Tensor powers of Rank 1 Sign-normalized Drinfeld Modules and Zeta Values

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Motivation

- Anderson and Thakur: Explicit formulas for log/exp coefficients for tensor powers of Carlitz allow them to connect polylogarithm values to zeta values
- Same goal for tensor powers of Drinfeld modules, but many additional complications arise!
- Need new techniques to obtain log/exp coefficients, and zeta values

Overview

1. Notation and background
2. Tensor powers of Drinfeld modules
3. Coefficients of exponential function
4. Coefficients of logarithm function
5. Zeta values

Notation

- $q = p^r$, p prime (for talk assume $p > 3$)
- $\mathbb{F}_q$ field with q elements
- $E/\mathbb{F}_q$, elliptic curve given by the equation

$$E : y^2 = t^3 + at + b$$

- $\mathbf{A} = \mathbb{F}_q[t, y]$ affine coordinate ring of E
- $\mathbf{K} = \mathbb{F}_q(t, y)$ its fraction field

Notation

- $A = \mathbb{F}_q[\theta, \eta]$ (note θ, η also satisfy E)
- $K = \mathbb{F}_q(\theta, \eta)$
- Map $\iota : \mathbf{A} \to A$, canonical isomorphism ($\iota(t) = \theta$)
- $\overline{K}$ algebraic closure of K
- K_∞ completion of K at infinite place
- $\mathbb{C}_\infty$ completion of $\overline{K}_\infty$
- $H \subset K_\infty$ is Hilbert class field of K
- $\Xi = (\theta, \eta)$ is an K-rational point on E with weighted degree $(2, 3)$

Twisting

- For $a \in \mathbb{C}_\infty(t, y)$, let $a^{(i)}$ be the ith Frobenius twist of a.
- Extend twisting to matrices (and vectors) coordinate-wise
- Define twisting on $E(\mathbb{C}_\infty)$, e.g. $\Xi^{(1)} = (\theta^q, \eta^q)$
- $\mathrm{Mat}_n(\mathbb{C}_\infty)\{\tau\}$, skew polynomial ring: $\tau M = M^{(1)}\tau$
- $\mathrm{Mat}_n(\mathbb{C}_\infty)\{\tau\}$ acts on $\mathbf{a} \in \mathbb{C}_\infty^n$

$$\tau^i \mathbf{a} = \mathbf{a}^{(i)}$$

and $\mathrm{Mat}_n(\mathbb{C}_\infty)$ acts by multiplication

Rank 1 Drinfeld Modules

Let $\rho : A \to H\{\tau\}$ be a rank 1 sign-normalized Drinfeld module. Recall:

- Drinfeld divisor $V = (\alpha, \beta) \in E(H)$ such that $V - V^{(1)} = \Xi$
- Shtuka function $f \in H(t, y)$ has divisor

$$\mathrm{div}(f) = (V^{(1)}) - (V) + (\Xi) - (\infty),$$

- Can normalize f to get explicit formula (Thakur)

$$f(t, y) = \frac{y - \eta - m(t - \theta)}{t - \alpha} = \frac{\nu(t, y)}{\delta(t)},$$

where m is the slope between $V^{(1)}$ and Ξ.

Anderson $\mathbf{A}$-module

For $n \geq 1$, an n-dimensional Anderson $\mathbf{A}$-module is an $\mathbf{A}$-module homomorphism $\rho : \mathbf{A} \to \mathrm{Mat}_n(\mathbb{C}_\infty)\{\tau\}$

$$\rho_a = d[a] + M_1\tau + \cdots + M_i\tau^i,$$

such that $d[a] = \iota(a)I + N$ for some nilpotent matrix $N \in \mathrm{Mat}_n(\mathbb{C}_\infty)$. The map ρ provides an $\mathbf{A}$ action on $\mathbb{C}_\infty^n$.

The map $d[\,\cdot\,]$ always denotes the constant matrix of ρ_a (ring homomorphism).

$\mathbf{A}$-Motive and dual $\mathbf{A}$-Motive

Let $U = \operatorname{Spec} \mathbb{C}_\infty[t,y]$ and define the $\mathbf{A}$-motive and dual $\mathbf{A}$-motive

$$M_0 = \Gamma(U, \mathcal{O}_E(V)) = \bigcup_{i \geq 0} \mathcal{L}((V) + i(\infty)),$$

$$N_0 = \Gamma\big(U, \mathcal{O}_E(-(V^{(1)}))\big) \subseteq \mathbb{C}_\infty[t,y].$$

Tensor Powers

Define the nth tensor powers over $K[t,y]$,

$$M = M_0^{\otimes n}, \quad N = N_0^{\otimes n}.$$

Theorem from geometry: Tensor powers are also spaces of functions

$$M \cong \Gamma(U, \mathcal{O}_E(nV)) \quad \text{and} \quad N \cong \Gamma(U, \mathcal{O}_E(-nV^{(1)})).$$

τ- and σ-action on M and N

Define $\sigma = \tau^{-1}$ and define (noncommutative) rings $\mathbb{C}_\infty[t, y, \tau]$ and $\mathbb{C}_\infty[t, y, \sigma]$ such that τ and σ commute with t and y and that

$$\tau z = z^q \tau, \quad \sigma z = z^{1/q} \sigma.$$

- M is a $\mathbb{C}_\infty[t, y, \tau]$-module: For $a \in M$ set $\tau a = f^n a^{(1)}$
- N is an $\mathbb{C}_\infty[t, y, \sigma]$-module: For $b \in N$ set $\sigma b = f^n b^{(-1)}$
- Are motives in the sense of Anderson

$\mathbf{A}$-Motive Bases

Define functions $g_k \in M$ with "suitable" normalization for $1 \leq k \leq n$ with divisors

$$\begin{aligned}
\mathrm{div}(g_1) &= -n(V) + (n-1)(\infty) + ([n]V) \\
\mathrm{div}(g_2) &= -n(V) + (n-2)(\infty) + (\Xi) + (V^{(1)} + [n-1]V) \\
\mathrm{div}(g_3) &= -n(V) + (n-3)(\infty) + 2(\Xi) + ([2]V^{(1)} + [n-2]V) \\
&\vdots \\
\mathrm{div}(g_{n-1}) &= -n(V) + (\infty) + (n-2)(\Xi) + ([n-2]V^{(1)} + [2]V) \\
\mathrm{div}(g_n) &= -n(V) + (n-1)(\Xi) + ([n-1]V^{(1)} + V).
\end{aligned}$$

Fact: For $1 \leq i \leq n$ the functions g_i form a τ-basis for M.

Dual $\mathbf{A}$-Motive Bases

Define functions $h_k \in N$ with "suitable" normalization for $1 \leq k \leq n$ with divisors

$$\begin{aligned}
\mathrm{div}(h_1) &= n(V^{(1)}) - (n+1)(\infty) + (-[n]V^{(1)}) \\
\mathrm{div}(h_2) &= n(V^{(1)}) - (n+2)(\infty) + (\Xi) + (-[n-1]V^{(1)} - V) \\
\mathrm{div}(h_3) &= n(V^{(1)}) - (n+3)(\infty) + 2(\Xi) + (-[n-2]V^{(1)} - [2]V) \\
&\vdots \\
\mathrm{div}(h_{n-1}) &= n(V^{(1)}) - (2n-1)(\infty) + (n-2)(\Xi) + (-[2]V^{(1)} - [n-2]V) \\
\mathrm{div}(h_n) &= n(V^{(1)}) - (2n)(\infty) + (n-1)(\Xi) + (-V^{(1)} - [n-1]V).
\end{aligned}$$

Fact: For $1 \leq i \leq n$ the functions h_i form a σ-basis for N.

The map ε

For $g \in N = \Gamma(U, \mathcal{O}_E(-nV^{(1)}))$, define the map

$$\varepsilon : N \to \mathbb{C}_\infty^n,$$

by writing g in the σ-basis for N,

$$g = \sum_{j=0}^{m} \sum_{i=1}^{n} d_{i,j} \sigma^j(h_i) = \sum_{j=0}^{m} \sum_{i=1}^{n} d_{i,j} (f f^{(-1)} \dots f^{(1-j)})^n h_i^{(-j)},$$

where $d_{i,j} \in \mathbb{C}_\infty$, then defining

$$\varepsilon(g) = \sum_{j=0}^{m} ((d_{n,j}, d_{n-1,j}, \dots, d_{1,j})^{(i)})^\top.$$

$\mathbf{A}$-Motive to $\mathbf{A}$-Module

Fixing the σ-basis for N makes a construction of Anderson explicit:

$$\begin{array}{ccc} N/(1-\sigma)N & \xrightarrow{\varepsilon} & \mathbb{C}_\infty^n \\ \Big\downarrow a & & \Big\downarrow \rho_a^{\otimes n} \\ N/(1-\sigma)N & \xrightarrow{\varepsilon} & \mathbb{C}_\infty^n \end{array}$$

- The map ε is a $\mathbb{F}_q$-vector space isomorphism
- Using ideas of Hartl and Juschka: $\rho^{\otimes n} : \mathbf{A} \to \mathrm{Mat}_n(H)\{\tau\}$ is an Anderson $\mathbf{A}$-module
- Call $\rho^{\otimes n}$ the nth tensor power of the Drinfeld module ρ

Example: Anderson $\mathbf{A}$-Module

The t-action for $\rho^{\otimes n}$ can be written as

$$\rho_t^{\otimes n} = \begin{pmatrix} \theta & a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \theta & a_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \theta & a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \theta & a_{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \theta & a_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ a_n & 1 & 0 & \dots & 0 \end{pmatrix} \tau,$$

for $a_i = \frac{2\eta}{\theta - t([i]V^{(1)} + [n-i]V)}$.

Exponential and Logarithm Functions

Define the exponential and logarithm function associated to $\rho^{\otimes n}$ as the vector of power series in $\mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty[[z_1,\ldots,z_n]])$

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)}, \quad \mathrm{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)}$$

with $Q_i = P_i = I$ and where $\mathbf{z} = (z_1,\ldots,z_n)^\top$. Note: $\mathrm{Exp}_\rho^{\otimes n}$ is the unique $\mathbb{F}_q$-linear power series satisfying

$$\mathrm{Exp}_\rho^{\otimes n}(d[a]\mathbf{z}) = \rho_a^{\otimes n}(\mathrm{Exp}_\rho^{\otimes n}(\mathbf{z}).)$$

Coefficients of $\mathrm{Exp}_\rho^{\otimes n}$: Preliminaries

For a fixed dimension n, for $1 \leq \ell \leq n$ and for $i \geq 0$, define the functions

$$\gamma_{i,\ell} = \frac{g_\ell}{(f f^{(1)} \dots f^{(i-1)})^n},$$

where for $i = 0$ we understand $\gamma_{0,\ell} = g_\ell$.

- Functions $\gamma_{i,\ell}$ related to "polyexponential" functions
- Come up naturally in certain residue calculations

Preliminaries

For $1 \leq \ell \leq n$, there exist constants $c_{\ell,1}, \cdots, c_{\ell,n} \in H$ and $d_{j,k} \in H$ such that

$$\frac{g_\ell}{(ff^{(1)} \dots f^{(i-1)})^n} = c_{\ell,1} g_1^{(i)} + c_{\ell,2} g_2^{(i)} + \dots c_{\ell,n} g_n^{(i)} + \sum_{j,k} d_{j,k} \alpha_{j,k},$$

where the functions $\alpha_{j,k}$ live in the Riemann-Roch space

$$\alpha_{j,k} \in \mathcal{L}(n(V^{(i)}) - n(\Xi^{(i)}) + k(\Xi^{(j-1)}) + \cdots + n(\Xi) - (n(j-1) + k - 1)(\infty)).$$

Define the matrix

$$C_i = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \dots & c_{n,n} \end{pmatrix}$$

Main Theorem on Exponential Function Coefficients (G.)

For dimension $n \geq 2$ and $\mathbf{z} \in \mathbb{C}_\infty$, if we write

$$\mathrm{Exp}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)},$$

then for $i \geq 0$, the exponential coefficients $Q_i = C_i$ and $Q_i \in \mathrm{Mat}_n(H)$.

Corollary on Exponential Function Coefficients (G.)

We get more exact information about the first column of Q_i. For $z \in \mathbb{C}_\infty$ we have the expression

$$\mathrm{Exp}_\rho^{\otimes n}\begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=0}^{\infty} \frac{z^{q^i}}{g_1^{(i)}(f f^{(1)} \dots f^{(i-1)})^n} \cdot \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}\Bigg|_{\Xi^{(i)}}.$$

Motivation and Proof

Motivation for theorem: Residue calculation for vector-valued Anderson generating functions.

Actual proof for theorem: Calculate recurrence formula that Q_i must satisfy, prove C_i satisfies it.

Coefficients of $\mathrm{Log}_\rho^{\otimes n}$

We study the coefficients of $\mathrm{Log}_\rho^{\otimes n}$ using a diagram of maps inspired by Sinha.

Rigid Analysis Definitions

- Tate algebras for $c \in A$: $\mathbb{T}_c = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; \left|c^i b_i\right| \to 0 \right\}$
- $\mathcal{E}$ as the rigid analytic variety associated to E
- $\mathcal{U} \subset \mathcal{E}$ be the inverse image under t of the closed disk in $\mathbb{C}_\infty$ of radius $|\theta|$ centered at 0
- $\mathcal{U}$ is the affinoid subvariety of $\mathcal{E}$ associated to $\mathbb{T}_\theta[y]$
- $\mathbb{B} := \Gamma\big(\mathcal{U}, \mathcal{O}_E(-n(V) + n(\Xi))\big)$

Modules Ω and Ω_0

Define $\mathbf{A}$-modules of rigid analytic functions

$$\Omega = \{h \in \mathbb{B} \mid h^{(1)} - f^n h = g \in N\},$$

$$\Omega_0 = \{h \in \mathbb{B} \mid h^{(1)} - f^n h = 0\}.$$

For a function $h(t,y) \in \Omega$, define the map $T : \Omega \to \mathbb{T}[y]^n$ by

$$T(h(t,y)) = \begin{pmatrix} h(t,y) \cdot g_1 \\ \vdots \\ h(t,y) \cdot g_n \end{pmatrix},$$

Map T provides a "vector version" of the spaces Ω_0 and Ω

The Map $\mathrm{RES}_{\Xi^{(i)}}$

Define $\mathcal{M}$ to be the submodule of $\mathbb{T}[y]$ consisting of all elements in $\mathbb{T}[y]$ which have a meromorphic continuation to all of $\mathbb{C}_\infty$. Then, for $(z_1, ..., z_n)^\top \in \mathcal{M}^n$ define the map $\mathrm{RES}_{\Xi^{(i)}} : \mathcal{M}^n \to \mathbb{C}_\infty^n$, as

$$\mathrm{RES}_{\Xi^{(i)}} \begin{pmatrix} z_1 \\ \vdots \\ z_3 \end{pmatrix} = \begin{pmatrix} \mathrm{Res}_{\Xi^{(i)}}(z_1\lambda) \\ \vdots \\ \mathrm{Res}_{\Xi^{(i)}}(z_n\lambda) \end{pmatrix},$$

where $\lambda = dt/2y$ is the invariant differential of E.

Interlude on Periods

Define the Anderson-Thakur function

$$\omega_\rho = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\xi^{q^i}}{f^{(i)}}, \quad \xi = -\left(m + \frac{\beta}{\alpha}\right),$$

and denote

$$\Pi_n = -\,\mathrm{RES}_\Xi(T(\omega_\rho^n)).$$

Theorem (G.)

The period lattice of $\mathrm{Exp}_\rho^{\otimes n}$ equals $\{d[a]\Pi_n \mid a \in \mathbf{A}\}$ and the last coordinate of $\Pi_n \in \mathbb{C}_\infty^n$ is

$$\frac{g_1(\Xi)}{a_1 a_2 \dots a_{n-1}} \cdot \pi_\rho^n,$$

where π_ρ is a fundamental period of ρ and the constants $a_i \in H$.

Main Diagram

Define the following diagram of maps:

$$\begin{array}{ccccc}
\Omega & \xrightarrow{\tau - f^n} & N & \xrightarrow{\varepsilon} & \mathbb{C}_\infty^n \\
\Big\downarrow{\scriptstyle T} & & & \nearrow{\scriptstyle \mathrm{Exp}_\rho^{\otimes n}} & \\
\mathcal{M}^n & \xrightarrow{\mathrm{RES}_\Xi} & \mathbb{C}_\infty^n & &
\end{array}$$

Theorem (G.): The diagram commutes.

Proof: Use Anderson generating functions

Using Diagram for Logarithm Coefficients

We will use the maps in the diagram to get formulas for the coefficients of $\mathrm{Log}_\rho^{\otimes n}$. For $d_j \in \mathbb{C}_\infty$ with $|d_j| \leq C$ for some constant $C > 0$ for $1 \leq j \leq n$, define

$$c(t,y) = d_n h_1 + \cdots + d_1 h_n \in N,$$

then define a rigid analytic function in $\Gamma(\mathcal{U}, \mathcal{O}_E(n(\Xi)))$

$$B(t,y;\mathbf{d}) = -\sum_{i=0}^{\infty} \frac{c(t,y)^{(i)}}{(f f^{(1)} f^{(2)} \dots f^{(i)})^n},$$

for the vector $\mathbf{d} = (d_1, \dots d_n)^\top \in \mathbb{C}_\infty^n$.

Using Diagram for Logarithm Coefficients

Telescoping series gives

$$(\tau - f^n)(B) = (\tau - f^n)\left(-\sum_{i=0}^{\infty} \frac{c(t,y)^{(i)}}{(f f^{(1)} f^{(2)} \dots f^{(i)})^n}\right) = c(t,y),$$

so $B(t,y,\mathbf{d}) \in \Omega$. Then using the diagram we find

$$\mathrm{Exp}_\rho^{\otimes n}(-\mathrm{RES}_\Xi(T(B))) = \varepsilon((\tau - f^n)(B)) = \varepsilon(c(t,y)),$$

but by definition

$$\varepsilon(c(t,y)) = (d_1, \dots d_n)^\top.$$

To summarize,

$$\mathrm{Exp}_\rho^{\otimes n}(-\mathrm{RES}_\Xi(T(B(t,y;\mathbf{d})))) = \mathbf{d}.$$

Using Diagram for Logarithm Coefficients

Take the logarithm of both sides (possibly shrinking radius of convergence) to get

$$-\mathrm{RES}_\Xi(T(B(t,y;\mathbf{d}))) = \mathrm{Log}_\rho^{\otimes n}(\mathbf{d})$$

After writing out $-\mathrm{RES}_\Xi(T(B(t,y;\mathbf{d})))$ as a power series in $\mathbf{d}$, we get a formula for the coefficients of $\mathrm{Log}_\rho^{\otimes n}$.

Main Theorem on Logarithm Coefficients (G.)

If we write

$$\mathrm{Log}_\rho^{\otimes n}(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)},$$

for $n \geq 2$, then for $i \geq 0$

$$P_i = \left\langle \mathrm{Res}_\Xi \left(\frac{g_j h_{n-k+1}^{(i)}}{(f f^{(1)} \dots f^{(i)})^n} \lambda \right) \right\rangle_{1 \leq j,k \leq n}.$$

Corollary on Logarithm Coefficients (G.)

For the coefficients P_i of the function $\mathrm{Log}_\rho^{\otimes n}$, the bottom row of P_i, for $i \geq 0$, can be written as

$$\left\langle \left. \frac{h_{n-k+1}^{(i)}}{h_1(f^{(1)} \dots f^{(i)})^n} \right|_{\Xi} \right\rangle_{1 \leq k \leq n} .$$

Definitions for Zeta Values

Recall the definitions:

- Extension of (1-dim) Drinfeld module ρ to integral ideals $\mathfrak{a} \subset A$ due to Hayes, which maps $\mathfrak{a} \mapsto \rho_\mathfrak{a} \in H[\tau]$
- $\partial(\rho_\mathfrak{a})$ is constant term of $\rho_\mathfrak{a}$ with respect to τ
- $\phi_\mathfrak{a} \in \mathrm{Gal}(H/K)$ is Artin automorphism associated to $\mathfrak{a}$
- B is integral closure of A in H

Define zeta function associated to ρ twisted by $b \in B$

$$\zeta_\rho(b; s) = \sum_{\mathfrak{a} \subseteq A} \frac{b^{\phi_\mathfrak{a}}}{\partial(\rho_\mathfrak{a})^s}$$

Main Theorem for Zeta Values (G.)

For $b \in B$ and for $n \leq q-1$, there exists a vector

$$(*, \ldots, *, C\zeta_\rho(b;n))^\top \in \mathbb{C}_\infty^n$$

such that

$$\mathbf{d} := \operatorname{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ \vdots \\ * \\ C\zeta_\rho(b;n) \end{pmatrix} \in H^n,$$

where $C = \frac{(-1)^{n+1} h_1(-\Xi)}{\theta - t([n]V^{(1)})} \in H$ and $\mathbf{d} \in H^n$ is explicitly computable.

Issue: Radius of convergence for $\operatorname{Log}_\rho^{\otimes n}$

Main Idea for Zeta Values Proof

We use ideas from Papanikolas and G. to realize the zeta value as a sum of elements which are *almost* the bottom row of P_i, for $i \geq 0$,

$$\zeta_\rho(b;n) = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\min(i,q+e)} \sum_{k=1}^{n} d_{k,j}^{(i)} h_{n-k+1}^{(i-j)}}{C \cdot h_1 \left(f^{(1)} \cdots f^{(i-j)}\right)^n} \Bigg|_{\Xi}.$$

where $d_{k,j}^{(i)}, C \in H$. Since $\mathrm{Exp}_\rho^{\otimes n}$ is the inverse power series of $\mathrm{Log}_\rho^{\otimes n}$ we get the theorem.

Example: Zeta Values $n = 2, q = 3$

Let $E/\mathbb{F}_3$ be defined by $y^2 = t^3 - t - 1$ (A has class number 1). In this case, the theorem gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathrm{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ C\zeta(2) \end{pmatrix},$$

Where $C = -\frac{\eta^3}{\eta^2+1} \in K$. Implies:

- $(*, C\zeta(2))^\top$ is in period lattice of $\mathrm{Exp}_\rho^{\otimes n}$
- $\zeta(2)/\pi_\rho^2 \in K$
- Since $(q-1)|n$, agrees with Goss - conditions under which zeta values in the period lattice.

Example: Zeta Values $n = 2, q = 4$

Let $E/\mathbb{F}_4$ be defined by $y^2 + y = t^3 + c$, where $c \in \mathbb{F}_4$ is a root of the polynomial $c^2 + c + 1 = 0$. Then the theorem gives

$$\begin{pmatrix} (\theta^4+\theta)^2 + (\theta^4+\theta)^4 \\ (\theta^4+\theta) + (\theta^4+\theta)^3 \end{pmatrix} = \mathrm{Exp}_\rho^{\otimes n} \begin{pmatrix} * \\ C\zeta(2) \end{pmatrix}.$$

Where $C = (\theta^4 + \theta)^{-1} \in K$. Implies:

- $(*, C\zeta(2))^\top$ is **not** in period lattice of $\mathrm{Exp}_\rho^{\otimes n}$
- Implies $\begin{pmatrix} (\theta^4+\theta)^2 + (\theta^4+\theta)^4 \\ (\theta^4+\theta) + (\theta^4+\theta)^3 \end{pmatrix}$ is not torsion.
- Can we show this explicitly?

Future Directions

- Extend formulas to zeta values for all $n \geq 1$. What to use for Anderson-Thakur polynomials?
- Use multivariable L-functions of Pellarin to get higher zeta values?
- Extend theory to curves of higher genus. Work over Jacobian?

Thank you for listening!