p.p. for some* $a > 1$,

(iii) $\gamma < 0$ *and* $\mathbb{G}(\mathbf{U}, a) < \infty$ *a.s. for all* $a > 1$.

(b) Analogously, the following are equivalent:

(i) $\mathbb{E}W = 0$

(ii) $\gamma > 0$ *or* $\mathbb{G}(\mathbf{U}, a) = \infty$ *a.s. for all* $a > 1$,

(iii) $\gamma > 0$ *or* $\mathbb{G}(\mathbf{U}, a) = \infty$ *w.p.p. for some* $a > 1$.

In particular, $\mathbb{E}W \in \{0, 1\}$.

The proof of the following theorem shows that the series $\mathbb{G}(\mathbf{U}, a)$ introduced in Theorem 2.4.1 converges a.s. for all $a > 1$ if $Z_1 \log^+ Z_1/\mu(U_0)$ is integrable. However, we will see in Example 2.4.22 that in general, the integrability of this random variable is not a necessary condition for W to be nondegenerate.

Theorem 2.4.6. *If* $-\infty \leq \gamma < 0$ *and* $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$, *then* $\mathbb{E}W = 1$.

Proof. We have seen in Lemma 2.3.3(b) that $\mathbb{E}\log^+ X_0 = \mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]$. Consequently, the stationarity of $(X_n)_{n\geq 0}$ implies that for arbitrary $a > 1$,

$$\begin{aligned}
\infty > 1 + \mathbb{E}\left[\frac{\log^+ X_0}{\log a}\right] &\geq \sum_{n\geq 0} \mathbb{P}(\log^+ X_0 > n \log a) \\
&= \sum_{n\geq 0} \mathbb{P}(X_n > a^n) \\
&= \sum_{n\geq 0} \int \mathbb{P}(X_n > a^n | \mathbf{U})\, d\mathbb{P} \\
&= \int \left(\sum_{n\geq 0} \mathbb{P}(X_n > a^n | \mathbf{U}) \right) d\mathbb{P} \\
&= \mathbb{E}\mathbb{G}(\mathbf{U}, a),
\end{aligned}$$

i.e. $\mathbb{P}\left(\mathbb{G}(\mathbf{U},a)<\infty\right)=1$. Moreover, the integrability of $\frac{Z_1\log^+ Z_1}{\mu(U_0)}$, the inequality $T_i\le Z_1$ for $i\ge 1$ and the fact that $x\mapsto \log^+ x$ is nondecreasing in $[0,\infty]$ ensure that

$$\begin{aligned}
0\le \gamma_+ &= \mathbb{E}\left[\sum_{i\ge 1}\frac{T_i}{\mu(U_0)}\log^+\frac{T_i}{\mu(U_0)}\right]\\
&\le \mathbb{E}\left[\sum_{i\ge 1}\frac{T_i}{\mu(U_0)}\log^+\frac{Z_1}{\mu(U_0)}\right]\\
&= \mathbb{E}\left[\frac{Z_1}{\mu(U_0)}\log^+\frac{Z_1}{\mu(U_0)}\right]\\
&= \mathbb{E}\left[\frac{Z_1}{\mu(U_0)}(\log Z_1-\log\mu(U_0))\mathbf{1}_{\{Z_1\ge\mu(U_0)\}}\right] \qquad (2.4.14)\\
&\le \mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+\mathbb{E}\left[\frac{Z_1|\log\mu(U_0)|}{\mu(U_0)}\right]\\
&= \mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+\mathbb{E}\left[\frac{|\log\mu(U_0)|}{\mu(U_0)}\cdot\mathbb{E}(Z_1|\mathbf{U})\right]\\
&= \mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]+\mathbb{E}|\log\mu(U_0)|<\infty.
\end{aligned}$$

Hence, the assertion follows from Theorem 2.4.1. □

Corollary 2.4.7. *Suppose that $-\infty\le\gamma<0$ and that there is some $\ell\ge 1$ such that*

$$\mathbb{P}(T_i=0 \textit{ for all } i>\ell)=1. \qquad (2.4.15)$$

Then $\gamma_-<\infty$, $\mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right]<\infty$ and $\mathbb{E}W=1$. The assertion remains true if (2.4.15) is replaced by the assumption $\|N\|_\infty=\text{ess sup}\,N<\infty$.

Proof. Put $\psi(x):=x\log x$, $x\ge 0$ and note that ψ is convex with $\sup_{0<x<1}|\psi(x)|=1/e$. Consequently, $\gamma_-\le \ell/e<\infty$, $-\infty<\gamma<0$ and additionally

$$\begin{aligned}
-\infty<\mathbb{E}\psi\left(\frac{Z_1}{\mu(U_0)}\right) &= \log\ell\cdot\mathbb{E}\left[\frac{Z_1}{\mu(U_0)}\right]+\ell\,\mathbb{E}\psi\left(\frac{Z_1}{\ell\mu(U_0)}\right)\\
&= \log\ell+\ell\,\mathbb{E}\psi\left(\frac{1}{\ell}\sum_{i=1}^{\ell}\frac{T_i}{\mu(U_0)}\right)\\
&\le \log\ell+\mathbb{E}\left[\sum_{i=1}^{\ell}\psi\left(\frac{T_i}{\mu(U_0)}\right)\right]\\
&= \log\ell+\gamma<\infty.
\end{aligned}$$

Together with (1.2.1), this gives

$$\begin{aligned}\mathbb{E}\left[\frac{Z_1 \log Z_1}{\mu(U_0)}\right] &= \mathbb{E}\psi\left(\frac{Z_1}{\mu(U_0)}\right) + \mathbb{E}\left[\frac{Z_1 \log \mu(U_0)}{\mu(U_0)}\right] \\ &= \mathbb{E}\psi\left(\frac{Z_1}{\mu(U_0)}\right) + \mathbb{E}\log \mu(U_0) \\ &= \mathbb{E}\psi\left(\frac{Z_1}{\mu(U_0)}\right) + \beta \in (-\infty, \infty),\end{aligned}$$

in particular $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$. Then Theorem 2.4.6 gives the claim. The second part follows by a similar argument. □

Remark 2.4.8. Using the decomposition $W_1 \log W_1 = \frac{Z_1 \log Z_1}{\mu(U_0)} - \frac{Z_1 \log \mu(U_0)}{\mu(U_0)}$ and the fact that $\mathbb{E}|\log \mu(U_0)| < \infty$, it can be checked that the moments $\mathbb{E}W_1 \log^+ W_1$ and $\mathbb{E}[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}]$ are commonly finite or infinite which is not obvious at first glance because $\mu(U_0)$ is a random variable.

The next result is an application of Birkhoff's ergodic theorem and says that on $\{W > 0\}$, $Z_n^{1/n}$ converges a.s. to some finite positive constant.

Theorem 2.4.9. *On the event* $\{W > 0\}$,

$$Z_n^{1/n} \underset{n\to\infty}{\longrightarrow} e^{-\beta} = e^{-\mathbb{E}\log \mu(U_0)} \in (0, \infty) \quad a.s..$$

In particular,

$$Z_n^{1/n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\mu} \quad a.s. \text{ on } \{W > 0\}$$

if $(Z_n)_{n\geq 0}$ *forms a WBP with reproduction mean* $\mu \in (0, \infty)$.

Proof. For any $\omega \in \Omega$ with $W(\omega) \in (0, \infty)$, it is evident that $\lim_{n\to\infty} W_n(\omega)^{1/n} = 1$. As seen in the proof of Theorem 2.4.1, $\lim_{n\to\infty} \mu_n(\mathbf{U})^{1/n} = e^{\beta} = \exp(\mathbb{E}\log \mu(U_0)) \in (0, \infty)$ with probability 1. Consequently,

$$\lim_{n\to\infty} Z_n^{1/n} = \lim_{n\to\infty} \mu_n(\mathbf{U})^{-1/n} \cdot \lim_{n\to\infty} W_n^{1/n} = e^{-\beta} \text{ on the event } \{W > 0,\ \mu_n(\mathbf{U})^{1/n} \to e^{\beta}\}.$$

□

2.4.2 Stationary and m-dependent random environment

As mentioned in Section 2.3, stationary m-dependent sequences of random variables are particular examples of ergodic sequences. In this special situation, the following theorem gives an improvement of Theorem 2.4.1. It says that (under mild additional assumptions) the integrability of $Z_1 \log^+ Z_1/\mu(U_0)$ is also a *necessary* condition for uniform integrability of the considered martingale $(W_n)_{n\geq 0}$. Once more, we emphasize the fact that this is an essential difference with the general case just treated as Example 2.4.22 will explicitly show. Note that as in the general case, we have to exclude the case $\gamma = 0$. The latter restriction can be removed in the case of i.i.d. random environment which is nothing but 0-dependence; the corresponding result will be given in the following subsection.

Theorem 2.4.10. *Let $m \geq 0$ and suppose that the sequence $\mathbf{U} = (U_n)_{n\geq 0}$ is stationary and consists of m-dependent random variables. If $\gamma_- < \infty$ and $\gamma \neq 0$, the following statements are equivalent:*

(i) $\mathbb{P}(W = 0) < 1$,

(ii) $\mathbb{E}W = 1$,

(iii) $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$ *and* $\gamma < 0$.

Proof. The implication "$(ii) \Rightarrow (i)$" is trivial.
"$(i) \Rightarrow (iii)$": If $\gamma > 0$, we have already found the contradiction $\mathbb{P}(W = 0) = 1$ in Theorem 2.4.3. It remains to prove the finiteness of $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right]$. For this purpose, assume $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] = \infty$, i.e. $\mathbb{E}\log^+ X_0 = \infty$ by Lemma 2.3.3. Since the sequence $(X_n)_{n\geq 0}$ is stationary and m-dependent (Lemma 2.3.1(d)), this guarantees (see Lemma A.1)

$$\limsup_{n\to\infty} \frac{1}{n} \log^+ X_n = \infty \quad \text{a.s.}$$

and therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log X_n = \infty \quad \text{a.s.}$$

as well. Now $W = 0$ a.s. follows from (2.4.13) in Remark 2.4.4.

Summarizing, we have proved that (i) implies $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$ and $\gamma < 0$.

"$(iii) \Rightarrow (ii)$": First, recall that by (2.4.14), $\gamma_+ \leq \mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] + \mathbb{E}|\log \mu(U_0)| < \infty$. We claim that $\lim_{n\to\infty} \frac{1}{n} \log^+ X_n = 0$ a.s.: As $\mathbb{E}\log^+ X_0 = \mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$ (Lemma 2.3.3(b)), the m-dependence and stationarity of $(X_n)_{n\geq 0}$ (Lemma 2.3.1(d)) in combination with Lemma A.1 yields

$$\limsup_{n\to\infty}\frac{1}{n}\log^+ X_n = \limsup_{n\to\infty}\frac{1}{n}\log^+\left[\sum_{i\geq 1}\hat{T}_i(n)\right] = 0 \quad \text{a.s.},$$

hence $\mathbb{E}W = 1$ by (2.4.9) in Remark 2.4.2 and the assumption $\gamma < 0$. The proof is now complete. □

If (2.4.15) holds (or $\|N\|_\infty < \infty$), we obtain the following simplified version of Theorem 2.4.10.

Corollary 2.4.11. *If $U_n, n \geq 0$ are m-dependent for some $m \geq 0$ and form a stationary sequence, $\gamma \neq 0$ and (2.4.15) holds, then*

$$\mathbb{P}(W = 0) < 1 \iff \mathbb{E}W = 1 \iff \gamma < 0.$$

Proof. We have seen in Corollary 2.4.7 that γ_- is finite and that $\mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right] < \infty$ is implied by $\gamma < 0$. Now Theorem 2.4.12(a) completes the proof. □

2.4.3 The case of i.i.d. environment

In this section, we focus on the situation of i.i.d. random variables U_n, $n \geq 0$. It is an immediate consequence of this model assumption that cells in different generations reproduce independently of one another because for any $n \geq 1$, $\sigma^1, ..., \sigma^n \in \mathcal{N}$ such that $|\sigma^1| < ... < |\sigma^n|$ and measurable sets $A_1, ..., A_n \subset \mathbb{R}^{\mathbb{N}}$,

$$\begin{aligned}
\mathbb{P}\left(T(\sigma^1) \in A_1, ..., T(\sigma^n) \in A_n\right) &= \mathbb{E}\left[U_{|\sigma^1|}(A_1)\cdot ... \cdot U_{|\sigma^n|}(A_n)\right] \\
&= \mathbb{E}U_{|\sigma^1|}(A_1)\cdot ... \cdot \mathbb{E}U_{|\sigma^n|}(A_n) \\
&= \mathbb{P}\left(T(\sigma^1) \in A_1\right)\cdot ... \cdot \mathbb{P}\left(T(\sigma^n) \in A_n\right).
\end{aligned}$$

The preceding section already contains a result for i.i.d. random environment under the additional assumption $\gamma \neq 0$, but by a simple application of the Chung-Fuchs theorem, this restriction can be dropped:

Theorem 2.4.12. *Suppose that the random variables $U_n, n \geq 0$ are i.i.d..*

(a) If $\gamma_- < \infty$ and $c = \mathbb{P}\left(\bigcap_{i\geq 1}\{T_i = 0 \text{ or } T_i = \mu(U_0)\}\right) < 1$, the following conditions are equivalent:

(i) $\mathbb{P}(W = 0) < 1$,

(ii) $\mathbb{E}W = 1$,

(iii) $\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] < \infty$ *and* $\gamma < 0$.

(b) If $c = 1$ and $\mathbb{P}(Z_1 = \mu(U_0)) = 1$, then $W = \mathbb{E}W = 1$ a.s..

(c) If $c = 1$ and $\mathbb{P}(Z_1 = \mu(U_0)|\mathbf{U}) < 1$ a.s., then $W = 0$ a.s..

Note that in the situation of *(b)* and *(c)*, we automatically have

$$\gamma = \sum_{i\geq 1} \mathbb{E}\left[\frac{T_i}{\mu(U_0)} \log \frac{T_i}{\mu(U_0)}\right] = 0.$$

The theorem does *not* cover the case $c = 1$, $\mathbb{P}(Z_1 = \mu(U_0)) < 1$, but $\mathbb{P}(Z_1 = \mu(U_0)|\mathbf{U}) = 1$ with positive probability.

Proof of Theorem 2.4.12. (a) Keeping the proof of Theorem 2.4.10 in mind, we merely have to show that $\mathbb{P}(W = 0) < 1$ implies $\gamma < 0$, or equivalently, that W is degenerate in case $\gamma \geq 0$. If $\gamma > 0$, this has already been done in the proof of Theorem 2.4.3. Turning to the case $\gamma = 0$, the assumption $c < 1$ and Lemma 2.3.3(c) ensure that

$$\mathbb{P}(\log \tilde{X}_0 = 0) = \mathbb{P}(\tilde{X}_0 = 1) < 1.$$

Consequently, the Chung-Fuchs theorem, Lemma 2.3.1(d) and (2.4.10) give

$$\limsup_{n\to\infty} \hat{W}_n \geq \limsup_{n\to\infty}\ \exp\left(\sum_{k=0}^{n-1} \log \tilde{X}_k\right) \geq \limsup_{n\to\infty} \sum_{k=0}^{n-1} \log \tilde{X}_k = \infty \quad \text{a.s.},$$

i.e. $W = 0$ a.s. in this case as well.

(b) As for all $A \in \mathbb{B}^{\mathbb{N}}, B \in \mathbb{B}$ and $v \in \mathcal{N}$

$$\begin{aligned}
\mathbb{P}\left(T(v) \in A, \mu(U_{|v|}) \in B\right) &= \mathbb{E}\left[\mathbf{1}_{\{\mu(U_{|v|})\in B\}} \mathbb{P}(T(v) \in A|\mathbf{U})\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\mu(U_{|v|})\in B\}} U_{|v|}(A)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\mu(U_0)\in B\}} U_0(A)\right] \\
&= \mathbb{P}\left(T \in A, \mu(U_0) \in B\right),
\end{aligned}$$

it follows that for all $v \in \mathcal{N}$, $\mathbb{P}\left(\Sigma \circ T(v) = \mu(U_{|v|})\right) = 1$. Hence, $c = 1$ implies $Z_n = \mu_n(\mathbf{U})$ a.s. for all $n \geq 0$ and $W = 1$ a.s..

(c) Let $T^*(v) := (T_i^*(v))_{i\geq 1} := T(v)/\mu(U_{|v|})$ and $U_{|v|}^* := \mathbb{P}(\Sigma \circ T^*(v) \in \cdot|\mathbf{U})$, $v \in \mathcal{N}$. Then $(W_n)_{n\geq 0}$ is the uniquely determined weighted branching process with the factors $T^*(v)$, $v \in \mathcal{N}$. Consequently, $\mathbb{P}(\Sigma \circ T^*(v) \in \mathbb{N}_0) = 1$ for all $v \in \mathcal{N}$. More precisely, we will see that $(W_n)_{n\geq 0}$ may be viewed as an ordinary branching process in stationary ergodic random environment. To be rigorous, let $\mathbf{U}^* := (U_n^*)_{n\geq 0}$ and observe that $\mathbf{U}^*$ is stationary ergodic, too. Additionally, let $(Y(v))_{v\in\mathcal{N}}$ be a family of $\mathbb{N}_0$-valued random variables on $(\Omega, \mathfrak{A}, \mathbb{P})$ that are conditionally independent given $\mathbf{U}^* := (U_n^*)_{n\geq 0}$ with

$$\mathbb{P}(Y(v) \in \cdot|\mathbf{U}^*) = U_{|v|}^* \quad \text{a.s. for all } v \in \mathcal{N},$$

and recursively define $Z_0^* := 1$, $I_0^* := \{(\emptyset)\}$,

$$I_{n+1}^* := \{(w, i) \in \mathbb{N}^{n+1} : w \in I_n^* \text{ and } 1 \leq i \leq Y(w)\}$$

and

$$Z_{n+1}^* := |I_{n+1}^*| = \sum_{w\in I_n^*} Y(w)$$

for $n \geq 0$. Thus (cf. the articles [103]–[106] by Tanny), $(Z_n^*)_{n\geq 0}$ forms a GWPRE with environmental sequence $\mathbf{U}^*$ fulfilling

$$\mu^*(U_0^*) := \mathbb{E}(Z_1^*|\mathbf{U}^*) = \sum_{j\geq 0} jU_0^*(\{j\}) = \mathbb{E}(\Sigma \circ T^*(\emptyset)|\mathbf{U}) = \frac{\mathbb{E}(Z_1|\mathbf{U})}{\mu(U_0)} = 1 \quad \text{a.s..}$$

To get the connection to the original process $(W_n)_{n\geq 0}$, note that for all finite $\mathcal{N}' \subset \mathcal{N}$ and all $A_v \subset \mathbb{N}_0$ $(v \in \mathcal{N}')$, the conditional independence of the random variables $(T(v), \mu(U_{|v|}))$, $v \in \mathcal{N}$ (given $\mathbf{U}$) on the one hand and of $Y(v)$, $v \in \mathcal{N}$ (given $\mathbf{U}^*$) on the other hand yields

$$\begin{aligned}
\mathbb{P}\left(\Sigma \circ T^*(v) \in A_v, v \in \mathcal{N}'\right) &= \mathbb{E}\mathbb{P}\left(\Sigma \circ T^*(v) \in A_v, v \in \mathcal{N}'|\mathbf{U}\right) \\
&= \mathbb{E}\left[\prod_{v\in\mathcal{N}'} \mathbb{P}\left(\Sigma \circ T^*(v) \in A_v|\mathbf{U}\right)\right] \\
&= \mathbb{E}\left[\prod_{v\in\mathcal{N}'} U_{|v|}^*\right] \\
&= \mathbb{E}\left[\prod_{v\in\mathcal{N}'} \mathbb{P}(Y(v) \in A_v|\mathbf{U}^*)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\mathbb{P}\left(Y(v) \in A_v, v \in \mathcal{N}' | \mathbf{U}^*\right) \\
&= \mathbb{P}\left(Y(v) \in A_v, v \in \mathcal{N}'\right),
\end{aligned}
$$

showing that $(\Sigma \circ T^*(v))_{v\in\mathcal{N}}$ and $(Y(v))_{v\in\mathcal{N}}$ have the same distribution. From this it easily follows that $(W_n)_{n\geq 0}$ and $(Z_n^*)_{n\geq 0}$ are identically distributed, in particular

$$\mathbb{P}(W = 0) = \mathbb{P}(W_n \to 0) = \mathbb{P}(Z_n^* \to 0). \tag{2.4.16}$$

In the next step we show that $A := \{U_0^*(\{0,1\}) = 1\}$ has $\mathbb{P}(A) = 0$. To see this, observe that $\mathbb{P}\Big(U_0^*(\{0,1\}) = 1, U_0^*(\{0\}) > 0\Big) = 0$ because $\sum_{j\geq 0} jU_0^*(\{j\}) = 1$ a.s.. Therefore,

$$\mathbb{P}(A) = \mathbb{P}\Big(U_0^*(\{1\}) = 1\Big) = \mathbb{P}\Big(\mathbb{P}(Z_1 = \mu(U_0)|\mathbf{U}) = 1\Big) = 0.$$

Then we may apply (2.4.16) and Theorem 1 in [10] to infer

$$\mathbb{P}(W = 0) = \mathbb{E}\mathbb{P}(Z_n^* \to 0|\mathbf{U}^*) = 1,$$

for $\mathbb{E}\log\mu^*(U_0^*) = 0$. □

Remark 2.4.13. The following example shows that the Chung-Fuchs theorem is not applicable (without further assumptions) to m-dependent sequences of random variables when $m \geq 1$: Given a sequence $(X_n)_{n\geq 0}$ of i.i.d. random variables which are nondeterministic and bounded, put $Y_n := X_n - X_{n-1}$, $n \geq 1$. Then $(Y_n)_{n\geq 1}$ obviously forms a 1-dependent sequence of nondeterministic, bounded and centered random variables, but

$$\limsup_{n\to\infty} \sum_{i=1}^{n} Y_i = \limsup_{n\to\infty}(X_n - X_0) < \infty \quad \text{a.s..}$$

Hence, the Chung-Fuchs theorem can not be used without further restrictions to relax the assumption $\gamma \neq 0$ in Theorem 2.4.10 when $m \geq 1$.

Since the following corollary can easily be proved by combining Corollary 2.4.11 and Theorem 2.4.12(a), we omit the details.

Corollary 2.4.14. *If $U_n, n \geq 0$ are i.i.d., $c < 1$ and (2.4.15) holds, then*

$$\mathbb{P}(W = 0) < 1 \Longleftrightarrow \mathbb{E}W = 1 \Longleftrightarrow \gamma < 0.$$

The next result finishes this section. If Z_1 is integrable, it shows that $(Z_n/\mathbb{E}Z_n)_{n\geq 0}$ is also a martingale which, however, tends to zero almost surely if $\mu(U_0)$ is not a.s. a constant. Recall that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(T(v) : v \in \mathcal{N},\ |v| < n)$ for $n \geq 1$.

Theorem 2.4.15. *Suppose that $U_n, n \geq 0$ are i.i.d. and $\varrho := \mathbb{E}Z_1 = \mathbb{E}\mu(U_0) \in (0,\infty)$. Then the sequence $(\overline{W}_n)_{n\geq 0}$, given by $\overline{W}_n := Z_n/\varrho^n$ for $n \geq 0$, is a nonnegative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. If $\mu(U_0)$ is nondeterministic (i.e. $0 < \mathrm{Var}\,\mu(U_0) \leq \infty$), $\overline{W}_n$ tends to zero a.s. as $n \to \infty$.*

Proof. We have already mentioned at the beginning of this subsection that by the independence of $U_0, U_1, \ldots$, individuals in different generations reproduce independently of one another. Hence if $|v| = n+1$ for some $n \geq 0$, $T(v)$ is independent of $\mathcal{F}_n$, and the martingale property of $(\overline{W}_n)_{n\geq 0}$ is easily established. By the strong law of large numbers,

$$\begin{aligned} \left(\frac{\mu_n(\mathbf{U})}{\varrho^n}\right)^{1/n} &= \exp\left(\frac{1}{n}\sum_{i=0}^{n-1}\log\mu(U_i) - \log\varrho\right) \\ &\underset{n\to\infty}{\longrightarrow} \exp\left(\mathbb{E}\log\mu(U_0) - \log\varrho\right) \\ &= \exp(\beta - \log\varrho) < 1 \quad \text{a.s.} \end{aligned}$$

because considering the strict concavity of the logarithm and the fact that $\mu(U_0)$ is nondeterministic, Jensen's inequality yields $\beta = \mathbb{E}\log\mu(U_0) < \log\mathbb{E}\mu(U_0) = \log\varrho$. Thus,

$$\mu_n(\mathbf{U}) = o(\varrho^n) \quad \text{a.s. as } n \to \infty$$

which together with the a.s. convergence of W_n to W entails

$$\overline{W}_n = W_n \cdot \frac{\mu_n(\mathbf{U})}{\varrho^n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s. as } n \to \infty,$$

as desired. □

Remark 2.4.16. (a) Informally speaking, the preceding theorem states that (without any additional assumption beyond $\varrho \in (0,\infty)$ and $\mathrm{Var}\,\mu(U_0) > 0$) Z_n grows much slower than ϱ^n. In other words, the sequence $(\varrho^n)_{n\geq 0}$ does not properly describe the rate of growth of $(Z_n)_{n\geq 0}$ and therefore cannot be used to normalize the underlying process in a sensible way.

(b) If $\mu(U_0)$ has a Dirac distribution, then necessarily $\mathbb{P}(\mu(U_0) \in \cdot) = \delta_\varrho$, and trivially $W_n = \overline{W}_n$ a.s. for all $n \geq 0$.

(c) If $(U_n)_{n\geq 0}$ is any stationary ergodic sequence, then in general, $(\overline{W}_n)_{n\geq 0}$ does not form a martingale. Nevertheless, the second assertion in Theorem 2.4.15 remains true as can be seen by using Birkhoff's ergodic theorem instead of the strong law of large numbers.

2.4.4 Applications to ordinary WBP

We now consider ordinary weighted branching processes (see for example [94]–[98]). In this special case, the environmental sequence $\mathbf{U} = (U_n)_{n\geq 0}$ is deterministic and nonvarying, i.e. $\mathbb{P}(U_0 \in \cdot) = \mathbb{P}(U_1 \in \cdot) = ... = \delta_\Gamma$ for some $\Gamma \in \mathbb{M}$ and

$$\mu(U_i) = \mu(\Gamma) = \int_{[0,\infty]} y \, \Gamma^{\Sigma}(dy) =: \mu \in (0,\infty) \quad \text{a.s. for all } i \geq 0.$$

Moreover, we know from Remark 2.4.16(b) that in this case $W_n = \overline{W}_n$ a.s. for all $n \geq 0$, and it is obvious that cells reproduce independently of each other. Furthermore, $\mathbb{E}(Z_n|\mathbf{U}) = \mathbb{E}Z_n = \mu^n \in (0,\infty)$ a.s.,

$$\gamma_\pm = \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu} \log^\pm \frac{T_i}{\mu}\right] \quad \text{and} \quad \gamma = \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu} \log \frac{T_i}{\mu}\right] \in \overline{\mathbb{R}}.$$

Here, the relevance of a $Z \log Z$-condition is evident because for any $a > 1$, $\mathbb{G}(\mathbf{U}, a)$ is deterministic and takes the form

$$\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} \mu^{-1} \int_{(a^n,\infty)} x \, \mathbb{P}^{Z_1}(dx) = \sum_{n\geq 0} \mu^{-1} \mathbb{E} Z_1 \mathbf{1}_{\{Z_1 > a^n\}} \quad \text{a.s.},$$

and by Proposition 2 in [95] or Proposition 13 in [97], the latter expression is finite for some/all $a > 1$ iff $\mathbb{E}Z_1 \log^+ Z_1 < \infty$.

In a slightly different form, the following result is due to Lyons [82] whose method proof we used in the more general situation of WBPRE. Weaker results (requiring finiteness of $\sum_{i\geq 1} \mathbb{E}T_i^\alpha$ for some $\alpha > 1$) are given by Rösler, Topchii and Vatutin in [97] and by Rösler in [95] (when $\alpha = 2$). In the context of branching random walks, a similar theorem is due to Biggins [17]. Liu [78] gives some bibliographical remarks on similar results in less general situations.

Theorem 2.4.17. *Suppose that $(Z_n)_{n\geq 0}$ is a weighted branching process with finite and positive reproduction mean μ.*

(a) If $\gamma_- < \infty$ and $c = \mathbb{P}(T \in \{0,\mu\}^{\mathbb{N}}) < 1$, then the following conditions are equivalent:

  (i) $\mathbb{P}(W = 0) < 1$,

  (ii) $\mathbb{E}W = 1$,

  (iii) $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ *and* $\gamma < 0$.

(b) If $c = 1$ and $\mathbb{P}(Z_1 = \mu) = 1$, then $W = \mathbb{E}W = 1$ a.s..

(c) If $c = 1$ and $\mathbb{P}(Z_1 = \mu) < 1$, then $W = 0$ a.s..

(d) If $-\infty \leq \gamma < 0$ and $\mathbb{E}Z_1 \log^+ Z_1 < \infty$, then $\mathbb{E}W = 1$.

Proof. (a) and (b) are immediate consequences of Theorem 2.4.12 as $\mu(U_0) = \mu$ a.s..

(c) By replacing $T(v), v \in \mathcal{N}$ with $\check{T}(v) := \mu^{-1}T(v), v \in \mathcal{N}$, we may suppose $\mu = 1$. Thus, $(Z_n)_{n\geq 0}$ forms a critical Galton-Watson process with $p_1 := \mathbb{P}(Z_1 = 1) < 1$. Consequently, Lemma I.3.1 in [12] gives

$$W = \lim_{n\to\infty} Z_n = 0 \quad \text{a.s..}$$

(d) has been proved in Corollary 2.4.14. □

Remark 2.4.18. (a) In the context of supercritical Galton-Watson processes, it is well-known (cf. Theorem II.2.1 in [7] or Theorem I.10.1 in [12]) that $\mathbb{E}W \in \{0,1\}$. However, we cannot ensure that this assertion is also true in full generality for weighted branching processes because Theorem 2.4.17 is not applicable if γ_- and $\mathbb{E}Z_1 \log^+ Z_1$ are both infinite (for instance, if $\gamma_+ = \gamma_- = \infty$). But at least we can state that if $\gamma_+ \vee \gamma_- < \infty$, then $\mathbb{E}W \in \{0,1\}$ (cf. Corollary 2.4.5).

(b) Rösler [95] has shown that if $\mu = 1$, $\mathbb{E}Z_1 \log^+ Z_1 < \infty$, $\sum_{i\geq 1} \mathbb{E}T_i^2 < 1$, and the underlying GWP $(Z'_n)_{n\geq 0}$ from Section 1.1 is supercritical with extinction probability q', then

$$\mathbb{P}(W = 0) = q' = \mathbb{P}(Z'_n \to 0) < 1,$$

i.e. $W > 0$ a.s. on the set where $(Z'_n)_{n\geq 0}$ survives.

The subsequent theorem gives a characterization of $\mathfrak{L}_1$-convergence for ordinary WBP which makes use of Theorem 1.1.6.

Theorem 2.4.19. *Given a WBP* $(Z_n)_{n\geq 0}$, *the following conditions are equivalent:*

(i) $\mathbb{E}W = 1$,

(ii) $\lim_{n\to\infty} \mathbb{E}|W_n - W| = 0$,

(iii) $(W_n)_{n\geq 0}$ *is uniformly integrable,*

(iv) $\mathbb{E}\left(\sup_{n\geq 0} W_n\right) < \infty$.

Proof. As the other results follow from standard results of martingale theory, we merely mention that $\sup_{n\geq 0} W_n$ is integrable if $\mathbb{E}W = 1$ as can be seen by applying Theorem 1.1.6 to $f(x) = x$. □

Theorem 2.4.17 gives raise to the question whether it is possible to find a suitable deterministic norming for $(Z_n)_{n\geq 0}$ if one of the conditions $\gamma < 0$ and $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ is not satisfied. This question is further motivated by a well-known result going back to Heyde [54] and Seneta [99] which says that for *any* nondeterministic supercritical Galton-Watson process $(Z'_n)_{n\geq 0}$ with reproduction mean $\mu' \in (1,\infty)$, there exists a sequence $(c'_n)_{n\geq 0}$ such that $\lim_{n\to\infty} c'_{n+1}/c'_n = \mu'$ and $(Z'_n/c'_n)_{n\geq 0}$ converges almost surely to a nondegenerate limit Z' with $\mathbb{P}(Z' > 0) = 1 - q'$, where q' is the extinction probability of the GWP. Notice that the condition $\mathbb{E}Z'_1 \log^+ Z'_1 < \infty$ is not required, but in that case c'_n can simply be chosen as $c'_n = \mu'^n$.

The following result is due to Biggins and Kyprianou [23, 24] who formulated it in terms of a BRW. It seems worth noting that the $Z \log Z$-condition is not required, but that the assumption $\gamma < 0$ is imposed. Moreover, the theorem only claims convergence in probability, rather than almost sure convergence. As vaguely indicated by Biggins and Kyprianou, their proof suggests that one cannot ensure a.s. convergence without further restrictions (without giving a counterexample, however).

Theorem 2.4.20. (Biggins, Kyprianou [23, 24])
Suppose that $(Z_n)_{n\geq 0}$ *is a WBP with* $\mu = \mathbb{E}Z_1 = 1$ *and associated Galton-Watson process* $(Z'_n)_{n\geq 0}$ *as in Section 1.1 (in particular,* $Z'_1 = N$*). Assume that the additional conditions*

(i) *There is some* $\eta \in (0,1)$ *such that* $\sum_{i\geq 1} \mathbb{E}T_i^\theta < \infty$ *for all* $\theta \in (1-\eta, 1+\eta)$ *(which implies* $\gamma_+ < \infty$ *as will turn out in Remark 3.6.3),*

(ii) $\mathbb{P}(N < \infty) = 1$ *(i.e. the WBP is in fact a BRW) and* $\mathbb{E}N = \sum_{i\geq 1} \mathbb{P}(T_i > 0) > 1$,

(iii) $\gamma = \sum_{i\geq 1} \mathbb{E}T_i \log T_i < 0$

are satisfied. Then there exists a sequence of positive constants $(c_n)_{n\geq 0}$ *such that*

$$\frac{Z_n}{c_n} \underset{n\to\infty}{\longrightarrow} \Delta \quad \text{in probability,}$$

where Δ *is a finite random variable which is strictly positive when the (supercritical) Galton-Watson process* $(Z'_n)_{n\geq 0}$ *survives, i.e.* $\Delta > 0$ *a.s. on the event* $\{Z'_n \not\to 0\} = \{Z'_n \to \infty\}$*. Moreover, on the same event,* $\frac{Z_{n+1}}{Z_n} \underset{n\to\infty}{\longrightarrow} 1$ *in probability.*

The following theorem is quoted from [78] and demonstrates how the WBP can suitably be normalized if $\gamma \geq 0$. It turns out that higher moment assumptions on Z_1 are required.

Theorem 2.4.21. (Liu [78])
Suppose that $\gamma_+ < \infty$, $c < 1$ *and* $\mathbb{E}N \in (1,\infty)$, *where* $N = \sum_{i\geq 1} \mathbf{1}_{\{T_i > 0\}}$ *(which implies* $\gamma_- < \infty$*).*

(a) *If* $\gamma = 0$, *then*

$$Z_n^* := -\sum_{|v|=n} L(v)\mathbf{1}_{\{L(v)>0\}} \log L(v), \qquad n \geq 1$$

is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$*. If additionally,* $\mathbb{E}N^{1+\varepsilon} + \mathbb{E}Z_1^{1+\varepsilon} < \infty$ *for some* $\varepsilon > 0$*, then* $(Z_n^*)_{n\geq 1}$ *converges a.s. to some nondegenerate random variable* $Z^* \geq 0$ *with infinite mean.*

(b) *If* $\gamma > 0$, *then*

$$Z_n^{(\beta)} := \sum_{|v|=n} L(v)^\beta, \qquad n \geq 1,$$

is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$*, where* β *is the unique number in* $(0,1)$ *satisfying* $\sum_{i\geq 1} \mathbb{E}T_i^\beta = 1$*. If in addition,* $\mathbb{E}Z_1^{(\beta)} \log^+ Z_1^{(\beta)} < \infty$*, then* $(Z_n^{(\beta)})_{n\geq 1}$ *converges a.s. to some nondegenerate random variable* $Z^{(\beta)} \geq 0$ *with mean* $\mathbb{E}Z^{(\beta)} = 1$.

The martingale $(Z_n^*)_{n\geq 1}$ satisfies $\mathbb{E}Z_n^* = 0$ for all $n \geq 1$ and is in general not nonnegative. However, as remarked by Liu, it is ultimately nonnegative in the sense that for almost every $\omega \in \Omega$, there is some $n_0 = n_0(\omega)$ such that $Z_n^*(\omega) \geq 0$ for all $n \geq n_0(\omega)$. The choice of $\beta \in (0,1)$ in (b) is based on the observation that if $\mathbb{E}N \in (1,\infty)$, then the function $\alpha \mapsto \sum_{i\geq 1} \mathbb{E}T_i^\alpha \cdot \mathbf{1}_{\{T_i>0\}}$ is strictly convex on $[0,1]$ (cf. Lemma 3.3.1) with derivative $\gamma^{(\alpha)} = \sum_{i\geq 1} \mathbb{E}T_i^\alpha \log T_i$ on $(0,1]$. Choosing β as described, it follows that $\gamma^{(\beta)} = \sum_{i\geq 1} \mathbb{E}T_i^\beta \log T_i$ is negative which makes Theorem 2.4.17 applicable to the process $(Z_n^{(\beta)})_{n\geq 1}$.

2.4.5 A counterexample and additional remarks

As has already been announced in Subsection 2.4.1 and Subsection 2.4.2, the following example shows that in the general case of stationary ergodic random environment, the finiteness of $\mathbb{E}[Z_1 \log^+ Z_1/\mu(U_0)]$ is not a *necessary* condition for W to be nondegenerate. Tanny [106] has given a similar construction in the context of GWPRE.

Example 2.4.22. Let $(X_n)_{n\geq 0}$ be a stationary ergodic sequence of $\mathbb{N}$-valued random variables on $(\Omega, \mathfrak{A}, \mathbb{P})$ such that

(i) $\mathbb{E}X_0 = \infty$ and

(ii) $\lim_{n\to\infty} \frac{X_n}{n} = 0$ a.s.

(see Example (a) in [102] for the construction of such a sequence). Moreover, let

$$\lambda_n := 2^{-n}\delta_{\mathbf{v}^{(n)}} + (1-2^{-n})\delta_{(1,0,0,\ldots)},$$

where $\mathbf{v}^{(n)} := (v_i^{(n)})_{i\geq 1}$ is defined by

$$v_i^{(n)} := \begin{cases} 1, & 1 \leq i \leq 2^{n+2}, \\ 0, & i > 2^{n+2} \end{cases} \qquad (n \geq 1),$$

and for $\mathbf{x} \in [0,\infty)^{\mathbb{N}}$, $\delta_{\mathbf{x}}$ is the Dirac measure in $\mathbf{x}$. Then $\mathbf{U} := (U_n)_{n\geq 0} := (\lambda_{X_n})_{n\geq 0}$ is stationary ergodic with

$$\mu(U_0) = \int_{[0,\infty]} x \; U_0^\Sigma(dx) = (1-2^{-X_n}) + 2^{X_n+2} \cdot 2^{-X_n} = 5 - 2^{-X_n} \in (4;5) \quad \text{a.s..} \qquad (2.4.17)$$

In the next step we check that the conditions of Theorem 2.4.1 are fulfilled: We will see (in a slightly more general situation) in Remark 2.4.23 that $\gamma_+ < \infty$ and $\gamma = -\mathbb{E}[\log \mu(U_0))]$, i.e. $-\infty < \gamma < 0$ by (2.4.17). Furthermore, it follows from (ii) that for almost every $\omega \in \Omega$ and all sufficiently large n that $\frac{X_n(\omega)}{n} < 1 - \frac{2}{n}$, i.e. $2^{X_n(\omega)+2} < 2^n$ and therefore

$$\int_{(2^n,\infty)} x\ U_n^{\Sigma}(\omega)(dx) = 0.$$

Consequently, $\mathbb{G}(\mathbf{U}, 2)$ is a.s. finite, whence $\mathbb{E}W = 1$ by Theorem 2.4.1.

It remains to show that $\mathbb{E}[Z_1 \log^+ Z_1/\mu(U_0)]$ is infinite. But (2.4.17) and (i) ensure that

$$\begin{aligned}
\mathbb{E}\left[\frac{Z_1 \log^+ Z_1}{\mu(U_0)}\right] &= \mathbb{E}\left[\frac{\mathbb{E}(Z_1 \log^+ Z_1|\mathbf{U})}{\mu(U_0)}\right] \\
&\geq \frac{1}{5}\mathbb{E}\left[\int_{[0,\infty]} x \log^+ x\ U_0^{\Sigma}(dx)\right] \\
&\geq \frac{1}{5}\mathbb{E}\left[2^{-X_0} \cdot 2^{X_0+2} \log 2^{X_0+2}\right] \\
&= \frac{4\log 2}{5}\mathbb{E}\left[X_0 + 2\right] = \infty,
\end{aligned}$$

as claimed.

At the end of this chapter we give some supplementary remarks relating our results to well-known results on GWP and GWPRE.

Remark 2.4.23. (a) Suppose that $\mathbb{P}(T \in \{0,1\}^{\mathbb{N}}) = 1$ and $\beta = \mathbb{E}(\log \mu(U_0)) \in (0,\infty)$. Then $(Z_n)_{n\geq 0}$ may be viewed as a supercritical GWP in random environment with stationary ergodic environmental sequence $\mathbf{U}^{\Sigma} := (U_n^{\Sigma})_{n\geq 0}$, for if $|\mathbf{v}_n| = n \geq 0$,

$$\mathbb{P}\left(\sum_{i\geq 1} T_i(\mathbf{v}_n) = k \middle| \mathbf{U}\right) = \mathbb{P}(\Sigma \circ T(\mathbf{v}_n) = k|\mathbf{U}) = U_n^{\Sigma}(\{k\}) \quad \text{a.s. for all } k \geq 0.$$

In addition,

$$\mu(U_n) = \sum_{k\geq 1} kU_n^{\Sigma}(\{k\}) \quad \text{a.s. for all } n \geq 0,$$

$$\begin{aligned}
\gamma_+ = \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{T_i}{\mu(U_0)}\right] &= \mathbb{E}\left[\sum_{i\geq 1} \frac{T_i}{\mu(U_0)} \log^+ \frac{1}{\mu(U_0)} \mathbf{1}_{\{T_i=1\}}\right] \\
&= \mathbb{E}\left[\log^+ \frac{1}{\mu(U_0)} \cdot \frac{Z_1}{\mu(U_0)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mu(U_0)^{-1}\log^+\frac{1}{\mu(U_0)}\cdot\mathbb{E}(Z_1|\mathbf{U})\right]\\
&= \mathbb{E}\left[\log^+\frac{1}{\mu(U_0)}\right]\\
&= \mathbb{E}\left[\log^-\mu(U_0)\right] < \infty,
\end{aligned}
$$

$$\gamma_- = \mathbb{E}\left[\log^-\frac{1}{\mu(U_0)}\right] = \mathbb{E}\left[\log^+\mu(U_0)\right] < \infty$$

and

$$\gamma = \gamma_+ - \gamma_- = \mathbb{E}\left[\log\frac{1}{\mu(U_0)}\right] = -\beta \in (-\infty, 0).$$

Now Corollary 2.4.5 says that

$$\mathbb{E}W = 1 \Longleftrightarrow \mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0}\frac{1}{\mu(U_n)}\int_{(a^n,\infty)} x\ U_n^{\Sigma}(dx) < \infty \text{ w.p.p. for some } a > 1,$$

and $\mathbb{E}W = 0$ otherwise, in full accordance with Theorem 7.1. in [83]. Moreover, by Theorem 2.4.6, $\mathbb{E}W = 1$ if $\mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right] < \infty$, confirming Theorem 3 of [106]. As

$$\lim_{n\to\infty}\mu_n(\mathbf{U})^{1/n} = \lim_{n\to\infty}\exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\log\mu(U_k)\right) = e^{\beta} > 1 \quad \text{a.s.}$$

by the ergodic theorem, we have that for almost every ω, there exist constants $a_1, a_2 \in (1,\infty)$ with

$$a_1^n \leq \mu_{n+1}(\mathbf{U})(\omega) \leq a_2^n$$

for all sufficiently large n. Since $\mathbb{G}(\mathbf{U}, a)$ is finite w.p.p. for some $a > 1$ if and only if it is finite a.s. for all $a > 1$ (Lemma 2.3.4(d)), this shows that

$$
\begin{aligned}
&\mathbb{G}(\mathbf{U}, a) < \infty \text{ w.p.p. for some } a > 1\\
&\Longleftrightarrow \sum_{n\geq 0}\frac{1}{\mu(U_n)}\int_{(\mu_{n+1}(\mathbf{U}),\infty)} x\ U_n^{\Sigma}(dx) < \infty \text{ a.s.},
\end{aligned}
$$

the latter condition appearing in Theorem 1 in [106]. Note that the additional technical assumption made there is dispensable.

Summarizing, we can state that our results from Subsection 2.4.1 form generalizations of known results on branching processes in stationary ergodic random environment.

(b) If in the situation of (a) the random variables U_n, $n \geq 0$ are even i.i.d., then $\gamma = -\beta < 0$, and Theorem 2.4.12 implies that

$$\mathbb{E}W = 1 \Longleftrightarrow \mathbb{E}\left[\frac{Z_1\log^+ Z_1}{\mu(U_0)}\right] < \infty,$$

and $\mathbb{E}W = 0$ otherwise, agreeing with Theorem 2 in [106] which deals with Galton-Watson processes in i.i.d. random environment. In view of Theorem 2.4.10, the result remains true if $(U_n)_{n\geq 0}$ is merely stationary and m-dependent for some $m \geq 1$.

(c) If additionally, the sequence $(U_n)_{n\geq 0}$ is deterministic and nonvarying, then $\gamma = -\log \mu$ is negative iff $\mu > 1$, and Theorem 2.4.17 says that for the classical Galton-Watson process $(Z_n)_{n\geq 0}$ with normalized limit $W := \lim_{n\to\infty} \frac{Z_n}{\mu^n}$ and offspring distribution $(p_k)_{k\geq 0}$ satisfying $p_1 = \mathbb{P}(Z_1 = 1) < 1$,

$$\mathbb{E}W = 1 \quad \Longleftrightarrow \quad \mathbb{E}Z_1 \log^+ Z_1 = \sum_{k\geq 2} p_k k \log k < \infty \text{ and } \mu > 1,$$

which is the classical result by Kesten and Stigum (see e.g. Theorem II.2.1 in [7] or Theorem I.10.1 in [12]).

Chapter 3

A characterization of $\mathfrak{L}_\alpha$-convergence for weighted branching processes

3.1 Motivation

The previous chapter has given an almost exhaustive characterization of $\mathfrak{L}_1$-convergence of the martingale obtained by normalizing a WBPRE with its conditional mean. We have seen that in the general case of a stationary ergodic random environment, necessary and sufficient for $\mathfrak{L}_1$-convergence (which is tantamount with uniform integrability) can be expressed in terms of the random variable $\mathbb{G}(\mathbf{U}, a) = \sum_{n\geq 0} \mu(U_n)^{-1} \int_{(a^n,\infty)} x \, U_n^{\Sigma}(dx)$ for fixed $a > 1$, and the expectation $\gamma = \sum_{i\geq 1} \mathbb{E}\frac{T_i}{\mu(U_0)} \log \frac{T_i}{\mu(U_0)}$. When the environment is given by an i.i.d. (or even deterministic) sequence, it has been shown in Theorem 2.4.12 that besides γ, the $Z \log Z$-condition $\mathbb{E}[Z_1 \log^+ Z_1/\mu(U_0)] < \infty$ is crucial. If $(Z_n)_{n\geq 0}$ forms a WBP, we may suppose without loss of generality that $\mu(U_0) = \mu = \mathbb{E}Z_1 = 1$ (possibly after replacing $T(v)$ with $\check{T}(v) = \mu^{-1}T(v)$ for each $v \in \mathcal{N}$). Thus $Z_n = W_n$ for each n, and $\mathfrak{L}_1$-convergence of $(Z_n)_{n\geq 0}$ is guaranteed by two conditions: The first condition $\gamma = \sum_{i\geq 1} \mathbb{E}T_i \log T_i$ depends on the sequence T_i, $i \geq 1$ itself whereas the second condition $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ depends on the weights T_i, $i \geq 1$ only via their sum. Thus, in contrast to the well-known case of supercritical GWP, an additional assumption has to be imposed explicitly: In that case it suffices to check whether the $Z \log Z$-condition is satisfied because γ is automatically negative as the weights can only take the values 0 or $1/\mathbb{E}Z_1 < 1$ (cf. part (c) of Remark 2.4.23).

In the rest of this thesis, we focus on weighted branching processes $(Z_n)_{n\geq 0}$ which form themselves a martingale, i.e. satisfy

$$\mathbb{E}Z_1 = \sum_{i\geq 1} \mathbb{E}T_i = 1 \tag{3.1.1}$$

and hence $\mathbb{E}Z_n = 1$ for all n. This chapter deals with the question of $\mathfrak{L}_\alpha$-convergence when $\alpha > 1$, i.e. we are searching for a characterization of the condition $\lim_{n\to\infty} \mathbb{E}|W_n - W|^\alpha = 0$ in terms of the generic weight sequence $(T_i)_{i\geq 1}$. For supercritical Galton-Watson processes, it is well-known that the condition $\mathbb{E}Z_1^\alpha < \infty$ is equivalent to $\mathfrak{L}_\alpha$-convergence (cf. [28] and [6]), but in view of what has been mentioned above, it is not surprising that a further assumption on $(T_i)_{i\geq 1}$ which does not only depend on $Z_1 = \sum_{i\geq 1} T_i$ is needed here. It will turn out in Theorem 3.4.3 that this condition reads $\sum_{i\geq 1} \mathbb{E}T_i^\alpha < 1$.

First of all, we mention that by basic facts of martingale theory (consult [53], Section 2.3), we have the characterizations

$$\begin{aligned} W_n \xrightarrow{\mathfrak{L}_\alpha} W \text{ as } n \to \infty \quad &\Longleftrightarrow \quad (W_n^\alpha)_{n\geq 0} \text{ is uniformly integrable} \\ &\Longleftrightarrow \quad \mathbb{E}\left(\sup_{n\geq 0} W_n^\alpha\right) < \infty \\ &\Longleftrightarrow \quad \sup_{n\geq 0} \mathbb{E}W_n^\alpha < \infty \end{aligned}$$

whenever $\alpha > 1$. As will turn out in the following, these assertions are also tantamount with $\mathbb{E}W^\alpha \in (0, \infty)$.

Before describing the method of proof that is going to be used in this chapter, we make some additional assumptions helping us to avoid trivialities and to ensure clearness of our account. The final section of this chapter will be devoted to a justification of these assumptions. In fact, it will come out that these assumptions do not form veritable restrictions.

In what follows, we shall always assume that

$$c = \mathbb{P}\left(T_i \in \{0, 1\} \text{ for all } i \geq 1\right) < 1 \tag{3.1.2}$$

which means that the WBP is *not* a (critical) GWP. Furthermore, recall from (1.1.4) that

$$N = \sum_{i\geq 1} \mathbf{1}_{\{T_i > 0\}}$$

denotes the number of positive weights in the first generation. Given $p > 1$, Liu [78] has provided necessary and sufficient conditions for $0 < \mathbb{E}W^p < \infty$ under the additional assumption

$$\mathbb{P}(N < \infty) = 1. \tag{3.1.3}$$

It should be mentioned that the case $\mathbb{P}(N = \infty) > 0$ is explicitly allowed in this thesis. However, we assume that

$$\mathbb{P}(N \geq 2) > 0 \tag{3.1.4}$$

unless stated otherwise, or equivalently

$$\sup_{j \neq k} \mathbb{P}(T_j \wedge T_k > 0) > 0, \tag{3.1.5}$$

which means that the WBP is not only a multiplicative random walk. Note that together with (3.1.1), this ensures $\Delta := \mathbb{E}\left(\sup_{i \geq 1} T_i\right) < 1$ because $\sup_{i \geq 1} T_i \leq \sum_{i \geq 1} T_i$ with strict inequality holding on a set of positive probability, whence for any sequence $(k_j)_{j \geq 1}$ of positive integers,

$$\mathbb{E}L(k_1, ..., k_n) \leq \Delta^n \underset{n \to \infty}{\longrightarrow} 0. \tag{3.1.6}$$

In particular, $L(k_1, ..., k_n)$ tends to zero in probabilty as $n \to \infty$. By Lemma 3.2 in [35], even $\sup_{|v|=n} L(v) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, and if additionally $\sum_{i \geq 1} \mathbb{E}T_i^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$, this convergence holds even almost surely by Lemma 3.2 of [35].

We close this section with an account of references. As mentioned above, Liu [78] has treated the question of $\mathfrak{L}_\alpha$-convergence for $\alpha > 1$ in the context of generalized multiplicative cascades where (3.1.3) is assumed. As to branching random walks, the same result is due to Biggins [20] if $1 < \alpha \leq 2$. For the same set of exponents α, Rösler, Topchii and Vatutin [97] give a complete result without the assumption (3.1.3), extending the result for $\alpha = 2$ given by Rösler in [95]. A somewhat weaker result can be found in [13] if $N \leq 2$ and $T_1, T_2 \leq 1$ a.s.. Similar conditions as the ones we have to impose for $\mathfrak{L}_\alpha$-convergence appear in the theory of stochastic fixed point equations (see for example [93] or [34]). Finally, note that independently of and simultaneously with our results, Iksanov has characterized $\mathfrak{L}_\alpha$-convergence in the framework of stochastic fixed point equations without assuming (3.1.3). His recent preprint [58] is given as a reference. It will be explained in the following section that his method of proof differs from ours. However, it is also a probabilistic one, being closely related to the spinal tree construction we used in the previous chapter.

3.2 The double martingale structure

The approach to be used in this chapter adapts the one used by Alsmeyer and Rösler [6] in the situation of a normalized supercritical Galton-Watson process. In contrast to analytic methods which have often been used for branching random walks or Galton-Watson processes, it avoids the investigation of the corresponding Laplace or Fourier transforms, but uses rather probabilistic arguments. It is based on the key observation that the underlying branching model bears a *double martingale structure*: Besides forming a martingale itself, $(W_n)_{n\geq 0}$ is related to a family of further martingales in the sense that their increments $D_n = W_n - W_{n-1}$, $n \geq 2$ can also be viewed as martingale limits which will be explained in detail below. Taking this as a starting point, our approach heavily relies on the repeated use of convex function inequalities for martingales due to Topchii and Vatutin, and Burkholder, Davis and Gundy (see Theorem B.4 and Theorem B.2). Note that in the context of generalized renewal measures, a similar approach has been used by Alsmeyer [3].

To begin with, some further notation is necessary: Given any $\alpha \in (0,\infty)$, put

$$g(\alpha) := \sum_{i\geq 1} \mathbb{E}T_i^\alpha \in (0,\infty] \quad \text{and} \quad \mu(\alpha) := \mathbb{E}Z_1^\alpha \in (0,\infty].$$

Due to the standing assumption (3.1.1), $g(1) = \mu(1) = \mu = 1$. Additionally, define

$$\begin{aligned}
Z_n^{(\alpha)} &:= \sum_{|v|=n} L(v)^\alpha, \\
W_n^{(\alpha)} &:= g(\alpha)^{-n} Z_n^{(\alpha)}, \quad W_n := W_n^{(1)} \\
D_n^{(\alpha)} &:= W_n^{(\alpha)} - W_{n-1}^{(\alpha)} = g(\alpha)^{-n} \sum_{|v|=n-1} L(v)^\alpha \left(\sum_{i\geq 1} T_i(v)^\alpha - g(\alpha) \right), \quad D_n := D_n^{(1)} \\
\text{and } \overline{D}_n^{(\alpha)} &:= g(\alpha)^n D_n^{(\alpha)}
\end{aligned}$$

for $n \geq 1$ and $\alpha \in (0,\infty)$ with $g(\alpha) < \infty$. All variables with index 0 are defined as 1 unless stated otherwise. Note that as $W_n^{(1)} = W_n = Z_n = Z_n^{(1)}$ for all $n \geq 0$, we will use both expressions W_n and Z_n interchangeably. Furthermore, observe that if $g(\alpha) < \infty$, $(Z_n^{(\alpha)})_{n\geq 0}$ is the WBP with generic weight sequence $(T_i^\alpha)_{i\geq 1}$, hence $\mathbb{E}Z_n^{(\alpha)} = \sum_{|v|=n} \mathbb{E}L(v)^\alpha = g(\alpha)^n$ for all $n \geq 0$. Similarly, $(W_n^{(\alpha)})_{n\geq 0}$ is the WBP with generic weight sequence $(T_i^\alpha / g(\alpha))_{i\geq 1}$.

Now the key observation is the following: Fix $n \geq 2$ and any enumeration $(\mathbf{v}_j)_{j\geq 1} = (\mathbf{v}_j^{(n-1)})_{j\geq 1}$ of $\{v \in \mathcal{N} : |v| = n-1\} = \mathbb{N}^{n-1}$. Moreover, define the filtration $(\mathcal{H}_m)_{m\geq 0} = (\mathcal{H}_m^{(n-1)})_{m\geq 0}$ by

$$\mathcal{H}_0 := \mathcal{F}_{n-1} \quad \text{and} \quad \mathcal{H}_m := \sigma(\mathcal{F}_{n-1}, T(\mathbf{v}_j)_{1\leq j\leq m}), \quad m \geq 1$$

Then the assumption of independence of $T(v)$, $v \in \mathcal{N}$ ensures that $(\Theta_m)_{m\geq 0}$ is a martingale with respect to $(\mathcal{H}_m)_{m\geq 0}$, where

$$\Theta_m := \sum_{j=1}^{m} L(\mathbf{v}_j)\left(\sum_{i\geq 1} T_i(\mathbf{v}_j) - 1\right) \qquad [= 0 \text{ if } m = 0]$$

yields an a.s. absolutely convergent series with limit D_n.

This martingale structure permits us to apply certain martingale inequalities, leading to expressions involving terms of the form

$$\overline{D}_n^{(s)} = g(s)^n D_n^{(s)} = \sum_{|v|=n-1} L(v)^s \left(\sum_{i\geq 1} T_i(v)^s - g(s)\right)$$

or

$$D_n^{(s)} = g(s)^{-n} \sum_{|v|=n-1} L(v)^s \left(\sum_{i\geq 1} T_i(v)^s - g(s)\right)$$

with $n \geq 0$ and some dyadic power $s \geq 1$. In the same way as above (when $s = 1$) it can be checked that both $\overline{D}_n^{(s)}$ and $D_n^{(s)}$ are martingale limits for fixed n and s. Finally, it is also easy to verify that for any s, the sequences $(D_n^{(s)})_{n\geq 1}$ and $(\overline{D}_n^{(s)})_{n\geq 1}$ constitute sequences of martingale differences, i.e. $\left(\sum_{k=0}^n D_k^{(s)}\right)_{n\geq 0}$ and $\left(\sum_{k=0}^n \overline{D}_k^{(s)}\right)_{n\geq 0}$ are again martingales.

Remark 3.2.1. The method of proof just sketched differs from the ideas Liu [78] and Iksanov [58] employed, respectively. Liu's article involves a change of measure method which is based on the construction of a probability measure $\mathbb{Q}$ (called *Peyrière measure*) on the product space $\Omega \times \mathbf{I}$ (equipped with a suitable σ-algebra), where Ω is the (original) probability space, and $\mathbf{I}$ denotes the set of all infinite sequences of positive integers. Random variables which are defined on Ω and indexed by (finite or infinite) sequences of integers are then understood as random variables on $\Omega \times \mathbf{I}$. This construction is useful because certain moment relations are passed on from the original model to the extended probability space, and additionally, some of these random variables turn out to be independent under the distribution $\mathbb{Q}$. Furthermore, this approach enables Liu to apply results from the theory of random difference equations. A similar construction for Galton-Watson processes can be found in [81].

The starting point for Iksanov's proof is given by the representation (1.1.6). He relates the situation of stochastic fixed point equations to Lemma 2.2.2 (in the case of ordinary weighted branching processes) and to that of so-called *perpetuities* which makes a result from the theory of perpetuities applicable. Note that by Lemma 2.2.2, $\mathbb{E}\phi(\hat{W}_n) = \mathbb{E}W_n\phi(W_n)$ for any $n \geq 0$ and any measurable nonnegative function ϕ, where $(\hat{W}_k)_{k\geq 0}$ is the sequence of random variables defined in Section 2.2.2. This particularly implies $\mathbb{E}W_n^p = \mathbb{E}\hat{W}_n^{p-1}$ whenever $n \geq 0$ and $p \in (1,\infty)$.

3.3 Preparatory moment results

Lemma 3.3.1. *(a) If $g(\alpha) < \infty$ for some $\alpha > 1$, then g is strictly convex in $[1,\alpha]$.*

(b) If $g(x) < 1$ for some $x > 1$, then $g(y) < 1$ for all $y \in (1,x)$.

Proof. (a) As for all $a > 0$, $x \mapsto a^x$ is convex on $[1,\infty)$ (and even strictly convex if $a \neq 1$) we obtain that for all $\lambda \in (0,1)$ and $x, y \in [1,\alpha]$ such that $x \neq y$,

$$\begin{aligned} g(\lambda x + (1-\lambda)y) &= \sum_{i\geq 1} \mathbb{E}T_i^{\lambda x+(1-\lambda)y} \\ &< \sum_{i\geq 1} \mathbb{E}(\lambda T_i^x + (1-\lambda)T_i^y) \\ &= \lambda g(x) + (1-\lambda)g(y) \end{aligned}$$

because $\sum_{i\geq 1}\mathbb{P}(T_i \notin \{0,1\}) > 0$ in view of (3.1.2).

(b) Pick $y \in (1,x)$ and put $\lambda := \frac{x-y}{x-1} \in (0,1)$. Then by (a) and the fact that $g(1) = 1$,

$$g(y) = g(\lambda + (1-\lambda)x) < \lambda + (1-\lambda)g(x) < 1.$$

□

Recall that for positive β with $g(\beta) < \infty$,

$$\overline{D}_0^{(\beta)} = 1 \text{ and } \overline{D}_n^{(\beta)} = g(\beta)^n D_n^{(\beta)} = \sum_{|v|=n-1} L(v)^\beta \left(\sum_{i\geq 1} T_i(v)^\beta - g(\beta)\right), \quad n \geq 1.$$

Moreover, if ψ is a nonnegative function defined on $[0,\infty)$, we say that ψ satisfies a *growth condition* if for some $C = C_\psi \in (0,\infty)$,

$$\psi(2x) \leq C\psi(x), \quad x \geq 0. \tag{3.3.1}$$

Note that if in addition, ψ is nondecreasing, this implies that for any $c > 0$, we can find some constant $C = C_{\psi,c} \in (0,\infty)$ such that

$$\psi(cx) \leq C\psi(x), \quad x \geq 0. \tag{3.3.2}$$

Unless stated otherwise, each function ψ on $[0,\infty)$ is extended to the real line by setting $\psi(x) := \psi(-x)$, $x < 0$. Denote by $\mathfrak{Z}$ the set of all even nonnegative functions ψ which are continuous, nondecreasing on $[0,\infty)$ and satisfy condition (3.3.2), and notice that for any $\psi \in \mathfrak{Z}$ and $m \in \mathbb{Z}$, the function $\mathbb{S}^m\psi$, defined through

$$\mathbb{S}^m f(x) := f(|x|^{2^m}), \quad x \in \mathbb{R},$$

also forms an element of $\mathfrak{Z}$.

Given two expressions A, B, we write $A \prec B$ if $B < \infty$ implies $A < \infty$.

Lemma 3.3.2. *Let $l \in \mathbb{N}$, $\psi \in \mathfrak{Z}$ and assume that $\mathbb{E}\psi(Z_1) < \infty$, $\mu(2^l) < \infty$ and $g(2^l) < 1$. Then*

$$\sup_{n\geq 0} \mathbb{E}\psi(W_n) \prec Q(l,\psi) := Q_1(l,\psi) + Q_2(l,\psi),$$

where

$$Q_1(l,\psi) := \mathbb{E}\mathbb{S}^{-l}\psi\left(\sum_{k\geq 0} \overline{D}_k^{(2^l)}\right), \qquad Q_2(l,\psi) := \sum_{m=0}^{l-1}\sum_{k\geq 0} \mathbb{E}\mathbb{S}^{-m}\psi(\overline{D}_k^{(2^m)}),$$

and for $s > 0$ with $g(s) < \infty$,

$$\overline{D}_0^{(s)} = 1 \qquad \text{and} \qquad \overline{D}_k^{(s)} = \sum_{|v|=k-1} L(v)^s \left(\sum_{i\geq 1} T_i(v)^s - g(s)\right), \quad k \geq 1.$$

Furthermore,

$$\sum_{k\geq 0} \overline{D}_k^{(2^l)} \in [0,\infty) \text{ a.s..}$$

Put $\mathcal{F}_{-1} := \mathcal{F}_0 = \{\emptyset, \Omega\}$ and recall that $\mathcal{F}_n = \sigma(T(v) : |v| \leq n-1)$, $n \geq 1$. In what follows, C always denotes a suitable finite and positive constant which may differ from line to line.

Proof. The proof runs by induction over l. If $l = 1$, an application of the Burkholder-Davis-Gundy inequality (Theorem B.2) yields

$$\mathbb{E}\psi(W) \leq \sup_{n\geq 0} \mathbb{E}\psi(W_n) \leq C\left[\mathbb{E}\mathbb{S}^{-1}\psi\left(\sum_{n\geq 0} \mathbb{E}(D_n^2|\mathcal{F}_{n-1})\right) + \sum_{n\geq 0} \mathbb{E}\psi(D_n)\right].$$

As $g(1) = 1$,

$$\sum_{n\geq 0} \mathbb{E}\psi(D_n) = Q_2(1,\psi).$$

Moreover, if $n \geq 1$, the independence of $T(v)$ and $T(w)$ for $v \neq w$ ensures

$$\begin{aligned}
\mathbb{E}(D_n^2|\mathcal{F}_{n-1}) &= \sum_{|v|=n-1}\sum_{|w|=n-1} L(v)L(w)\mathbb{E}\left[\left(\sum_{i\geq 1} T_i(v) - 1\right)\left(\sum_{i\geq 1} T_i(w) - 1\right)\right] \\
&= \operatorname{Var} Z_1 \sum_{|v|=n-1} L(v)^2 \\
&\leq \mu(2)g(2)^{n-1} W_{n-1}^{(2)} \\
&= \mu(2)g(2)^{n-1}\sum_{k=0}^{n-1} D_k^{(2)} \quad \text{a.s.,}
\end{aligned}$$

and therefrom

$$\begin{aligned}
\sum_{n\geq 1} \mathbb{E}(D_n^2|\mathcal{F}_{n-1}) &= \mu(2)\sum_{n\geq 0} g(2)^n \sum_{k=0}^{n} D_k^{(2)} \\
&= \frac{\mu(2)}{1-g(2)}\sum_{k\geq 0} g(2)^k D_k^{(2)} = \frac{\mu(2)}{1-g(2)}\sum_{k\geq 0}\overline{D}_k^{(2)} \quad \text{a.s..}
\end{aligned}$$

Note that

$$\sum_{n\geq 0}\sum_{k=0}^{n} g(2)^n |D_k^{(2)}| < \infty \text{ a.s.}$$

because $\sup_{k\geq 0}\mathbb{E}|D_k^{(2)}| \leq 2$ implies

$$\mathbb{E}\left(\sum_{n\geq 0}\sum_{k=0}^{n} g(2)^n|D_k^{(2)}|\right) = \sum_{n\geq 0}\sum_{k=0}^{n} g(2)^n\mathbb{E}|D_k^{(2)}| \leq 2\sum_{n\geq 0}(n+1)g(2)^n < \infty.$$

In particular, $\sum_{k\geq 0}\overline{D}_k^{(2)} \in [0,\infty)$ a.s.. In view of $D_0^2 = 1$ and (3.3.2),

$$\begin{aligned}
\sup_{n\geq 0}\mathbb{E}\psi(W_n) &\leq C\left[\mathbb{E}\mathbb{S}^{-1}\psi\left(1 + \frac{\mu(2)}{1-g(2)}\sum_{k\geq 0}\overline{D}_k^{(2)}\right) + Q_2(l,\psi)\right] \\
&\prec Q_1(1,\psi) + Q_2(1,\psi).
\end{aligned}$$

Now suppose that the claim is proved for some $l \in \mathbb{N}$, put $r := 2^l$ and assume that $g(2^{l+1}) < 1$ and $\mu(2^{l+1}) < \infty$. Furthermore, assume that $\sum_{k\geq 0}\overline{D}_k^{(r)} \in [0,\infty)$ a.s.. Note that Lemma 3.3.1 ensures $g(r) < 1$, and that $\mu(r) < \infty$. Hence, by the inductive hypothesis, $\sup_{n\geq 0}\mathbb{E}\psi(W_n) \prec Q_1(l,\psi) + Q_2(l,\psi)$. Now, another application of the Burkholder-

Davis-Gundy inequality together with Fatou's lemma gives the estimate

$$\begin{aligned}
Q_1(l,\psi) &= \mathbb{ES}^{-l}\psi\left(\sum_{k\geq 0}\overline{D}_k^{(r)}\right)\\
&\leq \liminf_{m\to\infty}\mathbb{ES}^{-l}\psi\left(\sum_{k=0}^{m}\overline{D}_k^{(r)}\right)\\
&\leq C\left[\mathbb{ES}^{-l-1}\left(\sum_{k\geq 0}\mathbb{E}\left(\overline{D}_k^{(r)2}\Big|\mathcal{F}_{k-1}\right)\right)+\sum_{k\geq 0}\mathbb{ES}^{-l}\left(\overline{D}_k^{(r)}\right)\right]
\end{aligned}$$

because the sequence $\left(\sum_{k=0}^{m}\overline{D}_k^{(r)}\right)_{m\geq 0}$ forms a martingale as well. Similarly to the case $l=1$, it follows that for $k\geq 1$,

$$\begin{aligned}
\mathbb{E}\left(\overline{D}_k^{(r)2}\Big|\mathcal{F}_{k-1}\right) &\leq \sum_{|v|=k-1}L(v)^{2r}\mathbb{E}\left(\sum_{i\geq 1}T_i^r-g(r)\right)^2\\
&\leq \sum_{|v|=k-1}L(v)^{2r}\mathbb{E}\left(\sum_{i\geq 1}T_i^r\right)^2\\
&\leq \mathbb{E}Z_1^{2r}\sum_{|v|=k-1}L(v)^{2r}\\
&= \mu(2r)g(2r)^{k-1}W_{k-1}^{(2r)} \quad \text{a.s.}
\end{aligned}$$

because $\sum_{i\geq 1}T_i^r\leq Z_1^r$ by Lemma B.1, and consequently

$$\begin{aligned}
\sum_{k\geq 1}\mathbb{E}\left(\overline{D}_k^{(r)2}\Big|\mathcal{F}_{k-1}\right) &\leq \mu(2r)\sum_{k\geq 0}g(2r)^k\sum_{m=0}^{k}D_m^{(2r)}\\
&= \frac{\mu(2r)}{1-g(2r)}\sum_{k\geq 0}g(2r)^kD_k^{(2r)}\\
&= \frac{\mu(2r)}{1-g(2r)}\sum_{k\geq 0}\overline{D}_k^{(2r)} \quad \text{a.s..}
\end{aligned}$$

By using $\sup_{k\geq 0}\mathbb{E}|D_k^{(2r)}|\leq 2$, $\sum_{k\geq 0}g(2r)^k\mathbb{E}|D_k^{(2r)}|<\infty$ which shows $\sum_{k\geq 0}\overline{D}_k^{(2r)}\in[0,\infty)$ with probability 1. To finish our proof, it suffices to show that

$$Q(l,\psi)\prec Q(l+1,\psi).$$

Suppose that $Q(l+1,\psi)<\infty$. Then the finiteness of $Q_2(l,\psi)$ is trivially obtained because

$$\infty > Q(l+1,\psi)\geq Q_2(l+1,\psi)\geq Q_2(l,\psi),$$

and we must only check that $Q_1(l,\psi)<\infty$. But as seen above,

$$Q_1(l,\psi) \le C(I_1(l,\psi)+I_2(l,\psi))$$

with

$$\begin{aligned} I_1(l,\psi) &= \mathbb{E}\mathbb{S}^{-l-1}\psi\left(1+\frac{\mu(2r)}{1-g(2r)}\sum_{k\ge 0}\overline{D}_k^{(2r)}\right)\\ &\prec \mu(2^{l+1})\vee \mathbb{E}\mathbb{S}^{-l-1}\psi\left(\sum_{k\ge 0}\overline{D}_k^{(2r)}\right)\\ &\prec Q_1(l+1,\psi)<\infty \end{aligned}$$

and

$$I_2(l,\psi)=\sum_{k\ge 0}\mathbb{E}\mathbb{S}^{-l}\psi\left(\overline{D}_k^{(r)}\right)\le Q_2(l+1,\psi)<\infty,$$

as demanded. □

Remark 3.3.3. The previous lemma reveals some additional technical difficulty that does not appear in the situation of supercritical Galton-Watson processes (see [6]): When estimating terms of the form $\mathbb{E}\psi(W)$ by means of the Burkholder-Davis-Gundy inequality, processes depending on the random variables $D_n^{(2^l)}$, $n\ge 0$ come into play, where $l\ge 1$. In the Galton-Watson case, the weights $L(v)$, $v\in\mathcal{N}$ only take the values 0 or 1, and the underlying process remains unchanged. This leads to the pleasant technical simplification that the result of such estimations can be expressed in terms of the original process instead of the random variables we have to introduce (cf. Lemmata 4.1 and 4.2 in [6]). However, our calculations comprise the case of supercritical GWP (see part (b) of Remark 3.6.2). Since we are in the general case of arbitrary nonnegative weights, it is an unfortunate effect that our estimates become more complicated.

Lemma 3.3.4. *Suppose that $(\overline{Z}_n)_{n\ge 0}$ is a weighted branching process with generic weights $(\overline{T}_i)_{i\ge 1}$ such that for some $q>1$,*

$$\overline{g}(1)\vee\overline{g}(q)<1,$$

where $\overline{g}(s):=\sum_{i\ge 1}\mathbb{E}\overline{T}_i^s$, $s\ge 1$. Then there exists another weighted branching process $(\hat{Z}_n)_{n\ge 0}$ with generic weights $(\hat{T}_i)_{i\ge 1}$ such that

$$\overline{Z}_n\le \hat{Z}_n \quad \textit{for all } n\ge 0,$$

$$\hat{Z}_1 = c + \overline{Z}_1 \quad a.s. \text{ for some } c > 0$$

and hence $\mathbb{E}f(\hat{Z}_1) \prec \mathbb{E}f(\overline{Z}_1)$ *whenever* $f \in \mathfrak{Z}$, *and*

$$\hat{g}(1) < 1,\ \hat{g}(q) < \hat{g}(1)^q,$$

where $\hat{g}(s) := \sum_{i\geq 1} \mathbb{E}\hat{T}_i^s$, $s \geq 1$.

Proof. To begin with, fix $c > 0$ such that $1 > (c + \overline{g}(1))^q > \frac{\overline{g}(q)+1}{2} \in (0,1)$. For $m \in \mathbb{N}$, let $(\hat{Z}_{m,n})_{n\geq 0}$ be the weighted branching process with weights

$$\hat{T}_{m,i}(v) := \begin{cases} \frac{c}{m} & \text{if } 1 \leq i \leq m \\ \overline{T}_{i-m}(v) & \text{if } i \geq m+1 \end{cases}, \quad v \in \mathcal{N}.$$

Then $\hat{Z}_{m,n} \geq \overline{Z}_n$ for all $n \geq 0$, and $\hat{Z}_{m,1} = c + \overline{Z}_1$. Put $\hat{g}_m(s) := \sum_{i\geq 1} \mathbb{E}\hat{T}_{m,i}^s$ which satisfies

$$\hat{g}_m(s) = m \cdot \left(\frac{c}{m}\right)^s + \overline{g}(s), \quad s \geq 1,$$

in particular $\hat{g}_m(1) = c + \overline{g}(1) < 1$. Now choose m so large that

$$\hat{g}_m(q) = \overline{g}(q) + \frac{c^q}{m^{q-1}} < \frac{\overline{g}(q)+1}{2} < (c + \overline{g}(1))^q = \hat{g}_m(1)^q.$$

The lemma follows if we choose $(\hat{Z}_n)_{n\geq 0} = (\hat{Z}_{m,n})_{n\geq 0}$ for such m because for each $f \in \mathfrak{Z}$,

$$\mathbb{E}f(\hat{Z}_1) = \mathbb{E}f(c + \overline{Z}_1) \prec \mathbb{E}f(\overline{Z}_1).$$

□

3.4 Results

We start with the following (rather weak) result which shows that if $\mathbb{P}(Z_1 \neq 1) > 0$ and $\mathbb{E}W = 1$, the integrability of W^α is not obvious because W will take arbitrary large values, implying that there is an increasing nonnegative function h with $\mathbb{E}h(W) = \infty$.

Theorem 3.4.1. *Suppose that* $\mathbb{E}W = 1$ *and* $\mathbb{P}(Z_1 = 1) < 1$. *Then* W *is unbounded in the sense that for all* $t > 0$,

$$\mathbb{P}(W \geq t) > 0.$$

Consequently, if ϕ *is a measurable, nonnegative and ultimately positive function on* $[0,\infty)$,

$$\mathbb{E}\phi(W) \in (0,\infty].$$

Proof. From the assumptions $\mathbb{E}Z_1 = 1$ and $Z_1 \neq 1$ w.p.p., it follows that $Z_1 > 1$ w.p.p., and consequently $\mathbb{P}(Z_1 > 1+\delta) > 0$ for some $\delta > 0$. Since $\{Z_1 > 1+\delta\} = \bigcup_{k\geq 1}\left\{\sum_{i=1}^k T_i > 1+\delta\right\}$, we can find some $m \geq 1$ such that

$$\lambda := \mathbb{P}\left(\sum_{i=1}^m T_i \geq 1+\delta\right) > 0. \tag{3.4.1}$$

Denote by $(Z_{n,m})_{n\geq 0}$ the weighted branching process with generic weight sequence $(T_{i,m})_{i\geq 1}$, where $T_{i,m} = T_i$ if $i \in \{1,...,m\}$ and $T_{i,m} = 0$ otherwise. In the following, we prove by induction that for all $n \geq 1$,

$$\mathbb{P}(Z_{n,m} \geq (1+\delta)^n) > 0.$$

The case $n = 1$ is just (3.4.1). Supposing that the claim has been proved for some $n \geq 1$, independence gives

$$\begin{aligned}
&\mathbb{P}(Z_{n+1,m} \geq (1+\delta)^{n+1}) \\
&= \mathbb{P}\left(\sum_{v\in\{1,...,m\}^n} L(v)\sum_{i=1}^m T_i(v) \geq (1+\delta)^{n+1}\right) \\
&\geq \mathbb{P}\left(Z_{n,m} \geq (1+\delta)^n, \sum_{i=1}^m T_i(v) \geq 1+\delta \text{ for all } v \in \{1,...,m\}^n\right) \\
&\geq \mathbb{P}\left(Z_{n,m} \geq (1+\delta)^n\right)\cdot \lambda^{m^n} > 0
\end{aligned}$$

because $Z_{n,m} = \sum_{v\in\{1,...,m\}^n} L(v)$. Now let $t > 1$ and choose $n_0 = n_0(t) \in \mathbb{N}$ satisfying $2t \leq (1+\delta)^{n_0}$. Then $Z_{n_0} \geq Z_{n_0,m}$ entails

$$\mathbb{P}(W^* \geq 2t) \geq \mathbb{P}(Z_{n_0} \geq 2t) \geq \mathbb{P}(Z_{n_0} \geq (1+\delta)^{n_0}) \geq \mathbb{P}(Z_{n_0,m} \geq (1+\delta)^{n_0}) > 0.$$

Finally, (1.1.10) (applied to $a = 1/2$ and $2t$ instead of t), shows that for some positive constant B,

$$\mathbb{P}(W > t) \geq B\mathbb{P}(W^* > 2t) > 0$$

which completes the proof. □

Remark 3.4.2. In the context of branching random walks (i.e. when (3.1.3) is assumed), Biggins and Grey [22] have shown (still supposing $\mathbb{E}W = 1$ and $\mathbb{P}(Z_1 \in \cdot) \neq \delta_1$) that the distribution of W has the following properties:

(a) For any $x > 0$, $\mathbb{P}(W = x) = 0$, i.e. $\Lambda := \mathbb{P}(W \in \cdot \cap (0, \infty))$ is a continuous measure on $(\mathbb{R}, \mathbb{B})$ with total mass $\|\Lambda\| = \mathbb{P}(W > 0) \leq 1$.

(b) The conditional distribution $\mathbb{P}(W \in \cdot | W > 0)$ is either singular or absolutely continuous.

(c) If $\mathbb{E}N = \sum_{i \geq 1} \mathbb{P}(T_i > 0) < \infty$, then the conditional distribution $\mathbb{P}(W \in \cdot | W > 0)$ has a Lebesgue density which is continuous on $(0, \infty)$. In the particular case of *homogeneous branching random walks* (cf. [77], Section 0 or [80], Section 8 for a description of the underlying model), Liu [77] has shown that the condition $\mathbb{E}N < \infty$ is superfluous. Concerning supercritical Galton-Watson processes, the corresponding result can be found in [12], Corollary I.10.4.

(d) Moreover, Liu [79] has shown that if $\mathbb{E}W = 1$, $1 \leq N < \infty$ a.s., $\mathbb{P}(N \geq 2) > 0$, $g(1) = 1$, $\mathbb{P}(Z_1 = 1) < 1$, and if the weights are ordered in such a way that for every $i \geq 1$,

$$\sup_{j \geq i} T_j = 0 \text{ a.s. on } \{T_i = 0\},$$

then $\mathbb{E}T_1^{-\varepsilon} < \infty$ for some $\varepsilon > 0$ implies that the law of W has a Lebesgue density on $(0, \infty)$. In addition, $\mathbb{P}(a < W \leq b) > 0$ for all $0 \leq a < b < \infty$.

The subsequent theorem is the main result of this chapter. As explained in Remark 3.2.1, it has been proved by Liu [78] under the additional hypothesis (3.1.3). Recently, Iksanov [58] has given a proof in the general case (by using different methods than we do), see Remark 3.2.1. Alternative proofs appear in [97] for the case $1 < \alpha \leq 2$ and in [95] for the case $\alpha = 2$. If $\alpha \in \mathbb{N}$ and $\sup_{i \geq 1} T_i \leq 1$ a.s., our result coincides with Theorem 2.1 of Mauldin and Williams [86] who used their result for the calculation of the Hausdorff dimension of the limit set in a random recursive construction.

Concerning branching random walks, the corresponding result for $\alpha \in (1, 2]$ is due to Biggins [19, 20]. Further references on similar results proved under weaker assumptions are given by Liu [78]. In Remark 3.6.2(b) we will see that Theorem 3.4.3 covers the well-studied case of supercritical GWP.

Theorem 3.4.3. *Let $\alpha > 1$. Then the following assertions are equivalent:*

(i) $\mathbb{E}Z_1^\alpha < \infty$ *and* $g(\alpha) = \sum_{i \geq 1} \mathbb{E}T_i^\alpha < 1$.

(ii) $\mathbb{E}W^\alpha \in (0, \infty)$.

Remark 3.4.4. The condition $g(\alpha) < 1$ is also of importance for the analysis of stochastic fixed point equations (see e.g. [93] or [58]). In this context, it might be interpreted as a contraction condition. As a further reference, a paper by Baringhaus and Grübel [13] could be mentioned. If $N \leq 2$, $\alpha \in (1,2)$ and $d \in \mathbb{R}$, they treated the class of fixed points ν of the mapping $\mathcal{K}$ from Subsection 1.1.3 which have mean d and finite absolute αth mean by imposing similar conditions and using contraction arguments with respect to the metric

$$d_\alpha(\nu_1, \nu_2) := \int_{\mathbb{R}} |\phi_{\nu_1}(t) - \phi_{\nu_2}(t)| \cdot |t|^{-\alpha-1} \, \lambda(dt),$$

where $\phi_\nu(t) = \int_{\mathbb{R}} e^{itx} \, \nu(dx)$ is the Fourier transform of ν. They even partly relax the assumption of nonnegative weights T_1, T_2.

Proof of Theorem 3.4.3. First, we prove that (i) implies $\sup_{n \geq 0} \mathbb{E} W_n^\alpha < \infty$. In analogy to the proof of Theorem 1.1 in [6], this is done by distinguishing the cases $\alpha \in (2^l, 2^{l+1}]$ for $l \geq 0$, and using an induction over l.

STEP 1.

If $\alpha \in (1,2]$, the proof is actually due to Rösler, Topchii and Vatutin (cf. Theorem 6 of [97]). In this case, $x \mapsto x^\alpha$ is convex with concave derivative, and the Topchii-Vatutin inequality (see Theorem B.4) and the fact that $W_0 = 1$ entail

$$\mathbb{E} W_n^\alpha \leq 1 + \sum_{k=1}^{n} \mathbb{E}|D_k|^\alpha$$

for any $n \geq 1$, hence

$$\mathbb{E} W^\alpha \leq \sup_{n \geq 0} \mathbb{E} W_n^\alpha \leq 1 + \sum_{k \geq 1} \mathbb{E}|D_k|^\alpha$$

since $(W_n)_{n \geq 0}$ is a nonnegative martingale. In the following, we will make use of the fact that each D_k may be viewed as a martingale limit (as described in Section 3.2). For this purpose, fix $k \geq 2$ and an enumeration $(\mathbf{v}_j)_{j \geq 1}$ of $\mathbb{N}^{k-1}$. Define $\Theta_0 := 0$,

$$\Theta_n := \sum_{j=1}^{n} L(\mathbf{v}_j) \left(\sum_{i \geq 1} T_i(\mathbf{v}_j) - 1 \right), \qquad n \geq 1$$

and observe that $D_k = \lim_{n \to \infty} \Theta_n$. The limit is independent of the chosen enumeration $(\mathbf{v}_j)_{j \geq 1}$ of $\mathbb{N}^{k-1}$ because the series converges absolutely. Moreover, it is easily checked that $(\Theta_n)_{n \geq 0}$ is a martingale with respect to $(\mathcal{H}_n)_{n \geq 0}$, where $\mathcal{H}_0 := \mathcal{F}_{k-1}$ and

$\mathcal{H}_n := \sigma(\mathcal{F}_{k-1}, (T(\mathbf{v}_j))_{1\le j\le n})$ if $n \ge 1$. Hence, another application of the Topchii-Vatutin inequality together with Fatou's lemma and (4.3.2) leads to

$$\begin{aligned}\mathbb{E}|D_k|^\alpha &\le \liminf_{n\to\infty}\mathbb{E}|\Theta_n|^\alpha \\ &\le 2\sum_{j\ge1}\mathbb{E}\left|L(\mathbf{v}_j)\left(\sum_{i\ge1}T_i(\mathbf{v}_j)-1\right)\right|^\alpha \\ &= 2\sum_{j\ge1}\mathbb{E}L(\mathbf{v}_j)^\alpha\cdot\mathbb{E}|Z_1-1|^\alpha \\ &= 2\mathbb{E}|Z_1-1|^\alpha\sum_{|v|=k-1}\mathbb{E}L(v)^\alpha \\ &= 2\mathbb{E}|Z_1-1|^\alpha g(\alpha)^{k-1}\end{aligned}$$

if $k \ge 2$, and this estimate trivially also holds in case $k = 1$ since $D_1 = Z_1 - 1$. Thus, as $\mathbb{E}|Z_1-1|^\alpha \prec \mathbb{E}Z_1^\alpha < \infty$ and $g(\alpha) < 1$,

$$\mathbb{E}W^\alpha \le \sup_{n\ge0}\mathbb{E}W_n^\alpha \le 1 + 2\mathbb{E}|Z_1-1|^\alpha\sum_{n\ge1}g(\alpha)^{n-1} < \infty.$$

STEP 2.

Now let $l \ge 0$ and suppose that the claim has been proved for all $\beta \in \cup_{k=0}^{l}(2^k, 2^{k+1}] = (1, 2^{l+1}]$, pick $\alpha \in (2^{l+1}, 2^{l+2}]$ and assume $g(\alpha) < 1$ and $\mathbb{E}Z_1^\alpha < \infty$. Since $\mathbb{E}Z_1^{2^{l+1}} \prec \mathbb{E}Z_1^\alpha < \infty$ and another appeal to Lemma 3.3.1 ensures $g(2^{l+1}) < 1$, Lemma 3.3.2 shows that it is enough to prove that

$$Q_1(l+1,\alpha) + Q_2(l+1,\alpha) := \mathbb{E}\left(\sum_{k\ge0}\overline{D}_k^{(2^{l+1})}\right)^{\alpha/2^{l+1}} + \sum_{m=0}^{l}\sum_{k\ge0}\mathbb{E}\left|\overline{D}_k^{(2^m)}\right|^{\alpha/2^m} < \infty.$$

For this purpose, abbreviate $s := 2^{l+1}$ and $\beta := \alpha/s \in (1,2]$. Then, as the sequence $((\sum_{k=0}^{n}\overline{D}_k^{(s)}, \mathcal{F}_n))_{n\ge0}$ forms a martingale with $\overline{D}_0^{(s)} = 1$, another application of Fatou's lemma and the Topchii-Vatutin inequality gives

$$\begin{aligned}Q_1(l+1,\alpha) &\le \liminf_{n\to\infty}\mathbb{E}\left|\sum_{k=0}^{n}\overline{D}_k^{(s)}\right|^\beta \\ &\le 1 + 2\liminf_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}\left|\overline{D}_k^{(s)}\right|^\beta = 1 + 2\sum_{k\ge1}\mathbb{E}\left|\overline{D}_k^{(s)}\right|^\beta.\end{aligned}$$

Now, viewing each $\overline{D}_k^{(s)}$ as a martingale limit, we can apply the Topchii-Vatutin inequality once more. Using similar arguments as in **STEP 1**, we conclude that for each $k \geq 1$,

$$\begin{aligned}
\mathbb{E}\left|\overline{D}_k^{(s)}\right|^\beta &\leq 2\sum_{|v|=k-1}\mathbb{E}L(v)^{s\beta}\cdot\mathbb{E}\left|\sum_{i\geq 1}T_i^s - g(s)\right|^\beta \\
&\leq 2\mathbb{E}\left(1+\sum_{i\geq 1}T_i^s\right)^\beta \sum_{|v|=k-1}\mathbb{E}L(v)^\alpha \\
&= 2\mathbb{E}\left(1+\sum_{i\geq 1}T_i^s\right)^\beta g(\alpha)^{k-1},
\end{aligned}$$

and

$$\mathbb{E}\left(1+\sum_{i\geq 1}T_i^s\right)^\beta \prec \mathbb{E}\left(\sum_{i\geq 1}T_i^s\right)^\beta \leq \mathbb{E}\left(\sum_{i\geq 1}T_i\right)^{s\beta} = \mathbb{E}Z_1^\alpha < \infty.$$

Putting these estimates together, we get

$$Q_1(l+1,\alpha) \leq 1 + 4\mathbb{E}\left(1+\sum_{i\geq 1}T_i^s\right)^\beta \cdot \sum_{k\geq 1} g(\alpha)^{k-1} < \infty$$

because $g(\alpha) < 1$.

As to $Q_2(l+1,\alpha)$, it suffices to show that for each $m \in \{0, ..., l\}$,

$$U(m,\alpha) := \sum_{k\geq 2}\mathbb{E}\left|\overline{D}_k^{(2^m)}\right|^{\alpha/2^m} \tag{3.4.2}$$

is finite because

$$\mathbb{E}\left|\overline{D}_1^{(2^m)}\right|^{\alpha/2^m} = \mathbb{E}\left|\sum_{i\geq 1}T_i^{2^m} - g(2^m)\right|^{\alpha/2^m} \leq \mathbb{E}\left(1+\sum_{i\geq 1}T_i^{2^m}\right)^{\alpha/2^m} \prec \mathbb{E}Z_1^\alpha < \infty.$$

If $k \geq 2$, we once more make use of the fact that each $\overline{D}_k^{(2^m)}$ can be seen as a martingale limit: Fix $k \geq 2$ and an enumeration $(\mathbf{v}_j)_{j\geq 1}$ of $\mathbb{N}^{k-1}$, put $p := 2^m$, $\delta := \alpha/p > 2$ and

$$Y_j := L(\mathbf{v}_j)^p\left(\sum_{i\geq 1}T_i(\mathbf{v}_j)^p - g(p)\right), \quad j \geq 1.$$

Then by the Burkholder-Davis-Gundy inequality, for some constant $C \in (0,\infty)$ which does not depend on k,

$$\mathbb{E}\left|\overline{D}_k^{(p)}\right|^\delta = \mathbb{E}\left|\sum_{j\geq 1}Y_j\right|^\delta \leq C\left(J_1(k,p,\delta) + J_2(k,p,\delta)\right), \tag{3.4.3}$$

where

$$\begin{aligned}
J_1(k,p,\delta) := \mathbb{E}\left[\sum_{j\ge 1}\mathbb{E}(Y_j^2|\mathcal{H}_{j-1})\right]^{\delta/2} &= \mathbb{E}\left[\sum_{j\ge 1} L(\mathbf{v}_j)^{2p}\mathbb{E}\left(\sum_{i\ge 1} T_i^p - g(p)\right)^2\right]^{\delta/2} \\
&\le \mathbb{E}\left[\sum_{|v|=k-1} L(v)^{2p}\mathbb{E}\left(\sum_{i\ge 1} T_i^p\right)^2\right]^{\delta/2} \\
&\le \mathbb{E}\left[\sum_{|v|=k-1} L(v)^{2p}\mathbb{E}Z_1^{2p}\right]^{\delta/2} \\
&= \mu(2p)^{\delta/2}\mathbb{E}\left[\sum_{|v|=k-1} L(v)^{2p}\right]^{\delta/2}
\end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned}
J_2(k,p,\delta) := \sum_{j\ge 1}\mathbb{E}|Y_j|^\delta &= \sum_{|v|=k-1}\mathbb{E}L(v)^{p\delta}\mathbb{E}\left|\sum_{i\ge 1} T_i^p - g(p)\right|^\delta \\
&\le \mathbb{E}\left(1+\sum_{i\ge 1} T_i^p\right)^\delta \sum_{|v|=k-1}\mathbb{E}L(v)^\alpha \\
&= \mathbb{E}\left(1+\sum_{i\ge 1} T_i^p\right)^\delta g(\alpha)^{k-1}.
\end{aligned} \tag{3.4.5}$$

Note that

$$\mathbb{E}\left(1+\sum_{i\ge 1} T_i^p\right)^\delta \prec \mathbb{E}\left(\sum_{i\ge 1} T_i^p\right)^\delta \le \mathbb{E}Z_1^{p\delta} = \mathbb{E}Z_1^\alpha < \infty.$$

Therefore,

$$U_2(m,\alpha) := \sum_{k\ge 2} J_2(k,p,\delta) \prec \sum_{k\ge 2} g(\alpha)^{k-1} < \infty,$$

and keeping (3.4.2) and (3.4.3) in mind, we still have to make sure that

$$U_1(m,\alpha) := \sum_{k\ge 2} J_1(k,p,\delta) < \infty.$$

For this purpose, observe that

$$J_1(k,p,\delta) \le \mu(2p)^{\delta/2}\mathbb{E}\left[\sum_{|v|=k-1} L(v)^{2p}\right]^{\delta/2} = \mu(2p)^{\delta/2}\mathbb{E}\overline{Z}_{k-1}^{\delta/2}$$

when $(\overline{Z}_n)_{n\geq 0}$ denotes the WBP with generic weight sequence $(\overline{T}_i)_{i\geq 1} = (T_i^{2p})_{i\geq 1}$. As in Lemma 3.3.4, write $\overline{g}(u) = \sum_{i\geq 1} \mathbb{E}\overline{T}_i^u = \sum_{i\geq 1} \mathbb{E}T_i^{2pu}$ for $u \geq 1$ and note that $\delta/2 = \alpha/2p > 1$. By assumption and Lemma 3.3.1, $\overline{g}(1) \vee \overline{g}(\delta/2) = g(2p) \vee g(\alpha) < 1$. Hence by Lemma 3.3.4, there exists another weighted branching process $(\hat{Z}_n)_{n\geq 0}$ with generic weight sequence $(\hat{T}_i)_{i\geq 1}$ such that

$$\overline{Z}_n \leq \hat{Z}_n \quad \text{for all } n \geq 0, \tag{3.4.6}$$

$$\mathbb{E}\hat{Z}_1^{\delta/2} \prec \mathbb{E}\overline{Z}_1^{\delta/2} \tag{3.4.7}$$

and

$$\hat{g}(1) < 1,\ \hat{g}(\delta/2) < \hat{g}(1)^{\delta/2}, \tag{3.4.8}$$

with $\hat{g}(\cdot)$ having the same meaning as in Lemma 3.3.4. Put

$$\hat{W}_n := \hat{Z}_n/\mathbb{E}\hat{Z}_n = \hat{Z}_n/\hat{g}(1)^n, \qquad n \geq 0$$

and note that $\mathbb{E}\hat{W}_n = 1$ for all $n \geq 0$. This leads to

$$\begin{aligned} J_1(k,p,\delta) &\leq \mu(2p)^{\delta/2}\mathbb{E}\overline{Z}_{k-1}^{\delta/2} \\ &\leq \mu(2p)^{\delta/2}\mathbb{E}\hat{Z}_{k-1}^{\delta/2} \qquad \text{by (3.4.6)} \\ &\leq \mu(2p)^{\delta/2}\hat{g}(1)^{(k-1)\delta/2} \cdot \sup_{n\geq 0} \mathbb{E}\hat{W}_n^{\delta/2}. \end{aligned}$$

Evidently, $(\hat{W}_n)_{n\geq 0}$ can also be viewed as a normalized WBP with generic weight sequence $(\hat{T}_i/\hat{g}(1))_{i\geq 1}$. Applying the inductive hypothesis to $(\hat{W}_n)_{n\geq 0}$ instead of $(W_n)_{n\geq 0}$, it follows that

$$\sup_{n\geq 0} \mathbb{E}\hat{W}_n^{\delta/2} < \infty$$

because

(I) $\delta/2 = \alpha/2p \in (1, 2^{l+1}]$,

(II) $\mathbb{E}\hat{W}_1^{\delta/2} = \hat{g}(1)^{-\delta/2}\mathbb{E}\hat{Z}_1^{\delta/2} \overset{(3.4.7)}{\prec} \mathbb{E}\overline{Z}_1^{\delta/2} = \mathbb{E}\left(\sum_{i\geq 1} T_i^{2p}\right)^{\delta/2} \leq \mathbb{E}Z_1^{p\delta} = \mathbb{E}Z_1^{\alpha} < \infty$ and

(III) $\sum_{i\geq 1} \mathbb{E}\left(\frac{\hat{T}_i}{\hat{g}(1)}\right)^{\delta/2} = \frac{\hat{g}(\delta/2)}{\hat{g}(1)^{\delta/2}} < 1$ by (3.4.8).

To finish the proof, observe that

$$U_1(m,\alpha) = \sum_{k\geq 2} J_1(k,p,\delta) \leq \mu(2p)^{\delta/2} \cdot \sup_{n\geq 0} \mathbb{E}\hat{W}_n^{\delta/2} \cdot \sum_{k\geq 1} \hat{g}(1)^{k\delta/2} < \infty$$

because $\mu(2p) \prec \mu(\alpha) < \infty$ and $\hat{g}(1)^{\delta/2} < \hat{g}(1) < 1$.

Now suppose that $\mathbb{E}W^\alpha \in (0,\infty)$. Using Lemma B.1 and the fixed point equation

$$W = \sum_{i\ge 1} T_i W_{(i)} \text{ a.s.}$$

from Theorem 1.1.3, where $W_{(1)}, W_{(2)}, ...$ are independent copies of W and also independent of $\mathcal{F}_1$, we obtain

$$W^\alpha \ge \sum_{i\ge 1} T_i^\alpha W_{(i)}^\alpha \text{ a.s..}$$

More precisely, strict inequality holds with positive probability in view of (3.1.5), the independence of the random variables $(T_1, T_2, ...)$, $W_{(1)}, W_{(2)}, ...$ and the fact that $\mathbb{E}W_{(i)} > 0$ for all $i \ge 1$. Thus,

$$\mathbb{E}W^\alpha > \sum_{i\ge 1} \mathbb{E}(T_i^\alpha W_{(i)}^\alpha) = \mathbb{E}W^\alpha g(\alpha),$$

i.e. $g(\alpha) < 1$ because $0 < \mathbb{E}W^\alpha < \infty$. We remark that in the context of stochastic fixed point equations, similar arguments are employed in [44] and [75]. Finally, the finiteness of $\mathbb{E}Z_1^\alpha$ is implied by the subsequent result.[1] □

In the previous chapter, we have given necessary and sufficient conditions for $\mathbb{E}W = 0$ or $\mathbb{E}W = 1$, respectively. However, without additional assumptions, we cannot guarantee that $\mathbb{E}W \in \{0,1\}$, for instance if $\gamma = -\infty$ and $\mathbb{E}Z_1 \log^+ Z_1 = \infty$ (cf. Theorem 2.4.17).

Theorem 3.4.5. *Suppose that* $\mathbb{E}W = \delta > 0$. *Then*

$$\mathbb{E}\psi(W) \ge \mathbb{E}\psi(\delta Z_1)$$

holds for any convex and increasing function $\psi : [0,\infty) \to [0,\infty)$. *If* $\alpha > 1$ *and* ϕ *is a nonnegative function satisfying* $0 < \mathbb{E}\phi(W) < \infty$ *and*

$$\liminf_{x\to\infty} \frac{\phi(x)}{x^\alpha} \in (0,\infty],$$

then $g(\alpha) < 1$.

Proof. If $\psi(x) = x^\alpha$ for some $\alpha > 1$ or $\psi(x) = e^{sx}$ for some $s > 0$, and (3.1.3) holds, the following proof is due to Liu [75, 78]. Again, we make use of the fixed point equation (1.1.6). Together with the fact that $\mathbb{E}W = \delta$, it implies

$$\mathbb{E}(W|\mathcal{F}_1) = \mathbb{E}\left(\sum_{i\ge 1} T_i W_{(i)} \middle| \mathcal{F}_1\right) = \sum_{i\ge 1} T_i \mathbb{E}(W_{(i)}|\mathcal{F}_1) = \sum_{i\ge 1} T_i \mathbb{E}W = \delta Z_1 \text{ a.s..}$$

[1] If $\mathbb{E}W = 1$, one could also use $Z_1 \le W^*$ and Theorem 1.1.6 to infer $\mathbb{E}Z_1^\alpha \le \mathbb{E}W^{*\alpha} \prec \mathbb{E}W^\alpha < \infty$.

Without loss of generality, we may assume that $\mathbb{E}\psi(W) < \infty$. Then Jensen's inequality gives

$$\mathbb{E}\psi(W) = \mathbb{E}[\mathbb{E}(\psi(W)|\mathcal{F}_1)] \geq \mathbb{E}[\psi(\mathbb{E}(W|\mathcal{F}_1))] = \mathbb{E}\psi(\delta Z_1).$$

The second assertion is an immediate consequence of Theorem 3.4.3, for $\mathbb{E}W^\alpha \prec \mathbb{E}\phi(W)$. □

Remark 3.4.6. The fact that $\mathbb{E}(W|\mathcal{F}_1) = Z_1$ a.s. if $\mathbb{E}W = 1$ which was established in the preceding proof by using the fixed point equation (1.1.6), can also be deduced from a more general result of martingale theory (e.g. [37], Theorem 7.4.3): Since $\mathbb{E}W = 1$ implies uniform integrability of the sequence $(W_n)_{n\geq 0}$ by Theorem 2.4.19, the latter can be extended to a martingale with index set $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$, i.e. writing $W_\infty := W$ and $\mathcal{F}_\infty := \sigma(\cup_{n\geq 0}\mathcal{F}_n) = \sigma(T(v) : v \in \mathcal{N})$, $(W_n)_{n\in\overline{\mathbb{N}}_0}$ forms a martingale with respect to $(\mathcal{F}_n)_{n\in\overline{\mathbb{N}}_0}$. In particular, $\mathbb{E}(W|\mathcal{F}_n) = W_n = Z_n$ a.s. for each $n \geq 0$.

The following lemma can be obtained by combining the previous results with standard results from martingale theory. Recall the standing assumptions (3.1.2) and (3.1.4).

Corollary 3.4.7. *For any $\alpha > 1$, the following conditions are equivalent:*

(i) $\mathbb{E}(\sup_{n\geq 0} W_n^\alpha) < \infty$,

(ii) $\sup_{n\geq 0} \mathbb{E}W_n^\alpha < \infty$,

(iii) $\mathbb{E}W^\alpha \in (0, \infty)$,

(iv) $\mathbb{E}Z_1^\alpha < \infty$ *and* $g(\alpha) < 1$,

(v) $W_n \xrightarrow{\mathfrak{L}_\alpha} W \quad (n \to \infty)$,

(vi) $(W_n^\alpha)_{n\geq 0}$ *is uniformly integrable,*

(vii) $\lim_{n\to\infty} \mathbb{E}W_n^\alpha = \mathbb{E}W^\alpha \in (0, \infty)$.

As a combination of Theorem 3.4.3 and Theorem 3.4.5 shows, $\mathbb{E}Z_1^\alpha = \infty$ implies that for each $m \geq 1$, the sequence $(\mathbb{E}Z_n^\alpha)_{n\geq m}$ is unbounded. The subsequent theorem shows that each of these moments is even infinite.

Theorem 3.4.8. *Suppose that $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing and satisfies (3.3.1) (and hence (3.3.2)) and $\mathbb{E}\psi(Z_1) = \infty$. Then $\mathbb{E}\psi(Z_n) = \infty$ for all $n \geq 1$. In particular, $\mathbb{E}Z_1^\alpha = \infty$ for some $\alpha > 1$ implies $\mathbb{E}Z_n^\alpha = \infty$ for all $n \geq 1$.*

Proof. Fix $n \geq 2$. Then we can find $w \in \mathbb{N}^{n-1}$ and $\varepsilon > 0$ such that $\mathbb{P}(L(w) \geq \varepsilon) > 0$, thus

$$\begin{aligned}
\mathbb{E}\psi(Z_n) &= \mathbb{E}\psi\left(\sum_{|v|=n-1} L(v) \sum_{i\geq 1} T_i(v)\right) \\
&\geq \mathbb{E}\psi\left(L(w)\sum_{i\geq 1} T_i(w)\right) \\
&\geq \int_{[\varepsilon,\infty)} \mathbb{E}\psi(sZ_1)\,\mathbb{P}(L(w)\in ds) \\
&\geq \mathbb{P}(L(w)\geq\varepsilon)\cdot\mathbb{E}\psi(\varepsilon Z_1) = \infty
\end{aligned}$$

because by (3.3.2), $\psi(Z_1) = \psi(\varepsilon^{-1}(\varepsilon Z_1)) \leq C\psi(\varepsilon Z_1)$ a.s. for some $C \in (0,\infty)$, whence $\mathbb{E}\psi(\varepsilon Z_1) = \infty$ as well. □

We have already seen in Theorem 3.4.3 that for arbitrary $\alpha > 1$, $0 < \mathbb{E}W^\alpha < \infty$ if $\mathbb{E}Z_1^\alpha < \infty$ and $g(\alpha) < 1$. In case $\alpha = 2$, a result by Caliebe and Rösler [34] helps us to compute $\mathbb{E}W^2$ explicitly. The result has also been stated by Rösler ([95], Section 4), but the explicit calculation is omitted there.

Theorem 3.4.9. *If $g(2) < 1$ and $\mu(2) = \mathbb{E}Z_1^2 < \infty$, then*

$$\operatorname{Var} W = \frac{\operatorname{Var} Z_1}{1-g(2)} < \infty \tag{3.4.9}$$

or equivalently,

$$\mathbb{E}W^2 = \frac{\mu(2)-g(2)}{1-g(2)} \in [1,\infty).$$

Proof. From Subsection 1.1.3 we know that $\mathbb{P}(W \in \cdot)$ is a fixed point of the mapping $\mathcal{K}$ introduced there. In case $\mathbb{P}(Z_1 = 1) = 1$, it is easy to see that $\mathbb{P}(W = 1) = 1$ as well, and the assertion is trivial because both sides of (3.4.9) are equal to 0. If $\mathbb{P}(Z_1 = 1) < 1$, observe that $(\sum_{i=1}^n T_i)_{n\geq 1}$ converges in $\mathfrak{L}_2$ because all T_i are a.s. nonnegative and Z_1^2 is integrable, hence $Z_1 - \sum_{i=1}^n T_i \xrightarrow{\mathfrak{L}_2} 0$ as $n \to \infty$ by the dominated convergence theorem.

In other words, Condition (A2b) in [34] is satisfied. Therefore, we may apply Theorem 4 in [34] to obtain

$$\operatorname{Var} W = \frac{(\mathbb{E}W)^2 \operatorname{Var} Z_1}{1-g(2)} = \frac{\operatorname{Var} Z_1}{1-g(2)} \in (0,\infty).$$

□

Remark 3.4.10. The proof just given exploits the connection between weighted branching processes and the theory of stochastic fixed point equations by relating the martingale limit W to solutions of certain fixed point equations. On the other hand, it is not difficult to prove Theorem 3.4.9 without any knowledge on stochastic fixed point equations, but by a straightforward calculation which yields that for any $n \geq 1$,

$$\begin{aligned}
\mathbb{E}Z_{n+1}^2 &= \sum_{|v|=n}\sum_{|w|=n} \mathbb{E}(L(v)L(w)) \cdot \mathbb{E}\left(\sum_{i\geq 1}\sum_{j\geq 1} T_i(v)T_j(w)\right) \\
&= \sum_{|v|=n} \mathbb{E}L(v)^2 \cdot \mathbb{E}\left(\sum_{i\geq 1}\sum_{j\geq 1} T_i(v)T_j(v)\right) + (\mathbb{E}Z_1)^2 \sum_{|v|=n}\sum_{|w|=n,\, w\neq v} \mathbb{E}(L(v)L(w)) \\
&= \mu(2)\sum_{|v|=n} \mathbb{E}L(v)^2 + \sum_{|v|=n}\sum_{|w|=n} \mathbb{E}(L(v)L(w)) - \sum_{|v|=n} \mathbb{E}L(v)^2 \\
&= (\mu(2)-1)g(2)^n + \mathbb{E}Z_n^2 \\
&= \operatorname{Var} Z_1 \cdot g(2)^n + \mathbb{E}Z_n^2,
\end{aligned}$$

whence by induction,

$$\mathbb{E}Z_n^2 = 1 + \operatorname{Var} Z_1 \cdot \sum_{k=0}^{n-1} g(2)^k, \qquad n \geq 1.$$

Since integrability of $\sup_{n\geq 0} Z_n^2$ implies $Z_n \xrightarrow{\mathfrak{L}_2} W$, we obtain

$$\mathbb{E}W^2 = \lim_{n\to\infty} \mathbb{E}Z_n^2 = 1 + \frac{\operatorname{Var} Z_1}{1-g(2)} = \frac{\mu(2)-g(2)}{1-g(2)}.$$

Obviously, the sequence $(\mathbb{E}Z_n^2)_{n\geq 0}$ is increasing which is also clear because $(Z_n^2)_{n\geq 0}$ forms a submartingale.

The following theorem deals with the moment generating function of W. It has been stated by Liu [78] subject to (3.1.3). Nevertheless, we will give a proof because Liu uses a result of Rösler (see Theorem 6 in [93]) which is not applicable without corrections. The relevant improvements concerning Rösler's article can be found in [55], Satz 2.3.18.

Given a nonnegative random variable Y on $(\Omega, \mathfrak{A}, \mathbb{P})$, put

$$\mathbb{D}(Y) := \{s > 0 : \mathbb{E}e^{sY} < \infty\}$$

and note that the set $\mathbb{D}(Y)$ is clearly always of the form $\emptyset$, $(0,\infty)$, $(0,\hat{s})$ or $(0,\hat{s}]$ for some $\hat{s} \in (0,\infty)$. Recall that $W^* = \sup_{n\geq 0} W_n$.

Theorem 3.4.11. *Assume that* $\mathbb{E}W = 1$. *Then the following statements are equivalent:*

(i) $\mathbb{D}(Z_1) \neq \emptyset$, *and* $T_i \leq 1$ *a.s. for all* $i \geq 1$.

(ii) $\mathbb{D}(W) \neq \emptyset$.

(iii) $\mathbb{D}(W^*) \neq \emptyset$.

More precisely, we have $\mathbb{D}(W^*) \subset \mathbb{D}(W) \subset \mathbb{D}(Z_1)$, *and* $\mathbb{D}(W)\setminus\mathbb{D}(W^*)$ *consists of at most one point.*

Proof. "$(i) \Rightarrow (ii)$": Obviously, $\mathbb{E}Z_1^2 < \infty$ and $g(2) < 1$ because $\sum_{i\geq 1} T_i^2 \leq \sum_{i\geq 1} T_i$ with strict inequality holding on a set of positive probability by (3.1.2). Hence, in view of (1.1.6), Satz 2.3.18 in [55] (which is a slight modification of Theorem 6 in [93]) gives $\mathbb{E}e^{s_0 W} < \infty$ for some $s_0 > 0$.
"$(ii) \Rightarrow (i)$": Let $s \in \mathbb{D}(W)$. Since $x \mapsto e^{sx}$ is convex and increasing, Theorem 3.4.5 yields $\mathbb{E}e^{sZ_1} \leq \mathbb{E}e^{sW} < \infty$. Moreover, $x^\alpha = o(e^{sx})$ for any $\alpha > 1$ shows $0 < \mathbb{E}W^\alpha < \infty$ for all $\alpha > 1$, hence

$$\sup_{\alpha>1} g(\alpha) \leq 1$$

by another appeal to Theorem 3.4.5. Now it is not difficult to see that $\sup_{i\geq 1} T_i \leq 1$ a.s.: If $A := \{\sup_{i\geq 1} T_i > 1\}$ had positive probability, then

$$g(p) = \sum_{i\geq 1} \mathbb{E}T_i^p \geq \mathbb{E}\left(\sup_{i\geq 1} T_i^p\right) \geq \int_A \left(\sup_{i\geq 1} T_i\right)^p d\mathbb{P} \underset{p\to\infty}{\longrightarrow} \infty$$

by the monotone convergence theorem, a contradiction.
As to the equivalence of (ii) and (iii), observe that $W \leq W^*$ and that by Theorem 1.1.6, for any $t > 1$ and $0 < a < 1$, there is some constant $B = B(a)$ such that

$$\mathbb{P}(W/a > t) \geq B(a)\mathbb{P}(W^* > t).$$

This implies

$$\mathbb{E}e^{(s/a)W} \succ \int_{(0,\infty)} e^{st}\mathbb{P}(W > at)\, \boldsymbol{\lambda}(dt) \succ \int_{(0,\infty)} e^{st}\mathbb{P}(W^* > t)\, \boldsymbol{\lambda}(dt) \succ \mathbb{E}e^{sW^*}$$

for any $s > 0$ and $a \in (0,1)$. Hence, $u \in \mathbb{D}(W)$ implies $u - \varepsilon \in \mathbb{D}(W^*)$ whenever $0 < \varepsilon < u$. Since $\mathbb{D}(W)$ is always of the form $\emptyset$, $(0,\infty)$, $(0,\hat{s})$ or $(0,\hat{s}]$ for some $\hat{s} \in (0,\infty)$, this completes the proof. □

Remark 3.4.12. In our context, the random variable C occurring in Theorem 6 of [93] or Satz 2.3.18 of [55] is identically 0.

Liu [79] has also analysed when W has moments of negative orders and obtained the following result.

Theorem 3.4.13. (Liu [79])
Suppose that the conditions of Remark 3.4.2(d) are satisfied, let $a > 0$ and denote by φ the Laplace transform of the law of W.

(a) If $\mathbb{E}T_1^{-a} < \infty$ and $\mathbb{E}T_1^{-a}\mathbf{1}_{\{N=1\}} < 1$, then

  (i) $\varphi(t) = O(t^{-a})$ as $t \to \infty$,

  (ii) $\mathbb{P}(W \le x) = O(x^a)$ as $x \downarrow 0$,

  (iii) $\mathbb{E}W^{-b} < \infty$ for all $b \in (0,a)$.

(b) If $\mathbb{E}T_1^{-a} < \infty$ and $\mathbb{E}T_1^{-a}\mathbf{1}_{\{N=1\}} = 1$, then $\mathbb{E}W^{-b} < \infty$ if $0 < b < a$, and $\mathbb{E}W^{-b} = \infty$ if $b \ge a$.

As outlined by Liu [77, 80], the theorem can be substantially simplified when focussing on homogeneous branching random walks, a situation which actually corresponds to the case where for fixed $v \in \mathcal{N}$, all positive weights $T_i(v)$ are equal.

3.5 Some remarks on the functions $\mu(\cdot)$ and $g(\cdot)$

The following remark is partly due to Liu [78] under the additional assumption (3.1.3).

Remark 3.5.1. (a) If $\sup_{i\geq 1} T_i \leq 1$ a.s. and $c = \mathbb{P}(T_i \in \{0,1\}$ for all $i) < 1$, then g is strictly decreasing on $[1,\infty)$, in particular $g < 1$ on $(1,\infty)$.[2] Thus, for any $\alpha > 1$,

$$\mathbb{E}W^\alpha \prec \mu(\alpha).$$

(b) If on the other hand $g(p) \leq 1$ for all $p \geq 1$, we have seen in the proof of Theorem 3.4.11 that $\sup_{i\geq 1} T_i \leq 1$ a.s.. Hence, $\mathbb{E}W = 1$ and $\mathbb{E}W^p < \infty$ for all $p \geq 1$ imply $\sup_{i\geq 1} T_i \leq 1$ a.s..

(c) A combination of (a) and (b) gives the following dichotomy concerning g: Either $\sup_{p>1} g(p) \leq 1$ or $\lim_{p\to\infty} g(p) = \infty$, according to whether $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) > 0$, where as in the proof of Theorem 3.4.11, $A = \{\sup_{i\geq 1} T_i > 1\}$. Note that in the latter case, there may be some $p_0 \geq 1$ such that $g(p) = \infty$ for all $p > p_0$. Recall that by convexity, $g(x) < \infty$ for some $x > 1$ implies $g(s) < \infty$ for all $s \in [1, x]$.

(d) Caliebe and Rösler (see Example 3.1 in [35]) have given an example where $g(1) = 1$, but $g(p) = \infty$ for all $p \in (0,\infty) \setminus \{1\}$.

(e) By putting $g(0) := \sum_{i\geq 1} \mathbb{E}T_i^0 \mathbf{1}_{\{T_i>0\}} = \sum_{i\geq 1} \mathbb{P}(T_i > 0) = \mathbb{E}N \in (0,\infty]$, the function g can be extended to the point 0. Now if $g(\beta) < 1$ for some $\beta > 1$, it follows that $\mathbb{E}N \in (1,\infty]$. To justify this assertion, assume w.l.o.g. that $g(0) = \mathbb{E}N < \infty$. Then a slight modification of Lemma 3.3.1 shows that g is strictly convex in $[0,\beta]$. Letting $\theta = \beta^{-1} < 1$, it follows that $1 = g(1) < \theta g(\beta) + (1-\theta)g(0)$, i.e. $g(0) > 1$ which means that the underlying GWP

$$\overline{Z}_n = \sum_{|v|=n} \mathbf{1}_{\{L(v)>0\}}, \quad n \geq 0$$

from Section 1.1 is a supercritical one (with possibly infinite reproduction mean $\overline{\mu} = g(0)$).

(f) If $T_1, T_2, ...$ are assumed to be independent, then $g(0) = \mathbb{E}N < \infty$ iff $N < \infty$ a.s. because $\mathbb{E}N = \sum_{i\geq 1} \mathbb{P}(T_i > 0) = \infty$ implies $\mathbb{P}(\limsup_{i\to\infty}\{T_i > 0\}) = 1$ by the Borel-Cantelli lemma, hence $\mathbb{P}(N = \infty) = 1$. In particular, $\mathbb{P}(N = \infty)$ can only take the values 0 or 1 in this special case.

The subsequent remark collects some relations between the functions $\mu(\cdot)$ and $g(\cdot)$.

[2] $c = 1$ obviously implies $g \equiv 1$ in $[1,\infty)$.

Remark 3.5.2. (a) $g(x) \le \mu(x) = \mathbb{E}Z_1^x$ for all $x > 1$,

(b) $g(x) \ge \mu(x)$ for all $x \in (0,1)$,

(c) If $T_i, i \ge 1$, are independent, Lemma 10.3.1 in [37] states that for any $x > 1$,

$$\mu(x) \prec g(x) \vee g(\tilde{x}),$$

where $\tilde{x} = 1$ if x is an integer and $\tilde{x} = x - \lfloor x \rfloor$ denotes the fractional part of x otherwise.

(d) If $x > 1$ and $N = \sum_{i\ge 1} \mathbf{1}_{\{T_i>0\}}$ is a.s. bounded, i.e. $\|N\|_\infty < \infty$, then $\mu(x) \prec g(x)$.

(a) and (b) follow immediately from the fact that $s \mapsto s^x$ is convex and therefore superadditive in $(0,\infty)$ if $x > 1$ (Lemma B.1), but concave, hence subadditive in case $0 < x < 1$. For (d), observe that

$$\mu(x) = \mathbb{E}Z_1^x \le \|N\|_\infty^x \cdot \mathbb{E}\left(\sup_{i\ge 1} T_i^x\right) \le \|N\|_\infty^x \cdot \mathbb{E}\left(\sum_{i\ge 1} T_i^x\right) = \|N\|_\infty^x \cdot g(x).$$

3.6 An a posteriori justification of the assumptions and some further remarks

Remark 3.6.1. Throughout this chapter, we have assumed that $\mathbb{P}(N \ge 2) > 0$ which means that with positive probability, there are at least two positive weights in the first generation (genuine branching). The purpose of this remark is to make clear that the case $\mathbb{P}(N \le 1) = 1$ is uninteresting. First of all, it is not difficult to see that in the latter case, $(Z_n)_{n\ge 0}$ forms a multiplicative random walk (cf. Example 1.1.2), i.e.

$$Z_n \stackrel{d}{=} \prod_{i=1}^{n} \Xi_i, \quad n \ge 1,$$

with independent copies $\Xi_1, \Xi_2, ...$ of Z_1. Consequently, $\mathbb{E}Z_n^\alpha = (\mu(\alpha))^n$ for each $n \ge 1$ and $\alpha \in (1,\infty)$. Now excluding the trivial case $\mathbb{P}(Z_1 = 1) = 1$ which means that $Z_n = 1$ a.s. for all n, Jensen's inequality gives $\mu(\alpha) > \mu(1)^\alpha = 1$ because $x \mapsto x^\alpha$ is strictly convex. Hence, $\mathbb{E}Z_n^\alpha = (\mu(\alpha))^n \uparrow \infty$ as $n \to \infty$.

The subsequent remark shows that Theorem 3.4.3 comprises the case of supercritical Galton-Watson processes.

Remark 3.6.2. (a) We have imposed the condition

$$c = \mathbb{P}(T_i \in \{0,1\} \text{ for all } i) < 1$$

which is a natural one because in case $c = 1$, $(Z_n)_{n\geq 0}$ forms a critical Galton-Watson process (cf. Example 1.1.1). Hence, $\lim_{n\to\infty} Z_n = 0$ a.s. unless $p_1 = \mathbb{P}(Z_1 = 1) = 1$ (see for example Proposition 1.2 in [7]). In the latter case, $\mathbb{P}(Z_n = W = 1 \text{ for all } n) = 1$, i.e. trivially $\sup_{n\geq 0} \mathbb{E}Z_n^\alpha < \infty$ for any $\alpha > 0$. Otherwise, if $\alpha > 1$, $\sup_{n\geq 0} \mathbb{E}Z_n^\alpha = \infty$ since boundedness of the sequence $(\mathbb{E}Z_n^\alpha)_{n\geq 0}$ would imply uniform integrability of $(Z_n)_{n\geq 0}$, hence $\mathbb{E}W = \lim_{n\to\infty} \mathbb{E}Z_n = 1$, which is a contradiction to $\mathbb{P}(W = 0) = 1$.

(b) Suppose that $(\tilde{Z}_n)_{n\geq 0}$ is a WBP with generic weights $(\tilde{T}_i)_{i\geq 1}$ satisfying

$$\tilde{c} := \mathbb{P}\left(\tilde{T}_i \in \{0,1\} \text{ for all } i \geq 1\right) = 1 \quad \text{and} \quad \tilde{\mu} := \sum_{i\geq 1} \mathbb{E}\tilde{T}_i \in (1,\infty),$$

and denote by $(Z_n)_{n\geq 0}$ the WBP with generic weight sequence $(T_i)_{i\geq 1} := (\tilde{\mu}^{-1}\tilde{T}_i)_{i\geq 1}$. Then $(\tilde{Z}_n)_{n\geq 0}$ forms a supercritical Galton-Watson process, whereas $(Z_n)_{n\geq 0} = (\tilde{Z}_n/\tilde{\mu}^n)_{n\geq 0}$ has $g(1) = 1$ and $g(\alpha) = \tilde{\mu}^{-\alpha}\sum_{i\geq 1} \mathbb{E}\tilde{T}_i = \tilde{\mu}^{1-\alpha} < 1$ for any $\alpha > 1$. As to $W = \lim_{n\to\infty} \tilde{Z}_n/\tilde{\mu}^n = \lim_{n\to\infty} Z_n$, Theorem 3.4.3 gives

$$\mathbb{E}W^\alpha \in (0,\infty) \quad \Leftrightarrow \quad \mathbb{E}Z_1^\alpha < \infty \quad [\Leftrightarrow \quad \mathbb{E}\tilde{Z}_1^\alpha < \infty]$$

for any $\alpha > 1$, in full accordance with the results obtained by Bingham and Doney [28] and by Alsmeyer and Rösler [6].

Remark 3.6.3. It is a well-known fact that $\mathfrak{L}_\alpha$-convergence implies $\mathfrak{L}_1$-convergence when $\alpha > 1$. Thus, $\mathbb{E}W = 1$ if one of the conditions of Theorem 3.4.3 holds. This can also be established by appealing to Theorem 2.4.17:

- The assumption $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ is obviously weaker than the condition $\mathbb{E}Z_1^\alpha < \infty$ for some $\alpha > 1$.

- Since $h(x) = x\log^+ x$ is convex in $[0,\infty)$ with $h(0)=0$, Lemma B.1 shows

$$\gamma_+ = \mathbb{E}\left(\sum_{i\geq 1} h(T_i)\right) \leq \mathbb{E}h(Z_1) < \infty.$$

  The finiteness of γ_+ is also clear if $g(\alpha) < \infty$ for some $\alpha > 1$ because $\log^+ x = o(x^{\alpha-1})$ as $x \to \infty$ yields $\gamma_+ = \sum_{i\geq 1}\mathbb{E}T_i\log^+ T_i \leq C\sum_{i\geq 1}\mathbb{E}T_i^\alpha = Cg(\alpha) < \infty$.

- In Remark 4.3.2 we will give another proof for the fact that $g(\alpha) < 1$ for some $\alpha > 1$ ensures $\gamma < 0$. This proof will be formulated in terms of a random variable having distribution $\xi = \sum_{i\geq 1}\mathbb{E}T_i\mathbf{1}_{\{T_i\in\cdot\}}$ (cf. Remark 2.3.2).

Remark 3.6.4. (a) The condition $\gamma < 0$ is also natural because if $g(1) = 1$ and $g(\alpha) < \infty$ for some $\alpha > 1$, it is not difficult to see that γ is the right derivative of g at 1: Choose $\beta_0 \in (1,\alpha)$ and let $1 < \beta < \beta_0$. Then

$$\frac{g(\beta)-g(1)}{\beta-1} = \sum_{i\geq 1}\frac{\mathbb{E}(T_i^\beta - T_i)\mathbf{1}_{\{T_i\geq 1\}}}{\beta-1} + \sum_{i\geq 1}\frac{\mathbb{E}(T_i^\beta - T_i)\mathbf{1}_{\{T_i<1\}}}{\beta-1} =: \Delta_1(\beta)+\Delta_2(\beta).$$

As to $\Delta_1(\beta)$, pick $i \geq 1$ and observe that for all $\omega \in \{T_i \geq 1\}$, there is some $\theta_i(\omega) \in [1,\beta_0]$ such that

$$\frac{T_i(\omega)^\beta - T_i(\omega)}{\beta-1} = T_i(\omega)^{\theta_i(\omega)}\log T_i(\omega) \leq T_i(\omega)^{\beta_0}\log T_i(\omega) \leq CT_i(\omega)^\alpha$$

with $C = \sup_{x\geq 1}\log x/x^{\alpha-\beta_0} < \infty$. Since $g(\alpha)$ is finite, a double application of the dominated convergence theorem gives

$$\begin{aligned}
\lim_{\beta\downarrow 1}\Delta_1(\beta) &= \sum_{i\geq 1}\left(\lim_{\beta\downarrow 1}\frac{\mathbb{E}(T_i^\beta - T_i)\mathbf{1}_{\{T_i\geq 1\}}}{\beta-1}\right)\\
&= \sum_{i\geq 1}\mathbb{E}\left(\lim_{\beta\downarrow 1}\frac{(T_i^\beta - T_i)\mathbf{1}_{\{T_i\geq 1\}}}{\beta-1}\right)\\
&= \sum_{i\geq 1}\mathbb{E}T_i\log^+ T_i\\
&= \gamma_+ \ [<\infty].
\end{aligned}$$

Turning to $\Delta_2(\beta)$, convexity of $\beta \mapsto T_i(\omega)^\beta$ for any $\omega\in\Omega$ and $i\geq 1$ together with Lemma VI.1.1(d) in [45] ensures that

$$\frac{(T_i^\beta - T_i)\mathbf{1}_{\{T_i<1\}}}{\beta-1}$$

is nonpositive and decreasing as $\beta \downarrow 1$. Consequently,

$$\lim_{\beta\downarrow 1} \Delta_2(\beta) = \sum_{i\geq 1} \mathbb{E}\left(\lim_{\beta\downarrow 1} \frac{(T_i^\beta - T_i)\mathbf{1}_{\{T_i<1\}}}{\beta - 1}\right) = \sum_{i\geq 1} \mathbb{E}T_i \log T_i \mathbf{1}_{\{T_i<1\}} = -\gamma_- \in [-\infty, 0]$$

by a double appeal to the monotone convergence theorem. Hence,

$$g'_+(1) := \lim_{\beta\downarrow 1} \frac{g(\beta) - g(1)}{\beta - 1} = \lim_{\beta\downarrow 1}(\Delta_1(\beta) + \Delta_2(\beta)) = \gamma_+ - \gamma_- = \gamma,$$

as claimed above.

Now if $g < 1$ in a right vicinity of 1, the condition $g'_+(1) = \gamma < 0$ comes out naturally.

(b) By similar calculations it can be shown that if $g(1) = 1$ and $g(\alpha) < \infty$ for some $\alpha > 1$,

- $g \in \mathcal{C}^\infty((1, \alpha))$, i.e. g is infinitely often differentiable in $(1, \alpha)$ with derivatives

$$\frac{d^k}{d\beta^k} g(\beta) = \sum_{i\geq 1} \mathbb{E}T_i^\beta (\log T_i)^k \in (-\infty, \infty), \quad 1 < \beta < \alpha,\ k \geq 1$$

  and

- $\lim_{\beta\downarrow 1} g'(\beta) = g'_+(1) = \gamma \in [-\infty, \infty)$.

Chapter 4

On the existence of ϕ-moments of the martingale limit of a weighted branching process

4.1 Introduction

The preceding chapter has provided a complete characterization of $\mathfrak{L}_\alpha$-convergence for ordinary weighted branching processes whenever $\alpha > 1$. In fact, putting $\phi_\alpha(x) := x^\alpha$, we have seen that W_n converges in $\mathfrak{L}_\alpha$ to its a.s. limit W if and only if $\mathbb{E}\phi_\alpha(W) \in (0, \infty)$ which is also equivalent to the pair of conditions $\mathbb{E}\phi_\alpha(Z_1) < \infty$ and $\sum_{i \geq 1} \mathbb{E}T_i^\alpha < 1$. These facts suggest to search for a similar characterization of $0 < \mathbb{E}\phi(W) < \infty$ when ϕ forms an element of a larger class of functions. More precisely, it is plausible to consider the class of regularly varying functions since these functions can be viewed as interpolations of the family of functions $\phi_\alpha, \alpha \geq 0$ (in a sense which will be explained in Section 4.2). The relevance of regularly varying functions also becomes apparent by the fact that under mild technical assumptions as in Theorem 2.4.17, $\mathfrak{L}_1$-convergence is tantamount with the pair of conditions $\gamma < 0$ and $\mathbb{E}H(Z_1) < \infty$, where $H(x) = x \log^+ x$ is regularly varying.

Before going into detail, let us mention that the question of existence of ϕ-moments of the limit of a (normalized) branching process has been treated in various contexts and by various authors. Concerning regularly varying functions ϕ, Bingham and Doney considered supercritical Galton-Watson processes in their article [28] as well as Crump-Mode and Jirina processes in [29]. They obtained necessary and sufficient conditions for a large class of regularly varying functions. It seems worth noting that their methods were of purely analytical nature. Alsmeyer and Rösler [6] confirmed these results and even dealt with a larger class of (convex) functions. Moreover, they used a completely different approach based on the inherent double martingale structure (as we also did in Chapter 3). As to weighted branching processes, Rösler, Topchii and Vatutin [97] considered the class of so-called *weakly convex* functions which also satisfy some submultiplicative property, the main examples being $x \mapsto x^p$ for $p \in (1,2]$, however.

In what follows, we adapt the approach used by Alsmeyer and Rösler, but unless stated otherwise, we restrict to regularly varying functions. As in the previous chapter, the double martingale structure of the underlying WBP is essential. The methods we use are very similar to that of Chapter 3, but we could not treat both situations simultaneously because in the following, we have to make use of the results obtained for the functions $x \mapsto x^\alpha$, $\alpha > 1$. In particular, Lemma 3.3.2 and Theorem 3.4.3 will play a key role.

To begin with, some further notation is necessary: We call two nonnegative functions f_1 and f_2 on $[0,\infty)$ *asymptotically equivalent* and write $f_1 \asymp f_2$ if

$$0 < \liminf_{x\to\infty} \frac{f_1(x)}{f_2(x)} \le \limsup_{x\to\infty} \frac{f_1(x)}{f_2(x)} < \infty,$$

while $f_1 \sim f_2$ has the usual meaning $\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = 1$. It is a trivial but useful observation that if ϕ and ψ are nonnegative, locally bounded and satisfy $\phi \asymp \psi$, and X is an arbitrary nonnegative random variable, then $\mathbb{E}\phi(X)$ and $\mathbb{E}\psi(X)$ are commonly finite or infinite, i.e. $\mathbb{E}\phi(X) \prec \mathbb{E}\psi(X)$ and $\mathbb{E}\psi(X) \prec \mathbb{E}\phi(X)$, where as in the previous chapter, we write $A \prec B$ if finiteness of B implies that of A. In particular, if $\phi(x) \asymp x^\alpha$ for some $\alpha > 1$, $\mathbb{P}^{Z_1} \neq \delta_1$, and if the basic assumptions of Chapter 3 hold, a combination of Theorem 3.4.1 and Theorem 3.4.3 implies that $\mathbb{E}\phi(W) \in (0,\infty)$ if and only if $g(\alpha) < 1$ and $\mathbb{E}Z_1^\alpha < \infty$.

For fixed $m \in \mathbb{Z}$, we maintain the notation $\mathbb{S}^m f(x) = f(|x|^{2^m})$, $x \in \mathbb{R}$, and C always denotes a finite positive constant which may differ from line to line.

Furthermore, we stick to the assumptions (3.1.1), (3.1.2) and (3.1.4) imposed in Chapter 3.

The further organization of this chapter is as follows. In the subsequent section, we recall some basic facts on regularly varying functions and introduce an important subclass (denoted $\mathfrak{R}^{\mathrm{IC}}$) of this general class of functions which consists of those regularly varying functions whose slowly varying part ℓ is increasing and ultimately log-concave, i.e. $\log \ell$ is ultimately concave. Since we want to apply similar convex function inequalities as in Chapter 3, in particular Lemma 3.3.2 (thus adapting the approach of Alsmeyer and

Rösler [6] for supercritical Galton-Watson processes), Subsection 4.2.1 finds a related class of convex functions. This will allow us to apply the Topchii-Vatutin inequality and the Burkholder-Davis-Gundy inequality in Section 4.4. Furthermore, we will see in Subsection 4.2.2 that any element of the subclass $\mathfrak{R}^{\mathrm{IC}}$ mentioned above is essentially submultiplicative, helping us to use similar arguments as in the case of the multiplicative functions $x \mapsto x^\alpha$ (cf. Theorem 3.4.3). Section 4.3 introduces and analyzes a useful multiplicative random walk related to the underlying process $(Z_n)_{n\geq 0}$. Section 4.4 presents the main result when dealing with the subclass $\mathfrak{R}^{\mathrm{IC}}$. We use a similar approach as for the case $\phi(x) = x^\alpha$ ($\alpha > 1$), but Theorem 3.4.3 is itself an important tool. Note that no additional assumptions on the underlying WBP $(Z_n)_{n\geq 0}$ have to be imposed. On the other hand, if the underlying weight sequence $T_1, T_2, ...$ is a.s. bounded by 1 which implies that along each fixed line of descent, the weights are decreasing, we find an almost complete solution without too strong assumptions on the regularly varying functions considered. This will be demonstrated in Section 4.5. The final section is devoted to applications of the results from Section 4.4 and Section 4.5.

4.2 Functions of regular variation

Recall that a nonnegative and ultimately positive function f defined on $[0,\infty)$ is called *regularly varying of order* $\alpha > 0$ if for any positive t,

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\alpha,$$

i.e. $f(tx) \sim t^\alpha f(x)$ for any $t > 0$. If this relation holds with $\alpha = 0$, f is called *slowly varying*. Plainly, h is regularly varying of order $\alpha > 0$ iff $x \mapsto h(x)x^{-\alpha}$ is slowly varying.

Every nonnegative and measurable slowly varying function ℓ can be written in the form

$$\ell(x) = c(x) \exp\left(\int_{[1,x]} \frac{\varepsilon(u)}{u} \boldsymbol{\lambda}(du)\right), \quad x \geq 1,$$

where

- c is nonnegative and measurable with $\lim_{x\to\infty} c(x) = c \in (0,\infty)$, and
- $u \mapsto \varepsilon(u)/u$ is locally integrable with $\lim_{x\to\infty} \varepsilon(x) = 0$

(see Theorem 1.3.1 in [30]). Consequently,

$$\ell(x) \asymp \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\,\boldsymbol{\lambda}(du)\right),$$

so that, as we are dealing with moments, we may restrict ourselves to considering slowly varying functions of this kind. Given any $\alpha \geq 0$, denote by $\mathfrak{R}_\alpha$ the class of locally bounded, measurable functions from $[0,\infty)$ to $[0,\infty)$ which are regularly varying at infinity with exponent α (slowly varying in case $\alpha = 0$).

Given any slowly varying function ℓ, we put

$$\mathbb{U}\ell(x) := \int_{(0,x]} \frac{\ell(s)}{s}\,\boldsymbol{\lambda}(ds) \quad \text{and} \quad \mathbb{H}\ell(x) := \int_{[x,\infty)} \frac{\ell(s)}{s}\,\boldsymbol{\lambda}(ds) \tag{4.2.1}$$

and stipulate the everywhere finiteness of such functions wherever they appear. Obviously, this is guaranteed for $\mathbb{U}\ell$ if $\frac{\ell(s)}{s}$ is locally integrable in $(0,\infty)$, and for $\mathbb{H}\ell$ if $\frac{\ell(s)}{s}$ is integrable in $[0,\infty)$. Note that $\mathbb{U}\ell$ is nondecreasing, while $\mathbb{H}\ell$ is a nonincreasing function. Furthermore, if $\ell_0, \ell_1 \in \mathfrak{R}_0$ satisfy $\ell_0 \asymp \ell_1$, and both $\mathbb{U}\ell_0$ and $\mathbb{U}\ell_1$ are everywhere finite, then $\mathbb{U}\ell_0$ and $\mathbb{U}\ell_1$ are asymptotically equivalent, i.e. $\mathbb{U}\ell_0 \asymp \mathbb{U}\ell_1$. Finally, if $\mathbb{U}\ell$ is everywhere finite, then by Karamata's theorem (Proposition 1.5.9a in [30]), $\mathbb{U}\ell$ is also slowly varying and grows stronger than ℓ, i.e. $\lim_{x\to\infty} \frac{\mathbb{U}\ell(x)}{\ell(x)} = \infty$. Invoking Proposition 1.5.9b of [30], we infer that the same assertion is also valid for $\mathbb{H}\ell$ instead of $\mathbb{U}\ell$, but we will use this fact only for $\mathbb{U}\ell$.

It has been loosely indicated in the Introduction that one might consider regularly varying functions as interpolations of the family of functions $\phi_\alpha(x) = x^\alpha$, $\alpha \geq 0$. To be precise, let $0 \leq \alpha_0 < \alpha_1 < \alpha_2 < \infty$ and choose $\phi \in \mathfrak{R}_{\alpha_1}$. Then by Proposition 1.3.6 in [30],

$$\lim_{x\to\infty} \frac{\phi(x)}{x^{\alpha_0}} = \lim_{x\to\infty} \frac{x^{\alpha_2}}{\phi(x)} = \infty,$$

i.e. $\phi(x)$ grows stronger than x^{α_0}, but weaker than x^{α_2}. In this sense, $\phi(x)$ lies asymptotically between x^{α_0} and x^{α_2}.

Finally, denote by $\mathfrak{R}_\alpha^{\mathrm{IC}}$ the subclass of functions $\phi \in \mathfrak{R}_\alpha$ of the form

$$\phi(x) = x^\alpha \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\,\boldsymbol{\lambda}(du)\right) \tag{4.2.2}$$

such that ε is nonnegative and ultimately nonincreasing with

$$\int_{[1,\infty)} \frac{\varepsilon(u)}{u}\,\boldsymbol{\lambda}(du) = \infty$$

(IC stands for *increasing concave* which is justified because choosing $a > 0$ sufficiently large, $x \mapsto \int_{[a,x]} \frac{\varepsilon(u)}{u} \boldsymbol{\lambda}(du)$ is increasing and concave in $[a, \infty)$.). Furthermore, put $\mathfrak{R}^{\mathrm{IC}} = \cup_{\alpha>0}\mathfrak{R}_\alpha^{\mathrm{IC}}$.

Obviously, each $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ is continuous, and satisfies $x^\alpha = o(\phi(x))$ and $\phi(x) = o(x^{\alpha+\delta})$ as $x \to \infty$ whenever δ is positive. In particular, $\phi(x) \not\asymp x^\beta$ for any $\beta > 0$.

4.2.1 Regular variation and convexity

A regularly varying function ϕ is in general neither convex nor smooth. Since we want to apply certain convex function inequalities in this chapter, we give some preparatory remarks which link regular variation, convexity and other useful properties. Most of these remarks are essentially taken from the first two sections in [6].

Let us stipulate hereafter that the usual primed notation for derivatives of a convex or concave function on $(0, \infty)$ is always to be understood in the right sense whenever right and left derivatives differ. Whenever necessary and without further notice, any function ψ defined on $[0, \infty)$ is extended to the real line by putting $\psi(x) = \psi(-x)$ for $x < 0$.

Now let $\mathfrak{C}_0$ be the class of convex differentiable functions ψ on $[0, \infty)$ which are (strictly) increasing on $[0, \infty)$ with $\psi(0) = 0$ and concave derivative ψ' on $(0, \infty)$ fulfilling $\lim_{x\downarrow 0} \psi'(x) = 0$. Obviously, $x \mapsto x^\alpha$ is an element of $\mathfrak{C}_0$ if $1 < \alpha \leq 2$, but the identity function does not belong to $\mathfrak{C}_0$. Given $n \geq 1$, we put

$$\mathfrak{C}_n := \mathbb{S}^n\mathfrak{C}_0 = \{\mathbb{S}^n\phi : \phi \in \mathfrak{C}_0\},$$

where it should be recalled that $\mathbb{S}^n\phi(x) = \phi(|x|^{2^n})$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, and finally

$$\mathfrak{C} := \bigcup_{n\geq 0} \mathfrak{C}_n.$$

Plainly, each $\psi \in \mathfrak{C}$ is convex with $\psi(0) = 0$ and therefore superadditive by Lemma B.1. As two further useful properties of functions in $\mathfrak{C}$, we mention (see Lemmata 3.3 and 3.4 in [6])

$$\psi(2x) \leq C\phi(x), \quad x \geq 0 \tag{4.2.3}$$

for some $C = C_\psi \in (0, \infty)$, and

$$\psi \in \mathfrak{C}_n \Rightarrow \liminf_{x\to\infty} \frac{\psi(x)}{x^{2^n}} > 0 \tag{4.2.4}$$

for any $n \geq 0$. Note that (4.2.3) and the monotonicity of ψ yield

$$\psi(cx) \leq C\psi(x), \quad x \geq 0 \tag{4.2.5}$$

for any $c > 0$ and some $C = C_{\psi,c} \in (0, \infty)$. Particularly, $\mathfrak{C}$ is contained in the function class $\mathfrak{Z}$ considered in Section 3.3.

Furthermore, define

$$\mathfrak{C}_0^* := \{\phi \in \mathfrak{C}_0 : \phi''(0) \in (0, \infty)\}$$

and $\mathfrak{C}^* := \cup_{n \geq 0} \mathfrak{C}_n^*$, where $\mathfrak{C}_n^* := \mathbb{S}^n \mathfrak{C}_0^* = \{\mathbb{S}^n \phi : \phi \in \mathfrak{C}_0^*\}$ for $n \geq 1$. It should be mentioned that this notation slightly differs from that used in [6]. Note that if $\psi \in \mathfrak{C}_n^*$ for some $n \geq 1$, then $\psi''(0) = 0$. Given any element ϕ of $\mathfrak{C}_0$, Lemma 3.3 in [6] asserts that there is some $\hat{\phi} \in \mathfrak{C}_0^*$ such that $\phi \sim \hat{\phi}$. Consequently, if $n \geq 1$ and $\psi \in \mathfrak{C}_n$, then $\psi \sim \hat{\psi}$ for some $\hat{\psi} \in \mathfrak{C}_n^*$.

Now the following lemma links regularly variation and the function classes just introduced, permitting us to allow convex function inequalities in the course of this chapter.

Lemma 4.2.1. *Given $\phi(x) = x^\alpha \ell(x) \in \mathfrak{R}_\alpha$ for some $\alpha \geq 1$, the following assertions hold true:*

(a) If $2^n < \alpha < 2^{n+1}$ for some $n \geq 0$, then $\phi \asymp \psi$ for some $\psi \in \mathfrak{C}_n^* \cap \mathfrak{R}_\alpha$.

(b) If $\alpha = 2^n$ for some $n \geq 0$ and $\ell \asymp \mathbb{U}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$, then $\phi \asymp \psi$ for some $\psi \in \mathfrak{C}_n^* \cap \mathfrak{R}_{2^n}$.

(c) If $\alpha = 2^n$ for some $n \geq 1$ and $\ell \asymp \mathbb{H}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$, then $\phi \asymp \psi$ for some $\psi \in \mathfrak{C}_{n-1}^* \cap \mathfrak{R}_{2^n}$.

Proof. This follows immediately by combining Lemma 2.1 and Lemma 3.3 of [6]. □

The following lemma collects further important properties shared by all elements of $\mathfrak{C}^*$, summarizing Lemma 3.3 and Lemma 3.4 of [6].

Lemma 4.2.2. *Let $\psi \in \mathfrak{C}_n^*$ for some $n \geq 0$. Then the following assertions hold true:*

(a) $\frac{\psi(x)}{x^{2^n}}$ is nondecreasing and $\frac{\psi(x)}{x^{2^{n+1}}}$ is nonincreasing in $x \geq 0$.

(b) $\lim_{x\downarrow 0}\frac{\psi(x)}{x^{2^n}} = (\mathbb{S}^{-n}\psi)'(0) = 0$ *and* $\lim_{x\downarrow 0}\frac{\psi(x)}{x^{2^{n+1}}} = \frac{(\mathbb{S}^{-n}\psi)''(0)}{2} \in (0,\infty)$.

In other words, any $\psi \in \mathfrak{C}_n^*$ $(n \geq 0)$ satisfies

$$\psi(s) = O(s^{2^{n+1}}), \quad s \downarrow 0, \tag{4.2.6}$$

so a fortiori

$$\psi(s) = o(s^{\alpha}), \quad s \downarrow 0 \tag{4.2.7}$$

whenever $0 < \alpha < 2^{n+1}$.

Given any nondecreasing convex function $\phi : [0,\infty) \to [0,\infty)$, we next define the associated function $\mathbb{L}\phi$ through

$$\mathbb{L}\phi(z) := \int_{(0,z]}\int_{(0,s]}\frac{\phi'(r)}{r}\,\boldsymbol{\lambda}(dr)\,\boldsymbol{\lambda}(ds), \quad z \geq 0. \tag{4.2.8}$$

If $\phi \in \mathfrak{C}^*$, then $\mathbb{L}\phi$ is everywhere finite, i.e. $\mathfrak{C}^* \subset \{\phi \in \mathfrak{C} : \mathbb{L}\phi(z) < \infty \text{ for all } z \geq 0\}$ since $\frac{\phi'(r)}{r}$ is integrable at 0. In case $\phi \in \mathfrak{C}_0^*$, the function $\mathbb{L}\phi$ will be of importance in our analysis. Therefore, the subsequent lemma collects a number of properties of the function $\mathbb{L}\phi$ associated with any $\phi \in \mathfrak{C}^*$ and relates it to the function ϕ itself. As a reference for the lemma, we mention Lemma 2.2 and Lemma 3.5 of [6].

Lemma 4.2.3. *Let $\phi \in \mathfrak{C}_n^*$ for some $n \geq 0$. Then $\mathbb{L}\phi$ is everywhere finite and convex and satisfies*

$$\liminf_{x\to\infty}\frac{\mathbb{L}\phi(x)}{\phi(x)} > 0$$

as well as

$$\liminf_{x\to\infty}\frac{\mathbb{L}\phi(x)}{x\log x} > 0.$$

If $n \geq 1$, then $2\phi(x/2) \leq \mathbb{L}\phi(x) \leq \phi(x)$ for all $x \geq 0$, in particular $\mathbb{L}\phi \asymp \phi$ by (4.2.3), whereas in case $n = 0$, $\mathbb{L}\phi \geq \phi$. More specifically, if $\phi(x) = x^{\alpha}\ell(x) \in \mathfrak{C}^* \cap \mathfrak{R}_{\alpha}$ for some $\alpha > 1$, then

$$\mathbb{L}\phi(x) \sim \frac{\phi(x)}{\alpha - 1},$$

while in case $\phi(x) = x\ell(x) \in \mathfrak{C}_0^* \cap \mathfrak{R}_1$,

$$\mathbb{L}\phi(x) \sim x\mathbb{U}\ell(x) = x\int_{(0,x]}\frac{\ell(s)}{s}\,\boldsymbol{\lambda}(ds)$$

and

$$\lim_{x\to\infty}\frac{\mathbb{L}\phi(x)}{\phi(x)} = \infty.$$

Remark 4.2.4. Any increasing convex function $\phi : [0,\infty) \to [0,\infty)$ satisfies the relation $\phi(x) \le x\phi'(x) \le \phi(2x)$, $x \ge 0$ (see [6]). In case $\phi \in \mathfrak{C}$, ϕ also satisfies (4.2.3) and therefore

$$\phi(x) \asymp x\phi'(x). \tag{4.2.9}$$

Moreover, if φ and ψ are two asymptotically equivalent elements of $\{\phi \in \mathfrak{C}_0 : \mathbb{L}\phi < \infty\}$, then $\mathbb{L}\varphi, \mathbb{L}\psi$ also belong to $\mathfrak{C}_0$ by Lemma 3.5 of [6], and $\mathbb{L}\varphi \asymp \mathbb{L}\psi$. To justify the last assertion, observe that by (4.2.9), $\mathbb{L}\varphi \asymp \mathbb{L}\psi$ if and only if

$$(\mathbb{L}\varphi)'(x) = \int_{(0,x]} \frac{\varphi'(s)}{s}\,\boldsymbol{\lambda}(ds) \asymp \int_{(0,x]} \frac{\psi'(s)}{s}\,\boldsymbol{\lambda}(ds) = (\mathbb{L}\psi)'(x)$$

which is readily established when using (4.2.9) and $\varphi \asymp \psi$ because $\frac{\varphi'(s)}{s} \asymp \frac{\varphi(s)}{s^2}$ and $\frac{\psi'(s)}{s} \asymp \frac{\psi(s)}{s^2}$.

4.2.2 Regular variation and submultiplicativity

Moment results concerning $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ will be proved by similar arguments as for the case $\phi(x) = x^\alpha$. Therefore, it is necessary to find estimates for expressions of the form $\phi(xy)$ in terms of $\phi(x)$ and $\phi(y)$. The following lemma shows that each $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ is essentially submultiplicative. More precisely, ϕ turns out to be asymptotically equivalent to a submultiplicative function.

Given $a \ge 0$ and $\varepsilon : [0,\infty) \to \mathbb{R}$ with $\lim_{x\to\infty} \varepsilon(x) = 0$ such that $\varepsilon(u)/u$ is locally integrable, we define

$$\ell_{a,\varepsilon}(x) := \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(s+a)}{s}\,\boldsymbol{\lambda}(ds)\right), \quad x \ge 0$$

which is of type IC whenever ε is nonnegative and ultimately nonincreasing.

Lemma 4.2.5. *Let* $\ell(x) = \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(s)}{s}\,\boldsymbol{\lambda}(ds)\right) \in \mathfrak{R}_0^{\mathrm{IC}}$ *and* $\alpha > 0$.

(a) For each $a > 0$, $\ell_{a,\varepsilon} \asymp \ell_{0,\varepsilon} = \ell$.

(b) If ε *is nonincreasing and nonnegative in* $[a+1,\infty)$, *then* $\ell_{a,\varepsilon}$ *is submultiplicative and increasing, i.e.*

$$\ell_{a,\varepsilon}(xy) \le \ell_{a,\varepsilon}(x)\cdot\ell_{a,\varepsilon}(y) \text{ for all } x,y \ge 0.$$

(c) *Each* $\ell \in \mathfrak{R}_0^{\mathrm{IC}}$ *satisfies* $\ell \asymp \tilde{\ell}$ *for some increasing submultiplicative* $\tilde{\ell} \in \mathfrak{R}_0^{\mathrm{IC}}$. *Consequently, each* $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ *is asymptotically equivalent to some submultiplicative* $\varphi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ *with increasing slowly varying part* $\tilde{\ell}$.

Proof. (a) For all $x \geq a+1$,

$$\begin{aligned}
\frac{\ell_{a,\varepsilon}(x)}{\ell_{0,\varepsilon}(x)} &= \exp\left(\int_{[1,x]} \left(\frac{\varepsilon(s+a)}{s} - \frac{\varepsilon(s)}{s}\right) \boldsymbol{\lambda}(ds)\right) \\
&\geq \exp\left(\int_{[1,x]} \left(\frac{\varepsilon(s+a)}{s+a} - \frac{\varepsilon(s)}{s}\right) \boldsymbol{\lambda}(ds)\right) \\
&= \exp\left(\int_{[1+a,x+a]} \frac{\varepsilon(s)}{s} \boldsymbol{\lambda}(ds) - \int_{[1,x]} \frac{\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \\
&= \exp\left(\int_{[x,x+a]} \frac{\varepsilon(s)}{s} \boldsymbol{\lambda}(ds) - \int_{[1,1+a]} \frac{\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \\
&\geq \exp\left(-\int_{[1,1+a]} \frac{\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \\
&= \ell(1+a)^{-1} \in (0,\infty).
\end{aligned}$$

On the other hand, choose $b \geq 1$ such that ε is nonincreasing in $[b,\infty)$, and fix $x \geq b$. Then

$$\begin{aligned}
\frac{\ell_{a,\varepsilon}(x)}{\ell_{0,\varepsilon}(x)} &= \exp\left(\int_{[1,x]} \frac{\varepsilon(s+a)-\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \\
&= \exp\left(\int_{[1,b]} \frac{\varepsilon(s+a)-\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \cdot \exp\left(\int_{[b,x]} \frac{\varepsilon(s+a)-\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) \\
&\leq \exp\left(\int_{[1,b]} \frac{\varepsilon(s+a)-\varepsilon(s)}{s} \boldsymbol{\lambda}(ds)\right) < \infty
\end{aligned}$$

because $\varepsilon(s+a) - \varepsilon(s) \leq 0$ for all $s \geq b$.

(b) Put $\mathbf{L}(x) := \log \ell_{a,\varepsilon}(x)$ for which we have to verify

$$\mathbf{L}(xy) \leq \mathbf{L}(x) + \mathbf{L}(y) \text{ for all } x,y \geq 0.$$

But since

$$\log(1 \vee xy) \leq \log(1 \vee x) + \log(1 \vee y) \text{ for all } x,y \geq 0$$

and $\varepsilon_a := \varepsilon(a+\cdot)$ is nonincreasing in $[1,\infty)$, this follows from

$$\begin{aligned}
\mathbf{L}(xy) &= \int_{[1,1\vee xy]} \frac{\varepsilon_a(s)}{s} \boldsymbol{\lambda}(ds) = \int_{[0,\log(1\vee xy)]} \varepsilon_a(e^u) \boldsymbol{\lambda}(du) \\
&\leq \int_{[0,\log(1\vee x)]} \varepsilon_a(e^u) \boldsymbol{\lambda}(du) + \int_{[\log(1\vee x),\log(1\vee x)+\log(1\vee y)]} \varepsilon_a(e^u) \boldsymbol{\lambda}(du)
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{[0,\log(1\vee x)]} \varepsilon_a(e^u)\,\boldsymbol{\lambda}(du) + \int_{[0,\log(1\vee y)]} \varepsilon_a(e^u)\,\boldsymbol{\lambda}(du) \\
&= \mathbf{L}(x) + \mathbf{L}(y)
\end{aligned}
$$

for all $x, y \ge 0$.

(c) follows from (a), (b) and the multiplicativity of $x \mapsto x^\alpha$.

□

The following lemma provides estimates for expressions of the form $\psi(xy)$ when ψ is an element of $\mathfrak{C}^*$ and also asymptotically equivalent to an element of $\mathfrak{R}_\alpha^{\mathrm{IC}}$. Note that by the preceding auxiliary result, ψ is also asymptotically submultiplicative.

Lemma 4.2.6. *Given $\alpha \in (2^n, 2^{n+1})$ for some $n \ge 0$, assume that $\psi(x) = x^\alpha \hat{\ell}(x) \in \mathfrak{C}_n^* \cap \mathfrak{R}_\alpha$ and $\psi \asymp \phi$ for some $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$. Then the following assertions hold true:*

(a) There is a finite positive constant $C > 0$ such that

$$\psi(xy) \le C[\psi(x) + x^\alpha] \cdot [(\psi(y) + y^\alpha) \vee 1] \text{ for all } x, y \ge 0.$$

(b) Put $\overline{\psi}(x) := x^{-1}\psi(x)$, $x > 0$. Then

$$\psi(xy) \le C x^\alpha (\psi(y) + y^\alpha) \text{ for all } (x, y) \in (0,1) \times [1,\infty)$$

and

$$\overline{\psi}(xy) \le C x^{\alpha-1} (\overline{\psi}(y) + y^{\alpha-1}) \text{ for all } (x, y) \in (0,1) \times [1,\infty).$$

Proof. (a) Put $q(x,y) := [\psi(x) + x^\alpha] \cdot [(\psi(y) + y^\alpha) \vee 1]$ and write ϕ in the form $\phi(x) = x^\alpha \ell(x)$. By Lemma 4.2.5(c),

$$\phi(x) \asymp \varphi(x) = x^\alpha \tilde{\ell}(x) \in \mathfrak{R}_\alpha \tag{4.2.10}$$

such that φ is submultiplicative in $[0,\infty)$. Since $\ell \asymp \hat{\ell} \asymp \tilde{\ell}$ and $\lim_{s\to\infty} \ell(s) = \infty$, the same is true for $\hat{\ell}$ and $\tilde{\ell}$, and we can pick $C_1 > 1$ such that

$$\inf_{s \ge C_1} \left(\ell(s) \wedge \hat{\ell}(s)\right) \ge 1,$$

$$0 < \inf_{s \ge C_1} \frac{\psi(s)}{\varphi(s)} \le \sup_{s \ge C_1} \frac{\psi(s)}{\varphi(s)} < \infty \tag{4.2.11}$$

and

$$0 < \inf_{s \geq C_1} \frac{\psi(s)}{\phi(s)} \leq \sup_{s \geq C_1} \frac{\psi(s)}{\phi(s)} < \infty \tag{4.2.12}$$

because $\psi \asymp \varphi$ and $\psi \asymp \phi$.

Fix $\eta \in (0,1)$. From the uniform convergence theorem for regularly varying functions (see Theorem 1.5.2 in [30]), we know that

$$\lim_{y \to \infty} \frac{\psi(xy)}{\psi(y)} = x^\alpha \text{ uniformly in } x \in [\eta/2, C_1].$$

As $x \mapsto x^{-\alpha}$ is bounded in $[\eta/2, C_1]$, we conclude that

$$\lim_{y \to \infty} \frac{\psi(xy)}{\psi(y) x^\alpha} = 1 \text{ uniformly in } x \in [\eta/2, C_1],$$

i.e. there is some $C_2 \geq C_1$ such that

$$\psi(xy) \leq 2x^\alpha \psi(y) \leq 2q(x,y) \text{ for all } x \in [\eta/2, C_1] \text{ and all } y \geq C_2. \tag{4.2.13}$$

We now distinguish various locations of x and y :

Case 1 $(x \geq C_1, y \geq C_2)$

If $x \geq C_1$ and $y \geq C_2$, in particular $xy \geq C_1$, we obtain from (4.2.11) that

$$\psi(xy) \leq C\varphi(xy) \leq C\varphi(x)\varphi(y) \leq C\psi(x)\psi(y) \leq Cq(x,y).$$

Case 2 $(y < C_2)$

As ψ is nondecreasing, (4.2.5) gives

$$\psi(xy) \leq \psi(C_2 x) \leq C\psi(x) \leq Cq(x,y).$$

Case 3 $(x < C_1, y \geq C_2)$

This is the most difficult case and requires some more work. If $x \geq \eta/2$, (4.2.13) gives the assertion, so we may assume $0 < x < \eta/2$, $y \geq C_2$ and distinguish three subcases.

(i) If $xy \geq C_2 \geq C_1$, (4.2.12) shows

$$\frac{\psi(xy)}{\psi(y)} \leq \frac{C\phi(xy)}{\phi(y)} = \frac{Cx^\alpha \ell(xy)}{\ell(y)} \leq Cx^\alpha$$

because ℓ is nondecreasing in $[C_2, \infty)$ and $x < 1$. Hence,

$$\psi(xy) \leq Cx^\alpha \psi(y) \leq Cq(x,y).$$

(ii) If $\eta < xy < C_2$,

$$\frac{\psi(xy)}{\psi(y)} = \frac{x^\alpha \hat{\ell}(xy)}{\hat{\ell}(y)} \le x^\alpha \sup_{s \in [\eta, C_2]} \hat{\ell}(s) = C x^\alpha$$

because $\hat{\ell}(y) \ge 1$ and $\hat{\ell}(s) = s^{-\alpha}\psi(s)$ is continuous and therefore bounded on $[\eta, C_2]$. Again, this yields $\psi(xy) \le Cq(x,y)$.

(iii) If $0 \le xy \le \eta\,(< 1)$, (4.2.6) implies

$$\psi(xy) \le C(xy)^{2^{n+1}} \le C(xy)^\alpha \le Cq(x,y).$$

The proof of (a) is herewith complete.

(b) is an immediate consequence of (a) and (4.2.7).

□

4.3 The related multiplicative random walk

In the following, the measure

$$\xi := \sum_{i \ge 1} \mathbb{E} T_i \mathbf{1}_{\{T_i \in \cdot\}}$$

will play an important role. Due to our assumption $g(1) = \sum_{i \ge 1} \mathbb{E} T_i = 1$, ξ is a distribution on $(0, \infty)$. Suppose that $X_1, X_2, ...$ are independent random variables with distribution ξ and denote by $(M_n)_{n \ge 0}$ the associated multiplicative random walk, i.e. $M_0 := 1$ and

$$M_n := \prod_{i=1}^n X_i, \quad n \ge 1.$$

The following result gives the connection of $(M_n)_{n \ge 0}$ to weighted branching processes. It is due to Biggins and Kyprianou if (3.1.3) holds (see [24], Lemma 4.1 (iii)). Although the proof in our situation does not require additional arguments, we give it in our terminology. In the case where $\|N\|_\infty = \operatorname{ess\,sup} N < \infty$, a similar random walk has been used by Durrett and Liggett [44].

Lemma 4.3.1. *(a) For all $n \ge 1$ and $x_1, ..., x_n > 0$,*

$$\mathbb{P}(M_1 \le x_1, ..., M_n \le x_n) = \mathbb{E}\left[\sum_{|v|=n} L(v) \prod_{k=1}^n \mathbf{1}_{(0,x_k]}(L(v_1, ..., v_k))\right].$$

This implies

$$\mathbb{E}f(M_n) = \mathbb{E}\left[\sum_{|v|=n} L(v) f(L(v))\right] \tag{4.3.1}$$

for all measurable $f \geq 0$, *in particular*

$$\sum_{|v|=n} \mathbb{E}L(v)^{\alpha} = g(\alpha)^n \tag{4.3.2}$$

for any $\alpha > 1$.

(b) *Suppose that* $\Psi : \mathbb{R}^2 \to [0, \infty)$ *is a measurable function. If for fixed* $n \geq 1$, $\Pi, \Pi(v)$ $(|v| = n)$ *are i.i.d. nonnegative random variables such that*

- $(\Pi(v))_{|v|=n}$ *is independent of* $\mathcal{F}_n$ *and*
- Π *is independent of* M_n,

then

$$\sum_{|v|=n} \mathbb{E}L(v)\Psi(L(v), \Pi(v)) = \mathbb{E}\Psi(M_n, \Pi). \tag{4.3.3}$$

Proof. (a) The proof runs by induction on n. The case $n = 1$ is simple because by definition,

$$\mathbb{E}\left[\sum_{|v|=1} L(v)\mathbf{1}_{\{L(v)\leq x_1\}}\right] = \mathbb{E}\left[\sum_{i\geq 1} T_i \mathbf{1}_{\{T_i \leq x_1\}}\right] = \xi((0, x_1]) = \mathbb{P}(M_1 \leq x_1).$$

Now suppose the claim is proved for some $n \geq 1$ and all $y_1, ..., y_n > 0$. Then using the inductive hypothesis in $(\star)$, we infer

$$\begin{aligned}
&\sum_{|v|=n+1} \mathbb{E}\left(L(v)\prod_{k=1}^{n+1} \mathbf{1}_{(0,x_k]}(L(v_1, ..., v_k))\right) \\
&= \sum_{|w|=n}\sum_{i\geq 1} \mathbb{E}\left(L(w)T_i(w)\prod_{k=1}^{n} \mathbf{1}_{(0,x_k]}((L(w_1, ..., w_k)) \cdot \mathbf{1}_{(0,x_{n+1}]}(L(w)T_i(w))\right) \\
&= \sum_{i\geq 1}\int_{(0,\infty)} s\mathbb{E}\left(\sum_{|w|=n} L(w)\prod_{k=1}^{n} \mathbf{1}_{(0,x_k]}((L(w_1, ..., w_k))\mathbf{1}_{(0,x_{n+1}]}(sL(w))\right) \mathbb{P}^{T_i}(ds) \\
&\overset{(\star)}{=} \sum_{i\geq 1}\int_{(0,\infty)} s\mathbb{P}(M_1 \leq x_1, ..., M_{n-1} \leq x_{n-1}, M_n \leq x_n \wedge x_{n+1}/s)\, \mathbb{P}(T_i \in ds)
\end{aligned}$$

$$\begin{aligned}
&= \int_{(0,\infty)} s\mathbb{P}(M_1 \le x_1, ..., M_{n-1} \le x_{n-1}, M_n \le x_n \wedge x_{n+1}/s)\, \xi(ds) \\
&= \int_{(0,\infty)} s\mathbb{P}(M_1 \le x_1, ..., M_{n-1} \le x_{n-1}, M_n \le x_n \wedge x_{n+1}/s)\, \mathbb{P}(X_{n+1} \in ds) \\
&= \mathbb{P}(M_1 \le x_1, ..., M_n \le x_n, M_{n+1} \le x_{n+1}).
\end{aligned}$$

(4.3.1) follows from this by the standard extension arguments and yields

$$\sum_{|v|=n} \mathbb{E}L(v)^\alpha = \mathbb{E}M_n^{\alpha-1} = \left(\mathbb{E}M_1^{\alpha-1}\right)^n = \left(\sum_{i\ge 1} \mathbb{E}T_i^\alpha\right)^n = g(\alpha)^n$$

for arbitrary $n \ge 1$, as desired.

(b) A simple calculation using (a) and the independence of the random variables encountered gives

$$\begin{aligned}
\mathbb{E}\Psi(M_n, \Pi) &= \int_{[0,\infty)} \mathbb{E}\Psi(M_n, s)\, \mathbb{P}(\Pi \in ds) \\
&= \sum_{|v|=n} \int_{[0,\infty)} \mathbb{E}L(v)\Psi(L(v), s)\, \mathbb{P}(\Pi(v) \in ds) \\
&= \sum_{|v|=n} \mathbb{E}L(v)\Psi(L(v), \Pi(v)).
\end{aligned}$$

□

Remark 4.3.2. The previous result allows us to give another proof of the implication

$$\text{"}g(\beta) < 1 \text{ for some } \beta > 1 \Rightarrow \gamma < 0\text{"}$$

(cf. Remark 3.6.3): Without loss of generality, assume $\gamma_- < \infty$. Since $\gamma_+ < \infty$ by Remark 3.6.3, $\mathbb{E}\log^\pm M_1 = \gamma_\pm$, hence $\mathbb{E}|\log M_1| < \infty$, and $\mathbb{E}M_1^{\beta-1} = g(\beta)$ by the preceding lemma, Jensen's inequality yields

$$(\beta - 1)\gamma = (\beta - 1)\mathbb{E}\log M_1 = \mathbb{E}\log M_1^{\beta-1} \le \log \mathbb{E}M_1^{\beta-1} = \log g(\beta) < 0.$$

The subsequent lemma will be of importance in Section 4.4 because it links the pair of conditions $\mathbb{E}\psi(Z_1) < \infty$ and $g(\alpha) < 1$ to the convergence of a certain series. Its proof is heavily based on the application of a result by Sgibnev [100] which gives sufficient conditions for the finiteness of the Φ-moment of the supremum of an additive random walk with negative drift, where Φ is submultiplicative in the sense that $\Phi(x + y) \le \Phi(x) \cdot \Phi(y)$ for all $x, y \ge 0$ (cf. [100]).

Lemma 4.3.3. *In the situation of Lemma 4.2.6, suppose that* $\mathbb{E}\psi(Z_1) < \infty$ *and* $g(\alpha) < 1$. *Then*

$$\sum_{k\geq 0} \mathbb{E}\overline{\psi}(M_k) = \sum_{k\geq 0} \mathbb{E}M_k^{\alpha-1}\hat{\ell}(M_k) < \infty.$$

Proof. We consider the additive random walk $(\log M_n)_{n\geq 0}$ first (Note that M_n is a.s. strictly positive). Since

$$\mathbb{E}\log^{\pm} M_1 = \sum_{i\geq 1} \mathbb{E}T_i \log^{\pm} T_i = \gamma_{\pm}$$

by the preceding lemma, and γ_+ is finite by Remark 3.6.3,

$$\mathbb{E}\log M_1 = \gamma = \sum_{i\geq 1} \mathbb{E}T_i \log T_i \in [-\infty, 0)$$

by Remark 4.3.2 and the fact that $g(\alpha) < 1$. Hence, the strong law of large numbers guarantees that

$$\lim_{n\to\infty} \log M_n = -\infty \text{ a.s.},$$

i.e. $\lim_{n\to\infty} M_n = 0$ a.s.. Consequently,

$$\sigma_1 := \sigma := \inf\{n > 0 : \log M_n < 0\} = \inf\{n > 0 : M_n < 1\}$$

is finite with probability 1 (and even integrable, see Korollar 14.2.3 in [2]).[1] Moreover, for $n \geq 1$, put

$$\sigma_{n+1} := \inf\{m > \sigma_n : \log M_m - \log M_{\sigma_n} < 0\} = \inf\{m > \sigma_n : M_m/M_{\sigma_n} < 1\}$$

and note that σ_{n+1} is also integrable (see Satz 1.4.4 and Korollar 14.2.3 in [2]) . This gives the decomposition

$$\sum_{n\geq 0} \mathbb{E}\overline{\psi}(M_n) = \mathbb{E}\left[\sum_{k=0}^{\sigma-1} \overline{\psi}(M_k)\right] + \sum_{n\geq 1} \mathbb{E}\overline{\psi}(M_{\sigma_n}) + \sum_{n\geq 1} \mathbb{E}\left[\sum_{k=\sigma_n+1}^{\sigma_{n+1}-1} \overline{\psi}(M_k)\right] =: J_1 + J_2 + J_3,$$

where empty sums are to be read as 0. As to J_1, observe that by Lemma 2 in [64] and the fact that $\gamma = \mathbb{E}\log M_1 < 0$,

$$\mathbb{E}f\left(\sup_{n\geq 0} \log M_n\right) = \frac{1}{\mathbb{E}\sigma} \sum_{k\geq 0} \mathbb{E}f(\log M_k)\mathbf{1}_{\{\sigma>k\}} = \frac{1}{\mathbb{E}\sigma}\mathbb{E}\left[\sum_{k=0}^{\sigma-1} f(\log M_k)\right]$$

[1] Here and in the following, we make use of the standard convention $\inf\emptyset := \infty$.

and

$$\mathbb{E}f\left(\sup_{n\ge 0} M_n\right) = \frac{1}{\mathbb{E}\sigma}\mathbb{E}\left[\sum_{k=0}^{\sigma-1} f(M_k)\right] \tag{4.3.4}$$

for each nonnegative, measurable f. Since $\mathbb{E}\sigma \in [1,\infty)$, this yields

$$J_1 = \mathbb{E}\sigma \cdot \mathbb{E}\overline{\psi}\left(\sup_{n\ge 0} M_n\right) \prec \mathbb{E}\overline{\psi}\left(\sup_{n\ge 0} M_n\right) =: R(\overline{\psi}).$$

Applying Lemma 4.2.6(b) for $y = 1$, we obtain $\overline{\psi}(x) \le Cx^{\alpha-1}$ for all $x \in (0,1)$, whence by Satz 1.4.4 in [2],

$$J_2 = \sum_{n\ge 1} \mathbb{E}\overline{\psi}(M_{\sigma_n}) \le C\sum_{n\ge 1} \mathbb{E}M_{\sigma_n}^{\alpha-1} = C\sum_{n\ge 1}(\mathbb{E}M_\sigma^{\alpha-1})^n < \infty$$

where $M_{\sigma_n} < 1$ a.s. and $\mathbb{E}M_\sigma^{\alpha-1} \in (0,1)$ have been utilized. We now turn to J_3. Evidently, if $\sigma_n < k < \sigma_{n+1}$, $M_k/M_{\sigma_n} \ge 1$. Putting

$$H(y) := (\overline{\psi}(y) + y^{\alpha-1}), \quad y \ge 1,$$

Lemma 4.2.6(b), (4.3.4) and Satz 1.4.4 in [2] show

$$\begin{aligned}
J_3 &= \sum_{n\ge 1}\mathbb{E}\left[\sum_{k=\sigma_n+1}^{\sigma_{n+1}-1} \overline{\psi}(M_k)\right] \\
&\prec \sum_{n\ge 1}\mathbb{E}\left[M_{\sigma_n}^{\alpha-1}\sum_{k=\sigma_n+1}^{\sigma_{n+1}-1} H(M_k/M_{\sigma_n})\right] \\
&\prec \sum_{n\ge 1}\mathbb{E}M_{\sigma_n}^{\alpha-1}\cdot\mathbb{E}\left[\sum_{k=0}^{\sigma-1} H(M_k)\right] \\
&\prec \mathbb{E}H\left(\sup_{n\ge 0} M_n\right)\cdot\sum_{n\ge 1}\mathbb{E}M_{\sigma_n}^{\alpha-1} \\
&\overset{(\bullet)}{\prec} \mathbb{E}\overline{\psi}\left(\sup_{n\ge 0} M_n\right)\cdot\sum_{n\ge 1}(\mathbb{E}M_\sigma^{\alpha-1})^n \\
&\prec R(\overline{\psi})
\end{aligned}$$

because $0 < M_{\sigma_n} < 1$ a.s. and $\mathbb{E}M_\sigma^{\alpha-1} < 1$. For $(\bullet)$, recall that

$$\lim_{x\to\infty} \overline{\psi}(x)x^{1-\alpha} = \lim_{x\to\infty} \psi(x)x^{-\alpha} = \lim_{x\to\infty} \hat{\ell}(x) = \infty,$$

i.e. $\mathbb{E}X^{\alpha-1} \prec \mathbb{E}\overline{\psi}(X)$, hence $\mathbb{E}H(X) \prec \mathbb{E}\overline{\psi}(X)$ for any nonnegative random variable X.

To show $J_1 + J_3 < \infty$, we are left with the proof of

$$R(\overline{\psi}) = \mathbb{E}\overline{\psi}\circ\exp\left(\sup_{n\ge 0}\log M_n\right) < \infty.$$

By assumption and Lemma 4.2.5, $\psi(x) \asymp \varphi(x) = x^{\alpha}\tilde{\ell}(x) \in \mathfrak{R}_\alpha$ which is submultiplicative with increasing slowly varying part $\tilde{\ell}$ such that $\tilde{\ell}(1) = 1$. Therefore, $\overline{\psi}(x) \asymp \overline{\varphi}(x) := x^{\alpha-1}\tilde{\ell}(x)$ which is also submultiplicative and increasing, and since $\overline{\psi}$ is locally bounded on $[0,\infty)$, we can confine ourselves to establishing that

$$\mathbb{E}\varphi^{\star}\left(\sup_{n\geq 0}\log M_n\right) < \infty,$$

where

$$\varphi^{\star}(x) := \overline{\varphi}(e^x) = e^{(\alpha-1)x}\tilde{\ell}(e^x), \quad x \in \mathbb{R},$$

which has $\varphi^{\star}(0) = 1$, is measurable, nondecreasing and submultiplicative in the sense that

$$\varphi^{\star}(x+y) \leq \varphi^{\star}(x)\cdot\varphi^{\star}(y) \text{ for all } x, y \in \mathbb{R}$$

(cf. [100]). To apply Theorem 2 in [100], it is necessary to determine

$$r_+(\varphi^{\star}) := \lim_{x\to\infty}\frac{\log\varphi^{\star}(x)}{x} = \alpha - 1 + \lim_{x\to\infty}\frac{\log\tilde{\ell}(e^x)}{x}.$$

But by Proposition 1.3.6 in [30], $\lim_{y\to\infty}\frac{\log\tilde{\ell}(y)}{\log y} = 0$ which in turn gives

$$r_+(\varphi^{\star}) = \alpha - 1 \in (0,\infty).$$

Write $\mathbb{F} := \mathbb{P}(\log M_1 \in \cdot) = \xi(\log \in \cdot)$ and denote by

$$\hat{\mathbb{F}}(s) := \int_{\mathbb{R}} e^{sx}\,\mathbb{F}(dx) = \int_{\mathbb{R}} e^{sx}\,\mathbb{P}(\log M_1 \in dx) = \mathbb{E}M_1^s = g(s+1) \in (0,\infty], \quad s \geq 0,$$

the moment generating function of $\mathbb{F}$, in particular

$$\hat{\mathbb{F}}(r_+(\varphi^{\star})) = \hat{\mathbb{F}}(\alpha-1) = g(\alpha) < 1.$$

As in [100], we introduce the associated measure $\mathbb{F}_{(1)}$ on $[0,\infty)$, defined by

$$\mathbb{F}_{(1)}(A) := \int_A \mathbb{F}((x,\infty))\,\boldsymbol{\lambda}(dx) \qquad (A \in \mathbb{B}_{(0,\infty)})$$

which is finite because Remark 3.6.3 and Remark 4.3.2 ensure

$$\mathbb{F}_{(1)}([0,\infty)) = \int_{[0,\infty)} \mathbb{P}(\log^+ M_1 > x)\,\boldsymbol{\lambda}(dx) = \mathbb{E}\log^+ M_1 = \gamma_+ < \infty.$$

We want to apply Theorem 2 in [100] which still requires finiteness of

$$\int_{[0,\infty)} \varphi^{\star}(x)\,\mathbb{F}_{(1)}(dx).$$

Recalling the fact that $\varphi^\star$ is increasing, i.e. locally bounded and that $\varphi^\star(x) \asymp \overline{\psi}(e^x)$, we infer

$$\int_{[0,\infty)} \varphi^\star(x)\,\mathbb{F}_{(1)}(dx) \prec I := \int_{[0,\infty)} \overline{\psi}(e^x)\,\mathbb{F}_{(1)}(dx).$$

It suffices to analyze the latter integral which can be rewritten as

$$\begin{aligned} I &= \int_{[0,\infty)} \overline{\psi}(e^x)\xi((e^x,\infty))\,\boldsymbol{\lambda}(dx) \\ &= \sum_{i\geq 1}\mathbb{E}\left(T_i\int_{[0,\infty)}\overline{\psi}(e^x)\mathbf{1}_{\{T_i>e^x\}}\,\boldsymbol{\lambda}(dx)\right) \\ &= \sum_{i\geq 1}\mathbb{E}\left(T_i\mathbf{1}_{\{T_i\geq 1\}}\int_{[1,T_i)}\frac{\psi(s)}{s^2}\,\boldsymbol{\lambda}(ds)\right). \end{aligned}$$

Since $\hat{\ell}(s) = s^{-\alpha}\psi(s)$ is continuous, hence locally bounded on $[1,\infty)$, and $\alpha - 2 > -1$, Karamata's theorem (Proposition 1.5.8 in [30]) enables us to find constants $C > 0$, $z_0 \geq 1$ such that

$$J(z) := \int_{[1,z]}\frac{\psi(s)}{s^2}\,\boldsymbol{\lambda}(ds) = \int_{[1,z]} s^{\alpha-2}\hat{\ell}(s)\,\boldsymbol{\lambda}(ds) \leq Cz^{\alpha-1}\hat{\ell}(z) = \frac{C\psi(z)}{z} \quad \text{for all } z \geq z_0.$$

Moreover, $J(z_0)$ is finite. Putting these estimates together, we get

$$\begin{aligned} I &\leq J(z_0)\sum_{i\geq 1}\mathbb{E}T_i\mathbf{1}_{\{1\leq T_i\leq z_0\}} + \sum_{i\geq 1}\mathbb{E}\left(T_i\mathbf{1}_{\{T_i>z_0\}}J(T_i)\right) \\ &\leq J(z_0) + C\sum_{i\geq 1}\mathbb{E}\psi(T_i) \\ &\leq J(z_0) + C\mathbb{E}\psi(Z_1) < \infty \end{aligned}$$

because ψ is superadditive in $[0,\infty)$ by Lemma B.1. Now an application of Theorem 2 in [100] completes the proof. □

The following remark collects two basic asymptotic properties of the sequence $(\xi_n)_{n\geq 0}$ which are immediate consequences of the strong law of large numbers and the central limit theorem, respectively.

Remark 4.3.4. Suppose that $\gamma_+ < \infty$ and $\gamma = \mathbb{E}\log X_1 \in [-\infty, 0)$.

(a) For any $s > 0$,

$$\xi_n([0,s]) = \mathbb{P}\left(\sum_{i=1}^n \log X_i \leq \log s\right) \underset{n\to\infty}{\longrightarrow} 1$$

because $\lim_{n\to\infty}\sum_{i=1}^n \log X_i = -\infty$ a.s. by the strong law of large numbers and the fact that $\gamma = \mathbb{E}\log X_1 < 0$.

(b) Suppose that $\tau^2 := \mathrm{Var}(\log X_1) = \sum_{i\geq 1} \mathbb{E}T_i \log^2 T_i - \gamma^2 \in (0,\infty)$. Then an application of the central limit theorem yields that for any $s \in \mathbb{R}$,

$$\begin{aligned}\xi_n\left([0, e^{n\gamma+s\tau\sqrt{n}}]\right) &= \mathbb{P}\left(\frac{\sum_{i=1}^n \log X_i - n\gamma}{\tau\sqrt{n}} \leq s\right)\\ &= \mathbb{P}\left(\frac{\sum_{i=1}^n (\log X_i - \mathbb{E}\log X_1)}{\sqrt{n\,\mathrm{Var}(\log X_1)}} \leq s\right)\\ &\underset{n\to\infty}{\longrightarrow} \Phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-y^2/2}\,dy.\end{aligned}$$

We have stated at the beginning of this section that $\sum_{i\geq 1} \mathbb{E}T_i \mathbf{1}_{\{T_i\in\cdot\}}$ renders a probability measure on $(0,\infty)$. This fact is complemented by the subsequent result which shows that *any* distribution ζ on $(0,\infty)$ has a representation of this kind.

Lemma 4.3.5. *Given any distribution ζ on $(0,\infty)$, there is a weighted branching process $(Z_n)_{n\geq 0}$ with independent generic weights $T_1, T_2, \ldots$ such that $\mathbb{P}(N \geq 2) > 0$ and*

$$\zeta(A) = \sum_{i\geq 1} \mathbb{E}T_i \mathbf{1}_{\{T_i\in A\}} \quad \textit{for all } A \in \mathbb{B}_{(0,\infty)}.$$

If $\int_{(0,\infty)} \frac{1}{s}\,\zeta(ds) < \infty$, $(T_i)_{i\geq 1}$ may be chosen such that $\|N\|_\infty = \mathrm{ess\ sup}\,N < \infty$.

Proof. We consider the non-zero measure Q on $(0,\infty)$, defined by

$$Q(A) := \int_A \frac{1}{s}\,\zeta(ds), \qquad A \in \mathbb{B}_{(0,\infty)},$$

which is σ-finite because $Q([1/n,n]) = \int_{[1/n,n]} \frac{1}{s}\,\zeta(ds) \leq n < \infty$ for all $n \geq 1$. In the rest of this proof, the total mass

$$\|Q\| := Q((0,\infty)) = \int_{(0,\infty)} \frac{1}{s}\,\zeta(ds)$$

plays a crucial role:

(I.) If $\|Q\| < \infty$, choose $K \in \mathbb{N}$ such that $K \geq 2 \vee \|Q\|$ and put

$$Q_1 := \ldots := Q_K := \frac{1}{K}Q, \quad Q_i := 0,\ i \geq K+1,$$

and $p := 1 - Q_1((0,\infty)) = 1 - \frac{\|Q\|}{K} \in [0,1)$. Furthermore, define

$$\hat{Q}_i := p\delta_0 + Q_i,\ 1 \leq i \leq K, \quad \text{and} \quad \hat{Q}_i := \delta_0,\ i \geq K+1.$$

(II.) In case $\|Q\| = \infty$, the σ-finiteness of Q enables us to find measurable, pairwise disjoint sets $A_j \subset (0, \infty)$, $j \geq 1$, such that for each $j \geq 1$, $Q_j := Q(\cdot \cap A_j)$ has $0 < \|Q_j\| < \infty$. Without loss of generality[2], we may assume that $\sup_{j \geq 1} \|Q_j\| \leq 1$. Now let $p_j := 1 - \|Q_j\| < 1$ and

$$\hat{Q}_j := p_j \delta_0 + Q_j, \quad j \geq 1.$$

In either case, note that $Q = \sum_{i \geq 1} Q_i$ and that each $\hat{Q}_j$ is a probability measure on $[0, \infty)$. Choose independent (nonnegative) random variables T_j, $j \geq 1$ such that each T_j has distribution $\hat{Q}_j$. This yields that by the definition of Q,

$$\begin{aligned} \sum_{i \geq 1} \mathbb{E} T_i \mathbf{1}_{\{T_i \in A\}} &= \sum_{i \geq 1} \int_A s \, \hat{Q}_i(ds) \\ &= \sum_{i \geq 1} \int_A s \, Q_i(ds) \\ &= \int_A s \, Q(ds) \\ &= \int_A s \cdot \frac{1}{s} \zeta(ds) = \zeta(A) \end{aligned}$$

for any $A \in \mathbb{B}_{(0,\infty)}$. To finish the proof, note that in case (I.),

$$\mathbb{P}(N \geq 2) \geq \mathbb{P}(T_1 \wedge T_2 > 0) = \mathbb{P}(T_1 > 0)\mathbb{P}(T_2 > 0) \geq (1-p)^2 > 0$$

and

$$\mathbb{P}(N \leq K) \geq \mathbb{P}(T_{K+1} = T_{K+2} = \ldots = 0) = 1,$$

while in case (II.),

$$\mathbb{P}(N \geq 2) \geq \mathbb{P}(T_1 > 0)\mathbb{P}(T_2 > 0) \geq (1-p_1)(1-p_2) > 0.$$

□

Given a measurable space $(\mathfrak{X}, \mathcal{A})$, denote by $\mathbb{M}(\mathfrak{X}, \mathcal{A})$ the collection of all probability measures on $(\mathfrak{X}, \mathcal{A})$. Furthermore, let $\pi_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the projection to the ith component ($i \geq 1$). Then the following remark paraphrases Lemma 4.3.5 in measure-theoretic terms.

[2] Otherwise, we may proceed as in (I.)

Remark 4.3.6. In a more measure-theoretic language, the previous result says that the mapping

$$\left\{\Gamma \in \mathbb{M}\left([0,\infty)^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}_{[0,\infty)}\right) : \sum_{i\geq 1}\int_{[0,\infty)^{\mathbb{N}}} \pi_i \, d\Gamma = 1\right\} \to \mathbb{M}((0,\infty), \mathbb{B}_{(0,\infty)}),$$

$$\Gamma \mapsto \sum_{i\geq 1}\int_{\{\pi_i\in\cdot\}} \pi_i \, d\Gamma,$$

is surjective. However, it is not injective because in case (I.) of the previous proof, K can be chosen arbitrarily large.

4.4 Results concerning $\phi \in \mathfrak{R}^{\mathrm{IC}}_\alpha$

The following theorem gives necessary and sufficient conditions for $0 < \mathbb{E}\phi(W) < \infty$ when α is not a dyadic power and ϕ belongs to the class $\mathfrak{R}^{\mathrm{IC}}_\alpha$. Its proof adapts the approach used for the case $\phi(x) = x^\alpha$ and applies Theorem 3.4.3. This is possible because $\phi(x)$ grows stronger than x^α as seen in Section 4.2. Moreover, the auxiliary results stated in Lemma 4.2.6 and Lemma 4.3.1 will enter into our arguments as well as Lemma 3.3.2. Evidently, we may exclude the case $Z_1 = 1$ a.s. in which $\mathbb{E}\phi(W) = \phi(1) = 1$.

Examples will be given in the final section of this chapter.

Theorem 4.4.1. *Suppose that $\mathbb{P}(Z_1 \neq 1) > 0$ and let $\phi \in \mathfrak{R}^{\mathrm{IC}}_\alpha$ for some $\alpha \in (1,\infty)$ which is not a dyadic power. Then the following assertions are equivalent:*

(i) $\mathbb{E}\phi(Z_1) < \infty$ *and* $g(\alpha) = \sum_{i\geq 1}\mathbb{E}T_i^\alpha < 1$,

(ii) $\mathbb{E}\phi(W) \in (0,\infty)$.

Proof. As in the proof of Theorem 3.4.3, we start by showing that $\sup_{n\geq 0}\mathbb{E}\phi(W_n) < \infty$ if $\mathbb{E}\phi(Z_1) < \infty$ and $g(\alpha) < 1$. For this purpose, choose $\psi \in \mathfrak{C}^* \cap \mathfrak{R}_\alpha$ according to Lemma 4.2.1 and $C, x_0 > 0$ such that $\phi(x) \leq C\psi(x)$ for all $x \geq x_0$. Now for arbitrary $n \geq 1$,

$$\mathbb{E}\phi(W_n) \leq \mathbb{E}\phi(W_n)\mathbf{1}_{\{W_n\leq x_0\}} + C\mathbb{E}\psi(W_n)\mathbf{1}_{\{W_n>x_0\}} \leq \sup_{x\in[0,x_0]}\phi(s) + C\mathbb{E}\psi(W_n),$$

so a fortiori, $\sup_{n\geq 0}\mathbb{E}\phi(W_n) \prec \sup_{n\geq 0}\mathbb{E}\psi(W_n)$ because $\sup_{x\in[0,x_0]}\phi(s) < \infty$. In other words, the crucial point is to prove that

$$\sup_{n\geq 0}\mathbb{E}\psi(W_n) < \infty.$$

In analogy to the proof of Theorem 3.4.3, this is carried out inductively, but now distinguishing the cases $\alpha \in (2^l, 2^{l+1})$, $l \geq 0$. Write $\psi(s) = s^\alpha \hat{\ell}(s)$, $s \geq 0$, and put $Y(v) := \sum_{i\geq 1} T_i(v) - 1$ and $\Pi(v) := 1 \vee \sum_{i\geq 1} T_i(v)$, $v \in \mathcal{N}$.

STEP 1.

Suppose that $\alpha \in (1,2)$ which means $\psi \in \mathfrak{C}_0^* \cap \mathfrak{R}_\alpha$. In particular, ψ is convex with concave derivative which implies by a double appeal to the Topchii-Vatutin inequality (Theorem B.4) and by Lemma 4.2.6 that

$$\begin{aligned}
\sup_{n\geq 0} \mathbb{E}\psi(W_n) &\leq \psi(1) + \sum_{n\geq 1} \mathbb{E}\psi(D_n) \\
&\leq \psi(1) + 2\sum_{n\geq 1}\sum_{|v|=n-1} \mathbb{E}\psi(L(v)|Y(v)|) \\
&\prec \sum_{n\geq 1}\sum_{|v|=n-1} \mathbb{E}\psi(L(v)\Pi(v)) \qquad (4.4.1)\\
&\prec \sum_{n\geq 1}\sum_{|v|=n-1} \mathbb{E}\left[\Big(\psi(L(v)) + L(v)^\alpha\Big)\cdot\Big(\psi(\Pi(v)) + \Pi(v)^\alpha\Big)\right] \\
&= \mathbb{E}[\psi(1\vee Z_1) + (1\vee Z_1)^\alpha]\cdot \sum_{n\geq 1}\sum_{|v|=n-1}\Big(\mathbb{E}\psi(L(v)) + \mathbb{E}L(v)^\alpha\Big) \\
&\overset{(\star)}{\prec} \sum_{n\geq 1}\sum_{|v|=n-1}\Big(\mathbb{E}\psi(L(v)) + \mathbb{E}L(v)^\alpha\Big) \\
&= \sum_{n\geq 1}\sum_{|v|=n-1} \mathbb{E}\psi(L(v)) + \sum_{n\geq 1} g(\alpha)^{n-1} \\
&= \frac{1}{1-g(\alpha)} + \sum_{n\geq 0} \mathbb{E}\overline{\psi}(M_n) \\
&\prec \sum_{n\geq 0} \mathbb{E}\overline{\psi}(M_n),
\end{aligned}$$

where as earlier $\overline{\psi}(s) = s^{-1}\psi(s) = s^{\alpha-1}\hat{\ell}(s)$, $s \geq 0$. In $(\star)$, we have utilized the fact that

$$\mathbb{E}\psi(1\vee Z_1) \prec \mathbb{E}\psi(Z_1) \prec \mathbb{E}\phi(Z_1) < \infty.$$

Furthermore, $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ implies $\mathbb{E}(1\vee Z_1)^\alpha \prec \mathbb{E}\phi(1\vee Z_1) < \infty$. For the penultimate line, we also used Lemma 4.3.1. So we have shown that

$$\sup_{n\geq 1} \mathbb{E}\psi(W_n) \prec \sum_{n\geq 0} \mathbb{E}\overline{\psi}(M_n),$$

and the latter sum is finite by Lemma 4.3.3.

STEP 2.

Now assume that the claim is proved whenever $\alpha \in \cup_{k=0}^{l}(2^k, 2^{k+1})$ for some $l \geq 0$, and pick $\alpha \in (2^{l+1}, 2^{l+2})$ in which case $\psi \in \mathfrak{C}_{l+1}^*$. As Lemma 3.3.2 shows, we may restrict ourselves to proving that

$$Q_1(l+1,\psi) = \mathbb{E}\mathbb{S}^{-l-1}\psi\left(\sum_{k\geq 0}\overline{D}_k^{(2^{l+1})}\right) \text{ and } Q_2(l+1,\psi) = \sum_{m=0}^{l}\sum_{k\geq 0}\mathbb{E}\mathbb{S}^{-m}\psi\left(\overline{D}_k^{(2^m)}\right)$$

are finite, where for $s > 0$ with $g(s) < \infty$,

$$\overline{D}_0^{(s)} = 1 \quad \text{ and } \quad \overline{D}_k^{(s)} = \sum_{|v|=k-1} L(v)^s \left(\sum_{i\geq 1} T_i(v)^s - g(s)\right), \quad k \geq 1.$$

Note that Lemma 3.3.2 is applicable because $\mu(2^{l+1}) \prec \mathbb{E}\phi(Z_1) < \infty$ and $g(2^{l+1}) < 1$. Moreover, it is obvious that $\mathfrak{C}^*$ is a subset of the function class $\mathfrak{Z}$ defined in Section 3.3. Put $s := 2^{l+1}$. Turning to $Q_1(l+1,\psi)$ first, observe that $\mathbb{S}^{-l-1}\psi \in \mathfrak{C}_0^*$ is convex with concave derivative. Hence, a similar argument as in the proof of Theorem 3.4.3 (when estimating $Q_1(l+1,\alpha)$) ensures

$$\begin{aligned}
Q_1(l+1,\psi) &= \mathbb{E}\mathbb{S}^{-l-1}\psi\left(\sum_{k\geq 0}\overline{D}_k^{(s)}\right) \\
&\leq \psi(1) + 4\sum_{k\geq 1}\sum_{|v|=k-1}\mathbb{E}\mathbb{S}^{-l-1}\psi\left(L(v)^s\left(1 \vee \sum_{i\geq 1} T_i(v)^s\right)\right) \qquad (4.4.2) \\
&\leq \psi(1) + 4\sum_{k\geq 1}\sum_{|v|=k-1}\mathbb{E}\mathbb{S}^{-l-1}\psi\left(L(v)^s\left(1 \vee \sum_{i\geq 1} T_i(v)\right)^s\right) \\
&\prec \sum_{k\geq 1}\sum_{|v|=k-1}\mathbb{E}\psi(L(v)\Pi(v)) < \infty.
\end{aligned}$$

For the last estimate, recall that

$$\sum_{k\geq 1}\sum_{|v|=k-1}\mathbb{E}\psi(L(v)\Pi(v)) \prec \sum_{n\geq 0}\mathbb{E}\overline{\psi}(M_n) < \infty$$

by the same argument as in (4.4.1) of **STEP 1**.

To show $Q_2(l+1,\psi) < \infty$, or equivalently,

$$U(m,\psi) := \sum_{k\geq 1}\mathbb{E}\mathbb{S}^{-m}\psi\left(\overline{D}_k^{(2^m)}\right) < \infty \text{ for all } m \in \{0,...,l\},$$

fix m and note first that $\mathbb{E}\psi(Z_1) < \infty$ gives

$$\mathbb{E}\mathbb{S}^{-m}\psi\left(\overline{D}_1^{(2^m)}\right) \leq \mathbb{E}\mathbb{S}^{-m}\psi\left(1 \vee \sum_{i\geq 1} T_i^{2^m}\right) \leq \mathbb{E}\mathbb{S}^{-m}\psi\left((1 \vee Z_1)^{2^m}\right) = \mathbb{E}\psi(1 \vee Z_1) < \infty.$$

Concerning $\sum_{k\geq 2} \mathbb{E}\mathbb{S}^{-m}\psi\left(\overline{D}_k^{(2^m)}\right)$, another appeal to the Burkholder-Davis-Gundy inequality (Theorem B.2) shows that it is sufficient to prove

$$\sum_{k\geq 2} J_1(k,m,\psi) + \sum_{k\geq 2} J_2(k,m,\psi) < \infty, \tag{4.4.3}$$

where

$$J_1(k,m,\psi) = \mathbb{E}\mathbb{S}^{-m-1}\psi\left(\mu(2^{m+1}) \sum_{|v|=k-1} L(v)^{2^{m+1}}\right)$$

and

$$J_2(k,m,\psi) = \sum_{|v|=k-1} \mathbb{E}\mathbb{S}^{-m}\psi\left(L(v)^{2^m}\left|\sum_{i\geq 1} T_i(v)^{2^m} - g(2^m)\right|\right)$$

(cf. (3.4.3)–(3.4.5) in the proof of Theorem 3.4.3). But

$$\begin{aligned}
\sum_{k\geq 2} J_2(k,m,\psi) &\leq \sum_{k\geq 2}\sum_{|v|=k-1} \mathbb{E}\mathbb{S}^{-m}\psi\left(L(v)^{2^m}\left(1\vee\sum_{i\geq 1} T_i(v)^{2^m}\right)\right) \\
&\leq \sum_{k\geq 2}\sum_{|v|=k-1} \mathbb{E}\mathbb{S}^{-m}\psi\left(L(v)^{2^m}\left(1\vee\sum_{i\geq 1} T_i(v)\right)^{2^m}\right) \\
&= \sum_{k\geq 2}\sum_{|v|=k-1} \mathbb{E}\psi(L(v)\Pi(v)) < \infty
\end{aligned} \tag{4.4.4}$$

since the convergence of the last sum has been obtained in (4.4.1). As to $\sum_{k\geq 2} J_1(k,m,\psi)$, note that for some constant $C \in (0,\infty)$ not depending on k,

$$J_1(k,m,\psi) \leq C\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\sum_{|v|=k-1} L(v)^{2^{m+1}}\right) = C\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\overline{Z}_{k-1}\right) \tag{4.4.5}$$

in view of (4.2.5) and the fact that $\mu(2^{m+1}) \prec \mathbb{E}Z_1^\alpha \prec \mathbb{E}\phi(Z_1) < \infty$. (4.2.5) is applicable because $\mathbb{S}^{-m-1}\psi \in \mathfrak{C}_{l-m}^*$, and $(\overline{Z}_n)_{n\geq 0}$ is the weighted branching process with generic weights $(\overline{T}_i)_{i\geq 1} = (T_i^{2^{m+1}})_{i\geq 1}$ satisfying

$$\overline{g}(1) \vee \overline{g}(\alpha/2^{m+1}) = g(2^{m+1}) \vee g(\alpha) < 1$$

by Lemma 3.3.1. As in Lemma 3.3.4 and in the proof of Theorem 3.4.3, we have written $\overline{g}(u) = \sum_{i\geq 1} \mathbb{E}\overline{T}_i^u$ for $u \geq 1$. According to Lemma 3.3.4, we can choose $(\hat{Z}_n)_{n\geq 0}$ with generic weights $(\hat{T}_i)_{i\geq 1}$ such that

$$\overline{Z}_n \leq \hat{Z}_n \quad \text{for all } n \geq 0, \tag{4.4.6}$$

$\hat{Z}_1 = c + \overline{Z}_1$ a.s. for some $c > 0$, hence

$$\mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{Z}_1) \prec \mathbb{E}\mathbb{S}^{-m-1}\psi(\overline{Z}_1) = \mathbb{E}\mathbb{S}^{-m-1}\psi\left(\sum_{i\geq 1} T_i^{2^{m+1}}\right), \tag{4.4.7}$$

and

$$\hat{g}(1) < 1,\ \hat{g}(\alpha/2^{m+1}) < \hat{g}(1)^{\alpha/2^{m+1}} \tag{4.4.8}$$

with $\hat{g}(\cdot)$ as in Lemma 3.3.4. For (4.4.7), just note that $\mathfrak{C}^*$ is a subset of the function class $\mathfrak{Z}$ considered in Lemma 3.3.4. Consequently, we can estimate

$$J_1(k,m,\psi) \leq C\mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{Z}_{k-1}) \leq C\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\hat{g}(1)^{k-1}(1 \vee \hat{W}_{k-1})\right),$$

$(\hat{W}_j)_{j\geq 0}$ denoting the normalization of $(\hat{Z}_j)_{j\geq 0}$ with generic weights $\hat{T}_i/\hat{g}(1)$, $i \geq 1$. Furthermore, $\hat{g}(1) < 1$ and Lemma 4.2.6(b) give

$$\begin{aligned} &\mathbb{S}^{-m-1}\psi\left(\hat{g}(1)^{k-1}(1 \vee \hat{W}_{k-1})\right) \\ &\leq\ C\hat{g}(1)^{\alpha(k-1)/2^{m+1}}\left(\mathbb{S}^{-m-1}\psi(1 \vee \hat{W}_{k-1}) + (1 \vee \hat{W}_{k-1})^{\alpha/2^{m+1}}\right) \end{aligned}$$

and thereby

$$J_1(k,m,\psi) \leq C\hat{g}(1)^{(k-1)\alpha/2^{m+1}} \sup_{j\geq 0} \mathbb{E}\left[\mathbb{S}^{-m-1}\psi(1 \vee \hat{W}_j) + (1 \vee \hat{W}_j)^{\alpha/2^{m+1}}\right], \quad k \geq 2.$$

For $\sum_{k\geq 2} J_1(k,m,\psi) < \infty$, it now suffices to establish

$$\sup_{j\geq 0} \mathbb{E}\left[\mathbb{S}^{-m-1}\psi(1 \vee \hat{W}_j) + (1 \vee \hat{W}_j)^{\alpha/2^{m+1}}\right] < \infty$$

which reduces to

$$\sup_{j\geq 0} \mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{W}_j) < \infty \tag{4.4.9}$$

because $\psi \asymp \phi$ implies $\mathbb{S}^{-m-1}\psi(x)/x^{\alpha/2^{m+1}} = \mathbb{S}^{-m-1}\hat{\ell}(x) \underset{x\to\infty}{\longrightarrow} \infty$. In order to prove (4.4.9), it is readily checked that

(I) $\mathbb{S}^{-m-1}\psi(x) \asymp \mathbb{S}^{-m-1}\phi(x) = x^{\alpha/2^{m+1}} \exp\left(\int_{[1,1\vee x]} \frac{\mathbb{S}^{-m-1}\varepsilon(s)}{2^{m+1}s}\,\boldsymbol{\lambda}(ds)\right) \in \mathfrak{R}^{\mathrm{IC}}_{\alpha/2^{m+1}}$,

(II) $\alpha/2^{m+1} \in \cup_{k=0}^{l}(2^k, 2^{k+1})$,

(III) $\mathbb{E}\mathbb{S}^{-m-1}\phi(\hat{W}_1) < \infty$ because $\phi \asymp \psi$, (4.2.5), (4.4.7) and Lemma B.1 imply

$$\mathbb{E}\mathbb{S}^{-m-1}\phi(\hat{W}_1) \prec \mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{W}_1) \prec \mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{Z}_1) \prec \mathbb{E}\mathbb{S}^{-m-1}\psi(\overline{Z}_1) \leq \mathbb{E}\psi(Z_1) < \infty,$$

and

(IV) $\sum_{i\ge 1} \mathbb{E}\left(\frac{\hat{T}_i}{\hat{g}(1)}\right)^{\alpha/2^{m+1}} = \frac{\hat{g}(\alpha/2^{m+1})}{\hat{g}(1)^{\alpha/2^{m+1}}} < 1$ by (4.4.8).

Thus, an application of the inductive hypothesis to $(\hat{W}_n)_{n\ge 0}$ and $\mathbb{S}^{-m-1}\phi \in \mathfrak{R}^{\mathrm{IC}}_{\alpha/2^{m+1}}$ guarantees

$$\sup_{j\ge 0} \mathbb{E}\mathbb{S}^{-m-1}\phi(\hat{W}_j) < \infty$$

and therefrom (4.4.9).

Summarizing, we have shown that

$$\begin{aligned}
\sup_{n\ge 0} \mathbb{E}\phi(W_n) &\prec Q_1(l+1,\psi) + Q_2(l+1,\psi) \\
&\prec \max_{0\le m\le l} U(m,\psi) \\
&\prec \max_{0\le m\le l} \sum_{k\ge 2} [J_1(k,m,\psi) + J_2(k,m,\psi)] \\
&\prec \max_{0\le m\le l} \sum_{k\ge 2} J_1(k,m,\psi) \\
&\prec \max_{0\le m\le l} \left[\sup_{j\ge 0} \mathbb{E}\mathbb{S}^{-m-1}\psi(\hat{W}_j) \cdot \sum_{k\ge 1} \left(\hat{g}(1)^{\alpha/2^{m+1}}\right)^k\right] < \infty
\end{aligned}$$

by (4.4.9) and the fact that $\max_{0\le m\le l} \hat{g}(1)^{\alpha/2^{m+1}} < 1$, as demanded.

Now the previously shown yields that $\mathbb{E}\phi(W) \le \sup_{n\ge 0} \mathbb{E}\phi(W_n) < \infty$ by the continuity of ϕ and Fatou's lemma, and also uniform integrability of the sequence $(W_n)_{n\ge 0}$ itself, i.e. $\mathbb{E}W = 1$ and $\mathbb{E}\phi(W) > 0$ by Theorem 3.4.1 since ϕ is ultimately positive.

On the other hand, if $0 < \mathbb{E}\phi(W) < \infty$, i.e. $0 < \mathbb{E}\psi(W) < \infty$, Theorem 3.4.5 shows $\mathbb{E}\phi(Z_1) \prec \mathbb{E}\psi(Z_1) < \infty$ by the convexity of ψ. Moreover, since $x^\alpha = o(\phi(x))$ as $x \to \infty$, we obtain $\mathbb{E}W^\alpha \in (0,\infty)$, whence $g(\alpha) < 1$ by Theorem 3.4.3. □

The following corollary is easily obtained by combining previous results.

Corollary 4.4.2. *Suppose that $\mathbb{E}W > 0$ and $\phi(x) = x^\alpha \ell(x) \in \mathfrak{R}_\alpha$ for some $\alpha \ge 1$. Then $\mathbb{E}\phi(Z_1) \prec \mathbb{E}\phi(W)$ whenever one of the following conditions is satisfied:*

(i) α is not a dyadic power, i.e. $\alpha \in (1,\infty) \setminus \{2^n : n \ge 1\}$,

(ii) $\alpha = 2^n$ for some $n \ge 0$ and $\ell \asymp \mathbb{U}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$,

(iii) $\alpha = 2^n$ for some $n \ge 1$ and $\ell \asymp \mathbb{H}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$.

Moreover, in either case $g(\beta) < 1$ *for all* $\beta \in (1, \alpha)$*, and (ii) even implies* $g(\alpha) < 1$ *if* $\alpha \geq 2$*. The same is true for arbitrary* $\alpha > 1$ *if* $\mathbb{E}f(W) < \infty$ *for some nonnegative function* f *with* $\liminf_{x\to\infty} f(x)x^{-\alpha} \in (0, \infty]$.

Proof. The assertion follows by combining Theorem 3.4.5, Lemma 4.2.1 and Theorem 3.4.3. We just have to mention that in either case, $x^\beta = o(\phi(x))$ $(x \to \infty)$ for $\beta < \alpha$ by Proposition 1.3.6 in [30], and if *(ii)* holds, even $x^\alpha = O(\phi(x))$ $(x \to \infty)$ is valid. □

Remark 4.4.3. By a similar argument as in the proof of Theorem 3.4.8, it follows for any $\phi \in \mathfrak{R}_\alpha$ $(\alpha \geq 1)$ that $\mathbb{E}\phi(Z_1) = \infty$ implies $\mathbb{E}\phi(Z_n) = \infty$ for all $n \geq 1$. We just mention that by Theorem 1.5.3 of [30], $\phi \sim \tilde{\phi}$ for some nondecreasing $\tilde{\phi}$ (which plainly also belongs to $\mathfrak{R}_\alpha$), and that by definition, $f(\varepsilon x) \asymp f(x)$ for all regularly varying functions f and all $\varepsilon > 0$.

Remark 4.4.4. Arguing along the lines of Theorem 3.4.3, one can prove for any submultiplicative $\phi \in \mathfrak{C}_0$ that $\sup_{n\geq 0} \mathbb{E}\phi(W_n) < \infty$ if $\mathbb{E}\phi(Z_1) < \infty$ and $\sum_{i\geq 1} \mathbb{E}\phi(T_i) < 1$. Since the only obvious applications are given by $\phi(x) = x^\alpha$ for $1 < \alpha \leq 2$, we omit the proof.

Remark 4.4.5. As Theorem 1.1.6 is evidently applicable to any function $\psi \in \mathfrak{C}$, it follows that provided $\mathbb{E}W = 1$, $\mathbb{E}\psi(\sup_{n\geq 0} W_n) < \infty$ if and only if $\mathbb{E}\psi(W) < \infty$. Consequently, if ϕ is regularly varying of order $\alpha > 1$, and $\phi \asymp \psi$ for some $\psi \in \mathfrak{C}$ (for instance, if one of the conditions *(i)–(iii)* of Corollary 4.4.2 is fulfilled), then the same equivalence holds with ϕ instead of ψ.

The following corollary is stated without proof. We just mention that by the continuity of $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$, $\lim_{n\to\infty} \phi(W_n) = \phi(W)$ with probability 1. Moreover, by choosing ψ as in the proof of Theorem 4.4.1,

$$\mathbb{E}\phi\left(\sup_{n\geq 0} W_n\right) \prec \mathbb{E}\psi\left(\sup_{n\geq 0} W_n\right) = \int_{(0,\infty)} \psi'(t)\mathbb{P}\left(\sup_{n\geq 0} W_n \geq t\right) \boldsymbol{\lambda}(dt)$$

which enables us to argue with Theorem 1.1.6.

Corollary 4.4.6. *Suppose that (3.1.2) and (3.1.4) hold. Given $\phi \in \mathfrak{R}_\alpha^{\mathrm{IC}}$ for some $\alpha > 1$ which is not a dyadic power, the following conditions are equivalent:*

(i) $\mathbb{E}\phi(\sup_{n\geq 0} W_n) < \infty$,

(ii) $\sup_{n\geq 0} \mathbb{E}\phi(W_n) < \infty$,

(iii) $\mathbb{E}\phi(W) \in (0,\infty)$,

(iv) $\mathbb{E}\phi(Z_1) < \infty$ *and* $g(\alpha) < 1$,

(v) $(\phi(W_n))_{n\geq 0}$ *is uniformly integrable,*

(vi) $\lim_{n\to\infty} \mathbb{E}\phi(W_n) = \mathbb{E}\phi(W) \in (0,\infty)$.

Additionally, since ϕ is continuous and ultimately nondecreasing with $\phi(0) = 0$, *(i)* ensures

$$\lim_{n\to\infty} \mathbb{E}\phi(|W_n - W|) = 0$$

by the dominated convergence theorem. Moreover, $W_n \xrightarrow{\mathfrak{L}_\alpha} W$ as $n \to \infty$.

4.5 Results when the weights are decreasing along each line of descent

In the preceding section, we restricted the class of regularly varying functions ϕ under consideration in order to provide necessary and sufficient conditions for the existence of ϕ-moments of the martingale limit W. However, no serious additional assumptions on the underlying WBP were imposed. This section is devoted to the opposite kind of restriction: It will turn out that if an additional assumption on the weights $T_1, T_2, ...$ is fulfilled, we obtain a result similar to Theorem 4.4.1 for a substantially larger class of regularly varying functions ϕ. When this additional assumption is in force, we only require that ϕ is asymptotically equivalent to some $\psi \in \mathfrak{C}^*$. For example, we can also treat the case of dyadic exponents α, and the corresponding slowly varying part does not necessarily have to tend to infinity (unless $\alpha = 1$). To begin with, let us stipulate for the rest of this section that

$$\mathbb{P}\left(\sup_{i\geq 1} T_i \leq 1\right) = 1 \quad \text{and} \quad c < 1. \tag{4.5.1}$$

This means that for any sequence $(k_j)_{j\geq 1}$ of positive integers, the sequence of weights $(L(k_1,...,k_n))_{n\geq 1}$ is nonincreasing and obviously tends to zero with probability 1 by (3.1.6). In other words, the weight of any cell/individual cannot be larger than the weight of its mother. Plainly, the hypothesis $c < 1$ excludes the trivial case of critical Galton-Watson processes. Furthermore, we know from Remark 3.5.1 that g is strictly decreasing in $[1, \infty)$, in particular $g(\alpha) < 1$ for any $\alpha > 1$. Together with Lemma 3.2 in [35], this even yields $\lim_{n\to\infty} \sup_{|v|=n} L(v) = 0$ almost surely. In view of Theorem 3.4.3 and Theorem 4.4.1, this emphasizes that (4.5.1) is an essential restriction. In addition, note that $\gamma_+ = 0$ and $-\infty \leq \gamma < 0$. A further technical amenity is ensured by the fact that the multiplicative random walk $(M_n)_{n\geq 0}$ from Section 4.3 has almost surely nonincreasing paths because M_1 is a.s. bounded by 1. Finally, in the language of branching random walks, hypothesis (4.5.1) corresponds to a branching random walk with positive steps, i.e. individuals can only move to the right.

Before formulating the main result of this section which is proved by modifying the proof of Theorem 4.4.1, we mention that when (4.5.1) is assumed, the techniques used in this chapter also provide a new proof of the implication

$$\text{“}\mathbb{E}Z_1 \log^+ Z_1 < \infty \Rightarrow \mathbb{E}W = 1\text{“}$$

(cf. Corollary 4.5.3) which was already obtained in Theorem 2.4.17 for arbitrary ordinary weighted branching processes. As stated above, the condition $\gamma \in [-\infty, 0)$ is automatically satisfied.

When the underlying branching process is a branching random walk (with not necessarily nonnegative steps), the case $\alpha = 1$ and $\ell(x) = (\log^+ x)^\delta$ for some $\delta > 0$ (and hence $\mathbb{U}\ell(x) \asymp (\log^+ x)^{1+\delta}$ as will be checked in Section 4.6) of the following result is due to Biggins [19]. Again, we exclude the trivial case $Z_1 = 1$ a.s.. Example 4.6.2 in the following section will point out that Theorem 4.5.1 comprises the case of supercritical GWP and therefore generalizes well-known results on this type of branching process. Notice that in case $\alpha = 1$, we only give a sufficient condition for $0 < \mathbb{E}W\ell(W) < \infty$, and recall that $\lim_{x\to\infty} \frac{\mathbb{U}\ell(x)}{\ell(x)} = \infty$ for any slowly varying function ℓ.

Theorem 4.5.1. *Suppose (4.5.1), $\mathbb{P}(Z_1 = 1) < 1$ and let $\phi(x) = x^\alpha \ell(x) \in \mathfrak{R}_\alpha$ for some $\alpha \geq 1$. Then*

$$0 < \mathbb{E}\phi(W) < \infty \quad \textit{iff} \quad \mathbb{E}\phi(Z_1) < \infty$$

for any $\alpha > 1$ which is not a dyadic power. The same equivalence holds if $\alpha = 2^n$ for some $n \geq 1$ and either $\ell(x) \asymp \mathbb{U}\ell_0(x)$ or $\ell(x) \asymp \mathbb{H}\ell_0(x)$ for some $\ell_0 \in \mathfrak{R}_0$. Finally, if

$\alpha = 1$ *and* ℓ *satisfies* $\lim_{x\to\infty}\ell(x) = \infty$ *and* $\ell(x) \asymp \mathbb{U}\ell_0(x)$ *for some* $\ell_0 \in \mathfrak{R}_0$*, then*

$$0 < \mathbb{E}W\ell(W) < \infty \quad \textit{if} \quad \mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty.$$

Before starting the proof, another appeal to Lemma 4.2.1 allows us to choose $\psi \in \mathfrak{C}^* \cap \mathfrak{R}_\alpha$ with slowly varying part $\hat{\ell}$ such that $\psi \asymp \phi$. Note that $\lim_{x\to\infty}\frac{\phi(x)}{x} = \lim_{x\to\infty}\frac{\psi(x)}{x} = \infty$ by Proposition 1.3.6 in [30] and the additional assumption on ℓ when $\alpha = 1$.

Proof. We start by showing that the conditions $\mathbb{E}\phi(Z_1) < \infty$ or $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$, respectively, are sufficient for $\mathbb{E}\psi(W) < \infty$ by establishing $\sup_{n\geq 0}\mathbb{E}\psi(W_n) < \infty$. This is once more done by distinguishing the cases $\psi \in \mathfrak{C}_j^*$, $j \geq 0$, and by using a similar procedure as in the proofs of Theorem 3.4.3 and Theorem 4.4.1.

CASE 1: $\psi \in \mathfrak{C}_0^*$

First suppose that $\psi \in \mathfrak{C}_0^*$ which evidently corresponds to one of the following cases:

(I) $1 < \alpha < 2$ and $\mathbb{E}\phi(Z_1) < \infty$,

(II) $\alpha = 2$, $\ell \asymp \mathbb{H}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$ and $\mathbb{E}\phi(Z_1) < \infty$,

(III) $\alpha = 1$, $\ell \asymp \mathbb{U}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$ and $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$.

By the choice of ψ, $\psi''(0) \in (0,\infty)$. Then the same arguments as in **STEP 1** of the proof of Theorem 4.4.1 (cf. (4.4.1)) give

$$\sup_{n\geq 0}\mathbb{E}\psi(W_n) \prec \sum_{n\geq 1}\sum_{|v|=n-1}\mathbb{E}\psi(L(v)\Pi(v)),$$

where $\Pi(v) = 1 \vee \sum_{i\geq 1}T_i(v)$, $v \in \mathcal{N}$. By an application of Lemma 4.3.1(b)[3], we obtain

$$\sum_{n\geq 0}\sum_{|v|=n}\mathbb{E}\psi(L(v)\Pi(v)) = \sum_{n\geq 0}\mathbb{E}M_n^{-1}\psi(M_n\Pi) = \sum_{n\geq 0}\mathbb{E}M_n^{\alpha-1}\Pi^\alpha\hat{\ell}(M_n\Pi),$$

where Π is copy of $1 \vee Z_1$ that is independent of $(M_n)_{n\geq 0}$. Now fix $a \in (0,1)$, recall that $\lim_{n\to\infty}M_n = 0$ a.s., define $\tau_0 := 0$ and for $n \geq 1$,

$$\tau_n := \inf\{k > \tau_{n-1} : M_k/M_{\tau_{n-1}}\} < a \qquad [\inf\emptyset = \infty]^4.$$

[3]Choose $\Psi(x,y) = x^{-1}\psi(xy)\mathbf{1}_{(0,\infty)\times[0,\infty)}(x,y)$ and recall that $\psi(0) = 0$ and $M_n > 0$ a.s..

[4]Writing $\tau_1 = \tau_1(a)$ for $a \in (0,1)$, it is well-known from renewal theory (cf. Satz 0.3.1 in [2]) that $\lim_{a\downarrow 0}\frac{\mathbb{E}\tau_1(a)}{-\log a} = \frac{1}{-\gamma}$ ($= 0$ if $\gamma = -\infty$) because $\tau_1(a) = \inf\{n \geq 1 : -\log M_n > -\log a\}$, and $(-\log M_n)_{n\geq 0}$ is a standard renewal process with drift $-\gamma \in (0,\infty]$. In particular, $\tau_n < \infty$ a.s. for all n.

Then

$$\begin{aligned}\sum_{n\ge 0}\mathbb{E}M_n^{\alpha-1}\Pi^\alpha\hat{\ell}(M_n\Pi) &= \sum_{n\ge 0}\mathbb{E}\left[\Pi\sum_{k=\tau_n}^{\tau_{n+1}-1}(M_k\Pi)^{\alpha-1}\hat{\ell}(M_k\Pi)\right]\\ &\overset{(\star)}{\le} \sum_{n\ge 0}\mathbb{E}\left[\Pi\sum_{k=\tau_n}^{\tau_{n+1}-1}(a^n\Pi)^{\alpha-1}\hat{\ell}(a^n\Pi)\right]\\ &= \sum_{n\ge 0}a^{-n}\mathbb{E}\left[(a^n\Pi)^\alpha\ell(a^n\Pi)(\tau_{n+1}-\tau_n)\right]\\ &= \mathbb{E}\tau_1\sum_{n\ge 0}a^{-n}\mathbb{E}\psi(a^n\Pi).\end{aligned}$$

For $(\star)$, observe that $x\mapsto\frac{\psi(x)}{x}=x^{\alpha-1}\hat{\ell}(x)$ is nondecreasing by Lemma 4.2.2, and that $M_k\le a^n$ if $k\ge\tau_n$ because $(M_k)_{k\ge 0}$ is a.s. nonincreasing. For the last line, just note that Π is also independent of $(\tau_n)_{n\ge 0}$ and that for each $n\ge 0$, $\tau_{n+1}-\tau_n$ has the same distribution as τ_1. Now we know from Lemma 4.3 in [6] and the fact that $\mathbb{E}\tau_1<\infty$ that

$$\mathbb{E}\tau_1\sum_{n\ge 0}a^{-n}\mathbb{E}\psi(a^n\Pi)\prec\mathbb{E}\mathbb{L}\psi(\Pi),$$

where it should be recalled from (4.2.8) that $\mathbb{L}\psi(z)=\int_{(0,z]}\int_{(0,s]}\frac{\psi'(r)}{r}\,\boldsymbol{\lambda}(dr)\,\boldsymbol{\lambda}(ds)$ for $z\ge 0$. Furthermore, we have stated in Lemma 4.2.3 that $\mathbb{L}\psi\asymp\psi\asymp\phi$ if **(I)** or **(II)** is valid, whereas in case **(III)**, $\mathbb{L}\psi(z)\asymp z\mathbb{U}\hat{\ell}(z)\asymp z\mathbb{U}\ell(z)$ by Lemma 4.2.3 and the fact that $\ell\asymp\hat{\ell}$. Hence,

$$\sup_{n\ge 0}\mathbb{E}\psi(W_n)\prec\mathbb{E}\mathbb{L}\psi(\Pi)\prec\left\{\begin{array}{ll}\mathbb{E}\phi(Z_1) & \text{if (I) or (II) holds}\\ \mathbb{E}Z_1\mathbb{U}\ell(Z_1) & \text{if (III) holds}\end{array}\right\}<\infty.$$

CASE 2: $\psi\in\mathfrak{C}_{l+1}^*$ for some $l\ge 0$

Now assume that $\psi\in\mathfrak{C}_{l+1}^*$ for some $l\ge 0$ which is tantamount with the validity of one of the following three conditions:

(I) $2^{l+1}<\alpha<2^{l+2}$ and $\mathbb{E}\phi(Z_1)<\infty$,

(II) $\alpha=2^{l+2}$, $\ell\asymp\mathbb{H}\ell_0$ for some $\ell_0\in\mathfrak{R}_0$ and $\mathbb{E}\phi(Z_1)<\infty$,

(III) $\alpha=2^{l+1}$, $\ell\asymp\mathbb{U}\ell_0$ for some $\ell_0\in\mathfrak{R}_0$ and $\mathbb{E}\phi(Z_1)<\infty$.

Since $\psi(x)/x^{2^{l+1}}$ is nondecreasing by Lemma 4.2.2, $\mathbb{E}\phi(Z_1)<\infty$ ensures $\mu(2^{l+1})<\infty$, and $g<1$ in $(1,\infty)$ by (4.5.1), Lemma 3.3.2 shows that it suffices again to prove that the terms $Q_1(l+1,\psi)$ and $Q_2(l+1,\psi)$ defined there are finite.

- As to $Q_1(l+1,\psi) = \mathbb{E}\mathbb{S}^{-l-1}\psi\left(\sum_{k\geq 0}\overline{D}_k^{(2^{l+1})}\right)$, (4.4.2) shows

$$Q_1(l+1,\psi) \leq \sum_{n\geq 1}\sum_{|v|=n-1}\mathbb{E}\psi(L(v)\Pi(v)).$$

Put $\overline{\psi}(x) = \frac{\psi(x)}{x}$ for $x > 0$ and observe that by another appeal to Lemma 4.2.2, $\overline{\psi}$ can be continuously extended to the point 0 via $\overline{\psi}(0) = 0$. Furthermore, the same lemma ensures that $\frac{\overline{\psi}(x)}{x}$ is increasing in $(0,\infty)$. Hence, Proposition B.9 in [85] implies that $\overline{\psi}$ is *star-shaped* which means that $\overline{\psi}(rx) \leq r\overline{\psi}(x)$ whenever $r \in [0,1]$ and $x \geq 0$, or equivalently, $\psi(rx) \leq r^2\psi(x)$ for all $(r,x) \in [0,1]\times[0,\infty)$, giving

$$\begin{aligned} Q_1(l+1,\psi) &\leq \sum_{n\geq 1}\sum_{|v|=n-1}\mathbb{E}\psi(L(v)\Pi(v)) \\ &\leq \mathbb{E}\psi(1\vee Z_1)\sum_{n\geq 1}\sum_{|v|=n-1}\mathbb{E}L(v)^2 \\ &= \mathbb{E}\psi(1\vee Z_1)\sum_{n\geq 1} g(2)^{n-1} < \infty \end{aligned}$$

because $\sup_{v\in\mathcal{N}} L(v) \leq 1$ a.s. and $g(2) < 1$.

- Turning to $Q_2(l+1,\psi) = \sum_{m=0}^{l}\sum_{k\geq 0}\mathbb{E}\mathbb{S}^{-m}\psi(\overline{D}_k^{(2^m)})$, it suffices to prove finiteness of $\sum_{k\geq 2} J_1(k,m,\psi)$ and $\sum_{k\geq 2} J_2(k,m,\psi)$ for all $m \in \{0,...,l\}$ (cf. (4.4.3)). But by (4.4.4) and the calculations used for $Q_1(l+1,\psi)$, it follows that

$$\sum_{k\geq 2} J_2(k,m,\psi) \leq \sum_{k\geq 2}\sum_{|v|=k-1}\mathbb{E}\psi(L(v)\Pi(v)) < \infty$$

for each m, whence we can focus on the expressions

$$\sum_{k\geq 2} J_1(k,m,\psi) = \mathbb{E}\mathbb{S}^{-m-1}\psi\left(\mu(2^{m+1})\sum_{|v|=k-1}L(v)^{2^{m+1}}\right), \quad 0 \leq m \leq l.$$

Recall from (4.4.5) that for each k,

$$J_1(k,m,\psi) \leq C\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\sum_{|v|=k-1}L(v)^{2^{m+1}}\right)$$

for some C depending only on m and ψ. Furthermore,

$$\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\sum_{|v|=k-1}L(v)^{2^{m+1}}\right) \leq \mathbb{E}\mathbb{S}^{-1}\psi\left(\sum_{|v|=k-1}L(v)^2\right)$$

by an application of Lemma B.1 to $x \mapsto x^{2^m}$, i.e. we can restrict to the case $m = 0$ which requires finiteness of

$$\sum_{k \geq 2} \mathbb{E}\mathbb{S}^{-1}\psi\left(\sum_{|v|=k-1} L(v)^2\right).$$

Now fix $\varepsilon > 0$ such that $1 < \beta := \alpha/2 + \varepsilon < \alpha$. Then

$$\frac{\mathbb{S}^{-1}\psi(x)}{x^\beta} = \frac{\mathbb{S}^{-1}\hat{\ell}(x)}{x^\varepsilon} \underset{x\to\infty}{\longrightarrow} 0$$

by Proposition 1.3.6 in [30] and the fact that $\mathbb{S}^{-1}\hat{\ell}$ is also slowly varying. As $\mathbb{S}^{-1}\psi$ belongs to $\mathfrak{C}_l^*$, $\lim_{x\downarrow 0} \frac{\mathbb{S}^{-1}\psi(x)}{x} = 0$ by Lemma 4.2.2. Hence, we obtain the estimate $\mathbb{S}^{-1}\psi(x) \leq Cx + Cx^\beta$ for some $C \in (0,\infty)$. Since $\sum_{k\geq 2}\sum_{|v|=k-1} \mathbb{E}L(v)^2 = \sum_{k\geq 2} g(2)^{k-1} < \infty$, finiteness of $\sum_{k\geq 2} J_1(k,m,\psi)$ reduces to that of

$$\sum_{k\geq 2} \mathbb{E}\left(\sum_{|v|=k-1} L(v)^2\right)^\beta = \sum_{k\geq 2} \mathbb{E}\overline{Z}_{k-1}^\beta,$$

when $(\overline{Z}_n)_{n\geq 0}$ denotes the WBP with generic weights $(\overline{T}_1, \overline{T}_2, ...) = (T_1^2, T_2^2, ...)$. Note that by (4.5.1), $\overline{g}(s) = \sum_{i\geq 1} \mathbb{E}\overline{T}_i^s = \sum_{i\geq 1} \mathbb{E}T_i^{2s}$ satisfies $\overline{g}(1) \vee \overline{g}(\beta) < 1$. Now another appeal to Lemma 3.3.4 allows us to choose $(\hat{Z}_n)_{n\geq 0}$ with generic weights $(\hat{T}_i)_{i\geq 1}$ such that $\overline{Z}_n \leq \hat{Z}_n$ a.s. for all $n \geq 0$, $\hat{Z}_1 = c + \overline{Z}_1$ a.s. for some $c > 0$, $\hat{g}(1) = \sum_{i\geq 1} \mathbb{E}\hat{T}_i < 1$ and $\hat{g}(\beta) = \sum_{i\geq 1} \mathbb{E}\hat{T}_i^\beta < \hat{g}(1)^\beta$. Hence,

$$\mathbb{E}\overline{Z}_k^\beta \leq \mathbb{E}\hat{Z}_k^\beta \leq \hat{g}(1)^{k\beta} \cdot \sup_{j\geq 0} \mathbb{E}\hat{W}_j^\beta \quad \text{for all } k \geq 1, \tag{4.5.2}$$

where $(\hat{W}_j)_{j\geq 0}$ is the WBP with generic weights $(\hat{T}_i/\hat{g}(1))_{i\geq 1}$. Notice that

$$\sup_{j\geq 0} \mathbb{E}\hat{W}_j^\beta < \infty$$

by Corollary 3.4.7 because $\sum_{i\geq 1} \mathbb{E}\left(\frac{\hat{T}_i}{\hat{g}(1)}\right)^\beta = \frac{\hat{g}(\beta)}{\hat{g}(1)^\beta} < 1$ and

$$\mathbb{E}\hat{W}_1^\beta \prec \mathbb{E}\hat{Z}_1^\beta \prec \mathbb{E}\overline{Z}_1^\beta \leq \mathbb{E}Z_1^\beta \prec \mathbb{E}\phi(Z_1) < \infty.$$

For the last line, we utilized $x^\beta = o(\phi(x))$ as $x \to \infty$ and the fact that by (4.5.1), $\overline{Z}_1 = \sum_{i\geq 1} T_i^2 \leq \sum_{i\geq 1} T_i = Z_1$ a.s.. Now use (4.5.2) and $\hat{g}(1) < 1$ to infer

$$\sum_{k\geq 1} \mathbb{E}\overline{Z}_k^\beta \leq \sum_{k\geq 1} \hat{g}(1)^{k\beta} \cdot \sup_{j\geq 0} \mathbb{E}\hat{W}_j^\beta < \infty.$$

Summarizing, we have shown that $\mathbb{E}\phi(Z_1) < \infty$ implies $\sup_{n\geq 0} \mathbb{E}\psi(W_n) < \infty$.

As ψ is continuous, $\sup_{n\geq 0}\mathbb{E}\psi(W_n)<\infty$ implies $\mathbb{E}\psi(W)<\infty$, whence $\sup_{n\geq 0}\mathbb{E}\phi(W_n)<\infty$ and $\mathbb{E}\phi(W)<\infty$ since $\phi\asymp\psi$. To show $\mathbb{E}\phi(W)>0$, note that $\lim_{x\to\infty}\frac{\psi(x)}{x}=\infty$ and $\sup_{n\geq 0}\mathbb{E}\psi(W_n)<\infty$ imply uniform integrability of $(W_n)_{n\geq 0}$ (cf. [47], p. 74). Then the claim merely requires an application of Theorem 3.4.1 since ϕ is ultimately positive. The converse assertions follow from Corollary 4.4.2 because if $\alpha>1$, $0<\mathbb{E}\phi(W)<\infty$ implies $0<\mathbb{E}W^\beta<\infty$ whenever $1<\beta<\alpha$, hence uniform integrability of $(W_n)_{n\geq 0}$ and $\mathbb{E}W=1$. □

Remark 4.5.2. (a) In the particular case $\phi(x)=x(\log^+ x)^\beta$ for some positive β one could also argue by using Lemma 6 in [21] instead of considering the embedded multiplicative random walk $(M_{\tau_n})_{n\geq 0}$.

(b) In the situation of **CASE 1** of the previous proof, suppose that subcase **(I)** is valid, i.e. $1<\alpha<2$. Then we can give an alternative argument for $\sup_{n\geq 0}\mathbb{E}\psi(W_n)<\infty$ by using a result on regularly varying functions: Choose $\delta>0$ such that $\beta=\alpha-\delta>1$, recall that $\psi(x)=x^\alpha\hat{\ell}(x)$ for $x\geq 0$, and put

$$\psi_\delta(x):=\frac{\psi(x)}{x^\beta}=x^\delta\hat{\ell}(x)\in\mathfrak{R}_\delta.$$

From Theorem 1.5.3 in [30] we know that $\psi_\delta(x)\sim\tilde{\psi}_\delta(x)$ for some increasing function $\tilde{\psi}_\delta:[0,\infty)\to[0,\infty)$ (which is plainly also an element of $\mathfrak{R}_\delta$), and we can find some $\overline{x}>0$ such that

$$\psi_\delta(x)\leq 2\tilde{\psi}_\delta(x)\leq 4\psi_\delta(x)\text{ for all }x\geq\overline{x}.$$

Note that

$$M:=\sup_{z\in[0,\overline{x}]}\psi_\delta(z)<\infty$$

since ψ is continuous with $\psi(z)=O(z^2)$ as $z\downarrow 0$ by (4.2.6), and $\psi_\delta(z)=z^{-\beta}\psi(z)$ for all $z\geq 0$. Consequently, we obtain that for all $x\in[0,1]$ and $y\geq 0$,

$$\begin{aligned}
\psi(xy) &= \psi(xy)\mathbf{1}_{\{xy\leq\overline{x}\}}+\psi(xy)\mathbf{1}_{\{xy>\overline{x}\}}\\
&= (xy)^\beta\psi_\delta(xy)\mathbf{1}_{\{xy\leq\overline{x}\}}+(xy)^\beta\psi_\delta(xy)\mathbf{1}_{\{xy>\overline{x}\}}\\
&\leq M(xy)^\beta+2(xy)^\beta\tilde{\psi}_\delta(xy)\mathbf{1}_{\{xy>\overline{x}\}}\\
&\leq M(xy)^\beta+2(xy)^\beta\tilde{\psi}_\delta(y)\mathbf{1}_{\{y>\overline{x}\}}\\
&\leq M(xy)^\beta+4(xy)^\beta\psi_\delta(y)\\
&= M(xy)^\beta+4x^\beta\psi(y)
\end{aligned}$$

because $xy \le y$ and $\tilde{\psi}_\delta$ is increasing. Fix $v \in \mathcal{N}$. Since $L(v) \in [0,1]$ a.s. and $L(v)$, $\Pi(v)$ are independent, this yields

$$\begin{aligned} \mathbb{E}\psi(L(v)\Pi(v)) &\le M\mathbb{E}L(v)^\beta \cdot \mathbb{E}\Pi(v)^\beta + 4\mathbb{E}L(v)^\beta \cdot \mathbb{E}\psi(\Pi(v)) \\ &\le [M\mathbb{E}(1 \vee Z_1)^\beta + 4\mathbb{E}\psi(1 \vee Z_1)]\mathbb{E}L(v)^\beta, \end{aligned}$$

whence

$$\sup_{n\ge 0} \mathbb{E}\psi(W_n) \prec \sum_{n\ge 1}\sum_{|v|=n-1} \mathbb{E}\psi(L(v)\Pi(v)) \prec \sum_{n\ge 1}\sum_{|v|=n-1} \mathbb{E}L(v)^\beta = \sum_{n\ge 0} g(\beta)^n < \infty$$

because $g(\beta) < 1$.

From Lemma 4.2.2 we know that $\frac{\psi(x)}{x}$ is nondecreasing, but $\frac{\psi(x)}{x^2}$ is nonincreasing in $x \ge 0$. If one knew that for some $\beta \in (1,\alpha)$, $\frac{\psi(x)}{x^\beta}$ is nondecreasing, one could use a similar argument as for $Q_1(l+1,\psi)$ in **CASE 1** of the previous proof since this would ensure by another application of Proposition B.9 in [85] that $\frac{\psi(x)}{x^{\beta-1}}$ is star-shaped, i.e. $\psi(rx) \le r^\beta \psi(x)$ for all $r \in [0,1]$ and $x \ge 0$. However, this growth condition cannot be guaranteed in general.

(c) A close look at **CASE 1** shows that it has not essentially been used that ψ is also regularly varying. Still assuming (4.5.1), it follows for any $\psi \in \mathfrak{C}_0^*$ that $\mathbb{E}\mathbb{L}\psi(Z_1) < \infty$ implies

$$\sup_{n\ge 0} \mathbb{E}\psi(W_n) < \infty \quad \text{and} \quad 0 < \mathbb{E}\psi(W) < \infty$$

because Lemma 4.3 in [6] is applicable to any $\psi \in \mathfrak{C}_0^*$.

(d) If $\alpha = 1$ and $(W_n)_{n\ge 0}$ is a normalized supercritical Galton-Watson process with a.s. limit W, Alsmeyer and Rösler [6] showed that in the situation of the preceding theorem, the condition $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$ is even *necessary* for $0 < \mathbb{E}\phi(W) < \infty$.

When the underlying branching process is nothing but a normalized supercritical Galton-Watson process, Alsmeyer and Rösler [6] also obtained a new proof of the fact that if $Z_1 \log^+ Z_1$ is integrable, then $(W_n)_{n\ge 0}$ is uniformly integrable (which is part of the classical Kesten-Stigum theorem). Note that by Theorem 2.4.17, the situation of weighted branching processes also requires the assumption $\gamma < 0$. By using the same argument as Alsmeyer and Rösler, we can give an alternative proof of the implication

$$\text{"}\mathbb{E}Z_1 \log^+ Z_1 < \infty \Rightarrow \mathbb{E}W = 1\text{"}$$

if the WBP fulfills (4.5.1), hence $\gamma_+ = 0$ and $\gamma \in [-\infty, 0)$.

Corollary 4.5.3. *Assume that (4.5.1) holds. Then* $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ *implies* $\mathbb{E}W = 1$.

Proof. By Lemma 4.5(b) in [6], there is some $\psi \in \mathfrak{C}_0$ such that $\mathbb{L}\psi(x) < \infty$ for all $x \geq 0$, $\lim_{x\to\infty} \frac{\psi(x)}{x} = \infty$ and $\mathbb{E}\mathbb{L}\psi(Z_1) < \infty$. By Lemma 3.3 in [6], we can find $\varphi \in \mathfrak{C}_0^*$ with $\varphi \asymp \psi$. Hence, $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$ as well and by Remark 4.2.4, $\mathbb{E}\mathbb{L}\varphi(Z_1) < \infty$. Now use Remark 4.5.2(c) to infer

$$\sup_{n\geq 0} \mathbb{E}\varphi(W_n) < \infty$$

and note that this condition ensures uniform integrability of $(W_n)_{n\geq 0}$ (cf. [47], p. 74). □

Remark 4.5.4. The proof of Corollary 4.5.3 has exploited the fact that for any nonnegative random variable X, $\mathbb{E}X \log^+ X < \infty$ implies finiteness of $\mathbb{E}\mathbb{L}\varphi(X)$ for some $\varphi \in \mathfrak{C}_0^*$ with $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$. On the other hand, Lemma 4.2.3 shows that for *any* $\varphi \in \mathfrak{C}_0^*$, finiteness of $\mathbb{E}\mathbb{L}\varphi(X)$ ensures the integrability of $X \log^+ X$. Applying this in the case $\alpha = 1$ of Theorem 4.5.1 (with ψ as in the proof of that theorem), we obtain $\mathbb{E}\mathbb{L}\psi(Z_1) < \infty$ because $\mathbb{L}\psi(x) \asymp x\mathbb{U}\ell(x)$, and therefrom $\mathbb{E}Z_1 \log^+ Z_1 < \infty$ and $\mathbb{E}W = 1$.

4.6 Examples

Example 4.6.1. The purpose of this example is to demonstrate the applicability of Theorem 4.4.1. No further assumption on $T_1, T_2, \ldots$ beyond $g(1) = \sum_{i\geq 1} \mathbb{E}T_i = 1$ and $\mathbb{P}(Z_1 = 1) < 1$ is imposed.

(a) Put $\phi_{\alpha,\beta}(x) := x^\alpha (\log^+ x)^\beta$, $x \geq 0$, where $\alpha > 1$ is not a dyadic power and $\beta > 0$. Obviously,

$$\begin{aligned} \phi_{\alpha,\beta}(x) &\sim x^\alpha (1 + \log x)^\beta \mathbf{1}_{[1,\infty)}(x) + x^\alpha \mathbf{1}_{[0,1)}(x) \\ &= x^\alpha \exp\left(\int_1^{1\vee x} \frac{\beta}{s(1+\log s)}\, ds \right) \end{aligned}$$

which belongs to $\mathfrak{R}_\alpha^{\mathrm{IC}}$ because $\varepsilon_\beta(s) := \frac{\beta}{1+\log s} \mathbf{1}_{[1,\infty)}(s)$ is nonnegative and ultimately nonincreasing. Thus, Theorem 4.4.1 gives

$$\mathbb{E}\phi_{\alpha,\beta}(W) \in (0,\infty) \quad \text{iff} \quad \mathbb{E}\phi_{\alpha,\beta}(Z_1) < \infty \quad \text{and} \quad g(\alpha) < 1.$$

(b) Now we consider $\hat{\phi}_{\alpha,\beta}(x) := x^{\alpha}(\log\log x)^{\beta}\mathbf{1}_{[3,\infty)}(x)$ for $x \geq 0$, keeping the assumptions on α and β. Since for $s \geq 3$,

$$\frac{d}{ds}\log(\log\log s)^{\beta} = \frac{\beta}{s \cdot \log s \cdot \log\log s} = \frac{\hat{\varepsilon}_{\beta}(s)}{s},$$

and $\hat{\varepsilon}_{\beta}(s) = \beta(\log s \cdot \log\log s)^{-1}\mathbf{1}_{[3,\infty)}(s)$ is nonnegative and ultimately nonincreasing, we obtain that

$$\hat{\phi}_{\alpha,\beta} \sim x^{\alpha}\exp\left(\int_1^{1\vee x}\frac{\hat{\varepsilon}_{\beta}(s)}{s}\,ds\right) \in \mathfrak{R}_{\alpha}^{\mathrm{IC}}.$$

Hence,

$$\mathbb{E}\hat{\phi}_{\alpha,\beta}(W) \in (0,\infty) \quad \text{iff} \quad \mathbb{E}\hat{\phi}_{\alpha,\beta}(Z_1) < \infty \quad \text{and} \quad g(\alpha) < 1,$$

once more appealing to Theorem 4.4.1.

Example 4.6.2. Given the situation of Remark 3.6.2(b), $T_i = \tilde{\mu}^{-1}\tilde{T}_i \in \{0, \tilde{\mu}^{-1}\} \subset [0,1)$ a.s. for all $i \geq 1$, i.e. (4.5.1) is satisfied. Hence, Theorem 4.5.1 can be applied in the context of supercritical Galton-Watson processes, confirming results obtained by Alsmeyer and Rösler in Corollary 2.3 of [6], and slightly improving results by Bingham and Doney [28] (Theorems 5–7). Note that by Corollary 2.3 in [6], the condition $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$ is also *necessary* for $\mathbb{E}W\ell(W) \in (0,\infty)$ in this case. In the special case $\ell(x) \asymp (\log x)^p$ for some $p > 0$ (whence $\mathbb{U}\ell(x) \asymp (\log x)^{p+1}$ as will be established further below), this fact is due to Athreya [8].

Example 4.6.3. The following examples are devoted to the situation of Section 4.5, i.e. we suppose that (4.5.1) holds and exclude the case $Z_1 = 1$ a.s.. From now on, fix $p > 0$ and $n \geq 1$.

(a) Let $\ell(x) = (\log^+ x)^p$, $\ell_0(x) = p(\log x)^{p-1}\mathbf{1}_{[2,\infty)}(x) \in \mathfrak{R}_0$ and observe that

$$\mathbb{U}\ell_0(x) = \int_2^{2\vee x}\frac{p(\log s)^{p-1}}{s}\,ds = \int_{\log 2}^{\log(2\vee x)} pu^{p-1}\,du \asymp (\log^+ x)^p = \ell(x).$$

Via Theorem 4.5.1,

$$\mathbb{E}W^{2^n}(\log^+ W)^p \in (0,\infty) \quad \text{iff} \quad \mathbb{E}Z_1^{2^n}(\log^+ Z_1)^p < \infty.$$

(b) In a similar manner, $\overline{\ell}(x) = (\log\log x)^p \mathbf{1}_{[3,\infty)}(x) \asymp \mathbb{U}\overline{\ell}_0(x)$, where the function $\overline{\ell}_0(s) = \frac{p(\log\log s)^{p-1}}{\log s}\mathbf{1}_{[3,\infty)}(s)$ is slowly varying. This yields

$$\mathbb{E}W^{2^n}\overline{\ell}(W) \in (0,\infty) \quad \text{iff} \quad \mathbb{E}W^{2^n}\overline{\ell}(Z_1) < \infty.$$

(c) Let $\check{\ell}(x) := (1 \vee \log x)^{-p} \in \mathfrak{R}_0$ and $\check{\ell}_0(x) := p(\log x)^{-p-1}\mathbf{1}_{[2,\infty)} \in \mathfrak{R}_0$. Then

$$\mathbb{H}\check{\ell}_0(x) = \int_{2\vee x}^{\infty} \frac{p}{y(\log y)^{p+1}}\,dy = \int_{\log(2\vee x)}^{\infty} pu^{-p-1}\,du \asymp \check{\ell}_0(x)$$

and therefrom by Theorem 4.5.1

$$\mathbb{E}W^{2^n}(1\vee \log W)^{-p} \in (0,\infty) \quad \text{iff} \quad \mathbb{E}Z_1^{2^n}(1\vee\log Z_1)^{-p} < \infty.$$

(d) Similarly, $\ell^*(x) = (\log\log x)^{-p}\mathbf{1}_{[3,\infty)} \in \mathfrak{R}_0$ satisfies

$$\ell^*(x) \asymp \int_{\log\log x}^{\infty} pu^{-p-1}\,du \asymp \int_x^{\infty} \frac{p}{y\log y(\log\log y)^{p+1}}\,dy,$$

and $\ell_0^*(s) = p(\log s)^{-1}(\log\log s)^{-p-1}\mathbf{1}_{[3,\infty)}(s) \in \mathfrak{R}_0$, hence

$$\mathbb{E}W^{2^n}\ell^*(W) \in (0,\infty) \quad \text{iff} \quad \mathbb{E}Z_1^{2^n}\ell^*(Z_1) < \infty.$$

Finally, we exemplify our result for the case $\alpha = 1$.

Example 4.6.4. Assume (4.5.1), $\mathbb{P}(Z_1 \in \cdot) \neq \delta_1$ and let $p > 0$.

(a) As seen in the previous examples, $\ell(x) = (\log^+ x)^p$ satisfies $\ell(x) \asymp \mathbb{U}\ell_0(x)$ with $\ell_0(x) = p(\log x)^{p-1}\mathbf{1}_{[2,\infty)}(x) \in \mathfrak{R}_0$. Furthermore,

$$\mathbb{U}\ell(x) = \int_1^{1\vee x} \frac{(\log s)^p}{s}\,ds = \int_0^{\log(1\vee x)} u^p\,du \asymp (\log^+ x)^{p+1},$$

consequently

$$0 < \mathbb{E}W(\log^+ W)^p < \infty \quad \text{if} \quad \mathbb{E}Z_1(\log^+ Z_1)^{p+1} < \infty$$

by Theorem 4.5.1.

(b) Analogously, $\overline{\ell}(x) = (\log\log x)^p\mathbf{1}_{[3,\infty)}(x) \asymp \mathbb{U}\overline{\ell}_0(x)$ for some $\overline{\ell}_0 \in \mathfrak{R}_0$. Moreover, for $x \geq 3$,

$$\mathbb{U}\overline{\ell}(x) = \int_3^x \frac{(\log\log s)^p}{s}\,ds = \int_{\log 3}^{\log x} (\log u)^p\,du.$$

Since $u \mapsto (\log^+ u)^p$ belongs to $\mathfrak{R}_0$, Karamata's Theorem (Proposition 1.5.8 in [30]) gives $\int_{\log 3}^y (\log u)^p \, du \sim y(\log y)^p$ as $y \to \infty$, i.e.

$$\mathbb{U}\bar{\ell}(x) = \int_{\log 3}^{\log x} (\log u)^p \, du \asymp \log x \cdot (\log\log x)^p \qquad (x \to \infty).$$

Together with Theorem 4.5.1, this implies

$$0 < \mathbb{E}W(\log\log W)^p \mathbf{1}_{\{W \geq 3\}} < \infty \quad \text{if} \quad \mathbb{E}Z_1 \log Z_1 \cdot (\log\log Z_1)^p \mathbf{1}_{\{Z_1 \geq 3\}} < \infty.$$

(c) More generally, putting $l_1(x) = \log x$ for $x > 1$ and $l_n(x) = \log(l_{n-1}(x)) = l_{n-1}(\log x)$ for $n \geq 2$ and $x > e^{n-1}$, it can be checked that $l_n^p \asymp \mathbb{U}l_{n,p}$ for some $l_{n,p} \in \mathfrak{R}_0$, and $\mathbb{U}l_n^p(x) \asymp \log x \cdot l_n(x)^p$, hence

$$0 < \mathbb{E}W l_n^p(W) \mathbf{1}_{\{W > e^{n-1}\}} < \infty \quad \text{if} \quad \mathbb{E}Z_1 \log Z_1 \cdot l_n^p(Z_1) \mathbf{1}_{\{Z_1 > e^{n-1}\}} < \infty.$$

Chapter 5

Some applications to tail probabilities

5.1 On the tail probabilities of $W - W_n$

Since a.s. convergence implies convergence in probability, the expressions $\mathbb{P}(|W_n - W| > x)$ clearly tend to zero as $n \to \infty$ for any fixed $x > 0$. In this section, we give estimates for these tail probabilities (in n and x). First of all, observe that for any nonnegative random variable X and any increasing nonnegative function φ on $[0, \infty)$, $\mathbb{E}\varphi(X) < \infty$ implies

$$\varphi(t)\mathbb{P}(X \geq t) \leq \int_{\{X \geq t\}} \varphi(X)\, d\mathbb{P} \underset{t \to \infty}{\longrightarrow} 0$$

by the dominated convergence theorem, i.e.

$$\mathbb{P}(X \geq t) = o(\varphi(t)^{-1}) \quad \text{as } t \to \infty. \tag{5.1.1}$$

Moreover, if $\mathbb{E}W = 1$ and $\mathbb{E}\psi(W) < \infty$ for some $\psi \in \mathfrak{C}$,

$$\mathbb{P}(|W - W_n| > x) \leq \frac{\mathbb{E}\psi(|W - W_n|)}{\psi(x)} = \frac{o(1)}{\psi(x)}, \quad x > 0 \tag{5.1.2}$$

holds by Markov's inequality and the dominated convergence theorem because

$$\psi(|W - W_n|) \leq \psi\left(\sup_{n \geq 0} W_n\right) \text{ a.s.}$$

and

$$\mathbb{E}\psi\left(\sup_{n \geq 0} W_n\right) \prec \mathbb{E}\psi(W) < \infty$$

by Theorem 1.1.6 which ensures $\mathbb{E}\psi(|W - W_n|) = o(1)$ as $n \to \infty$. In particular, the expression $o(1)$ depends only on n, but not on x.

In the special case $\psi(x) = x^p$ with $p \in (1, 2]$, we obtain the following more explicit result.

Theorem 5.1.1. *Suppose that for some $p \in (1,2]$, $\mathbb{E}Z_1^p < \infty$ and $g(p) < 1$. Then*

$$\mathbb{P}(|W - W_n| > x) \leq \frac{2g(p)^n \mathbb{E}|W-1|^p}{x^p}, \quad n \geq 1, x > 0.$$

In case $p = 2$, we even have the slightly better estimate

$$\mathbb{P}(|W - W_n| > x) \leq \frac{g(2)^n \operatorname{Var} W}{x^2}, \quad n \geq 1, x > 0.$$

Proof. By what has been mentioned before the theorem, we only have to estimate $\mathbb{E}|W - W_n|^p$ for any $n \geq 1$. For this purpose, we use the representation

$$W - W_n = \sum_{|v|=n} L(v) \left(W_{(v)} - 1\right)$$

which is a consequence of Theorem 1.1.4. Again, $\left(W_{(v)}\right)_{|v|=n}$ is a family of independent copies of W that is independent of $\mathcal{F}_n$. Similarly to the proof of Theorem 3.4.3, this sum can be viewed as a martingale limit, and another application of the Topchii-Vatutin inequality (with $C = 2$) gives

$$\mathbb{E}|W - W_n|^p \leq 2 \sum_{|v|=n} \mathbb{E}L(v)^p \left|W_{(v)} - 1\right|^p = 2\mathbb{E}|W-1|^p g(p)^n.$$

In case $p = 2$, it is a well-known fact (and easy to check by calculation) that the Topchii-Vatutin inequality holds with $C = 1$ which proves the theorem. □

Remark 5.1.2. Recall from Theorem 3.4.9 that $\operatorname{Var} W = \frac{\operatorname{Var} Z_1}{1-g(2)}$ and notice that by a double appeal to the Topchii-Vatutin inequality,

$$\mathbb{E}|W-1|^p \leq \frac{4\mathbb{E}|Z_1 - 1|^p}{1 - g(p)} \quad \text{if } 1 < p < 2.$$

Remark 5.1.3. In the situation of the preceding theorem, we now consider the martingale $(\tilde{W}_n)_{n \geq 0}$, defined by $\tilde{W}_n = W_n - 1$ for $n \geq 0$. Then it follows by Markov's inequality and Doob's inequality (Theorem 7.4.8 in [37]) that for $n \geq 1$ and $x > 0$,

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |\tilde{W}_i| > nx\right) \leq \mathbb{E}\left(\max_{1 \leq i \leq n} |\tilde{W}_i|\right)^p \frac{1}{(nx)^p} \leq \left(\frac{p}{p-1}\right)^p \frac{1}{(nx)^p} \sup_{j \geq 1} \mathbb{E}|\tilde{W}_j|^p.$$

Since uniform integrability of $(W_n^p)_{n\geq 0}$ ensures that of $(|\tilde{W}_n|^p)_{n\geq 0}$ and by convexity of $x \mapsto x^p$, we further obtain

$$\begin{aligned}\mathbb{P}\left(\max_{1\leq i\leq n}|\tilde{W}_i| > nx\right) &\leq \left(\frac{p}{p-1}\right)^p \frac{1}{(nx)^p}\lim_{j\to\infty}\mathbb{E}|\tilde{W}_j|^p \\ &= \left(\frac{p}{p-1}\right)^p \frac{1}{(nx)^p}\mathbb{E}|W-1|^p \\ &\leq \frac{4\mathbb{E}|Z_1-1|^p}{1-g(p)}\left(\frac{p}{p-1}\right)^p \frac{1}{(nx)^p}\end{aligned}$$

by the previous remark. In case $p=2$ we may even drop the constant 4. This is an improvement of a recent result by Li [72] for general $\mathfrak{L}_p$-martingales $(1<p\leq 2)$ which uses the rough estimate

$$\mathbb{E}|\tilde{W}_n|^p \leq B_p n \sup_{1\leq i\leq n}\mathbb{E}|D_i|^p$$

for some constant $B_p \in (0,\infty)$ depending only on p, and gives

$$\mathbb{P}\left(\max_{1\leq i\leq n}|\tilde{W}_i| > nx\right) \leq B_p\left(\sup_{1\leq i\leq n}\mathbb{E}|D_i|^p\right)x^{-p}n^{1-p}.$$

A similar result in the case $p\geq 2$ has been obtained by Lesigne and Volný [71] using Burkholder's inequality.

Remark 5.1.4. Recall that $W^* = \sup_{n\geq 0} W_n$ and suppose that (4.5.1) holds. If there is some $\theta>0$ such that $\mathbb{E}e^{\theta Z_1}<\infty$, Theorem 3.4.11 asserts that $\mathbb{E}e^{\lambda W^*}<\infty$ for some $\lambda>0$. Hence, it follows by Markov's inequality that for all $n\geq 1$, $x>0$ and some $C\in(0,\infty)$,

$$\mathbb{P}\left(\sup_{n\geq 0}W_n > nx\right) \leq Ce^{-\lambda nx} \quad\text{and}\quad \mathbb{P}\left(\sup_{n\geq 1}|\tilde{W}_n| > nx\right) \leq Ce^{-\lambda nx}. \tag{5.1.3}$$

In particular, $\mathbb{E}e^{\lambda|D_i|} = \mathbb{E}e^{\lambda|W_i-W_{i-1}|} \leq \mathbb{E}e^{\lambda W^*}<\infty$ for all $i\geq 1$. Thus, the above estimate (5.1.3) improves a result by Lesigne and Volný for general martingale differences $X_1, X_2, \ldots$ satisfying $\sup_{i\geq 1}\mathbb{E}e^{\lambda|X_i|}<\infty$ which asserts

$$\mathbb{P}(|\tilde{W}_n| > nx) < \exp\left(-\frac{1}{2}(1-\varepsilon)(\lambda x)^{2/3}n^{1/3}\right)$$

for all $x>0$, $0<\varepsilon<1$ and all sufficiently large n (cf. [71], Theorem 3.2). An extension of the latter estimate to the sequence of maxima $\sup_{1\leq i\leq n}|\sum_{i=1}^n X_i|$ has been obtained by Laib [70], further related results can be found in [40] and [46].

For the one-sided tail probabilities $\mathbb{P}(W_n - W > x)$, Theorem 5.1.1 particularly gives

$$\mathbb{P}(W - W_n > x) \le \frac{g(2)^n \operatorname{Var} W}{x^2}, \quad n \ge 1, x > 0, \tag{5.1.4}$$

provided $g(2) < 1$ and $\operatorname{Var} Z_1 < \infty$. The subsequent result improves this estimate by applying a recent result by Bentkus [16] for martingales whose increments are bounded from above. Once again, the fixed point equation (1.1.8) comes into play.

To formulate the result, we use the convention $e^{-y/0} = 0$ whenever $y > 0$.

Theorem 5.1.5. *Suppose that* $\operatorname{Var} Z_1 < \infty$ *and* $g(2) < 1$, *and put* $K := 1 \vee \operatorname{Var} W < \infty$. *Then for each* $n \ge 1$ *and* $x > 0$,

$$\mathbb{P}(W_n - W > x) \le \mathbb{E}\left[\exp\left(-\frac{x^2}{2KZ_n^{(2)}}\right) \wedge c_0 \overline{\Phi}\left(\frac{x}{\left(KZ_n^{(2)}\right)^{1/2}}\right)\right],$$

where

$$\overline{\Phi}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-s^2/2}\, ds$$

is the survival function of the standard normal distribution, $c_0 = 1/\overline{\Phi}(\sqrt{3})$ *and* $Z_n^{(2)} = \sum_{|v|=n} L(v)^2$.

Proof. Fix n, x and an enumeration $(\mathbf{v}_j)_{j\ge1}$ of $\mathbb{N}^n$. Then we have the representation

$$W_n - W = \sum_{j\ge1} L(\mathbf{v}_j)(1 - W_{(\mathbf{v}_j)}) \quad \text{a.s.}$$

from the proof of Theorem 5.1.1. Now abbreviate $\mathbf{c} := (c_j)_{j\ge1}$ if the latter is an arbitrary sequence of nonnegative real numbers, $A := \left\{\mathbf{c} \in [0,\infty)^{\mathbb{N}} : 0 < \sum_{j\ge1} c_j < \infty\right\}$ and $\Theta := \mathbb{P}\left((L(\mathbf{v}_j))_{j\ge1} \in \cdot\right)$. Then

$$\begin{aligned}
\mathbb{P}(W_n - W > x) &= \int_A \mathbb{P}\left(W_n - W > x \middle| (L(\mathbf{v}_j))_{j\ge1} = \mathbf{c}\right) \Theta(d\mathbf{c}) \\
&= \int_A \mathbb{P}\left(\sum_{j\ge1} c_j(1 - W_{(\mathbf{v}_j)}) > x\right) \Theta(d\mathbf{c}).
\end{aligned}$$

Here,

$$\begin{aligned}
\mathbb{P}\left(\sum_{j\ge1} c_j(1 - W_{(\mathbf{v}_j)}) > x\right) &\le \mathbb{E}\mathbf{1}_{\liminf_{m\to\infty}\{\sum_{j=1}^m c_j(1-W_{(\mathbf{v}_j)})>x\}} \\
&\le \liminf_{m\to\infty} \mathbb{P}\left(\sum_{j=1}^m c_j(1 - W_{(\mathbf{v}_j)}) > x\right)
\end{aligned}$$

by Fatou's Lemma. Since $\left(\sum_{j=1}^m c_j(1-W_{(\mathbf{v}_j)})\right)_{m\geq 1}$ is a martingale (with respect to an appropriate filtration) that has increments which are bounded from above, i.e.

$$c_j(1-W_{(\mathbf{v}_j)}) \leq c_j \quad \text{a.s.},$$

a recent result by Bentkus (Theorem 1.1 in [16]) says that for every $\mathbf{c} \in A$,

$$\mathbb{P}\left(\sum_{j=1}^m c_j(1-W_{(\mathbf{v}_j)}) > x\right) \leq \mathcal{D}(x/\sigma_m(\mathbf{c})),$$

where $\sigma_m(\mathbf{c}) := \left(K\sum_{j=1}^m c_j^2\right)^{1/2}$ and $\mathcal{D}(y) := e^{-y^2/2} \wedge c_0\overline{\Phi}(y)$. Since $\sigma_m(\mathbf{c}) \uparrow \sigma_\infty(\mathbf{c}) := \left(K\sum_{j\geq 1} c_j^2\right)^{1/2}$ as $m \to \infty$ and $\mathcal{D}$ is continuous,

$$\begin{aligned}
\mathbb{P}(W_n - W > x) &\leq \int_A \liminf_{m\to\infty} \mathbb{P}\left(\sum_{j=1}^m c_j(1-W_{(\mathbf{v}_j)}) > x\right) \Theta(d\mathbf{c}) \\
&\leq \int_A \liminf_{m\to\infty} \mathcal{D}(x/\sigma_m(\mathbf{c}))\, \Theta(d\mathbf{c}) \\
&= \int_A \mathcal{D}(x/\sigma_\infty(\mathbf{c}))\, \Theta(d\mathbf{c}) \\
&= \int_{\{0<Z_n^{(2)}<\infty\}} \mathcal{D}\left(\frac{x}{\left(KZ_n^{(2)}\right)^{1/2}}\right) d\mathbb{P} \\
&= \mathbb{E}\mathcal{D}\left(\frac{x}{\left(KZ_n^{(2)}\right)^{1/2}}\right)
\end{aligned}$$

because $\mathcal{D}(\infty) := \lim_{y\to\infty} \mathcal{D}(y) = 0$. □

The following remark discusses applications of Theorem 5.1.5. Therefore, we assume throughout that $g(2) < 1$ and $\operatorname{Var} Z_1 < \infty$.

Remark 5.1.6. It is easily checked that for any $u, y > 0$ and $\beta \geq 1$,

$$e^{-u/y} \leq \left(\frac{\beta}{eu}\right)^\beta y^\beta. \tag{5.1.5}$$

(a) Using this in case $\beta = 1$ together with $\mathcal{D}(y) \leq e^{-y^2/2}$, we obtain that in the situation of the preceding theorem,

$$\mathbb{P}(W_n - W > x) \leq \int_{\{Z_n^{(2)}>0\}} \exp\left(-x^2/2KZ_n^{(2)}\right) d\mathbb{P} \leq \frac{2K}{ex^2}\mathbb{E}Z_n^{(2)} = \frac{2K}{ex^2} g(2)^n.$$

(b) In the case $\beta > 1$, we apply the preceding result and inequality (5.1.5) in two situations:

- If $\mathbb{E}\left(\sum_{i\geq 1} T_i^2\right)^\beta < \infty$ and $g(2\beta) < g(2)^\beta$, i.e. $\Gamma_\beta := \sup_{n\geq 0} \mathbb{E}(W_n^{(2)})^\beta < \infty$ by Theorem 3.4.3, we achieve the better estimate

$$\begin{aligned}\mathbb{P}(W_n - W > x) &\leq \mathbb{E}\exp\left(-x^2/2KZ_n^{(2)}\right)\\ &\leq \left(\frac{2\beta K}{e}\right)^\beta x^{-2\beta}\mathbb{E}(Z_n^{(2)})^\beta \qquad (5.1.6)\\ &\leq \left(\frac{2\beta K}{e}\right)^\beta \Gamma_\beta x^{-2\beta} g(2)^{n\beta}\end{aligned}$$

  for $n \geq 1$ and $x > 0$.

- Assume (4.5.1) and $\mathbb{E}Z_1^\beta < \infty$. Then $(Z_n^{(2)})^\beta \leq Z_n^\beta \leq W^{*\beta}$ a.s., and $\mathbb{E}W^{*\beta} < \infty$ by Theorem 3.4.3 and the fact that $g(\beta) < 1$. Furthermore, $Z_n^{(2)} = g(2)^n W_n^{(2)} \to 0$ a.s. as $n \to \infty$ since $g(2) < 1$ and $(W_n^{(2)})_{n\geq 0}$ is a nonnegative martingale that converges almost surely to a finite limit. Thus, by the dominated convergence theorem,

$$\mathbb{E}(Z_n^{(2)})^\beta = o(1) \quad \text{as } n \to \infty,$$

  whence

$$\mathbb{P}(W_n - W > x) \leq \frac{o(1)}{x^{2\beta}} \quad \text{as } n \to \infty, \qquad (5.1.7)$$

  where the numerator $o(1)$ is independent of x. This means that in this special situation, we attain a slight improvement of (5.1.2) for not too small x because we do not require $\mathbb{E}Z_1^{2\beta} < \infty$, but only $\mathbb{E}Z_1^{\beta\vee 2} < \infty$. On the other hand, (5.1.6) or (5.1.7) ensure $\mathbb{E}\left((W_n - W)^+\right)^p < \infty$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \mathbb{E}\left((W_n - W)^+\right)^p = 0$$

  whenever $0 < p < 2\beta$.

(c) It is a well-known fact (see e.g. Lemma 7.20 in [63]) that the survival function $\overline{\Phi}$ satisfies the estimate

$$\frac{t}{\sqrt{2\pi}(1+t^2)} e^{-t^2/2} \leq \overline{\Phi}(t) \leq \frac{1}{\sqrt{2\pi}t} e^{-t^2/2}, \qquad t > 0.$$

Concerning an upper bound for the constant c_0, Bentkus [16] asserts that $c_0 \leq 25$. For the function $\mathcal{D}$ introduced in the proof of Theorem 5.1.5, this means

$$\mathcal{D}(t) \leq e^{-t^2/2}\left(1 \wedge \frac{25}{\sqrt{2\pi}t}\right), \qquad t > 0.$$

Remark 5.1.7. Let $\alpha \in (1,2]$ and suppose that $\mathbb{E}W = 1$, $g(\theta) < 1$ for some $\theta > \alpha$, and $\sum_{i\geq 1} \mathbb{E}T_i^\alpha \log T_i^\alpha < g(\alpha)\log g(\alpha)$. If Z_1 belongs to the domain of attraction of a stable law of index α which satisfies an additional condition, Rösler et al. [98] have evaluated the rate of convergence of $W - W_n$ to 0 in the following sense: There is a sequence of constants $(c_n)_{n\geq 0} = (c_n(\alpha))_{n\geq 0}$ such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1$ and

$$(g(\alpha)^n c_n)^{-1/\alpha}(W - W_n)$$

converges in distribution to a nondegenerate random variable X. More precisely, they determined the limit of the Fourier transforms

$$\mathbb{E}\exp\left[it(g(\alpha)^n c_n)^{-1/\alpha}(W - W_n)\right], \qquad t \in \mathbb{R},$$

as $n \to \infty$. In addition, they showed that the sequence $(c_n)_{n\geq 0}$ is a suitable Seneta-Heyde norming of the martingale $(W_n^{(\alpha)})_{n\geq 0} = \left(g(\alpha)^{-n}\sum_{|v|=n} L(v)^\alpha\right)_{n\geq 0}$, i.e. $c_n^{-1}W_n^{(\alpha)}$ converges in probability to a random variable Δ_α which is strictly positive whenever $W_n^{(\alpha)}$ survives, and express the above limit of Fourier transforms in terms of Δ_α. It ought to be emphasized that Rösler et al. also allow negative weights in their articles [97] and [98], while the BRW case is covered by Theorem 2.4.20.

5.2 Some relations between the tail probabilities of Z_1, W and W^*

At the end of this thesis, we describe some relations between the tail probabilities of the random variables Z_1, W and W^*. Note that if $\mathbb{E}W = 1$, W and W^* automatically show a similar tail behaviour by Theorem 1.1.6.

Remark 5.2.1. Let $\alpha > 1$ and suppose that $c < 1$.

(a) If $g(\alpha) < 1$ and Z_1 has *polynomially decreasing tails of order* α, i.e. for appropriate $C, x_0 > 0$,

$$\mathbb{P}(Z_1 \geq x) \leq Cx^{-\alpha} \text{ for all } x \geq x_0,$$

then for any $\beta \in (1,\alpha)$,

$$\mathbb{E}Z_1^\beta = \int_{(0,\infty)} \beta x^{\beta-1}\mathbb{P}(Z_1 \geq x)\, \lambda(dx) < \infty.$$

Therefore, by Theorem 3.4.3, $\mathbb{E}W^\beta < \infty$, and by (5.1.1),

$$\mathbb{P}(W \geq x) = o(x^{-\beta}) \quad \text{as } x \to \infty,$$

in particular, W has polynomially decreasing tails of order β whenever $\beta < \alpha$.

(b) On the other hand, if $\mathbb{E}W = 1$ and W has polynomially decreasing tails of order α, then by Theorem 3.4.5, $\mathbb{E}Z^\beta \leq \mathbb{E}W^\beta < \infty$ for any $\beta < \alpha$, whence by (5.1.1), Z_1 has polynomially decreasing tails of order β.

(c) Assume that $\mathbb{E}W = 1$ and recall that a nonnegative random variable Y is said to have *exponentially decreasing tails* if for some $C, r > 0$,

$$\mathbb{P}(Y \geq t) \leq Ce^{-rt} \text{ for all } t \geq 0.$$

By the Markov inequality, we easily conclude from Theorem 3.4.11 that if $c < 1$, the following statements are equivalent:

(i) Z_1 has exponentially decreasing tails, and $\sup_{i\geq 1} T_i \leq 1$ a.s.,

(ii) W has exponentially decreasing tails,

(iii) W^* has exponentially decreasing tails.

The following theorem has been proved by Rösler, Topchii and Vatutin [98], providing conditions which guarantee that W belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$ iff Z_1 belongs to the domain of attraction of a stable law of the same index α. This implies that both random variables show a similar tail behaviour.

It should be mentioned that Rösler et al. have also studied the case of not necessarily nonnegative weights $T_1, T_2, ...$ in [98]. However, as before, we focus on nonnegative weights.

For more background material on stable distributions, we refer to [57], Chapter 2.

Theorem 5.2.2. (Rösler, Topchii, Vatutin [98])
Let $\alpha \in (1, 2]$ and suppose that $\mathbb{E}W = 1$ and $g(\alpha) < 1 = g(1)$.

(i) Suppose that there exist constants $0 < C_1 < C_2 < \infty$ such that for all $i \geq 1$,

$$\mathbb{P}(T_i = 0 \text{ or } C_1 < T_i < C_2) = 1. \tag{5.2.1}$$

Then W belongs to the domain of attraction of a stable law of index α iff Z_1 belongs to the domain of attraction of a stable law of index α.

(ii) *If in the situation of (i), $\alpha < 2$ and H is slowly varying at infinity,*

$$\begin{aligned} \mathbb{P}(Z_1 > x) &= \frac{\eta + o(1)}{x^\alpha} H(x) \text{ as } x \to \infty \text{ for some } \eta > 0 \\ \text{iff } \mathbb{P}(W > x) &= \frac{\vartheta + o(1)}{x^\alpha} H(x) \text{ as } x \to \infty \text{ for some } \vartheta > 0. \end{aligned}$$

(iii) *Assume that for some $\theta > \alpha$, $g(\theta) < 1$. Then W belongs to the domain of normal attraction of a stable law of index α iff Z_1 belongs to the domain of normal attraction of a stable law of index α.*

(iv) *If in the situation of (iii), $\alpha < 2$ and H satisfies $\lim_{x\to\infty} H(x) = H \in (0, \infty)$ (so a fortiori, H is slowly varying), we have the same equivalence as in (ii).*

Remark 5.2.3. (a) Given the situation of *(i)*, suppose that one (hence each) of the random variables Z_1, W belongs to the domain of attraction of a stable law of index α. Then for any $\delta < \alpha$,

$$\mathbb{E}W^\delta \vee \mathbb{E}Z_1^\delta < \infty$$

(cf. [57], Theorem 2.6.4), a result which rounds off those of Chapter 3.

(b) If $g(1) = 1$ and (5.2.1) holds, $g(\theta) < \infty$ for all $\theta > 0$ because

$$g(\theta) \leq \begin{cases} C_1^{\theta-1} & \text{if } 0 < \theta < 1 \\ C_2^{\theta-1} & \text{if } \theta > 1. \end{cases}$$

Hence, g is convex and thus continuous in $(0, \infty)$ which ensures that if $g(\alpha) < 1$, we can find $\theta > \alpha$ with $g(\theta) < 1$.

The following result is due to Liu [78] in case $\mathbb{P}(N < \infty) = 1$, i.e. if there are only finitely many positive weights in the first generation. The general case has recently been studied by Iksanov [58].

Theorem 5.2.4. (Liu [78]; Iksanov [58])
Suppose that $c < 1$ and that for some $\chi > 1$, $g(\chi) = \sum_{i\geq 1} \mathbb{E}T_i^\chi = g(1) = 1$, $\mu(\chi) = \mathbb{E}Z_1^\chi < \infty$ and $\sum_{i\geq 1} \mathbb{E}T_i^\chi \log^+ T_i < \infty$. Then there is a constant $C_\chi \in (0, \infty)$ such that

$$\lim_{x\to\infty} x^\chi \mathbb{P}(W > x) = C_\chi. \tag{5.2.2}$$

Remark 5.2.5. The conditions $\mathbb{E}Z_1^\chi < \infty$ and $\sum_{i\ge1}\mathbb{E}T_i^\chi \log^+ T_i < \infty$ hold simultaneously if for some $\varepsilon > 0$, $\mu(\chi+\varepsilon) = \mathbb{E}Z_1^{\chi+\varepsilon} < \infty$ because $\log^+ x \le Cx^\varepsilon$ for all $x \ge 0$ and some $C > 0$, hence $\sum_{i\ge1}\mathbb{E}T_i^\chi \log^+ T_i \le Cg(\chi+\varepsilon) \le C\mu(\chi+\varepsilon) < \infty$. Notice that, by Theorem 3.4.3, the condition $g(\chi) = 1$ implies $\mathbb{E}W^\chi = \infty$, in accordance with the asymptotic result (5.2.2) which gives

$$\mathbb{E}W^\chi \succ \int_{(1,\infty)} x^{\chi-1}\mathbb{P}(W > x)\,\boldsymbol{\lambda}(dx) \succ \int_{(1,\infty)} \frac{1}{x}\,\boldsymbol{\lambda}(dx) = \infty.$$

On the other hand, strict convexity of g (cf. Lemma 3.3.1) yields $g(\chi-\eta) < 1$ whenever $0 < \eta < \chi - 1$, whence $0 < \mathbb{E}W^{\chi-\eta} < \infty$ by Theorem 3.4.3, again in full accordance with (5.2.2) which shows

$$\mathbb{E}W^{\chi-\eta} \prec \int_{(1,\infty)} \frac{1}{x^{1+\eta}}\,\boldsymbol{\lambda}(dx) < \infty.$$

Theorem 5.2.4 evidently covers the case $\mathbb{P}(\sup_{i\ge1} T_i > 1) > 0$ because if the contrary is satisfied, then $g < 1$ on $(1,\infty)$ by Remark 3.5.1. In this situation (still assuming $c < 1$ to avoid trivialities), the following result by Liu [74] complements this section, in particular part (c) of Remark 5.2.1. For any nonnegative random variable X, let $\|X\|_\infty$ be the essential supremum of X and recall that for $\vartheta > 0$, $(Z_n^{(\vartheta)})_{n\ge0}$ denotes the WBP with generic weights T_i^ϑ, $i \ge 1$.

For a simplified version in the context of homogeneous BRW, we refer to [77].

Theorem 5.2.6. (Liu [74])
Assume that the following conditions are satisfied:

(i) $\|N\|_\infty < \infty$, $\sup_{i\ge1} T_i \le 1$ *a.s. and* $c < 1$,

(ii) $g(1) = 1$ *and* $\mathbb{P}(Z_1 \ne 1) > 0$ *(*$\Rightarrow \|Z_1\|_\infty > 1$*)*

(iii) $\{\vartheta \in (1,\infty) : \|Z_1^{(\vartheta)}\|_\infty \le 1\} \ne \emptyset$,

(iv) For some constants $0 < \delta < 1$, $\kappa \ge 0$ *and* $C > 0$, *and all sufficiently small* $x \in (0,1)$,

$$\mathbb{P}\left(Z_1^{(\theta)} > 1 - x \text{ and } \sup_{i\ge1} T_i \le \delta\right) \ge Cx^\kappa,$$

with θ *denoting the smallest solution of the equation* $\|Z_1^{(\theta)}\|_\infty = \|\sum_{i\ge1} T_i^\theta\|_\infty = 1$ *in* $(1,\infty)$.

Then for some constants $0 < c_1 \leq c_2 < \infty$ *and all sufficiently large* $x > 0$,

$$\exp\left(-c_2 x^{\theta/(\theta-1)}\right) \leq \mathbb{P}(W \geq x) \leq \exp\left(-c_1 x^{\theta/(\theta-1)}\right).$$

Remark 5.2.7. Evidently, the moment generating function $t \mapsto \mathbb{E}e^{tZ_1}$ of Z_1 is everywhere finite because $\|Z_1\|_\infty \leq \|N\|_\infty < \infty$. Furthermore, the function $R(\vartheta) = \|\sum_{i\geq 1} T_i^\vartheta \mathbf{1}_{\{T_i>0\}}\|_\infty$ is nonincreasing on $[0,\infty)$ with $R(0) = \|N\|_\infty \in [2,\infty)$ and $R(1) = \|Z_1\|_\infty > 1$.

Finally, we quote a further result by Liu [79] dealing with the distribution function of W (cf. Theorem 3.4.13). An analogous result for homogeneous BRW can be found in [77] and [80].

Theorem 5.2.8. (Liu [79])
Suppose that the assumptions of Remark 3.4.2(d) hold. Denote by m *the essential infimum of* N *and suppose that* $m \geq 2$. *Moreover, assume that for some* $a > 0$,

$$\mathbb{P}\left(\inf_{1\leq i\leq m} T_i \geq a\right) = 1,$$

and put $\lambda := -\log m/\log a$. *Then* $0 < \lambda < 1$, *and the following assertions are valid:*

(i) There is a constant $C_1 > 0$ *such that for all sufficiently small* $x > 0$,

$$\mathbb{P}(W \leq x) \leq \exp\left(-C_1 x^{-\lambda/(1-\lambda)}\right).$$

(ii) Let $\varepsilon > 0$. *If*

$$\mathbb{P}\left(N = m \text{ and } \sup_{1\leq i\leq m} T_i \leq a+\varepsilon\right) > 0,$$

then there is a constant $C_2 > 0$ *such that for all sufficiently small* $x > 0$,

$$\mathbb{P}(W \leq x) \geq \exp\left(-C_2 x^{-\lambda(\varepsilon)/(1-\lambda(\varepsilon))}\right),$$

with $\lambda(\varepsilon) = -\log m/\log(a+\varepsilon)$.

Appendix A

Two asymptotic results for stationary ergodic sequences of random variables

The following asymptotic result for stationary m-dependent sequences is well known (see for example Lemma 1.1 in [83]) in the independent case ($m = 0$), but by decomposing the sequence into finitely many independent subsequences, it is easily extended to the more general setting of m-dependence. We mention that *(i)* is readily verified for any sequence of nonnegative and identically distributed random variables via the Borel-Cantelli lemma, whereas *(ii)* does not hold for arbitrary stationary ergodic sequences as a counterexample by Tanny [102] shows (cf. Example 2.4.22).

Lemma A.1. *Suppose that for some $m \geq 0$, $\mathbf{Y} = (Y_n)_{n\geq 0}$ is a stationary m-dependent sequence of nonnegative random variables.*

(i) If $\mathbb{E}Y_0 < \infty$, then

$$\lim_{n\to\infty} \frac{Y_n}{n} = 0 \quad a.s..$$

(ii) If $\mathbb{E}Y_0 = \infty$, then

$$\limsup_{n\to\infty} \frac{Y_n}{n} = \infty \quad a.s..$$

Proof. (i) Stationarity and m-dependence imply that for any $l \in \{0, ..., m\}$, the sequence $\mathbf{Y}^{(l)} := (Y_{j(m+1)+l})_{j\geq 0}$ consists of i.i.d. random variables with finite expectation. Thus, by an application of Lemma 1.1 in [83], we obtain

$$\lim_{j\to\infty} \frac{Y_{j(m+1)+l}}{j} = 0 \quad \text{a.s.}$$

for any such l, and this easily yields the first assertion.

(ii) Similarly,

$$\limsup_{n\to\infty} \frac{Y_n}{n} \geq \limsup_{j\to\infty} \frac{Y_{j(m+1)}}{j(m+1)} = \infty \quad \text{a.s.}$$

because $\mathbb{E}Y_0 = \infty$ and another appeal to Lemma 1.1 in [83] show

$$\limsup_{j\to\infty} \frac{Y_{j(m+1)}}{j} = \infty \quad \text{a.s..}$$

□

The subsequent lemma is a straightforward extension of Birkhoff's ergodic theorem (see, for instance, Theorem 6.28 in [31]) which has been stated for the case $\mathbb{E}|Y_1| < \infty$. We also allow $\mathbb{E}Y_1 \in \{-\infty, +\infty\}$.

Lemma A.2. *Given any stationary ergodic sequence* $\mathbf{Y} = (Y_n)_{n\geq 1}$ *of (real-valued) random variables such that* $\mathbb{E}Y_1$ *exists in* $\overline{\mathbb{R}}$,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \underset{n\to\infty}{\longrightarrow} \mathbb{E}Y_1 \ \text{a.s..}$$

Proof. Without loss of generality, we may assume that $\mathbb{E}Y_1 \in \{-\infty, +\infty\}$ because the case $\mathbb{E}|Y_1| < \infty$ is covered by Birkhoff's ergodic theorem (see for example Theorem 6.28 in [31]). We treat the case $Y_1 \geq 0$ a.s. and hence $\mathbb{E}Y_1 = \infty$ first. Given any $c > 0$, define the truncated random variables $Y_n^{(c)} := Y_n \mathbf{1}_{\{Y_n \leq c\}}$, $n \geq 1$. Then the truncated sequence $(Y_n^{(c)})_{n\geq 1}$ is stationary ergodic as well (and bounded), hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i^{(c)} = \mathbb{E}Y_1^{(c)} < \infty \text{ a.s.}$$

by Birkhoff's ergodic theorem. But for any $c > 0$,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \geq \frac{1}{n}\sum_{i=1}^{n} Y_i^{(c)},$$

whence by monotone convergence and the fact that $Y_1^{(c)} \uparrow Y_1$ a.s. as $c \to \infty$,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i \geq \sup_{c>0} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i^{(c)} = \sup_{c>0} \mathbb{E}Y_1^{(c)} = \mathbb{E}Y_1 = \infty.$$

The general case follows from an application of the first part (and the ergodic theorem) to the stationary ergodic sequences $(Y_n^+)_{n\geq 1}$ and $(Y_n^-)_{n\geq 1}$, respectively because either Y_1^+ or Y_1^- is integrable. □

Appendix B

Two convex function inequalities for martingales

The following lemma gives an important property of convex functions on $[0, \infty)$. A real-valued function ϕ defined on $[0, \infty)$ is called *superadditive* if $\phi(x + y) \geq \phi(x) + \phi(y)$ for all $x, y \geq 0$. As a reference for the lemma, we mention [85], p. 453.

Lemma B.1. *Any convex function $\phi : [0, \infty) \to \mathbb{R}$ with $\phi(0) \leq 0$ is superadditive. If $\phi(x) = x^\alpha$ for some $\alpha > 1$, ϕ is even* strictly superadditive *in $(0, \infty)$, i.e. ϕ satisfies*

$$\phi(x + y) > \phi(x) + \phi(y) \quad \text{for all } x, y > 0.$$

The subsequent theorem is Theorem 11.3.2 in [37]. In the case $\Phi(x) = x^p$ for some $p > 0$, it can also be found in [53] (Theorem 2.12).

Theorem B.2. (Burkholder, Davis, Gundy [37])
Let Φ be a nondecreasing and continuous function on $[0, \infty)$ fulfilling $\Phi(0) = 0$ and $\Phi(2x) \leq c\Phi(x)$ for all $x > 0$ and some $c \in (0, \infty)$. Then there exists a constant $C \in (0, \infty)$ such that for every martingale $((S_n, \mathcal{Z}_n))_{n \geq 0}$ with $S_0 = 0$ and $\mathcal{Z}_0 = \{\emptyset, \Omega\}$

$$\begin{aligned} \mathbb{E}\Phi\left(\sup_{n\geq 1}|S_n|\right) &\leq C\mathbb{E}\Phi\left(\Xi\right) + C\mathbb{E}\Phi\left(\sup_{n\geq 1}|S_n - S_{n-1}|\right) \\ &\leq C\mathbb{E}\Phi\left(\Xi\right) + C\sum_{n\geq 1}\mathbb{E}\Phi\left(|S_n - S_{n-1}|\right), \end{aligned}$$

where $\Xi := \left(\sum_{n\geq 1}\mathbb{E}\left((S_n - S_{n-1})^2|\mathcal{Z}_{n-1}\right)\right)^{1/2}$ and $\Phi(\infty) := \lim_{x\to\infty}\Phi(x)$.

Remark B.3. We emphasize that the constant C does *not* depend on the martingale $(S_n)_{n\geq 0}$ but only on the constant c from the *growth condition* $\Phi(2x) \leq c\Phi(x)$, $x \geq 0$.

The following inequality is a reformulation of results given in [4] and [107].

Theorem B.4. (Alsmeyer, Rösler [4]; Topchii, Vatutin [107])
Suppose that $\Phi : \mathbb{R} \to [0, \infty)$ *is an even convex function with* $\Phi(0) = 0$ *and a concave derivative on* $(0, \infty)$. *Then for any martingale* $(S_k)_{k\geq 0}$ *and* $n \geq 1$, *the inequality*

$$\mathbb{E}\Phi(S_n) \leq \mathbb{E}\Phi(S_0) + C \sum_{k=1}^{n} \mathbb{E}\Phi(S_k - S_{k-1}) \tag{B.1}$$

holds with $C = 2$. *If* $(S_k)_{k\geq 0}$ *is nonnegative or* $\Phi(x) = x^2$, *then the inequality holds with* $C = 1$.

Remark B.5. If $(S_k)_{k\geq 0}$ is Φ-integrable, i.e. $\mathbb{E}\Phi(S_k) < \infty$ for all k, Theorem B.4 can be found as Theorem 1 in [4]. If the right-hand side of (B.1) is infinite, there is nothing to prove. Otherwise, an application of Theorem 2 in [107] shows that $\mathbb{E}\Phi(S_n) < \infty$ which implies that $(S_k)_{0\leq k\leq n}$ is a Φ-integrable martingale, whence again (B.1) can be obtained from Theorem 1 in [4].

List of abbreviations

WBP	weighted branching process
WBPRE	weighted branching process in random environment
BRW	branching random walk
GWP	Galton-Watson process
GWPRE	Galton-Watson process in random environment
i.i.d.	independent and identically distributed
i.o.	infinitely often
w.p.p.	with positive probability
a.s.	almost surely
w.l.o.g.	without loss of generality
w.r.t.	with respect to

Bibliography

[1] Agresti, A. On the extinction times of varying and random environment branching processes. *J. Appl. Probab.* **12**, 39-46 (1975).

[2] Alsmeyer, G. *Erneuerungstheorie.* Teubner, Stuttgart (1991).

[3] Alsmeyer, G. On generalized renewal measures and certain first passage times. *Ann. Probab.* **20**, 1229-1247 (1992).

[4] Alsmeyer, G. and Rösler, U. The best constant in the Topchii-Vatutin-inequality for martingales. *Stat. Prob. Lett.* **65**, 199-206 (2003).

[5] Alsmeyer, G. and Rösler, U. A stochastic fixed point equation related to weighted branching with deterministic weights. *Technical Report, University of Münster* (2003).

[6] Alsmeyer, G. and Rösler, U. On the existence of ϕ-moments of the limit of a normalized supercritical Galton-Watson process. To appear in *J. Theor. Probab.* (2004).

[7] Asmussen, S. and Hering, H. *Branching processes.* Birkhäuser, Boston (1983).

[8] Athreya, K.B. A note on a functional equation arising in Galton-Watson branching processes. *J. Appl. Probab.* **8**, 589-598 (1971).

[9] Athreya, K.B. Change of measures and the $L \log L$ theorem for branching processes. *Bernoulli* **6**, 323-338 (2000).

[10] Athreya, K.B. and Karlin, S. Branching processes with random environments, I: Extinction probabilities. *Ann. Math. Statist.* **42**, 1499-1520 (1971).

[11] Athreya, K.B. and Karlin, S. Branching processes with random environments, II: Limit theorems. *Ann. Math. Statist.* **42**, 1843-1858 (1971).

[12] ATHREYA, K.B. and NEY, P.E. *Branching Processes.* Springer, New York (1972).

[13] BARINGHAUS, L. and GRÜBEL, R. On a class of characterization problems for random convex combinations. *Ann. Inst. Statist. Math.* **49**, 555-567 (1997).

[14] BARRAL, J. Moments, continuité, et analyse multifractale des martingales de Mandelbrot. *Probab. Theory Relat. Fields* **113**, 535-569 (1999).

[15] BARRAL, J. Generalized vector multiplicative cascades. *Adv. Appl. Probab.* **33**, 874-895 (2001).

[16] BENTKUS, V. An inequality for tail probabilities of martingales with differences bounded from one side. *J. Theor. Probab.* **16**, 161-173 (2003).

[17] BIGGINS, J.D. Martingale convergence in the branching random walk. *J. Appl. Probab.* **14**, 25-37 (1977).

[18] BIGGINS, J.D. Chernoff's theorem in the branching random walk. *J. Appl. Probab.* **14**, 630-636 (1977).

[19] BIGGINS, J.D. Growth rates in the branching random walk. *Z. Wahrsch. Verw. Gebiete* **48**, 17-34 (1979).

[20] BIGGINS, J.D. Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20**, 137-151 (1992).

[21] BIGGINS, J.D., COHN, H. and NERMAN, O. Multi-type branching in varying environment. *Stoch. Proc. Appl.* **83**, 357-400 (1999).

[22] BIGGINS, J.D. and GREY, D.R. Continuity of limit variables in the branching random walk. *J. Appl. Probab.* **16**, 740-749 (1979).

[23] BIGGINS, J.D. and KYPRIANOU, A.E. Branching random walk: Seneta-Heyde norming. In B. CHAUVIN, S. COHEN and A. ROUAULT (eds), *Trees: Proceedings of a Workshop, Versailles June 14-16, 1995.* Birkhäuser, Basel (1996).

[24] BIGGINS, J.D. and KYPRIANOU, A.E. Seneta-Heyde norming in the branching random walk. *Ann. Probab.* **25**, 337-360 (1997).

[25] BIGGINS, J.D. and KYPRIANOU, A.E. Martingale convergence and the stopped branching random walk. *Probab. Theory Relat. Fields* **116**, 405-419 (2000).

[26] BIGGINS, J.D. and KYPRIANOU, A.E. Measure change in multitype branching. To appear in *Adv. Appl. Probab.* **36** (2004).

[27] BIGGINS, J.D. and KYPRIANOU, A.E. Fixed points of the smoothing transform; the boundary case. *Technical Report, University of Sheffield and University of Utrecht* (2003).

[28] BINGHAM, N.H. and DONEY, R.A. Asymptotic properties of supercritical branching processes I: The Galton-Watson process. *Adv. Appl. Probab.* **6**, 711-731 (1974).

[29] BINGHAM, N.H. and DONEY, R.A. Asymptotic properties of supercritical branching processes II: Crump-Mode and Jirina processes. *Adv. Appl. Probab.* **7**, 66-82 (1975).

[30] BINGHAM, N.H., GOLDIE, C.M. and TEUGELS, J.L. *Regular variation.* Cambridge Univ. Press, Cambridge (1987).

[31] BREIMAN, L. *Probability.* Addison-Wesley, Reading, MA (1968).

[32] BROWN, J.R. *Ergodic theory and topological dynamics (Pure and applied mathematics).* Academic Press, New York, San Francisco, London (1976).

[33] CALIEBE, A. Symmetric fixed points of a smoothing transform. *Adv. Appl. Probab.* **35**, 377-394 (2003).

[34] CALIEBE, A. and RÖSLER, U. Fixed points with finite variance of a smoothing transformation. *Stoch. Proc. Appl.* **107**, 105-129 (2003).

[35] CALIEBE, A. and RÖSLER, U. Fixed points of a smoothing transformation with finite expectation: closing a gap. *Technical Report, University of Kiel* (2003).

[36] CHAUVIN, B. Arbres et processus de Bellman-Harris. *Ann. Inst. H. Poincaré* **22**, 209-232 (1986).

[37] CHOW, Y.S. and TEICHER, H. *Probability Theory: Independence, Interchangeability, Martingales (3rd Edition).* Springer, New York (1997).

[38] COFFEY, J. and TANNY, D. A necessary and sufficient condition for noncertain extinction of a branching process in a random environment. *Stoch. Proc. Appl.* **16**, 189-197 (1984).

[39] Dekking, F.M. On the survival probability of a branching process in a finite state i.i.d. environment. *Stoch. Proc. Appl.* **27**, 151-157 (1987).

[40] de La Peña, V.H. A general class of exponential inequalities for martingales and ratios. *Ann. Probab.* **27**, 537-554 (1999).

[41] Doney. R.A. Age-dependent birth and death processes. *Z. Wahrsch. Verw. Gebiete* **22**, 69-90 (1972).

[42] Doney. R.A. A limit theorem for a class of supercritical branching processes. *J. Appl. Probab.* **9**, 707-724 (1972).

[43] Durrett, R. *Probability: Theory and Examples.* Wadsworth, Pacific Grove, CA (1991).

[44] Durrett, R. and Liggett, T.M. Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete* **64**, 275-301 (1983).

[45] Elstrodt, J. *Maß- und Integrationstheorie.* Springer, Berlin (1996).

[46] Freedman, D.A. On tail probabilities for martingales. *Ann. Probab.* **3**, 100-118 (1975).

[47] Gänssler, P. and Stute, W. *Wahrscheinlichkeitstheorie.* Springer, Berlin (1977).

[48] Geiger, J. Elementary new proofs of classical limit theorems for Galton-Watson processes. *J. Appl. Probab.* **36**, 301-309 (1999).

[49] Geiger, J. and Kersting, G. The survival probability of a critical branching process in a random environment. *Theory Probab. Appl.* **45**, 517-525 (2000).

[50] Geiger, J., Kersting, G. and Vatutin, V.A. Limit theorems for subcritical branching processes in random environment. *Ann. Inst. H. Poincaré* **39**, 593-620 (2003).

[51] Graf, S., Mauldin, R.D. and Williams, S.C. The exact Hausdorff dimension in random recursive constructions. *Mem. Am. Math. Soc.* **71**, No. 381 (1988).

[52] Guivarc'h, Y. Sur une extension de la notion de loi semi-stable. *Ann. Inst. H. Poincaré* **26**, 261-285 (1990).

[53] HALL, P. and HEYDE, C.C. *Martingale Limit Theory and its Applications.* Academic Press, New York (1980).

[54] HEYDE, C.C. Extension of a result of Seneta for the supercritical Galton-Watson process. *Ann. Math. Statist.* **41**, 739-742 (1970).

[55] HIRSCH, A. Fixpunktsätze für Verteilungen. *Diploma Thesis, University of Münster* (1998).

[56] HOLLEY, R. and LIGGETT, T.M. Generalized potlatch and smoothing processes. *Z. Wahrsch. Verw. Gebiete* **55**, 165-195 (1981).

[57] IBRAGIMOV, I.A. and LINNIK, YU.V. *Independent and stationary sequences of random variables.* Wolters-Noordhoff, Groningen (1971).

[58] IKSANOV, A.M. Elementary fixed points of the BRW smoothing transforms with infinite number of summands. *Technical report, University of Kiev* (2003).

[59] JAGERS, P. *Branching processes with biological applications.* Wiley, London (1975).

[60] JAGERS, P. General branching processes as Markov fields. *Stoch. Proc. Appl.* **32**, 183-212 (1989).

[61] JAGERS, P. and NERMAN, O. The growth and composition of branching populations. *Adv. Appl. Probab.* **16**, 221-259 (1984).

[62] KAHANE, J.P. and PEYRIÈRE, J. Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.* **22**, 131-145 (1976).

[63] KARR, A.F. *Probability.* Springer, New York (1993).

[64] KEENER, R. A note on the variance of a stopping time. *Ann. Stat.* **15**, 1709-1712 (1987).

[65] KINGMAN, J.F.C. The first birth problem for an age-dependent branching process. *Ann. Probab.* **3**, 790-801 (1975).

[66] KUHLBUSCH, D. On weighted branching processes in random environment. *Stoch. Proc. Appl.* **109**, 113-144 (2004).

[67] KURTZ, T.G., LYONS, R., PEMANTLE, R. and PERES, Y. A conceptual proof of the Kesten-Stigum theorem for multi-type branching processes. In K.B. ATHREYA and P. JAGERS (eds), *Classical and Modern Branching Processes.* IMA Vol. Math. Appl. **84**, 181-185. Springer, New York (1997).

[68] KYPRIANOU, A.E. Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis. *Ann. Inst. H. Poincaré* **40**, 53-72 (2004).

[69] KYPRIANOU, A.E. and RAHIMZADEH SANI, A. Martingale convergence and the functional equation in the multi-type branching random walk. *Bernoulli* **7**, 593-604 (2001).

[70] LAIB, N. Exponential-type inequalities for martingale difference sequences. Application to nonparametric regression estimation. *Commun. Stat. Theory Methods* **28**, 1565-1576 (1999).

[71] LESIGNE, E. and VOLNÝ, D. Large deviations for martingales. *Stoch. Proc. Appl.* **96**, 143-159 (2001).

[72] LI, Y. A martingale inequality and large deviations. *Stat. Prob. Lett.* **62**, 317-321 (2003).

[73] LINDVALL, T. Almost sure convergence of branching processes in varying and random environments. *Ann. Probab.* **2**, 344-346 (1974).

[74] LIU, Q. The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale. In B. CHAUVIN, S. COHEN and A. ROUAULT (eds), *Trees: Proceedings of a Workshop, Versailles June 14-16, 1995.* Birkhäuser, Basel (1996).

[75] LIU, Q. Sur une équation fonctionelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement. *Adv. Appl. Probab.* **29**, 353-373 (1997).

[76] LIU, Q. Fixed points of a generalized smoothing transform and applications to the branching random walk. *Adv. Appl. Probab.* **30**, 85-112 (1998).

[77] LIU, Q. Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks. *Stoch. Proc. Appl.* **82**, 61-87 (1999).

[78] Liu, Q. On generalized multiplicative cascades. *Stoch. Proc. Appl.* **86**, 263-286 (2000).

[79] Liu, Q. Asymptotic properties and absolute continuity of laws stable by random weighted mean. *Stoch. Proc. Appl.* **95**, 83-107 (2001).

[80] Liu, Q. An extension of a functional equation of Poincaré and Mandelbrot. *Asian J. Math.* **6**, 145-168 (2002).

[81] Liu, Q. and Rouault, A. On two measures defined on the boundary of a branching tree. In K.B. Athreya and P. Jagers (eds), *Classical and Modern Branching Processes.* IMA Vol. Math. Appl. **84**, 187-201. Springer, New York (1997).

[82] Lyons, R. A simple path to Biggins' martingale convergence for branching random walk. In K.B. Athreya and P. Jagers (eds), *Classical and Modern Branching Processes.* IMA Vol. Math. Appl. **84**, 217-221. Springer, New York (1997).

[83] Lyons, R., Pemantle, R. and Peres, Y. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **25**, 1125-1138 (1995).

[84] Mandelbrot, B. Multiplications aléatoires iterées et distributions invariantes par moyenne ponderée aléatoire. *C. R. Acad. Sci. Paris* **278**, 289-292 (1974).

[85] Marshall, A.W. and Olkin, I. *Inequalities: Theory of majorization and its applications.* Mathematics in science and engineering, **143**. Academic Press, New York (1979).

[86] Mauldin, R.D. and Williams, S.C. Random recursive constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295**, 325-346 (1986).

[87] Nerman, O. On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57**, 365-395 (1981).

[88] Neveu, J. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré* **22**, 199-207 (1986).

[89] O'Brien, G.L. The occurrence of large values in stationary sequences. *Z. Wahrsch. Verw. Gebiete* **61**, 347-353 (1982).

[90] Olofsson, P. The $x \log x$ condition for general branching processes. *J. Appl. Probab.* **35**, 537-544 (1998).

[91] Rachev, S.T. and Rüschendorf, L. Probability metrics and recursive algorithms. *Adv. Appl. Probab.* **27**, 770-799 (1995).

[92] Rösler, U. A limit theorem for "Quicksort". *RAIRO, Inform. Théor. Appl.* **25**, 85-110 (1991).

[93] Rösler, U. A fixed point theorem for distributions. *Stoch. Proc. Appl.* **42**, 195-214 (1992).

[94] Rösler, U. The weighted branching process. *Dynamics of complex and irregular systems (Bielefeld, 1991)*, 154-165, *Bielefeld Encount. Math. Phys. VIII,* World Sci. Publishing, River Edge, NJ (1993).

[95] Rösler, U. The $X \ln X$ condition for the weighted branching process. *Technical Report, University of Kiel* (1999).

[96] Rösler, U. and Rüschendorf, L. The contraction method for recursive algorithms. *Algorithmica* **29**, 3-33 (2001).

[97] Rösler, U., Topchii, V.A. and Vatutin, V.A. Convergence conditions for weighted branching processes. *Discrete Math. Appl.* **10**, 5-21 (2000).

[98] Rösler, U., Topchii, V.A. and Vatutin, V.A. The rate of convergence for weighted branching processes. *Sib. Adv. Math.* **12**, 57-82 (2003).

[99] Seneta, E. On recent theorems concerning the supercritical Galton-Watson process. *Ann. Math. Statist.* **39**, 2089-2102 (1968).

[100] Sgibnev, M.S. Submultiplicative moments of the supremum of a random walk with negative drift. *Stat. Prob. Lett.* **32**, 377-383 (1997).

[101] Smith, W.L. and Wilkinson, W.E. On branching processes in random environments. *Ann. Math. Statist.* **40**, 814-827 (1969).

[102] Tanny, D. A zero-one law for stationary sequences. *Z. Wahrsch. Verw. Gebiete* **30**, 139-148 (1974).

[103] Tanny, D. Limit theorems for branching processes in a random environment. *Ann. Probab.* **5**, 100-116 (1977).

[104] Tanny, D. Normalizing constants for branching processes in random environments (B.P.R.E.). *Stoch. Proc. Appl.* **6**, 201-211 (1978).

[105] Tanny, D. On multitype branching processes in a random environment. *Adv. Appl. Probab.* **13**, 464-497 (1981).

[106] Tanny, D. A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stoch. Proc. Appl.* **28**, 123-139 (1988).

[107] Topchii, V.A. and Vatutin, V.A. Maximum of the critical Galton-Watson processes and left continuous random walks. *Theory Probab. Appl.* **42**, 17-27 (1997).

[108] Waymire, E. and Williams, S. A cascade decomposition theory with applications to Markov and exchangeable cascades. *Trans. Amer. Math. Soc.* **348**, 585-632 (1996).

Lebenslauf

Name:	Dirk Kuhlbusch
Geburtsdatum:	11. März 1977
Geburtsort:	Warendorf
Familienstand:	ledig
Eltern:	Franz Kuhlbusch, verstorben Mathilde Kuhlbusch, geb. Buss
Schulbildung:	1983–1987 Everword-Grundschule Freckenhorst 1987–1996 Mariengymnasium Warendorf
Allg. Hochschulreife:	11. Juni 1996
Studium:	Oktober 1996 bis Dezember 2001 Diplomstudiengang Mathematik mit Nebenfach Betriebswirtschaftslehre an der WWU Münster
Prüfungen:	Vordiplom in Mathematik am 29.10.1998 Diplom in Mathematik am 20.12.2001
Tätigkeiten:	Oktober 1999 bis Dezember 2001 studentische Hilfskraft am Institut für Mathematische Statistik und am Mathematischen Institut der WWU Münster seit Januar 2002 wissenschaftlicher Mitarbeiter am Institut für Mathematische Statistik der WWU Münster
Beginn der Promotion:	Januar 2002 am Institut für Mathematische Statistik der WWU Münster bei Prof. Dr. G. Alsmeyer