Von Neumann and higher set theory

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Exercise

Take a G_δ set $A \subseteq \mathbb{R}^3$. Project A to xy-plane and let B denote the complement. Is the projection of B to x-axis Lebesgue-measurable?

Projective ses

They are formed by closing the family of Borel subsets of $\mathbb{R}^n$ for $n \in \mathbb{N}$ under projections and complements.

> *One does not know and one will never know of the family of projective sets, although it has cardinality $2^{\aleph_0}$ and consists of effective sets, whether every member has cardinality $2^{\aleph_0}$ if uncountable, has the Baire property, or is even Lebesgue measurable.*

N. Luzin, 1925

1. Borel sets are measurable (Lebesgue).
2. Continuous images of sets in (1) are measurable (Suslin).
3. Complements of sets in (2) are measurable (immediate).
4. Are continuous images of sets in (3) measurable?

Formalizing metamathematics: First-order logic

Given a language L recursively define L-formulas.

Then define proofs as certain finite sequences of L-formulas.

Theory is a set of axioms.

$T \vdash \varphi$: "there is a proof of formula φ from theory T"

T is *inconsistent* if $T \vdash \varphi$ for all formulas φ.

Otherwise T is *consistent* and we write $\mathrm{Con}(T)$.

ZFC stands for Zermelo–Fraenkel axioms for set theory, with the Axiom of Choice.

One formal interpretation of Luzin's quote

ZFC$\nvdash$ "all projective sets are Lebesgue measurable"
and
ZFC$\nvdash$ "some projective sets are not Lebesgue measurable"

Von Neumann's cumulative hierarchy

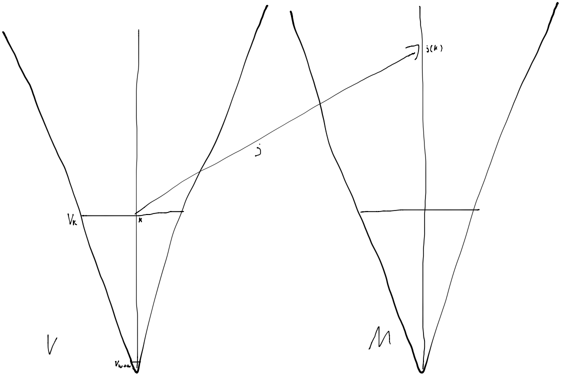
The canonical 'model' for ZFC is constructed by transfinite recursion on all ordinals α.

$$
\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\
V_\beta &= \bigcup_{\alpha<\beta} V_\alpha, \text{ if } \beta \text{ is a limit ordinal} \\
V &= \bigcup_{\alpha \text{ is an ordinal}} V_\alpha
\end{aligned}
$$

Every mathematical object has an isomorphic copy in V (ω denotes the least limit ordinal):

V_ω	finite mathematics (number theory, combinatorics)
$V_{\omega+1}$	countable mathematics (counable groups)
$V_{\omega+2}$	calculus
$V_{\omega+\omega}$	all of the non-set-theoretic mathematics.

Measurable cardinals



Gödel's Incompleteness theorem

Theorem (Gödel, 1930)

Assume T is a consistent, recursively axiomatizable theory that includes a sufficiently large fragment of the theory of $(\mathbb{N}, +, \cdot)$. Then there exists a statement φ in the language of T such that

$$T \nvdash \varphi \qquad \text{and} \qquad T \nvdash \neg\varphi.$$

Back to Luzin

One does not know and one will never know of the family of projective sets, although it has cardinality $2^{\aleph_0}$ and consists of effective sets, whether every member has cardinality $2^{\aleph_0}$ if uncountable, has the Baire property, or is even Lebesgue measurable.

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Answering Luzin's question...

Theorem (Gödel, 1939)

Con(ZFC) implies Con(ZFC + "not all projective sets are Lebesgue measurable").

Theorem (Solovay, 1967)

Con(ZFC + there exists an inaccessible cardinal) implies Con(ZFC + all projective sets have all regularity properties from Luzin's question).

…or showing it has no answer?

Theorem (Shelah, 1984)

Con(ZFC + all projective sets are Lebesgue measurable) implies Con(ZFC + there exists an inaccessible cardinal). In particular, Con(ZFC) does not imply Con(ZFC+all projective sets are Lebesgue measurable).

Theorem (Martin–Steel, 1988)

If there exist infinitely many Woodin cardinals then Projective Determinacy holds and therefore all projective sets have all regularity properties from Luzin's question.

Giving a complete answer to Luzin's question required both a method of proving independence results and an assumption that transcends ZFC.

Exercise

Take a G_δ set $A \subseteq \mathbb{R}^3$. Project A to xy-plane and let B denote the complement. Is the projection of B to x-axis Lebesgue-measurable?

Negative answer is relatively consistent with ZFC.
Positive answer is relatively consistent with ZFC, modulo large cardinals.

(The following remark was not present in my original slides.)
All the regularity properties of projective sets mentioned in Luzin's question are captured by a single axiom. It is the axion of *Projective Determinacy*, or PD. It asserts that for every projective subset A of the Cantor set (identified with $\{0, 1\}^{\mathbb{N}}$), every infinite perfect information game with the payoff set A is *determined*. This means that one of the players has a winning strategy for this game. PD follows from the existence of infinitely many Woodin cardinals by a result of Martin and Steel.

There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any established physical theory.

(K. Gödel, 1947.)

Cantor's Continuum Hypothesis

CH: There is a well-ordering of all separable objects of any given type (reals, Borel sets, C*-algebras,...) such that all of its proper initial segments are countable

Theorem (Gödel 1939)

Con(ZFC) implies Con(ZFC+CH)

Theorem (Cohen 1963)

Con(ZFC) implies Con(ZFC+$\neg$CH)

Cohen's *forcing* adds a generic element G to the universe of sets. One can customize properties of G, to some extent.

Real-valued measurable cardinal, RVM

Question (Banach)

Is there a σ-additive extension of the Lebesgue measure to the power-set of $\mathbb{R}$?

Lemma

RVM implies that CH fails.

Two theories T_1 and T_2 are *equiconsistent* if $\mathrm{Con}(T_1)$ implies $\mathrm{Con}(T_2)$ and $\mathrm{Con}(T_2)$ implies $\mathrm{Con}(T_1)$.

Theorem (Solovay, 1969)

The following theories are equiconsistent.

1. *There is a σ-additive extension of the Lebesgue measure to the power-set of $\mathbb{R}$.*
2. *There exists a measurable cardinal.*

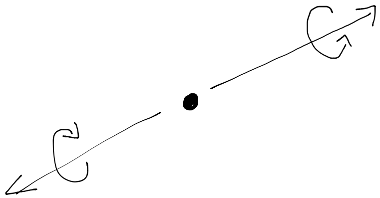
Science fiction?

Question

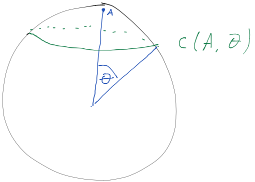
Is there a statement independent of ZFC whose truth can be verified by a physical experiment?

No such statement is presently known.

Einstein–Podolsky–Rosen paradox



Hidden variables and no-go theorems



Function $F\colon S^2 \to \{-1/2, 1/2\}$ is a spin-1/2 function if

1. $F(a) = -F(-a)$ for all a,
2. $\int_{c(a,\theta)} F(x)\, d\mu_{a,\theta}(x) = F(a) \cos\theta$ for all a and θ.

Theorem (J.L. Bell, 1965)

A spin-1/2 function cannot be Lebesgue measurable.

Hidden variables and no-go theorems 2

Function $G\colon S^2 \to \{-1, 0, 1\}$ is a KS-function if

1. $G(a) = -G(-a)$ for all a,
2. $G(a)^2 + G(b)^2 + G(c)^2 = 2$ whenever

$$\angle(a, b) = \angle(b, c) = \angle(a, c) = \pi/2.$$

Theorem (Kochen–Specker, 1967)

There is a finite subset $\mathbb{S}_0$ of S^2 such that no function G can satisfy the KS-function requirements on $\mathbb{S}_0$.

Pitowsky spin functions

Theorem (Pitowsky, 1982)

Assume Continuum Hypothesis. Then there exists a function $F\colon S^2 \to \{-1/2.1/2\}$ *such that*

1. *the restriction of* F *to* $c(a,\theta)$ *is measurable,*
2. $F(a) = -F(-a)$,
3. $\int_{c(a,\theta)} F(x)\, d\mu_{a,\theta}(x) = F(a)\cos\theta$ *for all* a *and* θ*.*

Function satisfying (1) is said to be *P-measurable*.

KSP functions

Theorem (Pitowsky, 1985)

Assume Continuum Hypothesis. Then there exists a function $F\colon S^2 \to \{-1, 0, 1\}$ *such that*

1. *the restriction of* F *to* $c(a, \pi/2)$ *is measurable,*
2. $F(a) = -F(-a)$,
3. $F(a)^2 + F(b)^2 + F(c)^2 = 2$ *for almost all* b *and* c *on* $c(a, \pi/2)$ *such that* $\angle(b, c) = \pi/2$.

Function satisfying (1) is said to be *KSP-measurable*.

Theorem (F.–Magidor, 2012)

Assume RVM. Then there are no Pitowsky spin functions and there are no KSP-functions.

The result for spin functions was announced by J. Shipman (1990).

RVM and P-measurability

Say $A \subseteq S^2$ is *P-full* if

$$\mu_{a,\theta}(A \cap c(a,\theta)) = 1$$

for all a and θ.
A function $F\colon S^2 \to [0,1]$ is *P-approximated* if there exists a Borel function $g\colon S^2 \to [0,1]$ which agrees with F on a P-full set.

Lemma

Assume RVM. P-measurable implies P-approximated.

Lemma

If F is a P-approximated spin-1/2 function then it is Borel.

Analogous results holds for KSP-measurable functions.

Questions

Question

Is there a statement independent of ZFC whose truth can be verified by a physical experiment?

Mean ergodic theorems : from amenable to non-amenable groups

June 7, 2012

von-Neumann conference, Muenster, May 2012

Amos Nevo, Technion

Averaging in dynamical systems

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$$\beta_t f(x) = \frac{1}{t} \int_0^t f(a_u x) du$$

- Do these averages converge ? and if so, what is their limit ?

Ergodicity

- For a domain $D \subset M$, the fraction of the time $0 \leq u \leq t$ that the orbit $a_u x$ of x spends in D is given by $\frac{1}{t} \int_0^t \chi_D(a_{-u}x)du$.

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- For the visiting times to converge to $\mathrm{vol}(D)/\mathrm{vol}(M)$, it is necessary that invariant sets must be null or co-null.
- Flows satisfying this condition are called ergodic flows.

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- For the proof, von-Neumann utilized his recently established spectral theorem.

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- To conclude the proof that $\lim_{T\to\infty} \beta_t f = \mathcal{E}_I f$, note that if f is invariant, then $\beta_t f = f$ for all t,

- and finally that the span of $\{a_s h - h\,;\ s \in \mathbb{R}, h \in \mathcal{H}\}$ is dense in the orthogonal complement of the space of invariants.

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- This class is a common generalization of compact groups and Abelian groups. The existence of Haar measure for compact groups, and Banach limits for general Abelian groups, both appear in the lectures.
- A group admitting a left-invariant mean is called an amenable group (or a meanable group ?)

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- Given any lcsc group G, define a sequence of sets $F_n \subset G$ of positive finite measure to be asymptotically invariant under left translations, if it satisfies for any fixed $g \in G$

$$\frac{|gF_n \Delta F_n|}{|F_n|} \longrightarrow 0 \text{ as } T \to \infty$$

Amenable groups

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- The main focus of ergodic theory has traditionally been on amenable groups, and asymptotically invariant sequences played a crucial role in many of the arguments.
- We will briefly recall some of these arguments, but first let us introduce the general set-up of ergodic theorems.

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- (X, μ) an ergodic probability measure preserving action of G.

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- $B_t \subset G$ a growing family of sets, for example an asymptotically invariant sequence, or $B_t = \{g \in G\,;\, N(g) \le t\}$ for some distance function N,
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- Define the averages β_t supported on B_t, absolutely continuous w.r.t. left Haar measure with density χ_{B_t}.
- Basic problem : Do these averages

$$\pi_X(\beta_t)f(x) = \frac{1}{|B_t|}\int_{B_t} f(g^{-1}x)dm(g)$$

converge, for a given function f on X ? If so, what is their limit ?

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- Shannon-McMillan-Breiman convergence theorem (1950's) in classical entropy theory, and Ornstein-Weiss entropy theory for general amenable groups (1980's)

- The correspondence principle between sets of positive density and ergodic systems in Multiple recurrence theory (Furstenberg 1970's).

Non-amenable groups

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- Note that another consequence of the existence of asymptotically invariant sequence G is that in any properly ergodic action of G, the space X admits an asymptotically invariant sequence of unit vectors $f_k \in L^2$, namely such that $\|\pi_X(g) f_k - f_k\| \to 0$ for all $g \in G$.

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- In fact this property characterizes amenable groups (Schmidt, Connes-Feldman-Weiss, Rosenblatt 1980's). It follows that G is amenable if and only if the Koopman operators satisfy $\|\pi_X(\beta_t)\|_{L_0^2(X)} = 1$ in every properly ergodic action (here $L_0^2(X) = 1^\perp$).

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- Thus, when G is non-amenable, at least in some actions, the operators $\pi_X(\beta_t)$ are strict contractions on $L^2_0(X)$.
- Now, go back to von-Neumann's spectral theory and try to prove ergodic theorems via spectral methods !

Some steps in the spectral approach

- The problem of ergodic theorems for general discrete groups was raised already by Arnol'd and Krylov (1962). They proved an equidistribution theorem for dense free subgroups of isometries of the unit sphere $\mathbb{S}^2$ via a spectral argument similar to Weyl's equidistribution theorem on the circle (1918).

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- Guivarc'h has established a mean ergodic theorem for radial averages on the free group, using von-Neumann's original approach via the spectral theorem (1968).
- Tempelman has proved mean ergodic theorems for averages on semisimple Lie group using spectral theory, namely the Howe-Moore vanishing of matrix coefficients theorem (1980's),
- The exciting, and distinctly non-amenable, possibility of ergodic theorems with quantitative estimates on the rate of convergence was realized by the Lubotzky-Phillips-Sarnak construction of a dense free group os isometries of $\mathbb{S}^2$ which has a spectral gap (1980's).

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- For every absolutely continuous generating probability measure β on G
$$\left\|\pi_X(\beta)f - \int_X f d\mu\right\| < (1-\eta)\|f\|$$
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- G has property T if in every ergodic action it has a spectral gap (Connes-Weiss, 1982).
- Most simple algebraic groups have property T (Kazhdan 1967), The only simple Lie groups that do not are the isometry groups of real and complex hyperbolic spaces.

Spectral transfer for semisimple algebraic groups

- Consider the case where G is a connected (semi)simple algebraic group, e.g. $SO^0(n,1)$, $Sp(n,\mathbb{C})$, $SU(p,q)$, $SL_n(F)$, $F=\mathbb{R},\mathbb{C},\mathbb{Q}_p$......

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- This method reduces the norm estimate of the operator $\pi(\beta)$ in a general representation π, to a norm estimate for the convolution operator $\lambda_G(\beta)$ in the regular representation.
- Thus it serves as an analog in this context, of the transference principle for amenable groups.

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- First, an irreducible non-trivial unitary representations σ of a simple algebraic group has the property that a sufficiently high tensor power representation $\sigma^{\otimes N}$ embeds as a subrepresentation of the regular representation of G. This fact follows since σ has a dense set of matrix coefficients which belong to $L^p(G)$ (Cowling, Howe, Howe-Moore, Li, Borel-Wallach, 1980's).

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- Second, convolution operators in the regular representation obey the Kunze-Stein phenomenon. For $\beta \in L^p(G)$, $f \in L^2(G)$, $1 \le p < 2$

$$\|\beta * f\|_2 \le C_p \|\beta\|_{L^p(G)} \|f\|_2 \ .$$

 This convolution inequality for $SL_2(\mathbb{R})$ is due to Kunze-Stein (1960's), and in general to Cowling (1980).

- Given the tensor power argument and the Kunze-Stein phenomenon, Jensen's inequality implies that for every irreducible unitary representation π of G and every family of absolutely continuous probability measures β_t

$$\|\pi(\beta_t)\| \leq \|\lambda_G(\beta_t)\|^{1/N} \leq C_\varepsilon \mathrm{vol}(B_t)^{-1/(2N)+\varepsilon} .$$

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- Quantitive mean ergodic theorem (N 98). For any ergodic action of a simple algebraic group which has a spectral gap, the convergence of the averages along the orbit to the space average takes place at the following rate :

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- Note that the result requires only that $\mathrm{vol}(B_t) \to \infty$!!

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- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$ such that for all t sufficiently large and ε sufficiently small

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

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- When B_t are admissible, pointwise almost sure convergence holds for the orbit averages, with a prescribed rate of convergence (N, Margulis+N+Stein, Gorodnik+N, 2000's)

Applications of the quantitative mean ergodic theorem

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- Assume that the error term in the mean ergodic theorem for β_t in $L^2(m_{G/\Gamma})$ satisfies

$$\left\| \pi(\beta_t)f - \int_{G/\Gamma} f d\mu \right\|_{L^2} \leq C m(B_t)^{-\theta} \|f\|_{L^2}$$

Then

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- The mean ergodic theorem is an efficient approach : for higher rank simple algebraic groups, this results generalizes previous lattice point counting results, and matches or improves the error estimate.

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- We emphasize that this result holds for all Γ-actions. The only connection to the original embedding of Γ in the group G is in the definition of the sets Γ_t.

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- Under this condition, the quantitative pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

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Examples

- In particular, if Γ has property T, for example $\Gamma = SL_n(\mathbb{Z})$, $n \geq 3$, then the fast mean and pointwise ergodic theorems hold in every ergodic measure-preserving action with a fixed $\theta > 0$.

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- Specializing further, in every action of Γ on a finite homogeneous space X, we have the following norm bound for the averaging operators

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- This estimate goes well beyond the basic one guaranteed by property T, and holds uniformly over families of finite-index subgroups provided they satisfy property T, or more generally Lubotzky-Zimmer's property τ (1985).

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- Discrete groups have non-type I representation theory, and spectral arguments are not available.
- However, both amenable groups and non-amenable groups admit amenable actions, in the sense of Zimmer (1976)

Ergodic theorems for hyperbolic groups

- Let Γ act discretely and co-compactly on a $CAT(-1)$ space M. Fix $m \in M$ with trivial stabilizer, and let $\Gamma_t = \Gamma \cdot m \cap B_t(m)$ be the discrete ball, and $A_t = \Gamma_{t+c} \setminus \Gamma_t$ be the discrete annuli of width c.

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  for almost every x, and in the L^p-norm.

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4. Define a horospherical equivalence relation $\mathcal{R}_{\partial\Gamma}$, which has a finite invariant measure, and such that the horospherical balls have the doubling property and thus are asymptotically invariant,

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6. Compute the limit, namely the conditional expectation on the space of relation-invariant sets,
7. compute the weights arising on Γ itself when integrating out the dependence on the boundary point. These weights give a pointwise ergodic theorem for measure-preserving actions of Γ.