Westfälische Wilhelms-Universität Münster
Fachbereich Mathematik und Informatik
Institut für Mathematische Statistik

Limit Results for Iterated Random Lipschitz Functions via Regenerative Methods

Diplomarbeit

Thema gestellt von

Prof. Dr. G. Alsmeyer

vorgelegt von

Gerd Hölker

Münster, den 10. April 2006

Introduction

Iterated Function Systems have a wide range of applications, for example simulation algorithms, control theory, queuing theory and other branches of applied probability. Moreover, Markov chains can be constructed by iterated random functions which makes it possible to consider this theory from another point of view. Diaconis and Freedman give an excellent review on the theory of iterated random functions and its applications including an extensive list of relevant literature. The following two examples are described there in detail, see [DF]. We will use them for an introduction in this thesis.

Consider the open unit interval $(0,1)$ and choose any point $x \in (0,1)$. Then pick one of the intervals $(0,x)$ and $(x,1)$ and move to a random y in the interval you have chosen. We can describe this procedure by the functions

$$\phi_u(x) = ux \quad \text{and} \quad \psi_u(x) = x + u(1-x)$$

where on the one hand u is uniformly chosen on $(0,1)$, and on the other hand we choose one of the two functions with probability $1/2$. Now we continue this procedure with starting point y. If $(u_n)_{n\geq 1}$ is a sequence of uniformly distributed random variables and $F_n \in \{\phi_{u_n}, \psi_{u_n}\}$, $n \geq 1$, each with probability $1/2$, we can describe this system as follows: $M_0 = x$, $M_1^x = F_1(x)$, $M_2^x = F_2 \circ F_1(x)$ and inductively

$$M_n^x = F_n(M_{n-1}^x), \quad n \geq 1.$$

In the second example let $\mathbb{X} = \mathbb{R}$, $a \in (0,1)$ and consider the functions $F_{\zeta_n} = ax + \zeta_n$ where $x \in \mathbb{X}$ and $\zeta_n = \pm 1$, each with probability $1/2$ for all $n \geq 1$. Also in this case we can consider a system of iterations, namely $M_0 = x$,

$$M_n^x = F_{\zeta_n}(M_{\zeta_{n-1}}^x), \quad n \geq 1.$$

More generally, let $(\mathbb{X}, d)$ be a complete separable metric space and $(F_n)_{n\geq 1}$ a sequence of independent, identically distributed functions from $\mathbb{X}$ to $\mathbb{X}$. Furthermore, assume that these functions are uniform Lipschitz, i.e.

$$l(F_n) = \sup_{x \neq y} \frac{d(F_n(x), F_n(y))}{d(x,y)} < \infty \quad \text{a.s.}$$

We are interested in analysing the system $M_0 = x$,

$$M_n^x = F_n(M_{n-1}^x) = F_n \circ \ldots \circ F_1(x), \quad n \geq 1,\ x \in \mathbb{X}.$$

Provided that the Liapunov exponent $l^* \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log l(F_n \circ \ldots \circ F_1) < 0$ a.s., $\mathbb{E} \log^+ L_1 < \infty$ and $\mathbb{E} \log^+ d(F_1(x_0), x_0) < \infty$ the forward iteration M_n^x, $n \geq 1$, converges weakly to a unique stationary distribution π for each $x \in \mathbb{X}$, while the associated backward iterations $\hat{M}_n^x = F_1 \circ \ldots \circ F_n(x)$ converges a.s. to a random variable $\hat{M}_\infty$ which is independent of x and has distribution π. Going back to our second example the forward and backward iterations are given by

$$\begin{aligned} M_n^x &= a^n x + a^{n-1}\zeta_1 + \ldots + a\zeta_{n-1} + \zeta_n \\ \hat{M}_n^x &= a^n x + \zeta_1 + a\zeta_2 + \ldots + a^{n-1}\zeta_n \end{aligned}$$

for all $n \geq 1$. With this discription it becomes clear why we have different kinds of convergence: Considering the forward process at stage n new randomness is introduced by ζ_n. Otherwise, the new randomness at stage n, with respect to the backward process is damped by a^{n-1} (note that $a \in (0,1)$).

Alsmeyer and Fuh establish [AF] limit theorems under slightly stronger assumptions, namely the mean contratction assumption $\mathbb{E} \log L_1 < 0$ and $\mathbb{E} \log^+ d(F_1(x_0), x_0) < \infty$ for some $x_0 \in \mathbb{X}$. Particularly, they analyse for each $\gamma \in (\gamma^*, 1)$ the probability

$$\mathbb{P}(d(\hat{M}_\infty, \hat{M}_n^x) > \gamma^n)$$

for large n, where γ^* is a suitable constant arising from their analysis. The key to the proof of these results is the analysis of the inequality

$$d(\hat{M}_{n+m}^x, \hat{M}_n^x) \leq \exp\left(\sum_{k=1}^n l(F_k)\right) d(F_{n+1} \circ \ldots \circ F_{n+m}(x), x) \quad \text{a.s.}$$

which holds true for all $n, m \geq 0$. Since $\mathbb{E} \log L_1 < 0$, the sequence $(\sum_{k=1}^n \log L_k)_{n\geq 1}$ constitutes an ordinary random walk with negative drift. By imposing additional polynomial or exponential moment conditions on $\log(1 + L_1)$ and $\log(1 + d(F_1(x_0), x_0))$ it is possible to provide suitable estimates for the factor $d(F_{n+1} \circ \ldots \circ F_{n+m}(x), x)$ so as to ensure that the right hand side of the foregoing inequality converges to 0 a.s. It is in fact shown in [AF] that the decrease of the Prokhorov distance between $P^n(x, \cdot)$ and π is of polynomial, respectively exponential order where $P^n(x, \cdot)$ denotes the n-step transition kernel of the Markov chain of the forward iterations.

A closer look at the approach chosen by Alsmeyer and Fuh shows that their results can be improved in several aspects. First, it seems quite natural to work with the weaker conditions "$l^* < 0$ a.s. and $\mathbb{E}\log^+ L_1 < \infty$" instead of "$\mathbb{E}\log L_1 < 0$" so as to ensure contraction of the given Iterated Function System. Second, and indeed more important, the constant γ^* arising in various results in [AF] should be e^{l^*} but is actually larger there caused by the chosen estimate

$$l(F_{1:n}) \leq \sum_{k=1}^{n} l(F_k)$$

at the outset of the analysis. This estimate is too crude to give $\gamma^* = e^{l^*}$. As a consequence, we will here provide a refined analysis involving both, $(l(F_{1:n}))_{n\geq 1}$ and $(\sum_{k=1}^{n} l(F_k))_{n\geq 1}$.

The thesis is organised as follows: Iterated Function Systems including necessary notation are formally introduced in Chapter 1, which also sketches some of the central ideas of our approach as opposed to that in [AF]. Chapter 2 contains necessary technical results which serve as prerequisites thereafter. It may be skipped at first reading. Chapter 3 contains our main results including their proofs and is followed by three Appendices collecting some facts from renewal theory, the definition of the Prokhorov metric and formulas for calculating expected values.

I would like to thank all, who supported me by writing my diploma thesis. In particular, thanks to my supervisor Prof. Dr. Alsmeyer. He has spared a lot of time for me answering every question I had and has given me many helpful comments.

Contents

Chapter 1

Iterated Function System and its properties

The brief examples we have seen in the introduction may serve as a motivation to define an Iterated Function System in the following way. Let $(\mathbb{X}, d)$ be a complete separable metric space with Borel-σ-field $\mathfrak{B}(\mathbb{X})$ and assume $M_0, \vartheta_1, \vartheta_2, \ldots$ are random variables defined on a given probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Furthermore, the $\vartheta_1, \vartheta_2, \ldots$ have identical distribution Λ and take values in a measurable space $(\Theta, \mathcal{A})$. Let $F : (\Theta \times \mathbb{X}, \mathcal{A} \otimes \mathfrak{B}(\mathbb{X})) \to (\mathbb{X}, \mathfrak{B}(\mathbb{X}))$ be a jointly measurable function which is Lipschitz continuous in the second argument, i.e., if $x, y \in \mathbb{X}$ and $\vartheta \in \Theta$, the inequality

$$d(F(\vartheta, x), F(\vartheta, y)) \leq K_\vartheta d(x, y) \tag{1.1}$$

holds true for some K_ϑ depending on ϑ. Then

$$M_n = F(\vartheta_n, M_{n-1}), \quad n \geq 1, \tag{1.2}$$

is called an *iterated function system* (*IFS*) *of i.i.d. Lipschitz maps*. Since $M_0, \vartheta_1, \vartheta_2, \ldots$ are independent, $(M_n)_{n\geq 0}$ constitutes a homogeneous Markov chain with state space $\mathbb{X}$ and transition kernel given by

$$P(x, B) = \Lambda(F(\cdot, x) \in B)$$

for $x \in \mathbb{X}$ and $B \in \mathfrak{B}(\mathbb{X})$. P^n denotes the n-step transition kernel, respectively. We further write $\mathbb{P}_x$ and $\mathbb{E}_x$ for $\mathbb{P}(\cdot|M_0 = x)$ and $\mathbb{E}(\cdot|M_0 = x)$, respectively. If ν is a probability distribution on $\mathbb{X}$, then $\mathbb{P}_\nu \stackrel{\text{def}}{=} \int \mathbb{P}_x \nu(dx)$ with expectation operator $\mathbb{E}_\nu$. For probabilities and expectations not depending on initial conditions we simply write $\mathbb{P}$ and $\mathbb{E}$, respectively.

We are naturally interested in the minimal K_ϑ satisfying (1.1). Hence we consider the function

$$l : \mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X}) \to \mathbb{R}_0^+ \cup \{\infty\},$$
$$f \mapsto l(f) \overset{\text{def}}{=} \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \tag{1.3}$$

where $\mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X})$ is the set of all Lipschitz continuous mappings $f : \mathbb{X} \to \mathbb{X}$.

It is necessary to make some comments about the measurable structure of $\mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X})$ and the measurability of l to ensure that the following expressions are well defined. To this end let $\mathbb{X}_0$ be a dense subset of $\mathbb{X}$ and $\mathfrak{M}(\mathbb{X}_0, \mathbb{X})$ the set of all mappings $f : \mathbb{X}_0 \to \mathbb{X}$. We endow $\mathfrak{M}(\mathbb{X}_0, \mathbb{X})$ with the product topology and product-σ-field. Then we have the following

Lemma 1.1. *$\mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X})$ is a Borel subset of $\mathfrak{M}(\mathbb{X}_0, \mathbb{X})$, the mappings*

$$\Phi : \mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X}) \times \mathbb{X} \to \mathbb{X}$$
$$(f, x) \mapsto f(x)$$

and l are Borel-measurable.

Proof. For a proof and further details see [DF, p. 58]. □

In our situation this means that

$$L_n \overset{\text{def}}{=} l(F(\vartheta_n, \cdot)), \quad n \geq 1, \tag{1.4}$$

is measurable and that $(L_n)_{n\geq 1}$ constitutes a sequence of i.i.d. random variables due to our model assumptions.

In order to simplify the notation, let us write $F_n(x)$ for $F(\vartheta_n, x)$. We define $F_{k:n} \overset{\text{def}}{=} F_k \circ \ldots \circ F_n$ and the reverse iteration $F_{n:k} \overset{\text{def}}{=} F_n \circ \ldots \circ F_k$ for all $1 \leq k \leq n$. Put $F_{0:1}(x) = F_{1:0}(x) \overset{\text{def}}{=} x$. By introducing this notation we obtain

$$M_n = F_n(M_{n-1}) = F_{n:1}(M_0) \tag{1.5}$$

for all $n \geq 1$.

The question now is: Under which conditions does an IFS converge to a stationary distribution π? A positive answer is that the random variable L_1 has to be less than 1 for typical ϑ_1, in other words: it has to be mean contractive. Alsmeyer and Fuh, see [AF], work with the conditions

$$\mathbb{E} \log L_1 < 0 \quad \text{and} \quad \mathbb{E} \log^+ d(F_1(x_0), x_0) < \infty \tag{1.6}$$

for some $x_0 \in \mathbb{X}$.

The idea to prove convergence is to analyse the *backwards iteration*

$$\hat{M}_n \stackrel{\text{def}}{=} F_{1:n}(M_0), \quad n \geq 1, \tag{1.7}$$

rather than the *forward iteration* M_n. This turns out to be fruitful because, on the one hand, the backward iterations exhibit a.s. convergence rather than only weak convergence and, on the other hand,

$$\mathbb{P}_x(M_n \in \cdot) = \mathbb{P}_x(\hat{M}_n \in \cdot) \tag{1.8}$$

for all $n \geq 1$ and $x \in \mathbb{X}$ which holds true by the independence of $M_0, \vartheta_1, \vartheta_2, \ldots$

Further it is useful to introduce the sequences

$$\begin{aligned} M_n^x &\stackrel{\text{def}}{=} F_{n:1}(x) \\ \hat{M}_n^x &\stackrel{\text{def}}{=} F_{1:n}(x) \end{aligned}$$

for $x \in \mathbb{X}$. The reason is that we will do comparisons between either M_n^x and M_n^y or $\hat{M}_n^x$ and $\hat{M}_n^y$ to prove our assertions. Also note the identity

$$\mathbb{P}((M_n^x, \hat{M}_n^x) \in \cdot) = \mathbb{P}_x((M_n, \hat{M}_n) \in \cdot).$$

Returning to our initial question, we will try more precisely to estimate the distance $d(\hat{M}_\infty^x, \hat{M}_n^{x_0})$ and show that it converges to 0 a.s. Therefore $\hat{M}_\infty$ is then a.s. the limit of $\hat{M}_n$ with distribution π, say. How this works and what the basic idea in this context is becomes clear if we consider the inequality

$$\begin{aligned} d(\hat{M}_{n+m}^x, \hat{M}_n^x) &= d(F_{1:n}(F_{n+1:n+m}(x)), F_{1:n}(x)) \\ &\leq l(F_{1:n}) d(F_{n+1:n+m}(x), x) \\ &\leq \left(\prod_{k=1}^n l(F_k)\right) d(F_{n+1:n+m}(x), x) \quad \text{a.s.} \end{aligned} \tag{1.9}$$

which holds true for all $n, m \geq 0$ and $x \in \mathbb{X}$. This suggests to provide further estimates by studying the zero-delayed random walk

$$\sum_{k=1}^n \log l(F_k) = \log\left(\prod_{k=1}^n l(F_k)\right), \quad n \geq 0,$$

as it is done in [AF].

Recalling the condition of contraction, $\mathbb{E} \log l(F_1) < 0$, the random walk has negative drift, i.e. $\sum_{k=1}^n \log l(F_k) \to -\infty$ as $n \to \infty$. In other words we may assume that by iterating sufficient many functions we have

$$\sum_{k=1}^n \log l(F_k) \leq \log \gamma$$

where γ is arbitrarily chosen in $(0,1)$. Moreover, we may assume that this really happens by iterating finitely many functions with respect to the expected value. This procedure makes clear that renewal theory might provide appropriate methods to analyse IFS which are contractive. Embedding this verbally formulated idea in a mathematically correct formalism means to consider the level $\log\gamma$ epochs $\sigma_0(\gamma) \stackrel{\text{def}}{=} 0$,

$$\sigma_n(\gamma) \stackrel{\text{def}}{=} \inf\left\{k > \sigma_{n-1}(\gamma) : \sum_{j=\sigma_{n-1}(\gamma)+1}^{k} \log l(F_j) \le \log\gamma\right\},\ n \ge 1. \quad (1.10)$$

The condition $\mathbb{E}\log L_1 < 0$ ensures that $\sigma_1(\gamma)$ is an a.s. finite passage time with finite expectation, say $\mu^*(\gamma)$, and $(\sigma_n(\gamma))_{n\ge 0}$ constitutes an ordinary discrete renewal process, see [AF] for further details.

Of course, $(M_{\sigma_n(\gamma)})_{n\ge 0}$ again forms an IFS of i.i.d. Lipschitz maps. But note that this IFS is *strictly contractive*, i.e.

$$\max\{l(F_{1:\sigma_1(\gamma)}), l(F_{\sigma_1(\gamma):1})\} \le \gamma < 1$$

by definition of σ_1. Using $\hat{M}^x_{\sigma_n(\gamma)} = F_{1:\sigma_n(\gamma)}(x)$ we infer from (1.9)

$$d(\hat{M}^x_{\sigma_{n+m}(\gamma)}, \hat{M}^x_{\sigma_n(\gamma)}) \le \gamma^n d(F_{\sigma_n(\gamma)+1:\sigma_{n+m}(\gamma)}(x), x) \quad (1.11)$$

for all $n, m \ge 0$ and $x \in \mathbb{X}$.

The proof of convergence results of $(M_n)_{n\ge 0}$ will now be done in two steps. First provide conditions under which the strictly contractive sequences $(M_{\sigma_n(\gamma)})_{n\ge 0}$, or $(\hat{M}_{\sigma_n(\gamma)})_{n\ge 0}$, converge for any $\gamma \in (0,1)$. In a second step ensure that the excursions between two successive ladder epochs are moderate enough to ensure convergence of the full sequences $(M_n)_{n\ge 0}$, respectivly $(\hat{M}_n)_{n\ge 0}$. If they are not given in the generality worked out in the first step adjust the results for the original sequences.

For instance, it is shown in [AF] that, given

$$\mathbb{E}\log L_1 < 0 \quad \text{and} \quad \mathbb{E}\log^+ d(F_1(x_0), x_0) < \infty,$$

the assertion

$$\lim_{n\to\infty} \mathbb{P}(d(\hat{M}^x_\infty, \hat{M}^{x_0}_n) > \gamma^n) = 0 \quad (1.12)$$

holds true for all $x \in \mathbb{X}$ and $\gamma \in (\gamma^*, 1)$ where γ^* is a constant defined as follows:

$$\log\gamma^* \stackrel{\text{def}}{=} \inf_{\gamma\in(0,1)} \frac{\log\gamma}{\mu^*(\gamma)} \quad (1.13)$$

(recall that $\mu^*(\gamma) = \mathbb{E}\sigma_1(\gamma)$).

The aim of this thesis is to restate the convergence results under slightly weaker conditions. Particularly, we drop the condition $\mathbb{E}\log L_1 < 0$ and just claim $\mathbb{E}\log^+ L_1 < \infty$. Of course, we then have to find a new condition which guarantees a suitable kind of contraction. To this end we introduce the Liapunov exponent.

Proposition and Definition 1.2. *Given an IFS of i.i.d. Lipschitz maps and $\mathbb{E}\log^+ L_1 < \infty$, there exists a constant l^* satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log l(F_{n:1}) = \inf_{n\geq 1} \frac{1}{n} \log l(F_{n:1}) = l^* \quad \text{a.s.} \tag{1.14}$$

as well as

$$\lim_{n\to\infty} \frac{1}{n} \int \log l(F_{n:1}) d\mathbb{P} = l^*. \tag{1.15}$$

l^* is called the Liapunov exponent*.*

Proof. The Liapunov exponent exists by Kingman's Subadditive Ergodic Theorem where it is to be noted that $\log l(\cdot)$ is subadditive, i.e. $\log l(F\circ G) \leq \log l(F) + \log l(G)$ for any two functions $F, G \in \mathfrak{C}_{Lip}(\mathbb{X}, \mathbb{X})$. For further details see Proposition 2 in [Elt, p. 40] and the subsequent remark. □

Let $l^* < 0$ a.s. Then there exists an $\varepsilon > 0$ satisfying $l^* + \varepsilon < 0$ a.s. Obviously, the inequality

$$\frac{1}{n} \log l(F_{n:1}) < l^* + \varepsilon \quad \text{a.s.} \tag{1.16}$$

for all $n \geq N$ where N is a suitable positive integer. By further calculations we observe

$$l(F_{n:1}) < \exp((l^* + \varepsilon)n) < 1 \quad \text{a.s.} \tag{1.17}$$

for all $n \geq N$.

Considering the last inequality it becomes clear that the condition $l^* < 0$ a.s. ensures contraction. Moreover, we again may expect that the first ladder epoch

$$\xi_1 \stackrel{\text{def}}{=} \inf\left\{k \geq 1 : \frac{1}{k} \log l(F_{1:k}) < l^* + \varepsilon\right\}$$

has finite mean. The idea of considering ladder epochs and the corresponding ladder heights here also turns out to be the right tool.

But we will improve [AF] in a second aspect, which is indeed more important. Most of the results in [AF] hold for any γ in the open interval $(\gamma^*, 1)$, see for example (1.12). It will turn out that the limit theorems are even valid for values in the open interval $(\exp(l^*), 1)$ which is bigger than $(\gamma^*, 1)$.

At the end of this chapter our considerations come back to the excursions between two ladder epochs we will analyse in the second step. It will come out of the proofs that the required properties with respect to the excursions will not hold in the desired generality. To this end additional restrictions are necessary. First we will study the excursion under the assumptions that for some $p > 0$ and $x_0 \in \mathbb{X}$

$$\mathbb{E} \log^{p+1}(1 + L_1) < \infty \quad \text{and} \quad \mathbb{E} \log^{p+1}(1 + d(F_1(x_0), x_0)) < \infty \tag{1.18}$$

hold true. In a second step we will replace these conditions by a new set of assumptions, namely,

$$\mathbb{E} L_1^p < \infty \quad \text{and} \quad \mathbb{E} d(F_1(x_0), x_0)^p < \infty \tag{1.19}$$

for some $p > 0$ and some $x_0 \in \mathbb{X}$. We will denote these sets of conditions also by (A) and (B) to provide associated notations.

Chapter 2

Prerequisites

In this chapter we will collect the main tools which are necessary to prove the limit theorems presented in Chapter 3. In the first section we introduce the excursions between two ladder epochs on the basis of the idea presenting in the chapter before and state some properties. In Section 2.2 we provide among other things necessary moment results for the excursions and an estimation for the distance $d(\hat{M}_\infty^{x_0}\hat{M}_n^x)$. We will distinguish between the sets of conditions (A) and (B).

2.1 The excursions

Throughout this chapter we make the standing assumptions

$$l^* < 0 \quad \text{a.s.}, \tag{2.1}$$

$$\mathbb{E}\log^+ L_1 < \infty \text{ and } \mathbb{E}\log^+ d(F_1(x_0), x_0) < \infty \tag{2.2}$$

for some $x_0 \in \mathbb{X}$.

Inequality (1.17) and the idea of introducing suitable ladder epochs to find a strictly contractive IFS turns out to be the right key to prove the convergence results given below. To this end we define for any $l \in (l^*, 0)$ the ladder epochs $\xi_0(l) \stackrel{\text{def}}{=} 0$,

$$\xi_n(l) \stackrel{\text{def}}{=} \inf\left\{k \geq \xi_{n-1} + 1 : \frac{1}{k - \xi_{n-1}} \log l(F_{\xi_{n-1}+1:k}) < l\right\}, \quad n \geq 1. \tag{2.3}$$

Since $\frac{1}{n}\log l(F_{1:n}) \to l^* < l$, ξ_1 is a.s. finite and by Proposition 27.3 j) in [ASP, p. 247], $(\xi_n)_{n\geq 0}$ constitutes an ordinary random walk. Moreover, $(\xi_n)_{n\geq 0}$ is also an ordinary renewal process because, obviously, the increments $\delta_n(l) \stackrel{\text{def}}{=} \xi_n(l) - \xi_{n-1}(l)$ are positive for all $n \geq 0$.

By the following lemma we want to ensure that $\mu(l) \stackrel{\text{def}}{=} \mathbb{E}\xi_1(l)$ is finite. In other words: is it possible, with respect to the expected value, to iterate the system finitely many times such that the process $l(F_{n:1})$ falls below the associated bound? Before we start our considerations it turns out to be helpful to simplify the notation: we will write ξ_n instead of $\xi_n(l)$ and δ_n for $\delta_n(l)$, respectively, for all $n \in \mathbb{N}$ and, similarly, $\mu = \mu(l)$.

Lemma 2.1. *Given the conditions* (2.1), (2.2), *let* $l \in (l^*, 0)$. *Then* $\mathbb{E}\ \xi_1 < \infty$. *If in addition* $\mathbb{E} \log^{p+1}(1 + L_1) < \infty$ *holds true for some* $p > 0$ *the* $(p+1)$*-st moment of* ξ_1 *is also finite. Finally, if* $\mathbb{E}L_1^p < \infty$, *there exists* $s_0 > 0$ *such that* $\mathbb{E}\exp(s\xi_1) < \infty$ *for all* $|s| < s_0$.

Proof. By Proposition 2 in [Elt] we know that there exists $m \in \mathbb{N}$ such that $\frac{1}{m}\mathbb{E} \log l(F_{m:1}) < l$, thus also $\mathbb{E}\ \frac{1}{m} \log l(F_{1:m}) < l$ because of (1.8). For any $n \geq 1$ we obtain

$$\begin{aligned}\frac{1}{nm} \log l(F_{1:nm}) &\leq \frac{1}{nm} \sum_{j=0}^{n-1} \log l(F_{jm+1:(j+1)m}) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{m} \log l(F_{jm+1:(j+1)m}).\end{aligned}$$

In the following we consider

$$\zeta \stackrel{def}{=} \inf\{n \in m\mathbb{N} :\ \frac{1}{n} \log l(F_{1:n}) < l\}.$$

We clearly have $\zeta \geq \xi_1$ and therefore $\mathbb{E}\ \xi_1 < \infty$ if $\mathbb{E}\zeta < \infty$. We further have

$$\begin{aligned}\frac{\zeta}{m} &= \inf\{n \geq 1 : \frac{1}{nm} \log l(F_{1:nm}) < l\} \\ &\leq \inf\{n \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{m} \log l(F_{jm+1:(j+1)m}) - l < 0\} \\ &= \inf\{n \geq 1 : \sum_{j=0}^{n-1} \frac{1}{m} \log l(F_{jm+1:(j+1)m}) - nl < 0\} \\ &= \inf\{n \geq 1 : \sum_{j=0}^{n-1} (\frac{1}{m} \log l(F_{jm+1:(j+1)m}) - l) < 0\}.\end{aligned}$$

The elements of the sequence $(\log l(F_{jm+1:(j+1)m}) - l)_{j \in \mathbb{N}_0}$ are i.i.d., and by definition of m we obtain

$$\mathbb{E}(\frac{1}{m} \log l(F_{jm+1:(j+1)m}) - l) = \mathbb{E}(\frac{1}{m} \log l(F_{1:m}) - l) < 0$$

which implies that the random walk $(\frac{1}{m}\log l(F_{jm+1:(j+1)m}) - l)_{j\in\mathbb{N}_0}$ has negative drift. Using Theorem $A.7$, $\mathbb{E}\ \frac{\zeta}{m} < \infty$ and therefore $\mathbb{E}\ \xi_1 < \infty$ as claimed.

To prove the second assertion first note that by Minkowski's inequality (recall $-l > 0$),

$$\left(\mathbb{E}\left(\frac{1}{m}\log l(F_{1:m}) - l\right)^{p+1}\right)^{1/(p+1)} \leq \left(\mathbb{E}\left(\frac{1}{m}\log l(F_{1:m})\right)^{p+1}\right)^{1/(p+1)} - l.$$

Using $\log l(F_{1:m}) \leq \sum_{j=1}^{m}\log L_j$, independence of the L_j and Minkowski's inequality a second time, we estimate the last expression as follows:

$$\begin{aligned}\left(\mathbb{E}\left(\frac{1}{m}\log l(F_{1:m})\right)^{p+1}\right)^{1/(p+1)} &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\mathbb{E}\log^{p+1} L_1\right)^{1/(p+1)} \\ &= \left(\mathbb{E}\log^{p+1} L_1\right)^{1/(p+1)}\end{aligned}$$

which is finite by assumption. Referring again to the above quoted theorem, the $(p+1)$-st moment of ζ exists and the assertion is proved.

For the last assertion we assume $\mathbb{E}L_1^p < \infty$. By utilising that the L_j are i.i.d. and $p/m \leq p$ we have

$$\begin{aligned}\mathbb{E}\exp\left(p\left(\frac{1}{m}\log l(F_{1:m}) - l\right)\right) &= e^{-pl}\mathbb{E}l(F_{1:m})^{p/m} \\ &\leq e^{-pl}\mathbb{E}\prod_{j=1}^{m} L_j^{p/m} \\ &\leq e^{-pl}\left(\mathbb{E}L_1^p\right)^m.\end{aligned}$$

The last expression is finite by assumption and by Theorem $A.8$ there exists s_0 such that $\mathbb{E}\exp(s\zeta) < \infty$ for all $|s| < s_0$. Since $\zeta \geq \xi_1$, $\mathbb{E}\exp(s\xi_1) < \infty$ for all $|s| < s_0$ and the proof is completed. □

The starting point of our considerations is to estimate $d(\hat{M}_\infty^{x_0}, \hat{M}_n^x)$, see Chapter 1. To this end we first introduce a stopping time with respect to the sequence $(\xi_n)_{n\geq 0}$, namely,

$$\tau(n) \stackrel{\text{def}}{=} \inf\{j \geq 0 :\ \xi_j(l) \geq n\}, \quad n \geq 0. \tag{2.4}$$

The random variables $\tau(n)$ turns out to be helpful for the subsequent considerations. Note that, by the Elementary Renewal theorem, we obtain

$$\lim_{n\to\infty}\frac{\tau(n)}{n} = \frac{1}{\mu} \quad \text{a.s.,} \tag{2.5}$$

see also Corollary $A.12$.

Using the triangle inequality,

$$d(\hat{M}^{x_0}_{\xi_{\tau(n)+m}}, \hat{M}^{x}_{n}) \leq d(\hat{M}^{x_0}_{\xi_{\tau(n)}}, \hat{M}^{x_0}_{n}) + d(\hat{M}^{x_0}_{\xi_{\tau(n)+m}}, \hat{M}^{x_0}_{\xi_{\tau(n)}}) + d(\hat{M}^{x}_{n}, \hat{M}^{x_0}_{n})$$

holds true for all $n, m \geq 0$. Furthermore, estimating every summand separately we obtain

$$\begin{aligned} d(\hat{M}^{x_0}_{\xi_{\tau(n)}}, \hat{M}^{x_0}_{n}) \leq l(F_{1:\xi_1}) \cdot \ldots \cdot l(F_{\xi_{\tau(n)-2}+1:\xi_{\tau(n)-1}}) \\ \cdot d(F_{\xi_{\tau(n)-1}+1:\xi_{\tau(n)}}(x_0), F_{\xi_{\tau(n)-1}+1:n}(x_0)) \text{ a.s.,} \end{aligned}$$

for the first one,

$$\begin{aligned} d(\hat{M}^{x_0}_{\xi_{\tau(n)+m}}, \hat{M}^{x_0}_{\xi_{\tau(n)}}) \leq l(F_{1:\xi_1}) \cdot \ldots \cdot l(F_{\xi_{\tau(n)-1}+1:\xi_{\tau(n)}}) \\ \cdot d(F_{\xi_{\tau(n)}+1:\xi_{\tau(n)+m}}(x_0), x_0) \text{ a.s.} \end{aligned}$$

for the second one, and

$$d(\hat{M}^{x}_{n}, \hat{M}^{x_0}_{n}) \leq l(F_{1:n}) d(x, x_0), \quad \text{a.s.}$$

for the last one $(n, m \geq 0)$.

At this point we are interested in the excursions, more precisely, in

$$d(F_{\xi_{\tau(n)-1}+1:\xi_{\tau(n)}}(x_0), F_{\xi_{\tau(n)-1}+1:n}(x_0)) \quad \text{and} \quad d(F_{\xi_{\tau(n)}+1:\xi_{\tau(n)+m}}(x_0), x_0).$$

The problem is complicated by the fact that m will later converge to infinity. We can estimate the excursions by the random varibles C_{n+1} and D_n defined through $C_0 \stackrel{\text{def}}{=} 0$,

$$\begin{aligned} C_{n+1} \stackrel{\text{def}}{=} \max\{ & d(F_{\xi_n+1:\xi_{n+1}}(x_0), x_0); \\ & d(F_{\xi_n+1:\xi_{n+1}}(x_0), F_{\xi_n+1:k}(x_0)), \xi_n < k < \xi_{n+1}\} \end{aligned} \tag{2.6}$$

and

$$D_n \stackrel{\text{def}}{=} \sum_{j \geq 0} \exp(lj) d(F_{\xi_{n+j}+1:\xi_{n+j+1}}(x_0), x_0), \tag{2.7}$$

respectively, for all $n \geq 0$.

First note that our model assumptions imply that $(C_n)_{n\geq 1}$ and $(F_{\xi_n+1:\xi_{n+1}})_{n\geq 0}$ are both consisting of i.i.d. random variables. This in turn further implies that $(D_n)_{n\geq 0}$ constitutes a stationary sequence, see for instance [Dur]. We will show that the condition $\mathbb{E}\log^+ d(F_1(x_0), x_0) < \infty$ implies the a.s. finiteness of D_n for all $n \geq 0$. Furthermore, $(D_n)_{n\geq 0}$ is ergodic, as defined in [Bre], and autoregressive of order 1 for $D_n = d(F_{\xi_n+1:\xi_{n+1}}(x_0), x_0) + e^l D_{n+1}$.

Observing the fact that $(F_{\xi_{\tau(n)}+k})_{k\geq 1}$ and $(F_k)_{k\geq 1}$ are identically distributed for any $n \geq 0$, we also have

$$D_{\tau(n)} \sim D_0 \tag{2.8}$$

for all $n \geq 0$, and $D_{\tau(n)}$ is independent of $(L_j, F_j)_{1\leq j\leq \xi_{\tau(n)}}$ and of $\tau(n)$. Further note that the random variables C_n and D_n are linked by the inequality

$$D_n \leq \sum_{j\geq 1} \exp(l(j-1))C_{n+j} \tag{2.9}$$

for all $n \geq 0$, as $d(F_{\xi_{n-1}+1:\xi_n}(x_0), x_0) \leq C_n$ for each $n \geq 1$.

Finally, $C_{\tau(n)}$ converges weakly to a limiting variable C_∞, see Lemma 2.3. For this reason, and also for some of the proofs, we obtain by renewal theoretic arguments:

Lemma 2.2. *Let $l \in (l^*, 0)$ and $H : [0,\infty) \to [0,\infty)$ be an arbitrary function satisfying $H(0) = 0$. Further let $\odot \in \{>, \leq\}$ and U denote the renewal measure of $(\xi_n)_{n\geq 0}$. For any $t \geq 0$ and $n \geq 0$ then*

$$\mathbb{P}(H(C_{\tau(n)}) \odot t) = \sum_{j=1}^{n} U(n-j)\mathbb{P}(\xi_1 \geq j, H(C_1) \odot t) \tag{2.10}$$

holds true.

Remark. For some useful informations on renewal measures see Appendix *A*.

Proof. First note that

$$\begin{aligned}\{\tau(n) = j\} &= \{\xi_j \geq n, \xi_{j-1} < n\}\\ &= \bigcup_{k=0}^{n-1}\{\xi_j \geq n, \xi_{j-1} = k\}\\ &= \bigcup_{k=0}^{n-1}\{\xi_j - \xi_{j-1} \geq n-k, \xi_{j-1} = k\}\end{aligned}$$

holds true for all $n, j \in \mathbb{N}$. Moreover, $\xi_{j-1} = \delta_{j-1} + \ldots + \delta_1$ is independent of both $\xi_j - \xi_{j-1} = \delta_j$ and C_j. Now choose an arbitrary $t \geq 0$. Since

$\xi_j - \xi_{j-1} \sim \xi_1$ we have

$$\begin{aligned}
\mathbb{P}(H(C_{\tau(n)}) \odot t) &= \sum_{j=0}^{n} \mathbb{P}(\tau(n) = j, H(C_j) \odot t) \\
&= \sum_{j=1}^{n} \mathbb{P}\left(\bigcup_{k=0}^{n-1} \{\xi_j - \xi_{j-1} \geq n-k, \xi_{j-1} = k\} \cap \{H(C_j) \odot t\} \right) \\
&= \sum_{j=1}^{n} \sum_{k=0}^{n-1} \mathbb{P}(\xi_{j-1} = k) \mathbb{P}(\xi_1 \geq n-k, H(C_1) \odot t) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\xi_{j-1} = n-k) \mathbb{P}(\xi_1 \geq k, H(C_1) \odot t) \\
&= \sum_{k=1}^{n} U(\{n-k\}) \mathbb{P}(\xi_1 \geq k, H(C_1) \odot t)
\end{aligned}$$

□

With the help of the previous lemma, we are now able to prove two important results:

Proposition 2.3. *$C_{\tau(n)}$ converges weakly to a limiting variable, say C_∞, with distribution function*

$$\mathbb{P}(C_\infty \leq t) = \frac{1}{\mathbb{E}\xi_1} \sum_{n \geq 1} \mathbb{P}(\xi_1 \geq n, C_1 \leq t), \quad t \geq 0. \tag{2.11}$$

Proof. Since $\xi_n \geq 0$ for all $n \geq 0$, $U(\{n-k\}) = 0$ for all $k \geq n+1$. By Blackwell's renewal theorem, see also $A.11$, we know $\lim_{n\to\infty} U(\{n\}) = 1/\mu$. Using (2.10) (with $H(x) = x$ on $[0, \infty)$) and the dominated convergence theorem, see [Sch, p. 409], we deduce

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(C_{\tau(n)} \leq t) &= \lim_{n\to\infty} \sum_{j=1}^{\infty} U(\{n-j\}) \mathbb{P}(\xi_1 \geq j, C_1 \leq t) \\
&= \sum_{j=1}^{\infty} \lim_{n\to\infty} U(\{n-j\}) \mathbb{P}(\xi_1 \geq j, C_1 \leq t) \\
&= \frac{1}{\mu} \sum_{j=1}^{\infty} \mathbb{P}(\xi_1 \geq j, C_1 \leq t) \overset{\text{def}}{=} \mathbb{P}(C_\infty \leq t)
\end{aligned}$$

for all $t \geq 0$. The weak convergence follows by Proposition 36.5 [AWT, p. 182]. □

Proposition 2.4. *In the situation of Lemma* 2.2 *the inequality*

$$\mathbb{P}(H(C_{\tau(n)}) > t) \le \mathbb{E}\xi_1 \mathbf{1}_{\{H(C_{\tau(n)})>t\}} \tag{2.12}$$

holds true for all $n \ge 0$ *and* $t \ge 0$. *Moreover,*

$$\sup_{n\ge 0} \mathbb{E}H(C_{\tau(n)}) \le \mathbb{E}\xi_1 H(C_1). \tag{2.13}$$

Proof. Since the ξ_k are strictly increasing, we clearly have $\sum_{k\ge 0} \mathbf{1}_{\{\xi_k=n\}} \le 1$ and therefore we obtain

$$\sup_{n\ge 0} U(\{n\}) = \sup_{n\ge 0} \mathbb{E} \sum_{k\ge 0} \mathbf{1}_{\{\xi_k=n\}} \le 1.$$

By combining this with (2.10), we deduce

$$\begin{aligned}
\mathbb{P}(H(C_{\tau(n)}) > t) &= \sum_{j=1}^{n} U(\{n-j\})\mathbb{P}(\xi_1 \ge j, H(C_1) > t) \\
&\le \sum_{j=1}^{n} \mathbb{P}(\xi_1 \ge j, H(C_1) > t) \\
&\le \sum_{j\ge 0} \mathbb{P}(\xi_1 > j, H(C_1) > t) \\
&= \mathbb{E}\xi_1 \mathbf{1}_{\{H(C_1)>t\}}
\end{aligned}$$

for all $t \ge 0$ and $n \ge 0$. Note that the last equality follows from Corollary C.2. Hence the first assertion is shown. Using this result and again Corollary C.2 we obtain

$$\begin{aligned}
\mathbb{E}H(C_{\tau(n)}) &= \int_0^\infty \mathbb{P}(H(C_{\tau(n)}) > t)dt \\
&\le \int_0^\infty \mathbb{E}\xi_1 \mathbf{1}_{\{H(C_1)>t\}} dt = \mathbb{E}\xi_1 H(C_1)
\end{aligned}$$

for all $n \ge 0$ which proves the second assertion. □

2.2 Auxiliary lemmata

We start by estimating $d(\hat{M}_\infty^{x_0}, \hat{M}_n^x)$ and will do so first under the stronger assumption that our given IFS is strongly contractive, i.e. $l(F_1) \le e^l$ a.s. for some $l \in (l^*, 0)$. As a consequence, $\xi_n = \xi_n(l) \equiv n$, $C_n = d(F_n(x_0), x_0)$ and $D_n = \sum_{j\ge 1} \exp(l(j-1))d(F_{n+j}(x_0), x_0)$.

Using the notation $d(F_{n+1:\infty}(x_0), x_0) \overset{\text{def}}{=} \lim_{m\to\infty} d(F_{n+1:n+m}(x_0), x_0)$ for all $n \in \mathbb{N}$ and $\hat{M}_\infty^{x_0} \overset{\text{def}}{=} \lim_{n\to\infty} \hat{M}_n^{x_0}$ we obtain

Lemma 2.5. *Let* $(M_n)_{n\geq 0}$ *be an IFS of i.i.d. Lipschitz maps. Suppose the conditions* (2.1) *and* (2.2) *are given. Assume that* $l(F_1) \leq e^l$ *a.s. for some* $l \in (l^*, 0)$*. Then the* D_n *are a.s. finite and*

$$d(F_{n+1:\infty}(x_0), x_0) \leq D_n \quad \text{a.s.} \tag{2.14}$$

holds true for all $n \geq 0$*. Furthermore,*

$$d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) \leq \exp(nl)(D_n + d(x, x_0)) \quad \text{a.s.} \tag{2.15}$$

for all $n \geq 0$ *and* $x \in \mathbb{X}$*.*

Proof. We first claim

$$\begin{aligned} d(F_{n+1:n+m}(x_0), x_0) \leq\ & d(F_{n+1}(x_0), x_0) \\ & + \sum_{i=2}^{m} d(F_{n+1:n+i}(x_0), F_{n+1:n+i-1}(x_0)) \text{ a.s.} \end{aligned} \tag{2.16}$$

for all $m \geq 2$ and $n \geq 0$.

This result can be obtained by induction over m. Let $n \geq 0$. For $m = 2$ we observe by the triangle inequality

$$d(F_{n+1:n+2}(x_0), x_0) \leq d(F_{n+1:n+1}(x_0), x_0) + d(F_{n+1:n+2}(x_0), F_{n+1:n+1}(x_0))$$

almost surely and since $F_{n+1:n+1}(x_0) = F_{n+1}(x_0)$ the assertion is proved for the case $m = 2$. We assume that the assertion holds true for all $m \geq 2$ and consider the inductive step $m \longrightarrow m + 1$:

$$\begin{aligned} d(F_{n+1:n+m+1}(x_0), x_0) &\leq d(F_{n+1:n+m+1}(x_0), F_{n+1:n+m}(x_0)) \\ &\quad + d(F_{n+1:n+m}(x_0), x_0) \\ &\leq d(F_{n+1:n+m+1}(x_0), F_{n+1:n+m}(x_0)) \\ &\quad + d(F_{n+1}(x_0), x_0) + \sum_{i=2}^{m} d(F_{n+1:n+i}(x_0), F_{n+1:n+i-1}(x_0)) \\ &= d(F_{n+1}(x_0), x_0) + \sum_{i=2}^{m+1} d(F_{n+1:n+i}(x_0), F_{n+1:n+i-1}(x_0)) \end{aligned}$$

almost surely where we get the first inequality by the triangle inequality and the second by the induction hypothesis.

Using the property $l(F_1) \le e^l$ a.s. we conclude

$$\begin{aligned} d(F_{n+1:n+m}(x_0), x_0) &\le d(F_{n+1}(x_0), x_0) + \sum_{i=2}^{m} d(F_{n+1:n+i}(x_0), F_{n+1:n+i-1}(x_0)) \\ &\le d(F_{n+1}(x_0), x_0) + \sum_{i=2}^{m} l(F_{n+1:n+i-1}) d(F_{n+i}(x_0), x_0) \\ &\le d(F_{n+1}(x_0), x_0) + \sum_{i=2}^{m} \exp((i-1)l) d(F_{n+i}(x_0), x_0) \\ &= \sum_{i=1}^{m} \exp((i-1)l) d(F_{n+i}(x_0), x_0) \quad \text{a.s.} \end{aligned}$$

for all $n \ge 0$ and $m \ge 2$. As m converges to infinity the first associated assertion follows.

On the other side we obtain with the help of the triangle inequality

$$\begin{aligned} d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) &\le d(\hat{M}_\infty^{x_0}, \hat{M}_n^{x_0}) + d(\hat{M}_n^{x_0}, \hat{M}_n^x) \\ &\le l(F_1) \cdot \ldots \cdot l(F_n) d(F_{n+1:\infty}(x_0), x_0) + l(F_1) \cdot \ldots \cdot l(F_n) d(x, x_0) \end{aligned}$$

a.s. for all $n \ge 0$ and $x \in \mathbb{X}$. Since $l(F_1) \le e^l$ a.s. and the F_i are i.i.d., we get the desired result for all $n \ge 0$ and $x \in \mathbb{X}$.

In the next step we will show that D_n are a.s. finite for all $n \ge 1$. First note that for any $\delta \in (-1, 0)$ there exists $i_0 \in \mathbb{N}$ such that $\frac{2\log i}{li} > \delta$ for all $i > i_0$ because $0 > \frac{2\log i}{li} \to 0$ as $i \to \infty$. Hence $1 + \frac{2\log i}{li} > 1 + \delta$ for all $i > i_0$. With these preparations and Corollary $C.2$(iv) we obtain

$$\begin{aligned} &\sum_{i\ge1} \mathbb{P}(e^{li} d(F_{n+i}(x_0), x_0) > i^{-2}) \\ &\qquad = \sum_{i\ge1} \mathbb{P}(\log d(F_1(x_0), x_0) > -li - 2\log i) \\ &\qquad = \sum_{i\ge1} \mathbb{P}\left(\log d(F_1(x_0), x_0) > i\left(1 + \frac{2\log i}{li}\right)(-l)\right) \qquad (2.17) \\ &\qquad \le \sum_{i=1}^{i_0} 1 + \sum_{i>i_o} \mathbb{P}(\log d(F_1(x_0), x_0) > i(1+\delta)(-l)) \\ &\qquad \le C(1 + \mathbb{E}(\log d(F_1(x_0), x_0))^+). \end{aligned}$$

for a suitable $C > 0$. Since $\mathbb{E}\log^+ d(F_1(x_0), x_0) < \infty$ the last expression of the foregoing inequality is finite. Applying the Borel-Cantelli lemma, we know that there are only finitely many i such that $e^{li}d(F_{n+i}(x_0), x_0) > i^{-2}$. W.l.o.g we may assume that $e^{li}d(F_{n+i}(x_0), x_0) \le i^{-2}$ holds true for all $i \in \mathbb{N}$. Hence we conclude

$$D_n = \sum_{i\ge 1} e^{l(i-1)} d(F_{n+i}(x_0), x_0) \le \frac{1}{e^l} \sum_{i \ge 1} i^{-2} = \frac{\pi^2}{6}\frac{1}{e^l}.$$

□

Lemma 2.6. *Given an IFS $(M_n)_{n\ge 0}$ of i.i.d. Lipschitz maps with the conditions* (2.1) *and* (2.2) *and an arbitrary $l \in (l^*, 0)$, the D_n are a.s. finite and $\mathbb{E}\log^+ C_n < \infty$. Moreover,*

$$\begin{aligned} d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) \\ \le \exp(l\xi_{\tau(n)-1})C_{\tau(n)} + \exp(l\xi_{\tau(n)})D_{\tau(n)} + l(F_{1:n})d(x, x_0) \quad \text{a.s.} \end{aligned} \tag{2.18}$$

holds true for all $n \ge 0$ and $x \in \mathbb{X}$.

Remark. In some cases we will apply a slightly different inequality. Since $\xi_{\tau(n)-1} < \xi_{\tau(n)}$ and $l < 0$ we get $\exp(l\xi_{\tau(n)})D_{\tau(n)} \le \exp(l\xi_{\tau(n)-1})D_{\tau(n)}$ and therefore

$$\begin{aligned} d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) \le \exp(l\xi_{\tau(n)-1})C_{\tau(n)} \\ + \exp(l\xi_{\tau(n)-1})D_{\tau(n)} + l(F_{1:n})d(x, x_0) \quad \text{a.s.} \end{aligned} \tag{2.19}$$

holds true for all $n \ge 0$ and $x \in \mathbb{X}$.

Before we prove Lemma 2.6, let us define $\log_* \stackrel{\text{def}}{=} \log(1 + x)$. In the following we will use $\log_*$ instead of $\log^+$. This ensures that $\log_* X$ is non-negative whenever this holds true for X. It will also be helpful to assert some properties of $\log_*$:

Lemma 2.7. *$\log_* x$ is subadditive, i.e. $\log_*(x + y) \le \log_* x + \log_* y$ for all $x, y \ge 0$, and satisfies $\log_*(xy) \le \log_* x + \log_* y$ for all $x, y \ge 0$.*

Proof. Since $xy \ge 0$, we have

$$\begin{aligned} &\log_*(x + y) = \log(1 + x + y) \\ &\le \log(1 + x + y + xy) = \log((1 + x)(1 + y)) = \log_* x + \log_* y \end{aligned}$$

which proves the fist assertion. We get the other inequality by a similar calculation:

$$\begin{aligned} &\log_*(xy) = \log(1 + xy) \\ &\le \log(1 + x + y + xy) = \log((1 + x)(1 + y)) = \log_* x + \log_* y \end{aligned}$$

since $x + y \ge 0$. □

Another helpful tool will be the random variable

$$U_n \stackrel{\text{def}}{=} \max\left\{\prod_{j=1}^{k} L_j, 0 \le k \le n\right\}, \quad n \ge 0.$$

In the next lemma we will state inequalities for U_n.

Lemma 2.8. *The following inequalities hold for all $n \ge 0$:*

$$U_n \le \prod_{k=1}^{n}(1+L_k) \quad \text{and} \quad \log_* U_n \le \sum_{k=1}^{n} \log_* L_k. \tag{2.20}$$

Proof. For each $n \ge 0$, we have

$$U_n \le \max\left\{\prod_{j=1}^{k}(1+L_j), 0 \le k \le n\right\} = \prod_{j=1}^{n}(1+L_j)$$

which proves the first inequality. But the second then follows from the subadditivity of $\log_*$ as stated in Lemma 2.7. □

Now we are able to prove Lemma 2.6:

Proof of Lemma 2.6. First transform the original IFS into one which is strongly contractive. Defining $F'_n \stackrel{\text{def}}{=} F_{\xi_{n-1}+1:\xi_n}$ we have by construction of the ξ_n

$$l(F'_n) = l(F_{\xi_{n-1}+1:\xi_n}) < \exp(l(\xi_n - \xi_{n-1})) \le \exp(l) \quad \text{a.s.} \tag{2.21}$$

for all $n \ge 1$ since $l \in (l^*, 0)$. The $(F'_n, l(F'_n))$ are i.i.d. and therefore

$$F'_{n:1}(M_0), \quad n \ge 0, \tag{2.22}$$

is a strongly contractive IFS of i.i.d. Lipschitz maps which satisfies

$$\hat{M}_{\xi_n} = F'_{1:n}(M_0), \quad n \ge 0,$$

i.e. the backward process of both, the original and the new strongly contractive one, are equal.

Using the triangle inequality we observe

$$d(\hat{M}^{x_0}_\infty, \hat{M}^x_n) \le d(\hat{M}^{x_0}_{\xi_{\tau(n)}}, \hat{M}^{x_0}_n) + d(\hat{M}^{x_0}_\infty, \hat{M}^{x_0}_{\xi_{\tau(n)}}) + d(\hat{M}^x_n, \hat{M}^{x_0}_n).$$

In the following we analyse the three summands on the right hand side with the help of the above defined IFS and Lemma 2.5. For the first one we obtain for each $n \ge 0$ the inequaltiy

$$\begin{aligned} d(\hat{M}^{x_0}_{\xi_{\tau(n)}}, \hat{M}^{x_0}_n) &\le l(F'_1) \cdot \ldots \cdot l(F'_{\xi_{\tau(n)-1}}) d(F_{\xi_{\tau(n)-1}+1:\xi_{\tau(n)}}(x_0), F_{\xi_{\tau(n)-1}+1:n}(x)) \\ &\le \exp(l\xi_{\tau(n)-1}) C_{\tau(n)} \quad \text{a.s.} \end{aligned}$$

Applying Lemma 2.5 we have for the second summand

$$\begin{aligned} d(\hat{M}_\infty^{x_0}, \hat{M}_{\xi_{\tau(n)}}^{x_0}) &\leq l(F_{1:\xi_1}) \cdot \ldots \cdot l(F_{\xi_{\tau(n)-1}+1:\xi_{\tau(n)}}) d(F'_{\xi_{\tau(n)}+1:\infty}(x_0), x_0) \\ &\leq \exp(l\xi_{\tau(n)}) D_{\tau(n)} \quad \text{a.s.} \end{aligned}$$

for all $n \geq 0$. For the last expression, obviously,

$$d(\hat{M}_n^x, \hat{M}_n^{x_0}) \leq l(F_{1:n}) d(x, x_0)$$

holds true a.s. for all $n \geq 0$. Therefore the asserted inequality (2.18) is shown.

Before we show the remaining assertions let us note that, by an appeal to Lemma 2.5 for the strictly contractive IFS $(F_{\xi_n:1})_{n\geq 0}$ (which defines D_n, see (2.7)), we infer $D_n < \infty$ a.s. for all $n \geq 0$ if $\mathbb{E}\log^+ d(F_{1:\xi_1}(x_0), x_0) < \infty$. Since $\log^+ d(F_{1:\xi_1}(x_0), x_0) \leq \log^+ C_1$ it is enough to prove $\mathbb{E}\log^+ C_1 < \infty$.

To this end first consider the following inequality:

$$\begin{aligned} d(F_{1:\xi_1}(x_0), x_0) &\leq d(F_{1:\xi_1}(x_0), F_{1:1}(x_0)) + d(F_{1:1}(x_0), x_0) \\ &\leq d(F_{1:\xi_1}(x_0), F_{1:2}(x_0)) + d(F_{1:2}(x_0), F_{1:1}(x_0)) + d(F_{1:1}(x_0), x_0) \\ &\leq \ldots \\ &\leq d(F_{1:\xi_1}(x_0), F_{1:\xi_1-1}(x_0)) + \sum_{j=1}^{\xi_1-1} d(F_{1:j}(x_0), F_{1:j-1}(x_0)) \\ &= \sum_{j=1}^{\xi_1} d(F_{1:j}(x_0), F_{1:j-1}(x_0)). \end{aligned}$$

Note that the first summand of each inequality is an element of the set $\{d(F_{1:\xi_1}(x_0), F_{1:k}(x_0)), 0 < k < \xi_1\}$. Altogether, we infer

$$C_1 \leq \sum_{j=1}^{\xi_1} d(F_{1:j}(x_0), F_{1:j-1}(x_0)). \tag{2.23}$$

By definition of U_n and its properties, see Lemma 2.8, we obtain

$$\begin{aligned} C_1 &\leq \sum_{j=1}^{\xi_1} d(F_{1:j}(x_0), F_{1:j-1}(x_0)) \\ &\leq \sum_{j=1}^{\xi_1} l(F_{1:j-1}) d(F_j(x_0), x_0) \\ &\leq \sum_{j=1}^{\xi_1} L_1 \cdot \ldots \cdot L_{j-1} d(F_j(x_0), x_0) \\ &\leq U_{\xi_1} \sum_{j=1}^{\xi_1} d(F_j(x_0), x_0). \end{aligned} \tag{2.24}$$

With the help of the above inequality, Lemmata 2.8 and 2.7 as well as Wald's first identity (see also Lemma A.9) we are now able to conclude

$$\begin{aligned} \mathbb{E} \log_* C_1 &\leq \mathbb{E} \log_* \left(U_{\xi_1} \sum_{j=1}^{\xi_1} d(F_j(x_0), x_0) \right) \\ &\leq \mathbb{E} \left(\log_* U_{\xi_1} + \log_* \sum_{j=1}^{\xi_1} d(F_j(x_0), x_0) \right) \\ &\leq \mathbb{E} \left(\sum_{j=1}^{\xi_1} \log_* L_j \right) + \mathbb{E} \left(\sum_{j=1}^{\xi_1} \log_* d(F_j(x_0), x_0) \right) \\ &= \mathbb{E}\xi_1 \left(\mathbb{E} \log_* L_1 + \mathbb{E} \log_* d(F_1(x_0), x_0) \right) \end{aligned}$$

which is finite by Lemma 2.1 and condition (2.2). □

2.2.1 Condition set (A)

Further results for the excursions will be proved hereafter under the additional conditions

$$\mathbb{E} \log^{p+1}(1 + L_1) < \infty \quad \text{and} \quad \mathbb{E} \log^{p+1}(1 + d(F_1(x_0), x_0)) < \infty \tag{2.25}$$

for some $p > 0$ and $x_0 \in \mathbb{X}$. The next lemma will provide necessary moment results to prove the limit theorems given in Chapter 3.

Lemma 2.9. *Let $l \in (l^*, 0)$ and $p > 0$. If*

$$\mathbb{E} \log^{p+1}(1 + L_1) < \infty \quad \text{and} \quad \mathbb{E} \log^{p+1}(1 + d(F_1(x_0), x_0)) < \infty, \tag{2.26}$$

then the following assertions hold for all $n \geq 0$:

$$\mathbb{E}\log^{p+1}(1+C_1) < \infty \tag{2.27}$$

$$\mathbb{E}\log^{p}(1+D_0) = \mathbb{E}\log^{p}(1+D_{\tau(n)}) < \infty. \tag{2.28}$$

The family $\{\log^p(1+C_{\tau(n)}), n \geq 0\}$ is uniformly integrable and satisfies

$$\sup_{n\geq 0}\mathbb{E}\log^{p}(1+C_{\tau(n)}) \leq \mathbb{E}\xi_1\log^{p}(1+C_1) < \infty. \tag{2.29}$$

Remark. Since the $D_{\tau(n)}$ are identically distributed by (2.8) we obtain the uniform integrability of $\{\log^p(1+D_{\tau(n)}), n \geq 0\}$ by (2.28). Furthermore, $\{\log^p(1+C_{\tau(n)}+D_{\tau(n)}), n \geq 0\}$ is uniformly integrable since both families $\{\log^{p+1}(1+C_{\tau(n)}), n \geq 0\}$ and $\{\log^p(1+D_{\tau(n)}), n \geq 0\}$ are uniformly integrable.

Proof. By Lemma 2.1, $\mathbb{E}\log^{p+1}(1+L_1) < \infty$ implies $\mathbb{E}\xi_1^{p+1} < \infty$. Together with Theorem $A.6$ we obtain

$$\mathbb{E}\left(\sum_{j=1}^{\xi_1}\log_* L_j\right)^{p+1} \leq N_{p+1}\mathbb{E}\xi_1^{p+1}\mathbb{E}\log_*^{p+1} L_1 < \infty$$

where N_{p+1} is a numerical constant depending on $(p+1)$. Since on the other hand $(\log_* d(F_n(x_0),x_0))_{n\geq 1}$ is a sequence of i.i.d. random variables and, by assumption, $\mathbb{E}\log_*^{p+1} d(F_1(x_0),x_0) < \infty$, there exists $M_{p+1} > 0$ such that

$$\mathbb{E}\left(\sum_{j=1}^{\xi_1}\log_* d(F_j(x_0),x_0)\right)^{p+1} \leq M_{p+1}\mathbb{E}\xi_1^{p+1}\mathbb{E}\log_*^{p+1} d(F_1(x_0),x_0) < \infty.$$

Moreover, by the proof of Lemma 2.6, we observe

$$\begin{aligned}\mathbb{E}(\log_* C_1)^{p+1} &\leq \mathbb{E}\left(\log_* U_{\xi_1}\sum_{j=1}^{\xi_1} d(F_j(x_0),x_0)\right)^{p+1}\\ &\leq \mathbb{E}\left(\sum_{j=1}^{\xi_1}\log_* L_j + \sum_{j=1}^{\xi_1}\log_* d(F_j(x_0),x_0)\right)^{p+1}.\end{aligned}$$

Using Minkowsky's inequality we finally deduce

$$\begin{aligned}&\left(\mathbb{E}(\log_* C_1)^{p+1}\right)^{1/(p+1)}\\ &\leq \left(\mathbb{E}\left(\sum_{j=1}^{\xi_1}\log_* L_j\right)^{p+1}\right)^{1/(p+1)} + \left(\mathbb{E}\left(\sum_{j=1}^{\xi_1}\log_* d(F_j(x_0),x_0)\right)^{p+1}\right)^{1/(p+1)}\end{aligned}$$

which is finite by the foregoing preparations and therefore the $(p+1)$-st moment of C_1 exists.

To show (2.28) it is sufficient to consider $\mathbb{E}\log_*^p D_0$ since $D_{\tau(n)}$ and D_0 are identically distributed. For this purpose let $a>1$ satisfying $ae^l<1$ and $\frac{1}{2e^l}\le b \overset{\text{def}}{=} \frac{1-ae^l}{ae^l}\le \frac{1}{e^l}$. Then

$$\begin{aligned}\mathbb{P}(D_0>e^t) &\le \mathbb{P}\left(\sum_{j\ge 1} e^{l(j-1)}C_j > e^t\right)\\ &\le \mathbb{P}(e^{l(j-1)}C_j > b(ae^l)^j e^t \quad \text{for some } j\ge 1).\end{aligned} \tag{2.30}$$

holds true by (2.9) for any $t\in(0,\infty)$. For the last step some more explanations are necessary. We claim that

$$\begin{aligned}A \overset{\text{def}}{=} \{\omega\in\Omega \;:\; &\sum_{j\ge 1} e^{l(j-1)}C_j > e^t\}\\ &\subset B \overset{\text{def}}{=} \{\omega\in\Omega \;:\; \exists j\ge 1 \quad e^{l(j-1)}C_j > b(ae^l)^j e^t\}.\end{aligned}$$

To this end let $\omega\in A$ and assume that $\omega\notin B$, i.e.

$$e^{l(j-1)}C_j \le b(ae^l)^j e^t$$

holds true for all $j\ge 1$. Then we obtain

$$\begin{aligned}\sum_{j\ge 1} e^{l(j-1)}C_j &\le \sum_{j\ge 1} b(ae^l)^j e^t\\ &= \sum_{j\ge 0} b(ae^l)^{j+1} e^t\\ &= b\frac{ae^l}{1-ae^l}e^t = e^t.\end{aligned}$$

But this is a contradiction to $\omega\in A$. Therefore ω must be an element of B and the assertion is shown. With this we obtain in (2.30)

$$\begin{aligned}\mathbb{P}(D_0>e^t) &\le \mathbb{P}(e^{l(j-1)}C_j > b(ae^l)^j e^t \quad \text{for some } j\ge 1)\\ &= \mathbb{P}(C_j > ba^j e^l e^t \quad \text{for some } j\ge 1)\\ &\le \mathbb{P}(2C_j > a^j e^t \quad \text{for some } j\ge 1)\\ &\le \mathbb{P}\left(\bigcup_{j\ge 1}(2C_j > a^j e^t)\right)\\ &\le \sum_{j\ge 1}\mathbb{P}(2C_1 > a^j e^t)\end{aligned} \tag{2.31}$$

where we have used that the C_j are i.i.d. By combining this with Corollary C.2(ii) we obtain

$$\begin{aligned}
\mathbb{E}((\log D_0)^+)^p &= \int_0^\infty pt^{p-1}\mathbb{P}(\log D_0 > t)dt \\
&= \int_0^\infty pt^{p-1}\mathbb{P}(D_0 > e^t)dt \\
&\le \int_0^\infty pt^{p-1}\sum_{j\ge1}\mathbb{P}(2C_1 > a^j e^t)dt \\
&= \sum_{j\ge1}\int_0^\infty pt^{p-1}\mathbb{P}(2C_1 > a^j e^t)dt \\
&= \sum_{j\ge1}\int_0^\infty pt^{p-1}\mathbb{P}(\log 2C_1 - j\log a > t)dt \\
&= \sum_{j\ge1}\mathbb{E}((\log 2C_1 - j\log a)^+)^p
\end{aligned} \tag{2.32}$$

where Fubini's theorem has been used in line 4. By the first part of this proof we have $\mathbb{E}((\log 2C_1)^+)^{p+1} \le \mathbb{E}\log_*^{p+1} 2C_1 < \infty$ and, in combination with Corollary C.3, we conclude $\mathbb{E}((\log D_0)^+)^p < \infty$ and therefore $\mathbb{E}\log_*^p D_0 < \infty$.

To prove the remaining assertions consider the function $H(t) \stackrel{\text{def}}{=} \log_*^p t$. By Hölder's inequality we obtain

$$\mathbb{E}\xi_1 \log_*^p C_1 \le (\mathbb{E}\xi_1^{p+1})^{1/(p+1)}(\mathbb{E}\log_*^{p+1} C_1)^{p/(p+1)}.$$

The expression on the right side is finite by Lemma 2.1 and (2.27). The uniform integrability of $\{\log^p(1 + C_{\tau(n)}), n \ge 0\}$ follows by (2.12) and Corollary 50.3 in [AWT, p. 278]. Furthermore, (2.29) easily follows from (2.13). □

In the previous lemma we provided necessary tools for the first two summands of inequality (2.18). The next lemma will be used to estimate the last term on the right hand side in (2.18).

Lemma 2.10. *Given $l^* < 0$ a.s. and $\mathbb{E}\log^{p+1}(1 + L_1) < \infty$ for some $p > 0$,*

$$\sum_{n\ge1} n^{p-1}\mathbb{P}(l(F_{1:n}) > \exp(ln)) < \infty \tag{2.33}$$

holds true as well as

$$\lim_{n\to\infty} n^p\mathbb{P}(l(F_{1:n}) > \exp(ln)) = 0 \tag{2.34}$$

for any $l \in (l^*, 0)$.

Proof. Let $l \in (l^*, 0)$. Choose m so large that $l' = \frac{1}{m}\mathbb{E}\log l(F_{1:m}) \in (l^*, l-\varepsilon)$ for some $\varepsilon \in (0, l-l^*)$, compare Proposition 1.2. By left truncation we may assume w.l.o.g. that $\mathbb{E}|\log l(F_{1:m})|^{p+1} < \infty$. Using $F_{nm+1:nm+r} \sim F_{1:r}$, we obtain

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}\mathbb{P}(l(F_{1:n}) > e^{ln}) &= \sum_{r=1}^{m}\sum_{n\geq 0}(nm+r)^{p-1}\mathbb{P}(l(F_{1:nm+r}) > e^{l(nm+r)}) \\ &\leq C\sum_{r=1}^{m}\sum_{n\geq 0} n^{p-1}\mathbb{P}(e^{-nml'}l(F_{1:nm})l(F_{nm+1:nm+r}) > e^{(l+\varepsilon n)m}) \\ &\leq C\sum_{r=1}^{m}\sum_{n\geq 1} n^{p-1}\mathbb{P}(l(F_{1:r}) > e^{(l+\varepsilon n)m}/2) \\ &\quad + Cm\sum_{n\geq 1} n^{p-1}\mathbb{P}(e^{-nml'}l(F_{1:nm}) > e^{(l+\varepsilon n)m}/2)\end{aligned}$$

where C is some positive constant. By (2.25) and $l(F_{1:r}) \leq L_1 \cdot \ldots \cdot L_r$, we observe

$$\begin{aligned}\sum_{n\geq 0} n^{p-1}\mathbb{P}(l(F_{1:r}) > e^{(l+\varepsilon n)m}/2) &= \sum_{n\geq 0} n^{p-1}\mathbb{P}(\log l(F_{1:r}) + \log 2 - lm > \varepsilon nm) \\ &\leq C'\,(1 + \mathbb{E}\log_*^p L_1) < \infty\end{aligned}$$

for some constant $C' > 0$.

Moreover, we have $\log(e^{-nml'}l(F_{1:nm})) \leq V_n$, where

$$V_n \overset{\text{def}}{=} \sum_{j=0}^{n-1}(\log l(F_{jm+1:(j+1)m}) - ml'), \quad n \geq 1,$$

is a zero-mean random walk with i.i.d. increments satisfying (again by (2.25))

$$\mathbb{E}|V_1|^{p+1} < \infty.$$

Consequently,

$$\begin{aligned}&\sum_{n\geq 1} n^{p-1}\mathbb{P}(e^{-nml'}l(F_{1:nm}) > e^{(l+\varepsilon n)m}/2) \\ &\qquad\qquad \leq \sum_{n\geq 1} n^{p-1}\mathbb{P}(V_n + \log 2 - lm > \varepsilon nm) < \infty\end{aligned}$$

follows by the one-sided tail estimates obtained by Chow and Lai [CL]. By the corollary to Theorem 1 in this article, see [CL, p. 56], we further infer from $\mathbb{E}|V_1|^{p+1} < \infty$ that

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\left(\sup_{k\geq n}|k^{-1}V_k| > \hat{\varepsilon}\right) < \infty$$

and thus

$$n^p \mathbb{P}\left(\sup_{k\geq n} |k^{-1}V_k| > \hat{\varepsilon}\right) \to \infty \quad \text{as} \quad n \to \infty$$

for all $\hat{\varepsilon} > 0$ using Lemma $C.4$. Now the desired conclusion follows from the following inequalities:

$$\begin{aligned}\mathbb{P}(l(F_{1:nm+r}) &> e^{l(nm+r)}) \\ &\leq \mathbb{P}(l(F_{1:r}) > e^{(l+\varepsilon n)m}/2) + \mathbb{P}(e^{-nml'} l(F_{1:nm}) > e^{(l+\varepsilon n)m}/2).\end{aligned}$$

Since $\mathbb{E}\log_*^{p+1} l(F_{1:r}) < \infty$, the first term on the right hand side is of order $o(n^{-p})$. For the second, we have for sufficiently large n

$$\begin{aligned}\mathbb{P}(e^{-nml'} l(F_{1:nm}) > e^{(l+\varepsilon n)m}/2) &\leq \mathbb{P}(V_n - lm + \log 2 > \varepsilon nm) \\ &\leq \mathbb{P}\left(\sup_{k\geq n} |k^{-1}V_k| > \varepsilon m/2\right)\end{aligned}$$

and thus from the above mentioned result

$$n^p \mathbb{P}(e^{-nml'} l(F_{1:nm}) > e^{(l+\varepsilon n)m}/2) \to 0 \quad \text{as} \quad n \to \infty.$$

□

We will repeatedly consider the probability of the right side of inequality (2.18). It turns out to be helpful to split this probability into a sum of two probabilities of the disjoint sets $\{\xi_{\tau(n)-1} \leq (1-\rho)n\}$ and its complement. For this set we observe results as stated below.

Lemma 2.11. *Let $l \in (l^*, 0)$, $p > 0$ and $\mathbb{E}\log^{p+1}(1 + L_1) < \infty$. Then*

$$\sum_{n\geq 1} n^{p-1} \mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) < \infty \tag{2.35}$$

holds true as well as

$$\lim_{n\to\infty} n^p \mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) = 0 \tag{2.36}$$

for all $\rho > 0$.

Proof. By Lemma (2.1), $\mathbb{E}\log^{p+1}(1 + L_1) < \infty$ implies $\mathbb{E}\xi_1^{p+1} < \infty$. In the following we consider the increments $\delta_j = \xi_j - \xi_{j-1}$ and note that $\mathbb{E}\delta_1^{p+1} < \infty$

since $\delta_1 = \xi_1$. Observe that $\tau(n) \le n$ since $0 = \xi_0 < \xi_1 < \xi_2 < \ldots$ Recalling that the δ_j are i.i.d. we have

$$\begin{aligned}\sum_{n\ge 0} n^{p-1}\mathbb{P}(\delta_{\tau(n)} > \rho n) &\le \sum_{n\ge 0} n^{p-1} \sum_{1\le k\le n} \mathbb{P}(\delta_k > \rho n)\\ &\le \sum_{n\ge 0} n^{p-1} n\mathbb{P}(\delta_1 > \rho n)\\ &= \sum_{n\ge 0} n^p\mathbb{P}(\delta_1 > \rho n)\\ &= C\mathbb{E}\delta_1^{p+1} < \infty\end{aligned}$$

for all $\rho > 0$ and some positive constant C. Since $\xi_{\tau(n)} \ge n$ by definition of $\tau(n)$, we are now able to conlude

$$\begin{aligned}\sum_{n\ge 0} n^{p-1}\mathbb{P}(\xi_{\tau(n)-1} \le (1-\rho)n) &= \sum_{n\ge 0} n^{p-1}\mathbb{P}(\xi_{\tau(n)} - \delta_{\tau(n)} \le (1-\rho)n)\\ &\le \sum_{n\ge 0} n^{p-1}\mathbb{P}(n - \delta_{\tau(n)} \le (1-\rho)n)\\ &= \sum_{n\ge 0} n^{p-1}\mathbb{P}(\delta_{\tau(n)} \ge \rho n) < \infty\end{aligned}$$

for all $\rho > 0$.

The second assertion also follows from the previous calculations. For this purpose note that the first inequality also provides

$$\sum_{n\ge 0} n^p\mathbb{P}(\delta_1 > \rho n) < \infty$$

for all $\rho > 0$ and therefore $n^p\mathbb{P}(\delta_1 \ge \rho n) \to 0$ as $n \to \infty$. By the same considerations as above then

$$0 \le \lim_{n\to\infty} n^p\mathbb{P}(\xi_{\tau(n)-1} \le (1-\rho)n) \le \lim_{n\to\infty} n^p\mathbb{P}(\delta_1 \ge \rho n) = 0$$

holds true for all $\rho > 0$ which completes the proof. □

2.2.2 Condition set (B)

In this subsection we will work out similar assertions as in the foregoing one, but now for slightly different conditions, namely,

$$\mathbb{E}L_1^p < \infty \quad \text{and} \quad \mathbb{E}d(F_1(x_0), x_0)^p < \infty \tag{2.37}$$

for some $p > 0$ and $x_0 \in \mathbb{X}$. The assertions will be similar because the theorems of both sets of conditions are of the same type and with it the idea how to prove them. Hence we will analyse the excursions with respect to the new situation in the following lemma.

Lemma 2.12. *Let $l \in (l^*, 0)$ and $p > 0$. Then*

$$\mathbb{E}L_1^p < \infty \quad \text{and} \quad \mathbb{E}d(F_1(x_0), x_0)^p < \infty \tag{2.38}$$

implies the following assertions for some $\eta > 0$:

$$\mathbb{E}C_1^{2\eta} < \infty \tag{2.39}$$

$$\mathbb{E}D_0^{2\eta} = \mathbb{E}D_{\tau(n)}^{2\eta} < \infty. \tag{2.40}$$

Moreover, the family $\{C_{\tau(n)}^\eta, n \geq 0\}$ is uniformly integrable and satisfies the inequality

$$\sup_{n \geq 0} \mathbb{E}C_{\tau(n)}^\eta \leq \mathbb{E}\xi_1 C_1^\eta < \infty. \tag{2.41}$$

Remark. Also in this case we have uniform integrability of $\{D_{\tau(n)}^{2\eta}, n \geq 0\}$ by (2.40) since the $D_{\tau(n)}$ are identically distributed. By the same argument as in the remark following Lemma 2.9 the family $\{(C_{\tau(n)} + D_{\tau(n)})^\eta, n \geq 0\}$ is uniformly integrable.

Proof. By assumption, $\mathbb{E}\exp(p \log L_1) = \mathbb{E}L_1^p < \infty$. On the other side, compare Lemma 2.1, there exists $s_0 > 0$ such that $\mathbb{E}\exp(s\xi_1) < \infty$ for all $0 < s < s_0$. This again implies that $\sum_{n \geq 0} e^{sn}\mathbb{P}(\xi_1 = n)$ converges, see Corollary C.2, and therefore

$$\mathbb{P}(\xi_1 = n) = o(e^{-sn}) \quad \text{as} \quad n \to \infty. \tag{2.42}$$

Furthermore, we obtain by Hölder's inequality

$$\begin{aligned}
\mathbb{E}\exp\left(4\eta \sum_{k=1}^{\xi_1} \log_* L_k\right) &= \sum_{n \geq 0} \int_\Omega \mathbf{1}_{\{\xi_1 = n\}} \exp\left(4\eta \sum_{k=1}^{n} \log_* L_k\right) d\mathbb{P} \\
&\leq \sum_{n \geq 0} (\mathbb{P}(\xi_1 = n))^{1/2} \left(\mathbb{E}\exp\left(8\eta \sum_{k=1}^{n} \log_* L_k\right)\right)^{1/2}
\end{aligned}$$

for some constant $C > 0$. Since

$$\mathbb{E}\exp\left(8\eta \sum_{k=1}^{n} \log_* L_k\right) \to 1 \quad \text{as} \quad \eta \to 0,$$

we can find in combination with (2.42) $\eta \le p/4$ sufficiently small such that

$$\mathbb{E}\exp\left(4\eta\sum_{k=1}^{\xi_1}\log_* L_k\right) < \infty.$$

Apply Theorem A.6 to see that

$$\mathbb{E}\left(\sum_{j=1}^{\xi_1} d(F_j(x_0), x_0)\right)^{4\eta} < \infty,$$

since $(d(F_j(x_0), x_0))_{j\in\mathbb{N}}$ constitutes a random walk and $4\eta \le p$. By (2.24) and Lemma 2.8 we write

$$\begin{aligned} C_1^{2\eta} &\le (U_{\xi_1})^{2\eta}\left(\sum_{j=1}^{\xi_1} d(F_j(x_0), x_0)\right)^{2\eta} \\ &\le \exp\left(2\eta\sum_{j=1}^{\xi_1}\log_* L_j\right)\left(\sum_{j=1}^{\xi_1} d(F_j(x_0), x_0)\right)^{2\eta} \quad \text{a.s.} \end{aligned}$$

Applying Hölder's inequality, we are now able to conclude

$$\mathbb{E}C_1^{2\eta} \le \left(\mathbb{E}\exp\left(4\eta\sum_{k=1}^{\xi_1}\log_* L_k\right)\right)^{1/2}\left(\mathbb{E}\left(\sum_{j=1}^{\xi_1} d(F_j(x_0), x_0)\right)^{4\eta}\right)^{1/2} < \infty.$$

To prove (2.40) we split our considerations into two cases. At first we assume that $2\eta \ge 1$. By the infinite version of Minkowski's inequality and $D_n \le \sum_{j\ge 1}\exp(l(j-1))C_{n+j}$, see also (2.9), we deduce

$$\begin{aligned} \left(\mathbb{E}D_0^{2\eta}\right)^{1/(2\eta)} &\le \left(\mathbb{E}\left(\sum_{j\ge 1}\exp(l(j-1))C_j\right)^{2\eta}\right)^{1/(2\eta)} \\ &\le \sum_{j\ge 1}\exp(l(j-1))\left(\mathbb{E}C_1^{2\eta}\right)^{1/(2\eta)} \\ &= \frac{\left(\mathbb{E}C_1^{2\eta}\right)^{1/(2\eta)}}{1-e^l} < \infty. \end{aligned}$$

If $0 < 2\eta < 1$, then $t \mapsto t^{2\eta}$ is subadditve and, again with (2.9), we deduce

$$\begin{aligned} \mathbb{E}D_0^{2\eta} \le \mathbb{E}\left(\sum_{j\ge 1}\exp(l(j-1))C_j\right)^{2\eta} &\le \sum_{j\ge 1}\exp(2\eta l(j-1))\mathbb{E}C_1^{2\eta} \\ &= \frac{\mathbb{E}C_1^{2\eta}}{1-\exp(2\eta l)} < \infty. \end{aligned}$$

This completes the proof of (2.40).

In the beginning of this proof we have already shown that $\mathbb{E}C_1^{2\eta} < \infty$ and $\mathbb{E}\exp(s\xi_1) < \infty$ for some $s > 0$. Also note that $\xi_1^2 \le \exp(s\xi_1)$ if ξ_1 is large enough. Then, by Hölder's inequality, we conclude

$$\mathbb{E}C_1^{\eta}\xi_1 \le (\mathbb{E}C_1^{2\eta})^{1/2}(\mathbb{E}\xi_1^2)^{1/2} \le C(\mathbb{E}C_1^{2\eta})^{1/2}(\mathbb{E}\exp(s\xi_1))^{1/2} < \infty$$

where C is some positive constant. Moreover, let $H(t) \overset{\text{def}}{=} t^{\eta}$. Now the uniform integrability follows by (2.12) and Corollary 50.3 in [AWT, p. 278]. Furthermore, the asserted inequality is a result from (2.13). □

Lemma 2.13. *Let $l \in (l^*, 0)$ and $\mathbb{E}L_1^p < \infty$ for some $p > 0$. Then there exists $\alpha \in (0,1)$ such that*

$$\mathbb{P}(l(F_{1:n}) > \varepsilon\alpha^n) \le \varepsilon^{-1}\alpha^n \tag{2.43}$$

holds true for all $n \ge 1$ and $\varepsilon > 0$.

Proof. Let $l \in (l^*, 0)$. By assumption, $\mathbb{E}L_1^r$ is a convex function of r on $[0,p)$ with negative derivative $\mathbb{E}\log L_1$ at 0. Hence there exists a $q \in (0, \min\{p,1\})$ with $m_q \overset{\text{def}}{=} \mathbb{E}L_1^q < 1$. Since $l(F_{1:n}) \le L_{1:n}$, $n \ge 1$, we have

$$\mathbb{E}l(F_{1:n})^q \le \mathbb{E}L_{1:n}^q = m_q^n$$

for all $n \ge 1$ and therefore

$$\mathbb{P}(l(F_{1:n}) > \varepsilon\beta^n) \le \varepsilon^{-1}\left(\frac{m_q}{\beta^q}\right)^n$$

for all $\varepsilon, \beta > 0$ and $n \ge 1$. We arrive at the desired conclusion by choosing any $\beta \in (m_q^{1/q}, 1)$ and then $\alpha = \max\{\beta, m_q/\beta^q\}$. □

Lemma 2.14. *Let $l \in (l^*, 0)$ and $p > 0$. If $\mathbb{E}L_1^p < \infty$ there exists $c > 0$ such that*

$$\lim_{n\to\infty} \alpha^{-n}\mathbb{P}(\xi_{\tau(n)-1} \le (1-\rho)n) = 0 \tag{2.44}$$

holds true for all $\rho > 0$ and $\alpha \in (c, 1)$ where α depends on l and ρ.

Proof. As $\mathbb{E}L_1^p < \infty$, Lemma 2.1 implies $\mathbb{E}\exp(\beta\xi_1) < \infty$ for some $\beta > 0$. By a similar argument to that used in the proof of Lemma 2.11 we have

$$\mathbb{P}(\xi_{\tau(n)-1} \le (1-\rho)n) \le \mathbb{P}(\delta_1 > \rho n)$$

$\rho > 0$. Further note that $\delta_1 = \xi_1$. Using Markov's inequality, we deduce

$$\mathbb{P}(\xi_{\tau(n)-1} \le (1-\rho)n) \le \frac{1}{\exp(\rho\beta n)}\mathbb{E}\exp(\beta\xi_1) < \infty$$

for all $\rho > 0$ and $n \ge 1$. The assertion now is given for all $\alpha \in (\exp(-\beta\rho), 1)$. □

Chapter 3

Limit Theorems

In the previous chapter we have collected the neccessary tools to obtain the results which hold for IFS with i.i.d. contractive Lipschitz maps.

Theorem 3.1. *Let* $(M_n)_{n\geq 0}$ *be an IFS of i.i.d. Lipschitz maps and assume that* $l^* < 0$, $\mathbb{E}\log^+ L_1 < \infty$ *and* $\mathbb{E}\log^+ d(F_1(x_0), x_0) < \infty$ *for some* $x_0 \in \mathbb{X}$. *Then the following assertions hold:*

(a) $\hat{M}_n$ *converges a.s. to a random variable* $\hat{M}_\infty$ *with distribution* π *which is independent of the initial distribution.*

(b) *For each* $l \in (l^*, 0)$, $\lim_{n\to\infty} \mathbb{P}_x(d(\hat{M}_\infty, \hat{M}_n) > e^{nl}) = 0$ *for all* $x \in \mathbb{X}$.

(c) M_n *converges in distribution to* π *under every* $\mathbb{P}_x$, $x \in \mathbb{X}$.

(d) π *is the unique stationary distribution of* $(M_n)_{n\geq 0}$ *and* $(\hat{M}_n)_{n\geq 0}$ *a stationary sequence under* $\mathbb{P}_\pi$.

(e) $(M_n)_{n\geq 0}$ *is ergodic under* $\mathbb{P}_\pi$.

Remark. The assertions (a), (c), (d) and (e) were proved earlier by Elton [Elt] for general stationary sequences $(F_n)_{n\geq 0}$.

Proof. It remains to prove part (b). Choose any $\hat{l} \in (l^*, 0)$. Then there exists an l such that $l^* < l < \hat{l}$. Hence we can write $\hat{l}$ as $\hat{l} = l + \varepsilon$ for some $\varepsilon > 0$. Using (2.19), we obtain

$$\mathbb{P}_x\left(d(\hat{M}_\infty, \hat{M}_n) > e^{n\hat{l}}\right) = \mathbb{P}\left(d(\hat{M}_\infty^{x_0}, \hat{M}_n^{x}) > e^{n\hat{l}}\right)$$
$$\leq \mathbb{P}\left(\exp\left(-n\hat{l} + l\xi_{\tau(n)-1}\right)(C_{\tau(n)} + D_{\tau(n)}) + e^{-n\hat{l}} l(F_{1:n}) d(x, x_0) > 1\right).$$

In the following we will show that the last expression converges to 0 for all $x \in \mathbb{X}$ as $n \to \infty$. To this end we consider each of the summands in the last line separately.

First note that, by the Elementary Renewal Theorem (see $A.12$) and the strong law of large numbers,

$$\frac{n}{\tau(n)-1} \to \mu \quad \text{and} \quad \frac{\xi_{\tau(n)-1}}{\tau(n)-1} \to \mu \quad \text{a.s.}$$

as $n \to \infty$. Therefore

$$l\xi_{\tau(n)-1} - n\hat{l} = (\tau(n)-1)\left(l\frac{\xi_{\tau(n)-1}}{\tau(n)-1} - \frac{n}{\tau(n)-1}(l+\varepsilon)\right) \to -\infty \tag{3.1}$$

a.s. as $n \to \infty$.

We have already shown that the $C_{\tau(n)}$ converges in distribution to a finite random variable. With Slutsky's theorem, see [AWT, p. 185], and Proposition 36.3. in [AWT, p. 181], we deduce

$$\exp\left(-n\hat{l} + l\xi_{\tau(n)-1}\right) C_{\tau(n)} \xrightarrow{P} 0 \tag{3.2}$$

where $\xrightarrow{P}$ means convergence in probability.

Using $D_{\tau(n)} \sim D_0$, we have for any $\kappa > 0$ and $t > 0$

$$\begin{aligned}
&\lim_{m\to\infty} \sup_{n\geq m} \mathbb{P}\left(\exp\left(-n\hat{l} + l\xi_{\tau(n)-1}\right) D_{\tau(n)} > \kappa\right) \\
&\qquad \leq \lim_{m\to\infty} \sup_{n\geq m} \mathbb{P}\left(\exp\left(-n\hat{l} + l\xi_{\tau(n)-1}\right) > \kappa/t\right) + \mathbb{P}(D_0 > t) \\
&\qquad = \mathbb{P}(D_0 > t).
\end{aligned}$$

Since D_0 is a.s. finite, $\mathbb{P}(D_0 > t) \to 0$ as $t \to \infty$ and we conclude

$$\exp\left(-n\hat{l} + l\xi_{\tau(n)-1}\right) D_{\tau(n)} \xrightarrow{P} 0. \tag{3.3}$$

At least note that, if $n \to \infty$,

$$\begin{aligned}
\log \exp(-n\hat{l}) l(F_{1:n}) &= -n\hat{l} + \log l(F_{1:n}) \\
&= n(-\hat{l} + \frac{\log l(F_{1:n})}{n}) \\
&\xrightarrow{P} -\infty \quad \text{a.s.}
\end{aligned}$$

since $\frac{\log l(F_{n:1})}{n} \to l^*$ a.s. □

3.1 Limit theorems for condition set (A)

Under additional assumptions we are now able to state limit theorems we already have mentioned in the introduction and Chapter 1.

Theorem 3.2. *Given the situation of Theorem* 3.1 *and the additional conditions*

$$\mathbb{E}\log^{p+1}(1+L_1)<\infty \quad \text{and} \quad \mathbb{E}\log^{p+1}(1+d(F_1(x_0),x_0))<\infty \tag{3.4}$$

for some $p>0$ and $x_0\in\mathbb{X}$, the following assertions hold true:

(a) *For each $l\in(l^*,0)$,*

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}_x(d(\hat{M}_\infty,\hat{M}_n)>\exp(ln))\leq c_l\,(1+\log^p(1+d(x,x_0))) \tag{3.5}$$

and

$$\lim_{n\to\infty} n^p\mathbb{P}_x(d(\hat{M}_\infty,\hat{M}_n)>\exp(ln))=0 \tag{3.6}$$

for all $x\in\mathbb{X}$ and some $c_l\in(0,\infty)$.

(b) *For each $l\in(l^*,0)$,*

$$\limsup_{n\to\infty} n^{(p-1)/p}\left(\frac{1}{n}\log d(\hat{M}_\infty,\hat{M}_n)-l\right)\leq 0 \quad \mathbb{P}_x\text{−}a.s. \tag{3.7}$$

for all $x\in\mathbb{X}$. In the case $0<p\leq 1$ this also remains true for $l=l^*$.

(c) *If $p\geq 1$, then*

$$\lim_{n\to\infty}\exp(-nl)d(\hat{M}_\infty,\hat{M}_n)=0 \quad \mathbb{P}_x\text{−}a.s.$$

for all $x\in\mathbb{X}$ and $l\in(l^*,0)$.

(d)

$$\rho(P^n(x,\cdot),\pi)\leq A_x(n+1)^{-p}$$

for all $n\geq 0$, $x\in\mathbb{X}$ and some positive constant A_x of the form $\max\{A,2d(x,x_0)\}$. Furthermore, A does neither depend on x nor on n.

(e) $\int_{\mathbb{X}}\log^p(1+d(x,x_0))\pi(dx)=\int_0^\infty pt^{p-1}\pi(x:\log(1+d(x,x_0))>t)dt<\infty.$

Remark. Note that both constants c_l and A_x depend on p for which the moment conditions hold.

The proof will take up a lot of space. Furthermore, these results will establish new assertions. For this reason we have shifted the proof to the end of this discussion, see Chapter 3.3.1.

The results of the foregoing theorem, in which the backward process was subject of our considerations, are applicable to the forward process. To this end we analyse the distance $d(M_n^x, M_n^y)$, $x, y \in \mathbb{X}$, which reflects the coupling rate of the forward iterations at time n with different starting points x and y. The connection is given by the fact that (M_n^x, M_n^y) and $(\hat{M}_n^x, \hat{M}_n^y)$ are identically distributed and

$$d(\hat{M}_n^x, \hat{M}_n^y) \leq d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) + d(\hat{M}_\infty^{x_0}, \hat{M}_n^y) \tag{3.8}$$

for all $x, y \in \mathbb{X}$ and $n \geq 0$.

Corollary 3.3. *Given the situation of Theorem 3.2, the following assertions hold:*

(a) *For each $l \in (l^*, 0)$,*

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}\mathbb{P}(d(M_n^x, M_n^y) > \exp(ln)) \\ \leq c_l\left(1 + \log^p(1 + d(x, x_0)) + \log^p(1 + d(y, x_0))\right)\end{aligned} \tag{3.9}$$

and

$$\lim_{n\to\infty} n^p\mathbb{P}(d(M_n^x, M_n^y) > \exp(ln)) = 0 \tag{3.10}$$

for all $x, y \in \mathbb{X}$ and some $c_l \in (0, \infty)$.

(b) *For each $l \in (l^*, 0)$,*

$$\limsup_{n\to\infty} n^{(p-1)/p}\left(\frac{1}{n}\log d(M_n^x, M_n^y) - l\right) \leq 0 \quad a.s. \tag{3.11}$$

for all $x, y \in \mathbb{X}$. In the case $0 < p \leq 1$ this remains true for $l = l^*$.

(c) *If $p \geq 1$, then*

$$\lim_{n\to\infty} \exp(-nl)d(M_n^x, M_n^y) = 0 \quad a.s.$$

for all $x, y \in \mathbb{X}$ and $l \in (l^*, 0)$.

Proof. (a) Let $l \in (l^*, 0)$. There exists $n_0 \in \mathbb{N}$ such that

$$l^* < -\frac{\log 2}{n_0} + l \leq -\frac{\log 2}{n} + l < l$$

for all $n \geq n_0$ and, obviously, $-\frac{\log 2}{n_0}+l \in (l^*, 0)$. Using the triangle inequality, we have

$$\begin{aligned}
&\mathbb{P}(d(\hat{M}_n^x, \hat{M}_n^y) > \exp(ln)) \\
&\qquad \leq \mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) + d(\hat{M}_\infty^{x_0}, \hat{M}_n^y) > \exp(ln)) \\
&\qquad \leq \mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) > \exp(ln)/2) + \mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^y) > \exp(ln)/2) \\
&\qquad \leq \mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) > \exp(n(l - (\log 2)/n_0))) \\
&\qquad\qquad + \mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^y) > \exp(n(l - (\log 2)/n_0)))
\end{aligned}$$

for all $n \geq n_0$ and $x, y \in \mathbb{X}$. Since (M_n^x, M_n^y) and $(\hat{M}_n^x, \hat{M}_n^y)$ are identically distributed, the assertion follows by the foregoing theorem.

(b) Given $l \in (l^*, 0)$, we have for any $n \geq 1$ and $x, y \in \mathbb{X}$

$$\begin{aligned}
\frac{1}{n}\log d(\hat{M}_n^y, \hat{M}_n^x) - l &\leq \frac{1}{n}\log(2\max\{d(\hat{M}_\infty^{x_0}, \hat{M}_n^x), d(\hat{M}_\infty^{x_0}, \hat{M}_n^y)\}) - 2l \\
&\leq \frac{1}{n}\log 2 + (\frac{1}{n}\log d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) - l) + (\frac{1}{n}\log d(\hat{M}_\infty^{x_0}, \hat{M}_n^y) - l).
\end{aligned}$$

Since $n^{(p-1)/p}\frac{1}{n}\log 2 \to 0$ as $n \to \infty$, the assertion follows by Theorem 3.2 part (b).

(c) The notes which lead to this corollary are sufficient. □

Denote by M_0^π a random variable with distribution π. The proceeding corollary will provide more informations on the distance of M_n^x for any $x \in \mathbb{X}$ to a stationary counterpart $M_n^\pi \stackrel{\text{def}}{=} F_{n:1}(M_0^\pi)$. The key for these considerations is the identity

$$\mathbb{P}(d(M_n^x, M_n^\pi) \in \cdot) = \int_{\mathbb{X}} \mathbb{P}(d(M_n^x, M_n^y) \in \cdot)\pi(dy) \tag{3.12}$$

for all $n \geq 0$ and $x \in \mathbb{X}$, compare the notations in Chapter 1. Doing this we obtain the following results:

Corollary 3.4. *Given the situation of Theorem 3.2, the following assertions hold:*

(a) *For each $l \in (l^*, 0)$*

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}(d(M_n^x, M_n^\pi) > \exp(nl)) \leq c_l(1 + \log^p(1 + d(x, x_0)))$$

holds true as well as

$$\lim_{n\to\infty} n^p\mathbb{P}(d(M_n^x, M_n^\pi) > \exp(nl)) = 0$$

for all $x \in \mathbb{X}$ and some constant $c_l \in (0, \infty)$.

(b) *For each* $l \in (l^*, 0)$,

$$\limsup_{n\to\infty} n^{(p-1)/p} \left(\frac{1}{n} \log d(M_n^x, M_n^\pi) - l \right) \le 0 \quad a.s. \tag{3.13}$$

for all $x \in \mathbb{X}$. *In the case* $0 < p \le 1$ *this remains also true for* $l = l^*$.

(c) *If* $p \ge 1$, *then*

$$\lim_{n\to\infty} \exp(-nl) d(M_n^x, M_n^\pi) = 0 \quad a.s.$$

for all $x \in \mathbb{X}$ *and* $l \in (l^*, 0)$.

Proof. (a) For any $x, y \in \mathbb{X}$, the sequence $(n^p \mathbb{P}(d(M_n^x, M_n^y) > \exp(ln)))_{n\ge1}$ is bounded by part (a) of the previous corollary. Hence the second assertion follows with the dominated convergence theorem and identity (3.12). On the other side, we have

$$\sum_{n\ge1} n^{p-1} \mathbb{P}(d(M_n^x, M_n^\pi) > \exp(nl)) \le \int_{\mathbb{X}} \hat{c}_l (1 + \log^p(1 + d(x, x_0)) + \log^p(1 + d(y, x_0))) \pi(dy)$$

for all $x \in \mathbb{X}$ and some $\hat{c}_l \in (0, \infty)$. Since $\int_{\mathbb{X}} \log(1 + d(y, x_0)) \pi(dy) < \infty$ by Theorem 3.2(e) and $\int_{\mathbb{X}} \pi(dy) = 1$, this completes the proof.

(b) This result is a consequence of

$$\mathbb{P}\left(\limsup_{n\to\infty} n^{(p-1)/p} \left(\frac{1}{n} \log d(M_n^x, M_n^y) - l \right) \le 0 \right) = 1$$

for each $l \in (l^*, 0)$ and $x, y \in \mathbb{X}$, $\int_{\mathbb{X}} \pi(dy) = 1$ and (3.12).

(c) Adapt the arguments of part (b). □

3.2 Limit theorems for condition set (B)

If we replace the set of conditions in Section 3.1 by the condition set (B) analogous results hold true:

Theorem 3.5. *Given the situation of Theorem* 3.1 *and additionally*

$$\mathbb{E}L_1^p < \infty \quad \text{and} \quad \mathbb{E}d(F_1(x_0), x_0)^p < \infty \tag{3.14}$$

for some $p > 0$ *and* $x_0 \in \mathbb{X}$, *the following assertions hold:*

(a) *For each* $l \in (l^*, 0)$,

$$\lim_{n\to\infty} \alpha_l^{-n} \mathbb{P}_x(d(\hat{M}_\infty, \hat{M}_n) > \exp(nl)) = 0$$

for all $x \in \mathbb{X}$ *and some* $\alpha_l \in (0,1)$.

(b) *There exists* $\eta > 0$ *such that for each* $q \in (0, \eta)$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{X}} \alpha_q^{-n}(1 + d(x, x_0))^{-q} \mathbb{E}_x d(\hat{M}_\infty, \hat{M}_n)^q = 0$$

for some $\alpha_q \in (0,1)$. *The same holds true for* $q = \eta$ *with* $\alpha_q = 1$.

(c)

$$\rho(P^n(x,\cdot), \pi) \le A_x r^n$$

for all $n \ge 0$, $x \in \mathbb{X}$, *some* $r \in (0,1)$ *and a constant* A_x *of the form* $\max\{A, 2d(x,x_0)\}$. *The constants* r *and* A *do neither depend on* x *nor on* n.

(d) $\int_{\mathbb{X}} d(x,x_0)^\eta \pi(dx) = \int_0^\infty \eta t^{\eta-1} \pi(x : d(x,x_0) > t) dt < \infty$ *for some* $\eta > 0$.

Remark. Note that the constants α_l, α_q, A_x and r depend on $p > 0$ for which the moment conditions hold.

Proof. See Subsection 3.3.2. □

As in Section 3.1, we use the foregoing theorem to state results for the distance $d(M_n^x, M_n^y)$.

Corollary 3.6. *Given the situation of Theorem* 3.5, *the following assertions hold:*

(a) *If* $l \in (l^*, 0)$

$$\lim_{n\to\infty} \alpha_l^{-n} \mathbb{P}(d(M_n^x, M_n^y) > \exp(nl)) = 0$$

for all $x, y \in \mathbb{X}$ *and some* $\alpha_l \in (0,1)$.

(b) *There exists* $\eta > 0$ *such that for each* $q \in (0, \eta)$,

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{X}} \alpha_q^{-n}(1 + \max\{d(x,x_0), d(y,x_0)\})^{-q} \mathbb{E} d(M_n^x, M_n^y)^q = 0$$

for some $\alpha_q \in (0,1)$. *The same holds true for* $q = \eta$ *with* $\alpha_q = 1$.

Proof. (a) Compare the proof to Corollary 3.3(a).

(b) Choose $\eta \in (0,1)$ such that Lemma 2.12 holds true. Obviously, we have $(1+\max\{d(x,x_0), d(y,x_0)\})^{-q} \le (1+d(x,x_0))^{-q}$ for all $x, y \in \mathbb{X}$. Then the assertion follows from Theorem 3.5 by the triangle inequality (3.8). □

By Fubini's theorem, we observe

$$\begin{aligned}\mathbb{E}d(M_n^x, M_n^\pi)^q &= \int_0^\infty qt^{q-1}\mathbb{P}(d(M_n^x, M_n^\pi) > t)dt \\ &= \int_0^\infty qt^{q-1}\int_{\mathbb{X}} \mathbb{P}(d(M_n^x, M_n^y) > t)\pi(dy)dt \\ &= \int_{\mathbb{X}}\int_0^\infty qt^{q-1}\mathbb{P}(d(M_n^x, M_n^y) > t)dt\ \pi(dy) \\ &= \int_{\mathbb{X}} \mathbb{E}d(M_n^x, M_n^y)^q\pi(dy)\end{aligned} \tag{3.15}$$

for all $n \geq 0$, $x, y \in \mathbb{X}$ and $q > 0$, and closure this section with

Corollary 3.7. *Given the situation of Theorem 3.5, the following assertions hold:*

(a) *For each $l \in (l^*, 0)$,*

$$\lim_{n\to\infty} \alpha_l^{-n}\mathbb{P}(d(M_n^x, M_n^\pi) > \exp(nl)) = 0$$

for all $x \in \mathbb{X}$ and some $\alpha_l \in (0, 1)$.

(b) *There exists $\eta > 0$ such that for each $q \in (0, \eta)$,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{X}} \alpha_q^{-n}(1 + d(x, x_0))^{-q}\mathbb{E}d(M_n^x, M_n^\pi)^q = 0$$

for some $\alpha_q \in (0, 1)$. The same holds true for $q = \eta$ with $\alpha_q = 1$.

Proof. (a) See the proof of the corresponding part of Corollary 3.4.

(b) Using

$$(1 + d(x, x_0))^q(1 + d(y, x_0))^q \geq (1 + \max\{d(x, x_0), d(y, x_0)\})^q$$

in combination with (3.15) and Theorem 3.5(d), we obtain

$$\begin{aligned}&\lim_{n\to\infty} \sup_{x\in\mathbb{X}} \alpha_q^{-n}(1 + d(x, x_0))^{-q}\mathbb{E}d(M_n^x, M_n^\pi)^q \\ &\leq \lim_{n\to\infty} \int_{\mathbb{X}} \sup_{x\in\mathbb{X}} \alpha_q^{-n}(1 + d(x, x_0))^{-q}\mathbb{E}d(M_n^x, M_n^y)^q\pi(dy) \\ &\leq \lim_{n\to\infty} \int_{\mathbb{X}} \sup_{x,z\in\mathbb{X}} \alpha_q^{-n}\frac{\mathbb{E}d(M_n^x, M_n^z)^q}{(1 + \max\{d(x, x_0), d(z, x_0)\})^q}(1 + d(y, x_0))^q\pi(dy) \\ &\leq C \lim_{n\to\infty} \sup_{x,z\in\mathbb{X}} \alpha_q^{-n}(1 + \max\{d(x, x_0), d(z, x_0)\})^q\mathbb{E}d(M_n^x, M_n^z)^q\end{aligned}$$

for some $C > 0$ and Corollary 3.6 gives the desired result. □

3.3 Proofs

3.3.1 Proof of Theorem 3.2

Proof of Theorem 3.2. (a) Choose any $\hat{l} \in (l^*, 0)$. Then one can find an l such that $l^* < l < \hat{l}$. Hence there exists $\varepsilon > 0$ with $\hat{l} = l + \varepsilon$. Note that $\exp(nl)d(x, x_0) \geq e^{n\hat{l}}/3$ iff $n \leq \frac{\log 3d(x,x_0)}{\varepsilon} \stackrel{\text{def}}{=} n_0$. Further, let $0 < \rho < -\frac{\varepsilon}{l}$ which implies $\widetilde{\eta} \stackrel{\text{def}}{=} l + \varepsilon - (1-\rho)l > 0$.

To make the following considerations easier first note that the following holds true for all sets $A, B \subset \Omega$:

$$\begin{aligned} \mathbb{P}(B) = \mathbb{P}((A \cup A^{\complement}) \cap B) = P((A \cap B) \cup (A^{\complement} \cap B)) \\ = \mathbb{P}(A \cap B) + \mathbb{P}(A^{\complement} \cap B) \leq \mathbb{P}(A) + \mathbb{P}(A^{\complement} \cap B). \end{aligned} \tag{3.16}$$

If we apply this inequality to

$$A \stackrel{\text{def}}{=} \{\omega \in \Omega : \ \xi_{\tau(n)-1} \leq (1-\rho)n\}$$

and

$$B \stackrel{\text{def}}{=} \{\omega \in \Omega : \ \exp(l\xi_{\tau(n)-1})(C_{\tau(n)} + D_{\tau(n)}) + l(F_{1:n})d(x, x_0) > \exp(\hat{l}n)\}$$

we obtain

$$\begin{aligned} &\mathbb{P}\Big(\exp\left(l\xi_{\tau(n)-1}\right) C_{\tau(n)} + \exp\left(l\xi_{\tau(n)-1}\right) D_{\tau(n)} + l(F_{1:n})d(x, x_0) > \exp(\hat{l}n)\Big) \\ &\leq \mathbb{P}\left(\xi_{\tau(n)-1} \leq (1-\rho)n\right) \\ &+ \mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} + \exp(l(1-\rho)n)D_{\tau(n)} + l(F_{1:n})d(x, x_0) > \exp(\hat{l}n)) \\ &\leq \mathbb{P}\left(\xi_{\tau(n)-1} \leq (1-\rho)n\right) + \mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} > \exp(\hat{l}n)/3) \\ &+ \mathbb{P}(\exp(l(1-\rho)n)D_{\tau(n)} > \exp(\hat{l}n)/3) + \mathbb{P}(l(F_{1:n})d(x, x_0) > \exp(\hat{l}n)/3) \end{aligned}$$

for all $n \geq 0$ and $\rho > 0$. Considering again inequality (3.16) for

$$A' \stackrel{\text{def}}{=} \{\omega \in \Omega : \ \exp(nl)d(x, x_0) > e^{n\hat{l}}/3\}$$

and

$$B' \stackrel{\text{def}}{=} \{\omega \in \Omega : \ l(F_{1:n})d(x, x_0) > \exp(\hat{l}n)/3\},$$

we have

$$\begin{aligned} \mathbb{P}(l(F_{1:n})d(x, x_0) &> \exp(\hat{l}n)/3) \\ &\leq \mathbf{1}(\exp(nl)d(x, x_0) > \exp(n\hat{l})/3) + \mathbb{P}(l(F_{1:n}) > \exp(nl)) \end{aligned}$$

for all $n \geq 0$ and all $x \in \mathbb{X}$. With the help of inequality (2.19), we finally observe

$$\begin{aligned}\mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) > \exp(\hat{l}n)) \leq \mathbb{P}\left(\xi_{\tau(n)-1} \leq (1-\rho)n\right) \\ + \mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} > \exp(\hat{l}n)/3) \\ + \mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} > \exp(\hat{l}n)/3) \\ + \mathbb{P}(l(F_{1:n}) > \exp(nl)) + \mathbf{1}(\exp(nl)d(x,x_0) > \exp(n\hat{l})/3)\end{aligned} \tag{3.17}$$

for all $x \in \mathbb{X}$ and $n \geq 0$.

This inequality turns out to be an useful tool to prove the given assertions. The idea is to analyse each summand on the right side separately.

That

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\left(\xi_{\tau(n)-1} \leq (1-\rho)n\right) \quad \text{and} \quad \sum_{n\geq 1} n^{p-1}\mathbb{P}(l(F_{1:n}) > \exp(nl))$$

are finite, is guaranteed by Lemmata 2.11 and 2.10, respectively. In the following let $C > 0$ be a generic constant which may differ from line to line. Note that C will always be independent of x. Using Hölder's inequality and (2.12) (with $H(y) \overset{\text{def}}{=} \log_* 3y$ and $t \overset{\text{def}}{=} n\widetilde{\eta}$), we have

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}\mathbb{P}(&\exp(n(1-\rho)l)C_{\tau(n)} > \exp(n\hat{l})/3) \\ &\leq \sum_{n\geq 1} n^{p-1}\mathbb{P}(\log_* 3C_{\tau(n)} > n(l+\varepsilon) - n(1-\rho)l) \\ &\leq \sum_{n\geq 1} n^{p-1}\mathbb{E}\xi_1 \mathbf{1}_{\{\log_* 3C_1 > n\widetilde{\eta}\}} \\ &\leq C\mathbb{E}\xi_1 \log_*^p 3C_1 \\ &\leq C(\mathbb{E}\xi_1^{p+1})^{1/(p+1)}(\mathbb{E}\log_*^{p+1} 3C_1)^{p/(p+1)}.\end{aligned}$$

The last expression is finite by (2.27) and Lemma 2.1. Recalling that $D_{\tau(n)} \sim D_0$, we obtain on the other side

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}\mathbb{P}(\exp(n(1-\rho)l)D_{\tau(n)} > \exp(n\hat{l}/3)) &\leq \sum_{n\geq 1} n^{p-1}\mathbb{P}(\log_* 3D_0 > n\widetilde{\eta}) \\ &\leq C\mathbb{E}\log_*^p D_0 < \infty\end{aligned}$$

which is finite by (2.28). By definition of n_0 we observe

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}\mathbf{1}(\exp(nl)d(x,x_0) > e^{n\hat{l}}/3) &= \sum_{n=1}^{n_0} n^{p-1}\mathbf{1}(\exp(nl)d(x,x_0) > e^{n\hat{l}}/3) \\ &\leq Cn_0^p \leq \hat{c}_l \log_*^p d(x,x_0)\end{aligned}$$

where $\hat{c}_l$ is some positive constant depending on l (recall that n_0 depends on ε which again depends on the choice of l).

To prove the second assertion of (a) we again consider inequality (3.17). By Markov's inequality, we observe

$$\begin{aligned}\mathbb{P}(\log_* 3C_{\tau(n)} > n\widetilde{\eta}) &\leq \frac{1}{n^p\widetilde{\eta}^p}\int_{\{\log_* 3C_{\tau(n)}\geq n\widetilde{\eta}\}} \log_*^p 3C_{\tau(n)} d\mathbb{P} \\ &\leq \frac{1}{n^p\widetilde{\eta}^p}\sup_{k\geq 0}\int_{\{\log_*^p 3C_{\tau(k)}\geq n^p\widetilde{\eta}^p\}} \log_*^p 3C_{\tau(k)} d\mathbb{P}\end{aligned}$$

for all $n \geq 1$. Therefore we obtain

$$\begin{aligned}&n^p\mathbb{P}(\exp(n(1-\rho)l)C_{\tau(n)} > \exp(n\hat{l})/3) \\ &\qquad\qquad \leq \frac{1}{\widetilde{\eta}^p}\sup_{k\geq 0}\int_{\{\log_*^p 3C_{\tau(k)}\geq n^p\widetilde{\eta}^p\}} \log_*^p 3C_{\tau(k)} d\mathbb{P}\end{aligned}$$

for all $n \geq 1$. Since $\{\log_*^p C_{\tau(n)}, n \geq 0\}$ is uniformly integrable the right side of the foregoing inequality converges to 0 as n goes to infinity and hence

$$n^p\mathbb{P}(\exp(n(1-\rho)l)C_{\tau(n)} > \exp(n\hat{l})/3) \to 0 \quad \text{as} \quad n \to \infty.$$

Recalling that $\{\log_*^p D_{\tau(n)}, n \geq 0\}$ is also uniformly integrable, we can prove along the same path

$$n^p\mathbb{P}(\exp(n(1-p)l)D_{\tau(n)} > \exp(n\hat{l})/3) \to 0 \quad \text{as} \quad n \to \infty.$$

Obviously, $n^p\mathbf{1}(\exp(nl)d(x,x_0) > \exp(n\hat{l})/3) = 0$ for all $n > n_0$. In combination with Lemmata 2.11 and 2.10 we have shown the desired assertion.

(b) Again we use inequality (2.19) to estimate $\frac{1}{n}\log d(\hat{M}_\infty^{x_0}, \hat{M}_n^x)$. By some further calculations we then observe

$$\begin{aligned}&\frac{1}{n}\log d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) \\ &\leq \frac{1}{n}\log(\exp(l\xi_{\tau(n)-1})C_{\tau(n)} + \exp(l\xi_{\tau(n)-1})D_{\tau(n)} + l(F_{1:n})d(x,x_0)) \qquad (3.18) \\ &\leq \frac{1}{n}l\xi_{\tau(n)-1} + \frac{1}{n}\log(C_{\tau(n)} + D_{\tau(n)} + \exp(-l\xi_{\tau(n)-1})l(F_{1:n})d(x,x_0))\end{aligned}$$

a.s. for all $n \geq 1$, $x \in \mathbb{X}$ and $l \in (l^*, 0)$.

Choose an arbitrary $\hat{l}$ in $(l^*, 0)$ and let $l \in (l^*, \hat{l})$. By the foregoing inequality we then obtain almost surely for each $n \geq 1$ and $x \in \mathbb{X}$

$$\begin{aligned} & n^{\frac{p-1}{p}} \left(\frac{1}{n} \log d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) - \hat{l} \right) \\ & \quad \leq n^{\frac{p-1}{p}} l \left(-\frac{\hat{l}}{l} + \frac{1}{n} \xi_{\tau(n)-1} \right) \\ & \quad + n^{-1/p} \log(C_{\tau(n)} + D_{\tau(n)} + \exp(-l\xi_{\tau(n)-1}) l(F_{1:n}) d(x, x_0)). \end{aligned} \tag{3.19}$$

The assertion is proved if the upper limit of the last expression is less then 0 $\mathbb{P}_x$-a.s. for all $x \in \mathbb{X}$. First, we claim

$$\lim_{n\to\infty} n^{-1/p} \log(C_{\tau(n)} + D_{\tau(n)} + \exp(-l\xi_{\tau(n)-1}) l(F_{1:n}) d(x, x_0)) = 0 \text{ a.s.} \tag{3.20}$$

for each $p > 0$. To this end note, that

$$\begin{aligned} & -l\xi_{\tau(n)-1} + \log l(F_{1:n}) \\ & \quad = (\tau(n) - 1) \left(-l \frac{\xi_{\tau(n)-1}}{\tau(n) - 1} + \frac{\log l(F_{1:n})}{n} \frac{n}{\tau(n) - 1} \right) \to -\infty \quad \text{a.s.} \end{aligned}$$

as $n \to \infty$. This holds true, because the bracket converges by the strong law of large numbers, Lemma 1.2 and the Elementary Renewal Theorem to $-l\mu + l^*\mu < 0$. Hence this gives

$$\lim_{n\to\infty} \exp(-l\xi_{\tau(n)-1}) l(F_{1:n}) d(x, x_0) = 0 \quad \text{a.s.}$$

and therefore there exists an n_0 such that

$$\exp(-l\xi_{\tau(n)-1}) l(F_{1:n}) d(x, x_0) < 1 \quad \text{a.s.} \tag{3.21}$$

for all $n \geq n_0$. W.l.o.g. we may assume $n_0 = 1$ which makes the following considerations easier.

Let $C > 0$ be a generic constant which may differ from line to line. Since $D_{\tau(n)} \sim D_0$, we obtain for each $\alpha > 0$

$$\sum_{n\geq 1} \mathbb{P}(\log_* D_{\tau(n)} > n^{1/p}\alpha/2) \leq \sum_{n\geq 1} \mathbb{P}(\log_*^p D_0 > n(\alpha/2)^p) \leq C\mathbb{E} \log_*^p D_0$$

which is finite by Lemma 2.9. Moreover, inequality (2.12) (with $H(t) =$

$\log_* t)$ and Hölder's inequality give

$$\begin{aligned}\sum_{n\geq 1}\mathbb{P}(\log_* C_{\tau(n)} > n^{1/p}\alpha/2) &\leq \sum_{n\geq 1}\mathbb{E}\xi_1\mathbf{1}_{\{\log_* C_1 > n^{1/p}\alpha/2\}}\\ &= \mathbb{E}\left(\xi_1\sum_{n\geq 1}\mathbf{1}_{\{\log_* C_1>n^{1/p}\alpha/2\}}\right)\\ &\leq \mathbb{E}\left(\xi_1\sum_{n\geq 1}\mathbf{1}_{\{\log_*^p C_1>n(\alpha/2)^p\}}\right)\\ &\leq C\mathbb{E}\xi_1\log_*^p C_1\\ &\leq C(\mathbb{E}\xi_1^{p+1})^{1/(p+1)}(\mathbb{E}\log_*^{p+1} C_1)^{p/(p+1)}.\end{aligned}$$

for all $\alpha > 0$. The last expression is finite by Lemmata 2.1 and 2.9.

Using the result of (3.21) and subadditivity of $\log_*$, we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{n\geq m} n^{-1/p}\log(C_{\tau(n)} + D_{\tau(n)} + \exp(-l\xi_{\tau(n)-1})l(F_{1:n})d(x,x_0)) > \alpha\right)\\ &\leq \mathbb{P}\left(\sup_{n\geq m} n^{-1/p}\log_*(C_{\tau(n)} + D_{\tau(n)}) > \alpha\right)\\ &\leq \mathbb{P}\left(\bigcup_{n\geq m}\{n^{-1/p}\log_*(C_{\tau(n)} + D_{\tau(n)}) > \alpha\}\right)\\ &\leq \sum_{n\geq m}\mathbb{P}(\log_*(C_{\tau(n)} + D_{\tau(n)}) > n^{1/p}\alpha)\\ &\leq \sum_{n\geq m}\mathbb{P}(\log_* C_{\tau(n)} + \log_* D_{\tau(n)} > n^{1/p}\alpha)\\ &\leq \sum_{n\geq m}\mathbb{P}(\log_* C_{\tau(n)} > n^{1/p}\alpha/2) + \sum_{n\geq m}\mathbb{P}(\log_* D_{\tau(n)} > n^{1/p}\alpha)\end{aligned}$$

for all $\alpha > 0$ and $m \geq 1$. Since both series are convergent, we consequently have

$$\mathbb{P}\left(\sup_{n\geq m} n^{-1/p}\log(C_{\tau(n)} + D_{\tau(n)} + \exp(-l\xi_{\tau(n)-1})l(F_{1:n})d(x,x_0)) > \alpha\right) \to 0$$

as $m \to \infty$ and the desired result follows by the $\mathbb{P} - \sup$ criteria, see Proposition 34.4 in [AWT, p. 167].

In a second step we consider the first summand of inequality (3.19). Note, that by the Elementary Renewal Theorem and the strong law of large numbers

$$\frac{\tau(n)-1}{n}\frac{\xi_{\tau(n)-1}}{\tau(n)-1} \to 1 \quad \mathbb{P}_x - \text{a.s.}$$

holds true as $n \to \infty$. Since $-\frac{\hat{l}}{l} > -1$, we have

$$-\frac{\hat{l}}{l} + \frac{\tau(n)-1}{n}\frac{\xi_{\tau(n)-1}}{\tau(n)-1} \geq 0 \quad \mathbb{P}_x - \text{a.s.}$$

for n large enough. This ensures

$$\limsup_{n\to\infty} n^{\frac{p-1}{p}} l \left(-\frac{\hat{l}}{l} + \frac{1}{n}\xi_{\tau(n)-1}\right)$$
$$= \limsup_{n\to\infty} n^{\frac{p-1}{p}} l \left(-\frac{\hat{l}}{l} + \frac{\tau(n)-1}{n}\frac{\xi_{\tau(n)-1}}{\tau(n)-1}\right) \leq 0 \quad \mathbb{P}_x - \text{a.s.}$$

for all $x \in \mathbb{X}$.

In conclusion, the assertion

$$\limsup_{n\to\infty} n^{\frac{p-1}{p}} \left(\frac{1}{n}\log d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) - \hat{l}\right) \leq 0 \quad \mathbb{P}_x - \text{a.s.} \tag{3.22}$$

holds true for all $x \in \mathbb{X}$ and $\hat{l} \in (l^*, 0)$. Since $n^{(p-1)/p} \to 0$ if $p \in (0,1)$, $n^{(p-1)/p} \equiv 1$ if $p = 1$, we may obviously replace $\hat{l}$ by l^* in (3.22).

(c) In part (b) we have shown that for $p = 1$

$$\limsup_{n\to\infty} \frac{1}{n}\log d(\hat{M}_\infty, \hat{M}_n) - l^* \leq 0 \quad \mathbb{P}_x\text{−a.s.}$$

for all $x \in \mathbb{X}$. Then we can find a suitable sequence $(R_n)_{n\geq 0}$ which satisfies

$$d(\hat{M}_\infty, \hat{M}_n) \leq (\exp(l^*)R_n)^n \quad \text{a.s.}$$

for all $n \geq 0$ with $\lim_{n\to\infty} R_n = 1$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{X}$. Hence we have for all $l \in (l^*, 0)$ and $x \in \mathbb{X}$

$$\exp(-nl)d(\hat{M}_\infty, \hat{M}_n) \leq (\exp(l^* - l)R_n)^n \to 0 \quad \mathbb{P}_x\text{−a.s.}$$

It remains to show that the assertion holds in the case $p > 1$. To this end choose any $\varepsilon > 0$ and $l \in (l^*, 0)$. By (3.6), the inequality

$$n^p \mathbb{P}_x(d(\hat{M}_\infty, \hat{M}_n) > \varepsilon \exp(nl)) \leq K(l, \varepsilon)$$

holds true for n large enough, where $K(l,\varepsilon)$ is a positive constant depending on l and ε. W.l.o.g. we may assume that this inequatlity is also true for every $n \geq 1$. Then we deduce

$$\sum_{n\geq 1} \mathbb{P}_x(d(\hat{M}_\infty, \hat{M}_n) > \varepsilon \exp(nl)) \leq K(l,\varepsilon) \sum_{n\geq 1} n^{-p} < \infty.$$

By the Borel-Cantelli lemma we have $\mathbb{P}_x(d(\hat{M}_\infty, \hat{M}_n) > \varepsilon \exp(nl) \quad i.o.) = 0$. Since ε was arbitrarily chosen the assertion is proved.

(d) By Lemma 2.10, we also have

$$\lim_{n\to\infty} (n+1)^p \mathbb{P}(l(F_{1:n}) > (n+1)^{-p}) = 0.$$

Now let $l \in (l^*, 0)$, $\rho \in (0,1)$ and finally A_x so large that, on the one side,

$$\mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) \leq A_x(n+1)^{-p}/2$$

for all $n \geq 0$, which is possible by Lemma 2.11, and

$$\begin{aligned}
&\mathbb{P}(\exp(l(1-\rho)n)(C_{\tau(n)} + D_{\tau(n)}) + l(F_{1:n})d(x,x_0) > A_x(n+1)^{-p}) \\
&\quad \leq \mathbb{P}(\log_*(C_{\tau(n)} + D_{\tau(n)}) > (-l)(1-\rho)n + \log(A_x/2) - p\log(n+1)) \\
&\quad + \mathbb{P}(l(F_{1:n})d(x,x_0) > A_x(n+1)^{-p}/2) \\
&\quad \leq A(n+1)^{-p}/2
\end{aligned}$$

for all $n \geq 1$ and some $A > 0$, where we have used Markov's inequality and the uniform integrability of $\{\log_*(C_{\tau(n)} + D_{\tau(n)}), n \geq 1\}$. Using a similar inequality as in part (a) we conclude

$$\begin{aligned}
&\mathbb{P}(d(\hat{M}^{x_0}_\infty, \hat{M}^x_n) > A_x(n+1)^{-p}) \\
&\quad \leq \mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) + \mathbb{P}(l(F_{1:n})d(x,x_0) > A_x(n+1)^{-p}/2) \\
&\quad + \mathbb{P}(\log_*(C_{\tau(n)} + D_{\tau(n)}) > (-l)(1-\rho)n + \log(A_x/2) - p\log(n+1)) \\
&\quad \leq A_x(n+1)^{-p}/2 + A(n+1)^{-p}/2 \\
&\quad \leq A_x(n+1)^{-p}
\end{aligned}$$

for all $n \geq 0$. The assertion now follows by Lemma $B.3$.

(e) By Corollary $C.2$(ii), we clearly have

$$\int_X \log^p(1+d(x,x_0))\pi(dx) = \int_0^\infty pt^{p-1}\pi(x : \log(1+d(x,x_0)) > t)dt.$$

Using inequality (2.18) in combination with $\tau(0) = C_0 = 0$, we obtain $d(\hat{M}_\infty^{x_0}, x_0) \le D_0$ and further calculations give

$$\begin{aligned}\int_0^\infty & pt^{p-1}\pi(x : \log_* d(x,x_0) > t)dt \\ &= \int_0^\infty pt^{p-1}\mathbb{P}(\log_* d(\hat{M}_\infty^{x_0}, x_0) > t)dt \\ &\le \int_0^\infty pt^{p-1}\mathbb{P}(\log_* D_0 > t)dt.\end{aligned}$$

The last expression is finite since $\mathbb{E}\log_*^p D_0 < \infty$ by (2.28). □

3.3.2 Proof of Theorem 3.5

Proof of Theorem 3.5. (a) Choose $\eta > 0$ such that Lemma 2.12 holds true. As in the proof of Theorem 3.2 part (a) we once more use inequality (3.17). In detail, let $\hat{l} \in (l^*, 0)$. Then there exists l with $l^* < l < \hat{l}$. Hence $\hat{l} = l + \varepsilon$ for some $\varepsilon > 0$. We further have $\exp(nl)d(x,x_0) \ge e^{n\hat{l}}/3$ iff $n \le \frac{\log 3d(x,x_0)}{\varepsilon} \stackrel{\text{def}}{=} n_0$. Choose $0 < \rho < -\frac{\varepsilon}{l}$ which implies $\widetilde{\eta} \stackrel{\text{def}}{=} l + \varepsilon - (1-\rho)l > 0$.

By Markov's inequality, we obtain

$$\begin{aligned}\frac{1}{\alpha^n}\mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} > \exp(\hat{l}n)/3) &= \frac{1}{\alpha^n}\mathbb{P}(3C_{\tau(n)} > \exp(n\widetilde{\eta})) \\ &\le \frac{1}{\alpha^n\exp(n\widetilde{\eta}\eta)} \sup_{k\ge 0}\mathbb{E}3^\eta C_{\tau(k)}^\eta\end{aligned}$$

for all $n \ge 0$. Since $\{C_{\tau(n)}^\eta, n \ge 1\}$ is uniformly integrable,

$$\alpha^{-n}\mathbb{P}(\exp(l(1-\rho)n)C_{\tau(n)} > \exp(\hat{l}n)/3) \to 0$$

as $n \to \infty$ for all $\alpha \in (\exp(-\widetilde{\eta}\eta), 1)$. Along the same path we show

$$\alpha^{-n}\mathbb{P}(\exp(l(1-\rho)n)D_{\tau(n)} > \exp(\hat{l}n)/3) \to 0$$

as $n \to \infty$ for all $\alpha \in (\exp(-\widetilde{\eta}\eta), 1)$. By Lemmata 2.14, 2.13 and the definition of n_0 the assertion follows.

(b) First choose $\eta \in (0,1)$ such that Lemma 2.12 holds true and $m_q = \mathbb{E}L_1^q < 1$ for $q \in (0,\eta]$, see also the proof of Lemma 2.13. We further have $1+d(x,x_0) \geq 1$ and $d(x,x_0)^q \leq (1+d(x,x_0))^q$. By inequality (2.19), we then obtain for all $l \in (l^*,0)$, $n \geq 1$ and $q \leq \eta$

$$(1+d(x,x_0))^{-q} d(\hat{M}_\infty^{x_0},\hat{M}_n^x)^q \leq \exp(ql\xi_{\tau(n)-1})(C_{\tau(n)}+D_{\tau(n)})^q + l(F_{1:n})^q \quad \text{a.s.}$$

Note that the right side does not depend on x, converges in probability to 0 and is uniformly integrable by Lemma 2.12. This implies the asserted result for $q=\eta$. If $q \in (0,1)$, we deduce by Hölder's inequality

$$\begin{aligned}&\mathbb{E}\exp(ql\xi_{\tau(n)-1})(C_{\tau(n)}+D_{\tau(n)})^q \\ &\qquad \leq \left(\mathbb{E}\exp(\eta ql\xi_{\tau(n)-1}/(\eta-q))\right)^{(\eta-q)/\eta}\left(\mathbb{E}(C_{\tau(n)}+D_{\tau(n)})^\eta\right)^{q/\eta}\end{aligned}$$

for all $n \geq 1$. By the uniform integrability and $\mathbb{E}l(F_{1:n})^q \leq \mathbb{E}L_{1:n} = m_q^n$ it remains to show that

$$\lim_{n\to\infty} \alpha_q^{-n}\left(\mathbb{E}\exp(\eta ql\xi_{\tau(n)-1}/(\eta-q))\right)^{(\eta-q)/\eta} = 0.$$

for some $\alpha_q \in (m_q,1)$. To this end first note that

$$\begin{aligned}&\left(\mathbb{E}\exp(\eta ql\xi_{\tau(n)-1}/(\eta-q))\right)^{(\eta-q)/\eta} \\ &\leq \left(\mathbb{E}\exp(\eta ql\xi_{\tau(n)-1}/(\eta-q))\mathbf{1}_{\{\xi_{\tau(n)-1}\leq(1-\rho)n\}}\right)^{(\eta-q)/\eta} + \exp(l(1-\rho)n)\end{aligned}$$

holds true for all $n \geq 1$ and $\rho \in (0,1)$ since $t \mapsto t^{(\eta-q)/\eta}$ is subadditive. Moreover, we obtain for the first summand of the last expression

$$\begin{aligned}&\left(\mathbb{E}\exp(\eta ql\xi_{\tau(n)-1}/(\eta-q))\mathbf{1}_{\{\xi_{\tau(n)-1}\leq(1-\rho)n\}}\right)^{(\eta-q)/\eta} \\ &\leq \left(\mathbb{E}\exp(2\eta ql\xi_{\tau(n)-1}/(\eta-q))\right)^{(\eta-q)/(2\eta)}\mathbb{P}(\xi_{\tau(n)-1}\leq(1-\rho)n)^{(\eta-q)/(2\eta)} \\ &\leq \mathbb{P}(\xi_{\tau(n)-1}\leq(1-\rho)n)^{(\eta-q)/(2\eta)}\end{aligned}$$

for all $n \geq 1$ and $\rho \in (0,1)$ using Hölder's inequality and the fact that $\exp(2\eta ql\xi_{\tau(n)-1}/(\eta-q)) \leq 1$. In conclusion,

$$\lim_{n\to\infty}\alpha_q^{-n}\mathbb{P}(\xi_{\tau(n)-1}\leq(1-\rho)n)^{(\eta-q)/(2\eta)} + \alpha_q^{-n}\exp(l(1-\rho)n) = 0$$

follows for some $\alpha_q \in (m_q,1)$ by Lemma 2.14 and part (b) is shown.

(c) Let $l \in (l^*, 0)$, $\rho \in (0,1)$ and $\beta \in (\exp((1-\rho)l), 1)$. Since the family $\{(C_{\tau(n)} + D_{\tau(n)})^\eta, n \geq 0\}$ is uniformly integrable we have $K_\eta = \sup_{n\geq 0} \mathbb{E}(C_{\tau(n)} + D_{\tau(n)})^\eta < \infty$. By Lemmata 2.13 and 2.14

$$\mathbb{P}(l(F_{1:n})d(x, x_0) > \varepsilon\alpha^n/2) \leq 2\varepsilon^{-1}d(x, x_0)\alpha^n$$

and

$$\mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) \leq c\alpha^n$$

holds true for all $n \geq 1$, $\varepsilon > 0$ and some suitable $c > 0$ and $\alpha \in (0,1)$. Note that c and α both depend on l chosen in the beginning of this proof. Moreover, using Markov's inequality, we obtain

$$\begin{aligned}
\mathbb{P}(\exp(l\xi_{\tau(n)-1})(C_{\tau(n)} + D_{\tau(n)}) > A\beta^n/2) \\
&\leq \mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) \\
&+ \mathbb{P}(\exp((1-\rho)nl)(C_{\tau(n)} + D_{\tau(n)}) > A\beta^n/2) \\
&\leq c\alpha^n + \frac{2^\eta \exp((1-\rho)\eta nl)K_\eta}{A\beta^{\eta n}} \\
&\leq Ar^n
\end{aligned}$$

for all $n \geq 0$, all $A > 0$ sufficiently large and with $r \stackrel{\text{def}}{=} \max\{\alpha, \beta, \frac{\exp((1-\rho)\eta l)}{\beta^\eta}\}$. Hence $r \in (0,1)$ and setting $A_x = \max\{A+1, 2d(x, x_0)\}$ we conclude

$$\begin{aligned}
\mathbb{P}(d(\hat{M}_\infty^{x_0}, \hat{M}_n^x) > A_x r^n) &\leq \mathbb{P}(l(F_{1:n})d(x, x_0) > A_x r^n/2) \\
&+ \mathbb{P}(\xi_{\tau(n)-1} \leq (1-\rho)n) \\
&+ \mathbb{P}(\exp((1-\rho)nl)(C_{\tau(n)} + D_{\tau(n)}) > A_x r^n/2) \\
&\leq r^n + Ar^n \leq A_x r^n
\end{aligned}$$

for all $n \geq 0$ and $x \in \mathbb{X}$. The assertion now follows by Lemma B.3.

(d) Suppose $l \in (l^*, 0)$ and $\eta > 0$ such that Lemma 2.12 holds true. A similar estimation as in Theorem 3.2 part (e) leads to

$$\int_0^\infty \eta t^{\eta-1}\pi(x : d(x, x_0) > t)dt \leq \int_0^\infty \eta t^{\eta-1}\mathbb{P}(D_0 > t)dt.$$

Since $\mathbb{E}D_0^\eta < \infty$ by Lemma 2.12, the proof is completed. □

Appendix A

Renewal Theory

The renewal theory turns out to be a very powerful tool to analyse IFS of i.i.d. random Lipschitz maps. For this reason we state some important results which will be used in our thesis. Essentially, this description is based on the presentation of renewal theory in [ASP] and [Gut]. For further details and proofs consult these references.

Throughout this chapter let $(\Omega, \mathfrak{A}, P)$ be a probability space. Consider the sequence $(X_n)_{n\geq 1}$ of i.i.d. random variables, also called *increments*, and the independent starting point S_0 which is called *delay*. We call $(S_n)_{n\geq 0}$ defined by $S_n \stackrel{\text{def}}{=} S_0 + \sum_{i=1}^{n} X_n$ a *random walk*. Further, let Q be the distribution of X_1 and Q_0 be the one of S_0. If $S_0 = 0$ a.s., we say $(S_n)_{n\geq 0}$ is a *standard random walk*. Moreover, if $S_0, X_1, X_2, ...$ are all a.s. nonnegative and $\mu = EX_1 > 0$, the random walk $(S_n)_{n\geq 0}$ is called *renewal process with positive drift* μ, respectively, *standard renewal process* if $S_0 = 0$.

We distinguish two kinds of random walks according to the lattice on which they are concentrated. For this purpose we define $\mathbb{G}_0 \stackrel{\text{def}}{=} \mathbb{R}$ which is an additive group and the closed subgroups $\mathbb{G}_\infty \stackrel{\text{def}}{=} \{0\}$ and $\mathbb{G}_d \stackrel{\text{def}}{=} d\mathbb{Z}$.

Definition A.1 ([ASP], Definition 26.2)**.** Let Q be a probability measure on $\mathbb{R}$. Then

$$d(Q) \stackrel{\text{def}}{=} \sup\{d \in [0,\infty] : \ Q(\mathbb{G}_d = 1)\}$$

is called the *span of* Q. Q is called *nonarithmetic* if $d(Q) = 0$ and *arithmetic with span* d, respectively d*-arithmetic*, if $d(Q) = d$ for some $d > 0$. In analogy, the random variable X is *nonarithmetic* / d*-arithmetic* if the induced probability measure P^X is nonarithmetic / d-arithmetic.

If the increments X_n, $n \geq 1$, a.s. take values in a pure subgroup of $\mathbb{R}$, this also holds true for any S_n and we have

Definition A.2 ([ASP], Definition 26.4)**.** A random walk $(S_n)_{n\geq 0}$ is called *nonarithmetic* if X_1 is nonarithmetic and *d-arithmetic* if X_1 is d-arithmetic and $P(S_0 \in \mathbb{G}_d) = 1$.

Furthermore, N is a *stopping time* with respect to an increasing sequence of sub-σ-algebras, $(\mathfrak{F}_n)_{n\geq 1}$, such that X_n is $\mathfrak{F}_n$-measurable and independent of $\mathfrak{F}_{n-1}$ for all n, that is, for every $n \geq 1$, we have

$$\{N = n\} \in \mathfrak{F}_n.$$

More precisely, we are interested in the stopping times when the process achives a temporary maximum or minimum, respectively, and sequences of maxima or minima.

Definition A.3. Let $(S_n)_{n\geq 0}$ be a random walk with i.i.d. increments $(X_n)_{n\geq 1}$ and $S_0 = 0$ a.s.

(a) $\nu(t) = \inf\{n \geq 0 : \ S_n \geq t\}$ is called the *first passage time*.

(b) The the random variables $\sigma_0 \stackrel{\text{def}}{=} 0$,

$$\sigma_n^{>} \stackrel{\text{def}}{=} \inf\{k > \sigma_{n-1} : \sum_{j=\sigma_{n-1}+1}^{k} X_j > 0\},$$
$$\sigma_n^{\geq} \stackrel{\text{def}}{=} \inf\{k > \sigma_{n-1} : \sum_{j=\sigma_{n-1}+1}^{k} X_j \geq 0\},$$

are called *strong ascending*, *weak ascending*, and analogous, $\sigma^{<}$ *strong descending*, $\sigma^{\leq}$ *weak descending ladder epochs* ($\inf\emptyset \stackrel{\text{def}}{=} \infty$). Moreover, the corresponding random variables

$$S_n^{>} \stackrel{\text{def}}{=} S_{\sigma_n^{>}}\mathbf{1}_{\{\sigma_n^{>}<\infty\}}, \quad S_n^{\geq} \stackrel{\text{def}}{=} S_{\sigma_n^{\geq}}\mathbf{1}_{\{\sigma_n^{\geq}<\infty\}},$$
$$S_n^{<} \stackrel{\text{def}}{=} S_{\sigma_n^{<}}\mathbf{1}_{\{\sigma_n^{<}<\infty\}}, \quad S_n^{\leq} \stackrel{\text{def}}{=} S_{\sigma_n^{\leq}}\mathbf{1}_{\{\sigma_n^{\leq}<\infty\}},$$

are called *strong ascending*, *weak ascending*, *strong descending*, *weak descending ladder heights*.

In the case that the increments are all nonnegative a.s., we obviously have $\sigma_n^{\geq} = n$ a.s. By the strong law of large numbers we obtain the following results:

Lemma A.4 ([ASP], Corollary 27.5)**.** *Let $(S_n)_{n\geq 0}$ be a standard random walk with $P(X_1 = 0) < 1$. Then the following assertions are equivalent:*

(a) $(\sigma_n^{\odot}, S_{\sigma_n^{\odot}})_{n\geq 0}$ *is a standard random walk and takes values in* $\mathbb{N}_0 \times \mathbb{R}$ *for* $\odot \in \{>, \geq\}$ $(\{<, \leq\})$.

(b) $\sigma_1^{\odot} < \infty$ *P-a.s. for* $\odot \in \{>, \geq\}$ $(\{<, \leq\})$.

(c) $\limsup_{n\to\infty} S_n = \infty$ *P-a.s.* ($\liminf_{n\to\infty} S_n = -\infty$ *P-a.s.*)

For a standard random walk $(S_n)_{n\geq 0}$ with increments $(X_n)_{n\geq 1}$ we have defined the *drift* by $\mu = EX_1$. In the following we assume that μ exists. Again by the strong law of large numbers the following holds true:

Theorem A.5 ([Gut], Theorem II.8.3)**.**

(a) *If* $0 < \mu \leq \infty$ *the random walk drifts to* $+\infty$*, i.e.* $S_n \to +\infty$ *a.s. as* $n \to \infty$.

(b) *If* $-\infty \leq \mu < 0$ *the random walk drifts to* $-\infty$*, i.e.* $S_n \to -\infty$ *a.s. as* $n \to \infty$.

(c) *If* $\mu = 0$ *and* $P(X_1 = 0) < 1$ *the random walk oscillates between* $-\infty$ *and* $+\infty$.

In the following we will state some moment results which will basically be used for the analysis of the excursions.

Theorem A.6 ([Gut], Theorems I.5.1 and I.5.2)**.** *Given a standard random walk $(S_n)_{n\geq 0}$ with increments $(X_n)_{n\geq 1}$ and a stopping time N, suppose that $E|X_1|^r < \infty$ for some $r \in (0, \infty)$.*

(a) *If* $0 < r \leq 1$,

$$E|S_N|^r \leq E|X_1|^r EN$$

holds true.

(b) *If* $1 \leq r < \infty$*, there exists a numerical constant* B'_r *depending on* r *only such that*

$$E|S_N|^r \leq B'_r E|X_1|^r EN^r.$$

Theorem A.7 ([Gut], Theorem III.3.1)**.** *Given a standard random walk $(S_n)_{n\geq 0}$ with increments $(X_n)_{n\geq 1}$ and positive drift μ. Furthermore, let $\nu(t) = \inf\{n \geq 0 : \ S_n \geq t\}$ and $r \geq 1$. Then the following assertions hold:*

(i) $E(X_1^-)^r < \infty \Leftrightarrow E(\nu(t))^r < \infty.$

(ii) $E(X_1^+)^r < \infty \Leftrightarrow E(S_{\nu(t)})^r < \infty.$

Theorem A.8 ([Gut], Theorem III.3.2)**.** *There exists* $s_0 > 0$ *such that* $Ee^{s\nu(t)} < \infty$ *for* $|s| < s_0$ *iff there exists* $s_1 > 0$ *such that* $Ee^{sX_1^-}$ *for* $|s| < s_1$.

Theorem A.9 ([ASP], Lemma 10.3)**.** *Given a standard random walk* $(S_n)_{n\geq 0}$ *with* $\mu = EX_1 > 0$.

(a) *For each stopping time* τ *with respect to* $(\mathfrak{F}_n)_{n\geq 0}$ Wald's identity

$$ES_\tau = \mu E\tau \tag{A.1}$$

holds true.

(b) *Furthermore, if* $\tau(t) = \inf\{n \geq 0 : \ S_n > t\}$, $t \geq 0$, *then*

$$\frac{\tau(t)}{t} \to \frac{1}{\mu} \ P - \text{f.s.} \quad \textit{and} \quad \frac{E\tau(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty.$$

Given a random walk $(S_n)_{n\geq 0}$, we define the random variable

$$N(B) \stackrel{\text{def}}{=} \sum_{n\geq 0} \mathbf{1}_{\{S_n \in B\}}$$

for all $B \in \mathbb{B}$, where $\mathbb{B}$ denotes the Borel-σ-field on $\mathbb{R}$. $N(B)$ counts the number of renewals in the set B.

Definition A.10. Given a random walk $(S_n)_{n\geq 0}$,

$$\mathbb{B} \ni B \mapsto U(B) \stackrel{\text{def}}{=} EN(B)$$

is called *renewal measure* and

$$U(t) \stackrel{\text{def}}{=} U((-\infty, t]), \quad t \in \mathbb{R}$$

renewal function. The stochastic process $(N_t)_{t\in\mathbb{R}}$ defined by $N_t \stackrel{\text{def}}{=} N((-\infty, t])$ is called *renewal counting process*.

The renewal function $U(B)$ is the expected number of renewals in the set B. Obviously,

$$U(B) = E(N(B)) = E\sum_{n\geq 0} \mathbf{1}_{\{S_n \in B\}} = \sum_{n\geq 0} P(S_n \in B)$$

holds true for each $B \in \mathbb{B}$. Moreover, the convergence results stated below hold.

Theorem A.11 (Blackwell's renewal theorem, see [ASP]). *Given a random walk* $(S_n)_{n\geq 0}$ *with drift* $\mu = EX_1 \in (0,\infty]$ *and span* d, *the following assertions hold:*

$$\lim_{t\to\infty} U([t,t+a]) = \frac{a}{\mu}, \quad \text{if } d = 0,$$

and

$$\lim_{n\to\infty} U(\{n\}) = \frac{1}{\mu}, \quad \text{if } d = 1.$$

Corollary A.12 (Elementary Renewal Theorem, see Corollaries 29.3 and 30.2 in [ASP]). *Given a random walk* $(S_n)_{n\geq 0}$ *with positive drift,*

$$\lim_{t\to\infty} \frac{U([0,t])}{t} = \frac{1}{\mu} \quad \text{and} \quad \lim_{\mathrm{t}\to\infty} \frac{\mathrm{U}([-\mathrm{t},0])}{\mathrm{t}} = 0$$

hold true. Furthermore, if $(S_n)_{n\geq 0}$ *is a renewal process,*

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{\mu} \quad \text{and} \quad \lim_{\mathrm{t}\to\infty} \frac{\mathrm{N_t}}{\mathrm{t}} = \frac{1}{\mu} \quad a.s.$$

hold true. By convention $\infty^{-1} = 0$.

Appendix B

Prokhorov metric

Two results stated in this thesis concern the distance of the n-step transition kernel $P^n(x, \cdot)$ for $x \in \mathbb{X}$ and π which is the associated distribution of $\hat{M}_\infty$. Doing this we have to introduce the Prokhorov metric. By representing the results below we follow the description of Dudley, see [Dud].

Consider the metric space $(\mathbb{X}, d)$ and let $\varepsilon > 0$. For any $B \in \mathfrak{B}$ let

$$B^\varepsilon \overset{\text{def}}{=} \{y \in \mathbb{X} : \ d(x, y) < \varepsilon \text{ for some } x \in B\}. \tag{B.1}$$

Definition B.1. Given a metric space $(\mathbb{X}, d)$, let λ_1 and λ_2 be two laws on $\mathbb{X}$. Then

$$\rho(\lambda_1, \lambda_2) \overset{\text{def}}{=} \inf\{\varepsilon > 0 : \ \lambda_1(B) \leq \lambda_2(B^\varepsilon) + \varepsilon \text{ for all } B \in \mathfrak{B}\} \tag{B.2}$$

is called the *Prokhorov metric.*

Theorem B.2 ([Dud], Theorem 11.3.1)**.** *For any metric space* $(\mathbb{X}, d)$*,* ρ *is a metric on the set of all laws on* $\mathbb{X}$*.*

To prove the results mentioned above the following lemma will be helpful.

Lemma B.3. *Let* $(\mathbb{X}, d)$ *be a metric space and* X_1, X_2 *be two* $\mathbb{X}$*-valued random variables with distributions* λ_1 *and* λ_2*. Then* $\mathbb{P}(d(X_1, X_2) \geq \varepsilon) < \varepsilon$ *implies* $\rho(\lambda_1, \lambda_2) \leq \varepsilon$*.*

Proof. Assume that B is an arbitrary set in $\mathfrak{B}$. If $X_1 \in B$, then either $X_2 \in B^\varepsilon$ or $d(X_1, X_2) \geq \varepsilon$ and we obtain the following inequality:

$$\begin{aligned} \lambda_1(B) = \mathbb{P}(X_1 \in B) &\leq \mathbb{P}(\{X_2 \in B^\varepsilon\} \ + \ \{d(X_1, X_2) \geq \varepsilon\}) \\ &\leq \mathbb{P}(X_2 \in B^\varepsilon) + \varepsilon = \lambda_2(B^\varepsilon) + \varepsilon. \end{aligned}$$

Since $\rho(\lambda_1, \lambda_2) \leq \varepsilon$ by definition of ρ the assertion is proved. $\square$

Appendix C

Auxiliary results

In this chapter we provide some helpful results to hold the proofs concise as far as possible. In particular, we give formulas for calculating expected values.

Proposition C.1 ([AWT], Proposition A.1)**.** *Let X be a nonnegative random variable on a probability space $(\Omega, \mathfrak{A}, P)$ and $\phi : [0,\infty) \to [0,\infty)$ be a continuous, monotone increasing and on $(0,\infty)$ continuously differentiable function with $\phi(0) = 0$. Then the indentity*

$$\int_A \phi(X)dP = \int_0^\infty \phi'(t)P(A \cap \{X > t\})dt$$

holds true for all $A \in \mathfrak{A}$.

Corollary C.2 ([AWT], Corollary A.2)**.** *Let X be a nonnegative random variable. Then the following assertions hold true:*

(i) $EX = \int_0^\infty P(X > t)dt$,

(ii) $EX^p = \int_0^\infty pt^{p-1}P(X > t)dt \quad (p > 0)$,

(iii) $E((X-a)^+)^p = \int_a^\infty p(t-a)^{p-1}P(X > t)dt \quad (a, p > 0)$.

Furthermore, if X is additionally a.s. integer, we also have

(iv) $EX = \sum_{n\geq 0} P(X > n)$,

(v) $EX^2 = \sum_{n\geq 0}(2n+1)P(X > n)$,

(vi) $E(e^{aX} - 1) = (e^a - 1)\sum_{n\geq 0} e^{an}P(X > n) \quad (a \in \mathbb{R})$,

(vii)

$$\sum_{n\geq 1} p(n-1)^{p-1}P(X>n) \leq EX^p \leq \sum_{n\geq 0} p(n+1)^{p-1}P(X>n) \quad (p\geq 1)$$

Corollary C.3. *Given a nonnegative random variable X and $p>0$, the condition $\mathbb{E}X^{p+1}<\infty$ implies*

$$\sum_{j\geq 1}\mathbb{E}((X-cj)^+)^p<\infty$$

for all $c>0$.

Proof. Let $c>0$. Using (ii) and (iii) from the previous corollary and Fubini's theorem, we obtain

$$\begin{aligned}
\sum_{j\geq 1}\mathbb{E}((X-cj)^+)^p &= \sum_{j\geq 1}\int_{cj}^{\infty} p(t-cj)^{p-1}\mathbb{P}(X>t)dt \\
&= \sum_{j\geq 1}\int_{cj}^{\infty}\left[\frac{d}{dt}(t-cj)^p\right]\mathbb{P}(X>t)dt \\
&= \int_{c}^{\infty}\sum_{j\in\mathbb{N},\ j\leq t/c}\left[\frac{d}{dt}(t-cj)^p\right]\mathbb{P}(X>t)dt \\
&= \int_{c}^{\infty}\left[\frac{d}{dt}\sum_{j\in\mathbb{N},\ j\leq t/c}(t-cj)^p\right]\mathbb{P}(X>t)dt \\
&\leq \int_{c}^{\infty}\left[\frac{d}{dt}\frac{t}{c}t^p\right]\mathbb{P}(X>t)dt \\
&\leq C\int_{0}^{\infty}(p+1)t^p\mathbb{P}(X>t)dt \\
&= C\mathbb{E}X^{p+1}
\end{aligned}$$

where C is some positive constant. By assumption the last expression is finite and the assertion is proved. □

Lemma C.4. *Let $p>0$ and $(a_n)_{n\geq 1}$ be a nonnegative, monotone decreasing sequence satisfying $\sum_{n\geq 1} n^{p-1}a_n<\infty$. Then*

$$n^p a_n \to 0 \quad \textit{as} \quad n\to\infty.$$

Proof. Write $a_n = \sum_{k\geq n}(a_k - a_{k+1})$ which is possible, because all differences under the sum are nonnegative. Now,

$$\begin{aligned}\sum_{n\geq 1} n^{p-1}a_n &= \sum_{n\geq 1} n^{p-1}\sum_{k\geq n}(a_k - a_{k+1}) = \sum_{k\geq 1}(a_k - a_{k+1})\sum_{n=1}^{k} n^{p-1} \\ &\geq C\sum_{k\geq 1} k^p(a_k - a_{k+1})\end{aligned}$$

for some suitable constant $C > 0$. Since $\sum_{n\geq 1} n^{p-1}a_n < \infty$, we also have $\sum_{k\geq 1} k^p(a_k - a_{k+1}) < \infty$ and conclude

$$n^p a_n = \sum_{k\geq n} n^p(a_k - a_{k+1}) \leq \sum_{k\geq n} k^p(a_k - a_{k+1}) \to 0$$

as $n \to \infty$. □

Bibliography

[AF] Alsmeyer, G. and Fuh, C.D. (2001). *Limit theorems for iterated random functions by regenerative methods*. Stoch. Proc. Appl. 96, pp. 123-142.

[ASP] Alsmeyer, G. (2002). *Stochastische Prozesse*. Skripten zur Mathematischen Statistik Nr. 33, 2. erweiterte Auflage.

[AWT] Alsmeyer, G. (2003). *Wahrscheinlichkeitstheorie*. Skripten zur Mathematischen Statistik Nr. 30, 3. Auflage.

[Bre] Breiman, L. (1993). *Probability*. Addison Wesley series in statistics, Reading, Mass. 2nd edition.

[CL] Chow, Y.S. and Lai, T.L. (1975). *Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess ob boundary crossings*. Trans. Amer. Math. Soc. 208, pp. 51 - 57.

[DF] Diaconis, P. and Freedman, D. (1999). *Iterated random functions*. SIAM Review Vol. 41, pp. 45-76.

[Dud] Dudley, R.M. (1989) *Real analysis and probability*. Wadsworth and Brooks/Cole mathematic series, Pacific Grove, California.

[Dur] Durrett, R. (1991). *Probability, theory and examples*. Wadsworth, Inc., Belmont, Calif.

[Elt] Elton, J.H. (1990). *A mulitplicative ergodic theorem for Lipschitz maps*. Stoch. Proc. Appl. 34, pp 39-47.

[Gut] Gut, A. (1988). *Stopped Random Walks. Limit Theorems and Applications*. Springer, New York.

[Sch] Schmitz, N. (1996). *Vorlesungen über Wahrscheinlichkeitstheorie*. Teubener Studienbücher, Stuttgart.

Ich versichere hiermit, die vorliegende Arbeit selbständig verfaßt und dabei keine anderen Hilfsmittel als die angegebenen verwendet zu haben.

Münster, den 10. April 2006