A note on hitting times for simple random walk on rooted, subcritical Galton-Watson trees

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Abstract

Consider a simple random walk on the set of vertices of a subcritical rooted Galton-Watson tree, denoted by $T$. We estimate the order of magnitude of the averaged expected number of steps this simple random walk uses for going to any vertex in a typical $T$ starting at its root, when the expected number of vertices is parametrized by a sequence $(a_n)_{n \geq 1}$ of natural numbers such that $a_n \to \infty$ as $n \to \infty$. Under this parametrization, the order of magnitude found is $\Theta(a_n^3)$.

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1 Introduction

Random walks have been used since a couple of years to investigate certain properties of finite or infinite graphs e.g. [DS84, Lov93, Woe00], partially as part of a larger program for developing a theory of probability on finite or infinite graphs which accounts for their geometrical structure, e.g. [Gri10, AF00, LP09, LPW09], partially as an independent and exciting research area with its own rights. Meanwhile, the study of random graph models has received attention not only within the probability community e.g. [CL06, Dur06, Bol01, JLR00, Kol99, VDH09], but also within the physics, biology, engineering, computer sciences and social sciences communities. In the present note we give a small contribution to this programme.

We will focus on the situation where also the graph is random.

Let $T = (V, E)$ be a rooted subcritical Galton-Watson tree with offspring distribution $\xi$ such that $E\xi < 1$ and $(X_m)_{m \in \mathbb{N}}$ a simple random walk on $T$. Let $o$ denote the root of $T$. We would like to estimate

$$\mathbb{E}_{GW} \left[ \sum_{i \in T} h_{oi} \right]$$

where $h_{ij}$ denotes the hitting time for $(X_m)_{m \in \mathbb{N}}$ between $i$ and $j$ (the expected number of steps $(X_m)_{m \in \mathbb{N}}$ takes before visiting $j$, starting from $i$, see (2.2) in our section 2) and $\mathbb{E}_{GW}$ denotes the expectation on the space of all rooted subcritical
Galton-Watson trees. In a sense, we just want to study $\sum_{i \in T} h_{oi}$ for a typical rooted Galton-Watson tree. Note that $h_{oi}$ is a random variable on this space of trees and is itself an expectation with respect to the law of $(X_m)_{m \in \mathbb{N}}$, therefore $(X_m)_{m \in \mathbb{N}}$ walking on such a random tree could also be seeing as a random walk on a random environment [Ofe03].

Taking $T$ as a rooted subcritical Galton-Watson tree is motivated by the following: In [LT14], we study $E_{GW}[\sum_{j \in \mathcal{C}_1} h_{ij}]$ on the largest connected component $\mathcal{C}_1$ of a critical and supercritical Erdős-Rényi random graph, for given $i \in \mathcal{C}_1$. Similarly, in [LT13] we study a very related quantity on the largest connected component of a dense Erdős-Rényi random graph, though by very different techniques. The situation in [LT13] is intrinsically different, roughly speaking because with high probability $\mathcal{C}_1$ has a lot of loops. In the critical regime, in contrast, an Erdős-Rényi at least locally looks like a tree, typically the next cycle (if there is any) is "far away" (details in [VDH09, Dur06, JLR00], among others).

The rest of this little note is organized as follows: In a Section 2 we formulate our main result, while its proof is given in Section 3. Though the proof is fairly simple, we believe the technique is new and allows applications for a wide range of offspring distributions in different random graph models where a subcritical Galton-Watson tree plays a role.

## 2 Notation and main results

Let $\xi$ be a random variable with values in the positive integers, mean $0 < \mu := E\xi < 1$ and variance $0 < \sigma^2 := \text{Var} \xi < \infty$. Let $T$ be the random tree generated by a Galton-Watson process starting with one single individual (the root) and branching according to the distribution of $\xi$. We denote the law of $T$ by $P_{GW}$ and its corresponding expectation by $E_{GW}$. Then the branching process a.s. dies out (e.g. [VDH09] section 3.6), thus the total population $N := 1 + \sum_{k \geq 1} Z_k$ is a.s. finite, where $Z_k$ denote the (random) number of individuals in generation $k$. Due to the martingale property we have

$$E_{GW}[Z_k] = \mu^k, \quad k \geq 1 \quad \text{and} \quad E_{GW}[N] = 1 + \sum_{k \geq 1} E_{GW}[Z_k] = \frac{1}{1 - \mu}. \quad (2.1)$$

Given $T$ a rooted Galton-Watson tree generated as before, let $(X_m)_{m \in \mathbb{N}}$ be a simple random walk taking values on the set of vertices of $T$. Its law is denote by $P$ and the corresponding expectation by $E$, and for a vertex $i$ in $T$, let $P_i(\cdot) = P(\cdot|X_0 = i)$ and $E_i(\cdot) = E(\cdot|X_0 = i)$. Moreover, for a vertex $i$ of $T$, let $\tau_i$ be the first time $(X_m)_{m \in \mathbb{N}}$ is at $i$, namely

$$\tau_i := \inf\{m > 0 : X_m = i\}.$$  

We denote the root of $T$ by $o$. Let us define the following two random variables:

$$H_o := E_{GW}\left[\sum_{i \in T} E_{o\tau_i}\right] \quad (2.2)$$

$$\hat{H}_o := E_{GW}\left[\frac{1}{E_{GW}[N]} \sum_{i \in T} E_{o\tau_i}\right].$$

We can think about $\hat{H}_o$ as a kind of averaged hitting time of $(X_m)_{m \in \mathbb{N}}$ between the root and any another node of a typical Galton-Watson tree.
Theorem 2.1. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers such that \(a_n^{-1} \to 0\) as \(n \to \infty\). If \(\mu = \mu_n = 1 - a_n^{-1}\) then \(H_\alpha = \Theta(a_n^4)\). In particular, \(\tilde{H}_\alpha = \Theta(a_n^3)\).

In the proof of Theorem 2.1, we will see that \(E_{GW} \left[ \sum_{i \in T} E_\alpha \tau_i \right] \) can be written as an explicit function of \(\mu\) - see (3.6) - which we call \(f(\mu)\). We can use that to get a formula for

\[
h_n := E_{GW} \left[ \sum_{i \in T} E_\alpha \tau_i \mid N = n \right] = E_{GW} \left[ \sum_{i=1}^{n} E_\alpha \tau_i \mid N = n \right].
\]

In this sum the root \(o\) corresponds to the vertex \(i = 1\) and therefore \(E_\alpha \tau_1 = 0\). Note that \(h_n\) does not depend on \(\mu\). This comes from the fact that under \(N = n\) fixed, the distribution of a subcritical Galton-Watson tree \(T\) with \(\mu < 1\) is the same as the distribution of a critical Galton-Watson tree \(T\) with \(\mu = 1\) (see e.g. [VDH09] Chapter 3 Theorem 3.7), so any function of the distribution of \(T\) - for example \(h_n\) - should not depend on the value of \(\mu\) neither. Inspired by the classical technique of \textit{Poissonization} (for example, see chapter E in [Ald89]), we have

\[
E_{GW} \left[ \sum_{i \in T} E_\alpha \tau_i \right] = \sum_{n \geq 1} E_{GW} \left[ \sum_{i \in T} E_\alpha \tau_i \mid N = n \right] \mathbb{P}_{GW}(N = n),
\]

therefore

\[
f(\mu) = \sum_{n \geq 1} h_n \mathbb{P}_{GW}(N = n). \tag{2.3}
\]

For example, we can consider the case \(P(\xi = 2) = p \in (0, 1/2)\). This corresponds to the so called (fully) binary tree (see [VDH09, Jan06]). Given the random tree generated in this way satisfies \(N = n\) for \(n = 1 \mod 2\), then in has \(n_i := (n - 1)/2\) internal nodes (the ones with two children) and \(n - n_i\) external nodes (the others). The total number of random binary trees having \(m \geq 1\) internal nodes is \(b_m = \binom{2m}{m}/(m + 1)\) the so called Catalan number [Jan06]. Therefore,

\[
\mathbb{P}_{GW}(N = n) = b_n, \quad p^n (1 - p)^{n-n_i} = \frac{2}{n+1} \left( \frac{n-1}{(n-1)/2} \right) p^{(n-1)/2} (1 - p)^{(n+1)/2}
\]

for every \(n = 1 \mod 2\). With \(E\xi = 2p\) we can combine all together in (2.3) to get

\[
\frac{f(2p)}{1 - p} = \sum_{n=1}^{\infty} h_n \frac{1}{2^{n-1}} \left( \frac{2^n}{2^{n-1}} \right) p^n (1 - p)^{2n-1}.
\]

Another example is to consider \(\xi \sim \text{Poi}(\lambda)\) with \(\lambda \in (0, 1)\). The next result is well known even in slightly more general situations, see e.g. [Dwa69], Lemma 2.1.3 in [Kol86] or [Pit98]:

Theorem 2.2. Let \(d\) be the span of \(\xi\) and let \((\xi_i)_{i \geq 1}\) be i.i.d. copies of \(\xi\). Then, for \(n = 1 \mod d\)

\[
\mathbb{P}_{GW}(N = n) = \frac{1}{n} P(\xi_1 + \cdots + \xi_n = n - 1).
\]

With this result, one can compute (posed as exercise 3.28 in [VDH09]):

\[
\mathbb{P}_{GW}(N = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}
\]

for every \(n \geq 1\). Therefore, once more combining all in (2.3) we get

\[
\lambda f(\lambda) = \sum_{n=1}^{\infty} h_n (\lambda e^{-\lambda})^n \frac{n^{n-1}}{n!}.
\]
3 Auxiliar results and proofs

Chen and Zhang (c.f. [HF04] Theorem 2.4) prove the following result, which is the base of our computations:

**Theorem 3.1.** For any \( i \) vertex of \( T \), let \( \Gamma_{oi} \) be the only path on \( T \) connecting \( o \) with \( i \). Denote by \( V_{oi} := V(\Gamma_{oi}) \) the set of its vertices and by \( E(\Gamma_{oi}) \) the set of its edges. For any \( v \in V_{oi} \), let \( m_v \) denote the (random) number of vertices of the subtree \( T \setminus E(\Gamma_{oi}) \) which is rooted at \( v \). Then

\[
\mathbb{E}_{o\tau_i} = (d(o,i))^2 + 2 \sum_{v \in V_{oi}} m_v \cdot d(v,i)
\]

(3.1)

where \( d(v,i) \) is the distance on \( T \) between \( v \) and \( i \).

**Proof or Theorem 2.1** By (3.1) we have that

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} \mathbb{E}_{o\tau_i} \right] = \left( \mathbb{E}_{GW} \left[ \sum_{i \in T} (d(o,i))^2 \right] + 2 \mathbb{E}_{GW} \left[ \sum_{i \in T} \sum_{v \in V_{oi}} m_v \cdot d(v,i) \right] \right).
\]

(3.2)

Let us look separately at each term:

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} (d(o,i))^2 \right] = \mathbb{E}_{GW} \left[ \sum_{k \geq 1} k^2 Z_k \right] = \sum_{k \geq 1} k^2 \mu^k = \frac{\mu}{(1-\mu)^3}.
\]

(3.3)

Let us construct a conditioned Galton-Watson tree \( \tilde{T} \) in the following way: Let \( \tilde{\xi} \) be an independent copy of \( \xi \). The construction of \( \tilde{T} \) is made under the event \( \{ \tilde{\xi} > 1 \} \), otherwise \( \tilde{T} \) is empty. We start by one individual, the root of \( \tilde{T} \). Its first generation is generated by \( \tilde{\xi} - 1 \) children. Then, each of these children reproduces in an i.i.d. fashion following the law of \( \xi \). This construction is important to compute the sum over \( v \in V_{oi} \) in

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} \sum_{v \in V_{oi}} m_v \cdot d(v,i) \right].
\]

Indeed, take the path \( \Gamma_{oi} \) from \( o \) to a fixed \( i \in T \). In this path, take a vertex \( v \in V_{oi} \setminus \{o,i\} \). The subtree with root at \( v \) is then a Galton-Watson tree with first generation having distribution \( \tilde{\xi} \) under the event \( \{ \tilde{\xi} > 1 \} \), which corresponds to \( \tilde{T} \). Let \( \tilde{Z}_k \) be the (random) number of children in generation \( k \geq 1 \) in \( \tilde{T} \) and \( \tilde{N} \) the (random) number of individuals in \( \tilde{T} \). Note that \( \{ \tilde{\xi} > 1 \} \) is equivalent to \( \{ \tilde{N} > 1 \} \). Always under \( \{ \xi > 1 \} \), one sees that \( \tilde{Z}_0 = Z_0 = 1 \), \( \tilde{Z}_1 = Z_1 - 1 \), \( \mathcal{L}(\tilde{Z}_k) = \mathcal{L}(Z_k) \) for \( k \geq 2 \) and that

\[
\tilde{N} = 1 + \tilde{Z}_1 + \sum_{k \geq 2} \tilde{Z}_k = Z_1 + \sum_{k \geq 2} \tilde{Z}_k
\]

\[
\Rightarrow \mathbb{E}_{GW}[\tilde{N} - 1 | \tilde{N} > 1] = \sum_{k \geq 1} \mathbb{E}_{GW}[Z_k] - 1 = \sum_{k \geq 0} \mathbb{E}_{GW}[Z_k] - 2 = \frac{1}{1-\mu} - 2 = \frac{2\mu - 1}{1-\mu}.
\]

(3.4)
Therefore,

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} \sum_{v \in V_{oi}} m_v \cdot d(v, i) \right] = \mathbb{E}_{GW} \left[ \sum_{k \geq 1} Z_k \sum_{j=1}^{k} j(\tilde{N} - 1) \bigg| \tilde{N} > 1 \right] \cdot P_{GW}(\tilde{N} > 1) \\
= P(\tilde{\xi} > 1) \cdot \mathbb{E}_{GW}[\tilde{N} - 1 | \tilde{N} > 1] \sum_{k \geq 1} \mathbb{E}_{GW}[Z_k] \frac{k(k + 1)}{2} \\
= P(\tilde{\xi} > 1) \cdot \mathbb{E}_{GW}[\tilde{N} - 1 | \tilde{N} > 1] \left( \sum_{k \geq 1} k^2 \mu^k + \sum_{k \geq 1} k \mu^k \right) \\
(\text{by definition of } \tilde{\xi}) = P(\xi > 1) \cdot \frac{2\mu - 1}{1 - \mu} \cdot \frac{\mu(2 - \mu)}{2(1 - \mu)^3} \\
= P(\xi > 1) \cdot \frac{\mu(2 - \mu)(2\mu - 1)}{2(1 - \mu)^4} \tag{3.5}
\]

where the last inequality comes from using (3.4). Therefore, putting together (2.1), (3.3) and (3.5) in (3.2) we get:

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} \mathbb{E}_{o_{\tau_i}} \right] = \frac{\mu}{(1 - \mu)^3} + P(\xi > 1) \cdot \frac{\mu(2 - \mu)(2\mu - 1)}{(1 - \mu)^4} \tag{3.6}
\]

We are going to choose \( \mu \) such that, for a given \( n \geq 1 \), we have \( \mathbb{E}_{GW}[N] = a_n \). This means, we consider trees with expected total progeny of \( a_n \). From (2.1) \( \mu = 1 - a_n^{-1} \), so finally in (3.6) with the additional definition of \( q := P(\xi > 1) \) we have:

\[
\mathbb{E}_{GW} \left[ \sum_{i \in T} \mathbb{E}_{o_{\tau_i}} \right] = \frac{1 - a_n^{-1}}{(a_n^{-1})^3} + q \frac{(1 - a_n^{-1})(1 + a_n^{-1})(1 - 2a_n^{-1})}{(a_n^{-1})^4} \\
= qa_n^4 + (1 - 2q)a_n^3 - (1 + q)a_n^2 + 2qa_n. \tag{3.7}
\]

which finishes the proof. \( \blacksquare \)

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**References**


