On the zeros and coefficients of certain weakly holomorphic Drinfeld modular forms

SoYoung Choi

Gyeongsang National University

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Classical case

Let

k an even integer,

f a nonzero weakly holomorphic modular form of weight k for $SL_2(\mathbb{Z})$,

$\mathfrak{F}$ the usual fundamental domain for $SL_2(\mathbb{Z})$.

Then we have the valence formula:

$$\frac{k}{12} = ord_\infty f + \frac{1}{2} ord_i f + \frac{1}{3} ord_\rho f + \sum_{\tau \in \mathfrak{F} \setminus \{i, \rho\}} ord_\tau f$$

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Remark

Write $k = 12l_k + r_k$ with unique $l_k \in \mathbb{Z}$ and $r_k \in \{0, 4, 6, 8, 10, 14\}$.

Then $ord_\infty f \leq l_k$.

Theorem (Duke and Jenkins, 2008)

For an even integer k, let

$k = 12l_k + r_k$ *with* $r_k \in \{0, 4, 6, 8, 10, 14\}$.

For each integer $-n \leq l_k$, we have a unique weakly holomorphic modular form $f_{k,n}$ of weight k for $SL_2(\mathbb{Z})$ having the q-expansion of the form :

$$f_{k,n}(z) = q^{-n} + O(q^{l_k+1}) \quad (q = e^{2\pi iz},\ z \in \mathbb{H}).$$

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the elements $f_{k,n}$ have integral Fourier coefficients.

the elements $f_{k,n}$ form a basis for $M_k^!$.

Theorem (Duke and Jenkins, 2008)

Let $f_{k,n}(z) = \sum_{i\geq -n} a_k(n,i)q^i$.
For $k = 4, 6, 8, 10, 14$, *we have that*

$$\text{if } (n,i) = 1, \text{ then } i^{k-1} | a_k(n,i).$$

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Theorem (Duke and Jenkins, 2008)

If $n \geq |l_k| - l_k$ then all of the zeros of $f_{k,n}$ in $\mathfrak{F}$ lie on the unit circle.

Theorem (Choi and Im, 2016)

The zeros in $\mathfrak{F}^+$ of almost all elements of a certain canonical basis for the space of weakly holomorphic modular forms for $\Gamma_0^+(2)$ lie on the circle with radius $1/\sqrt{2}$ which is a lower boundary arc of the fundamental domain $\mathfrak{F}^+$ for $\Gamma_0^+(2)$. Here $\Gamma_0^+(2) = < \Gamma_0(2), W_2 >$.

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- Define a weakly holomorphic Drinfeld modular form of weight k and type m for $GL_2(A)$.
- Construct a canonical basis for the space of weakly holomorphic Drinfeld modular forms of weight k and type m for $GL_2(A)$.
- investigate a generating function of the basis elements
- A duality of t-expansion coefficients of the basis elements
- Congruence properties of t-expansion coefficients of the basis elements
- The location of all zeros of the basis elements

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Let

- $\mathbb{F}_q$ be the finite field of characteristic p with q elements,
- $A = \mathbb{F}_q[T]$ be the ring of polynomials in an indeterminate T,
- $K = \mathbb{F}_q(T)$ be the field of rational functions in an indeterminate T,
- $K_\infty = \mathbb{F}_q((1/T))$ be the completion of K at its infinite place,
- C be the completed algebraic closure of K_∞,
- $\Omega = C - K_\infty$ be the Drinfeld upper half-plane,
- $\mathfrak{T} := \{z \in \Omega : |z| = inf\{|z - a| : a \in A\} \geq 1\}$ the fundamental domain for $GL_2(A)$.

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A weakly holomorphic Drinfeld modular form

From now on, fix an integer m with $0 \leq m < q-1$. Let $k \in \mathbb{Z}$ with $k \equiv 2m \pmod{q-1}$.

Definition

A weakly holomorphic Drinfeld modular form of weight k and type m for $GL_2(A)$ is a C-valued holomorphic function $f : \Omega \to C$ that satisfies:

(1) $f(\gamma z) = (det\gamma)^{-m}(cz+d)^k f(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(A)$,
(2) f is meromorphic at the cusp ∞.

For convenience we denote by $M^!_{k,m}$ the space of weakly holomorphic Drinfeld modular forms of weight k and type m.

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We denote by e the Carlitz exponential which is defined by

$$e(z) = z \prod_{\lambda \in L} (1 - \frac{z}{\lambda})$$

and consider the parameter t at infinity defined by

$$t(z) = 1/e(\tilde{\pi} z).$$

Here $L = \tilde{\pi} A$ is the lattice of periods of the Carlitz module

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If $f \in M^!_{k,m}$ then $f(z) = \sum_{n \geq n_0} a_n t^{(q-1)n+m}$ with $a_{n_0} \neq 0$, $(a_n \in C)$.

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Basis for the space of weakly holomorphic Drinfeld modular forms

Theorem (Choi)

For $f \neq 0 \in M^!_{k,m}$ *we have that* $\mathrm{ord}_\infty f \leq (q-1)l_k + m$.

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By this process we find the $f_{k,n}$ for each integer $n \geq -l_k$.

Remark

- $f_{k,n} = (-h)^{(q-1)l_k+m} g^{r_k} P_{k,n+l_k}(-j)$ for a unique polynomial $P_{k,n+l_k}(x) \in A[x]$ of degree $n + l_k$.
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For each integer k with $k \equiv 2m \pmod{q-1}$, we have that

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(1) *if* $1 < m < q-1$ *then* $-a_{2-k}(r, s-1) = a_k(s, r-1)$ *for all* r, s.

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(1) *if* $1 < m < q-1$ *then* $-a_{2-k}(r, s-1) = a_k(s, r-1)$ *for all* r, s.
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For a positive integer k with $k \equiv 0 \pmod{q-1}$,

$$E_k(z) = 1 - \frac{1}{E_k(L)} \sum_{a \text{ monic}} G_k(t(az)),$$

where $E_k(L) \in K$ is the Eisenstein series of weight k for the lattice L and G_k is the k-th Goss polynomial of L.

Then $E_k(z)$ is a Drinfeld modular form of weight k and type 0 for $GL_2(A)$.

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For a positive integer k with $k \equiv 0 \pmod{q-1}$,
let $p_k(T), q_k(T) \in A$ such that $(p_k(T), q_k(T)) = A$ and

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location of the zeros of $f_{k,n}$

Theorem (Choi and Im 2014)

Assume that $k \equiv 0 \pmod{q-1}$. Let l_k and r_k as before. Suppose either if $r_k = 0$ or if $r_k \neq 0$ and $n \geq -l_k$ is such that $\binom{n+l_k+r_k}{r_k} \neq 0$ in $\mathbb{F}_q$, then the zeros of $f_{k,n}$ in $\mathfrak{T}$ are on the unit circle $|z| = 1$.

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Example

Let $q = p^s$ with $s > 0$

For each i with $1 \leq i \leq r_k$, let $n(i)$ be the unique nonnegative integer such that

$$p^{n(i)} || i.$$

Let

$l = max\{n(i) | 1 \leq i \leq r_k\}$. For each integer $t > l$ and every integer $\beta > 0$ letting $n = \beta p^t - l_k$, we have $p^{n(i)} || n + l_k + i$. Hence

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Remark (Kazalicki 2008)

For each $n \geq 1$ let j_n be the Drinfeld modular function with the t-expansion of the form

$$j_n(z) = t^{-n(q-1)} + \sum_{i=1}^{\infty} c_n(i) t^{i(q-1)}.$$

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Thank you!