Quantum j invariant and Real Multiplication program for global function fields

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1. The **Hilbert class field** H_K of K: the maximal abelian extension of K, unramified at *every* place.
2. The **maximal abelian extension** K^{ab} of K.

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In particular, when $K = \mathbb{Q}$, we have that $H_{\mathbb{Q}} = \mathbb{Q}$.

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Let $d \in \mathbb{N} \setminus \{0\}$, squarefree, $\mu := \sqrt{-d}$, $\Lambda_\mu := \langle 1, \mu \rangle_{\mathbb{Z}}$ is a CM lattice in $\mathbb{C}$. To Λ_μ one associates therefore the CM elliptic curve:

$$E_\mu : Y^2 = 4X^3 - 60G_4(\mu)X - 140G_6(\mu);$$

where:

$$G_n(\mu) := \sum_{\lambda \in \Lambda_\mu \setminus \{0\}} \lambda^{-n};$$

are the corresponding Eisenstein series.

Imaginary quadratic number fields

Let:

$$j(\mu) := \frac{12^3}{1 - \frac{49}{20}\frac{G_6(\mu)^2}{G_4(\mu)^3}};$$

be the value taken in μ by the $j-$**invariant**:

$$j : Mod \to \mathbb{C};$$

defined on the moduli space:

$$Mod = (\mathbb{C} \setminus \mathbb{R})/GL_2(\mathbb{Z});$$

of elliptic curves defined over $\mathbb{C}$.

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Theorem of Fueter-Weber

1. There exists $\alpha \in \mathbb{Q}(\mu) \setminus \mathbb{Q}$ such that:

$$H_K = K(j(\alpha));$$

2. $$K^{ab} = H_K(h(E_{\alpha\, tors.}));$$

where $h : E_\alpha \to \mathbb{P}_1$ a Weber function.

Imaginary quadratic number fields

More precisely, by calling $g_2(\mu) := 60G_4(\mu)$ and $g_3(\mu) := 140G_4(\mu)$, and $z := [\wp_z : \wp'_z : 1]$ the general point of E_μ:

The Weber function h is in particular:

$$h(z) = \begin{cases} \frac{g_2(\mu)g_3(\mu)}{\Delta(\mu)}\wp_z, & \text{if } j(\mu) \neq 0, 1728 \\ \frac{g_2(\mu)^2}{\Delta(\mu)}\wp_z^2, & \text{if } j(\mu) = 1728 \\ \frac{g_3(\mu)}{\Delta(\mu)}\wp_z^3, & \text{if } j(\mu) = 0 \end{cases}$$

where $\Delta(\mu) = g_2(\mu)^2 - 27g_3(\mu)^3 \neq 0$.

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(Here the role of the Weierstrass $\wp$−function is played by the exponential $e^{2\pi i} : \mathbb{R}/\mathbb{Z} \to S^1$).

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$$\Pi(\theta) := \mathbb{R} / \langle 1, \theta \rangle_{\mathbb{Z}}$$

it is a **quantum** object.

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Eisenstein series can not be defined over a subgroup dense in $\mathbb{R}$.The project of developing a theory of "**real multiplication**" for quantum tori has been proposed by Y. Manin in order to solve Hilbert's 12th problem for real quadratic number fields in the spirit of Fueter-Weber theorem.

C. Castaño Bernard - T. M. Gendron results

Let us define $||x|| := \inf_{n\in\mathbb{Z}}\{d(x,n)\}$, for all $x \in \mathbb{R}$. $\forall\epsilon > 0$ it is defined:

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Such a set is not trivially $\{0\}$ (and actually infinite) if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ by Kronecker's Theorem.

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$$j_\epsilon(\theta) := \frac{12^3}{1 - \frac{49}{20}\frac{\zeta_{\theta,\epsilon}(6)^2}{\zeta_{\theta,\epsilon}(4)^3}}.$$

C. Castaño Bernard - T. M. Gendron results

Theorem (T. M. Gendron, C. Castaño Bernard)

The limit:

$$j^{qt}(\theta) := \lim_{\epsilon \to 0^+} j_\epsilon(\theta)$$

produces multiple values and it is a modular invariant defined on the moduli space of quantum tori.

Definition

The ***quantum*** $j-$***invariant*** *is the* ***multi-valued*** *function:*

$$j^{qt} : \mathbb{R}/GL_2(\mathbb{Z}) \multimap \mathbb{R} \cup \{\infty\}.$$

C. Castaño Bernard - T. M. Gendron results

PARI-GP *suggests* that if θ is a **fundamental unit** of $\mathcal{O}_K$ with fundamental discriminant D:

1. $$\sharp\{j^{qt}(\theta)\} = D;$$
2. $$\infty \notin j^{qt}(\theta).$$

If $\theta = (1+\sqrt{5})/2$ T. M. Gendron and C. Castaño Bernard computed explicit values.

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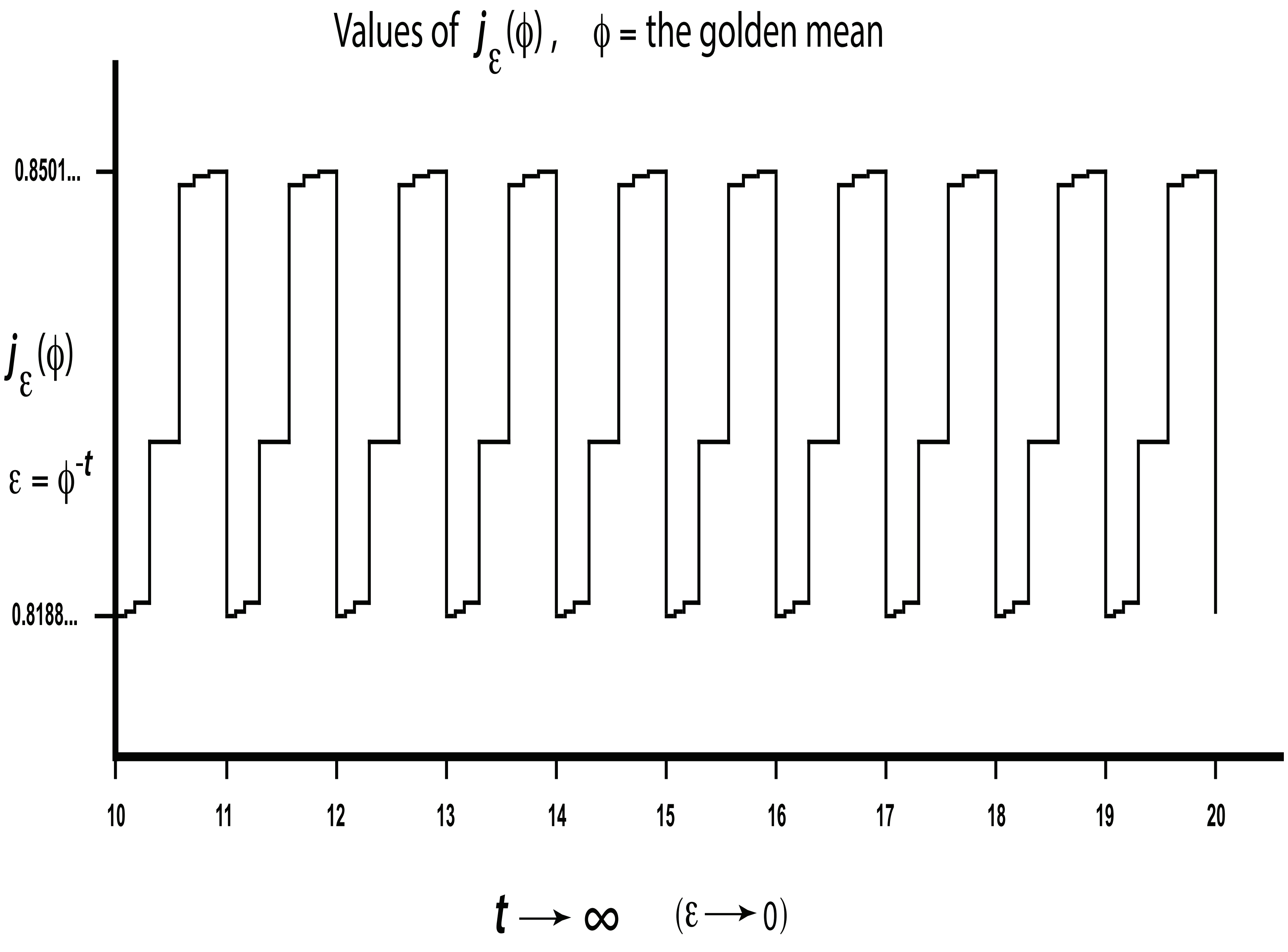
[1] Castaño Bernard C., Gendron T. M., "Modular invariant of quantum tori", *Proc. Lond. Math. Soc.* 109 (2014), Issue 4, 1014 - 1049



Values of $j_\varepsilon(\phi)$, ϕ = the golden mean
0.8501...
0.8188...
$j_\varepsilon(\phi)$
$\varepsilon = \phi^{-t}$
10
11
12
13
14
15
16
17
18
19
20
$t \longrightarrow \infty$ $(\varepsilon \longrightarrow 0)$

C. Castaño Bernard - T. M. Gendron results

Conjecture

Let θ be irrational real quadratic. One then has that:

1. $\sharp j^{qt}(\theta) < \infty$ *and* $\infty \notin j^{qt}(\theta)$;
2. $j^{qt}(\theta) \subset \overline{\mathbb{Q}}$;
3. *If* θ *is a fundamental unit:*

$$H_K = K(N(j^{qt}(\theta)));$$

for N *some weighted norm.*

Definitions

Let X be a smooth, projective curve over $\mathbb{F}_q$. Let $P \in X$.

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We know that if $X = \mathbb{P}_1(\mathbb{F}_q)$ and $P = \infty$:

$$A := A_P = \mathbb{F}_q[T] \qquad k := \mathbb{F}_q(T)$$

$$v_\infty = v_{1/T} := -\deg_T(\cdot) \qquad |\cdot| = |\cdot|_{1/T} := q^{\deg_T(\cdot)}$$

$$k_\infty = \mathbb{F}_q((1/T)) \qquad \mathcal{C} := (\overline{k_\infty})_\infty$$

Definitions

$$A \longleftrightarrow \mathbb{Z};$$

$$k \longleftrightarrow \mathbb{Q};$$

$$k_\infty \longleftrightarrow \mathbb{R};$$

$$\mathcal{C} \longleftrightarrow \mathbb{C}.$$

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If we restrict the definition of H_K to be *geometric*, then there are $h_K = \sharp Cl_K$ **distinct** abelian, unramified extensions of K which are all **maximal**.

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Note that:

$$\pi^{-1}(\infty) = \{\infty_1, \infty_2\}.$$

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If K/k is **real** (which means $\pi^{-1}(\infty) = \{\infty_1, \infty_2\}$), we have that $H_{A_{\infty_1}} \supsetneqq H_{\mathcal{O}_K}$.

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We may assume without loss of generality that f is a **fundamental unit**:

$$f^2 = a(T)f + b, \ \ a(T) \in A, \ \ b \in \mathbb{F}_q{}^*.$$

Let $d := \deg_T(discr(f)) = \deg_T(a(T))$.

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$$f \in A_{\infty_1}, \notin A_{\infty_2}, \;\; f^{-1} \in A_{\infty_2}, \notin A_{\infty_1}, \;\; T \in \mathcal{O}_K, \notin A_{\infty_1}, \notin A_{\infty_2}.$$

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$f \in {\mathcal{O}_K}^*, \notin {A^*}_{\infty_1}, \notin {A^*}_{\infty_2}$. Therefore **the group of units of $\mathcal{O}_K$ is infinite and no $\mathcal{O}_K-$lattice can be embedded in $\mathcal{C}$.** While on the other hand ${A^*}_{\infty_1} = {\mathbb{F}_q}^*$.

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We also note that both A_{∞_1} and $\mathcal{O}_K$ are Dedekind domains, so the ideal classes over them form actually a group.

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$A_{\infty_1}-$lattices can be embedded in $\mathcal{C}$. This gives rise to a **rank 1** theory which provides a generalization of the Theorem of Kronecker-Weber for any finite extension of $\mathbb{F}_q(T)$.

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There are $h_{A_{\infty_1}}$ distinct isomorphism classes of Hayes modules over A_{∞_1}.

Theorem of Hayes

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1. $H_{A_{\infty_1}}$ is the **smallest field of definition of a Hayes module over** A_{∞_1}.
2. For every given modulus $\mathfrak{m}$ in A_{∞_1} the (narrow) **absolute ray class field** of conductor dividing $\mathfrak{m}$ is:

$$K^{\mathfrak{m}}{}_{\infty_1} = H_{A_{\infty_1}}(\Psi[\mathfrak{m}]).$$

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Kronecker's Theorem for function fields

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Definition

The corresponding ***quantum torus*** *is the quotient:*

$$k_\infty / \langle 1, h \rangle_A .$$

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where $||x|| := \inf_{a \in A}\{|x - a|\}$, *for any given* $x \in k_\infty$.

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(Analogue of E. U. Gekeler's $j-$invariant for rank 2 Drinfeld modules).

Definitions

Definition

We call ***quantum*** $j-$***invariant*** *the following* ***multi-valued*** *function:*

$$j^{qt} : k_\infty / GL_2(A) \multimap k_\infty \cup \{\infty\};$$

$$j^{qt}(h) := \lim_{\epsilon \to 0^+} j_\epsilon(h).$$

Definitions

Theorem

1. $$\forall 0 < \epsilon < 1,\ \forall h \in k_\infty \setminus k,\ |j_\epsilon(h)| = q^{2q-1}.$$

   In particular, $\infty \notin j^{qt}(h)$.

2. $$h \in k \iff j^{qt}(h) = \infty.$$

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[2] T. M. Gendron, L. Demangos, "Quantum $j-$invariant in positive characteristic I: definition and convergence", *Archiv der Mathematik*, 107 (1), 25-35 (2016)

Definitions

Theorem

1. $$h \textit{ is quadratic } \Longrightarrow \sharp j^{qt}(h) < +\infty.$$

2. $$h \textit{ is quadratic unit} \Longrightarrow \sharp j^{qt}(h) = d = \deg_T(discr(h)).$$

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[3] T. M. Gendron, L. Demangos, "Quantum $j-$invariant in positive characteristic II: formulas and values at the quadratics", *Archiv der Mathematik*, 107 (2), 159-166 (2016)

Class field generation

Main theorem

Let K/k be quadratic and real, $h \in \mathcal{O}^*{}_K$ **unit**. Then:

$$H_{\mathcal{O}_K} = K(N(j^{qt}(h)));$$

$$N(j^{qt}(h)) := \prod_{\alpha \in j^{qt}(h)} \alpha.$$

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[4] T. M. Gendron, L. Demangos, "The quantum $j-$invariant and Hilbert class fields of real quadratic extensions in positive characteristic", https://arxiv.org/pdf/1607.03027.pdf

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Step 1: Diophantine approximation

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We define the sequence:

$$q_0 := 1,\ q_1 := a,\ \cdots,\ q_i := aq_{i-1} + bq_{i-2};$$

(best approximation of h if $b = 1$ by continued fraction).

Class field generation

Step 1: Diophantine approximation

Let $h \in k_\infty \setminus k$. Let assume $h^2 = a(T)h + b,\ b \in \mathbb{F}_q{}^*$.

We define the sequence:

$$q_0 := 1,\ q_1 := a,\ \cdots,\ q_i := aq_{i-1} + bq_{i-2};$$

(best approximation of h if $b = 1$ by continued fraction).

We have that $||q_n h|| = |q_n h - q_{n+1}| = \frac{|h'^{n+1}||h-h'|}{|a(T)|} = q^{-(n+1)d}$.

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This follows by Binet's formula:

$$q_n = \frac{h^{n+1} + (-h')^{n+1}}{\sqrt{D}},\ \forall n \in \mathbb{N}.$$

Class field generation

$\{T^{d-1}q_0, \ldots, Tq_0, q_0, T^{d-1}q_1, \ldots, Tq_1, q_1, \ldots\}$ is $\mathbb{F}_q$−basis of A, where the order is by **decreasing errors**:

$$||T^l q_n h|| = |T^l q_n h - T^l q_{n+1}| = q^{l-(n+1)d} < 1.$$

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One can prove that:

$$\Lambda_\epsilon(h) = \mathbb{F}_q\langle q_N, q_N T, ..., q_N T^{d-l-1}, q_{N+1}, q_{N+1}T, ..., q_{N+1}T^{d-1}, ...\rangle.$$

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By Binet's formula, if $N \to +\infty$, so $\epsilon \to 0^+$, we have that:

$$q_{N+i} \approx \frac{h^{N+1+i}}{\sqrt{D}}.$$

Class field generation

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$$\mathfrak{a}_i := (h, hT, ..., hT^i)_{A_{\infty_1}};$$

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we obtain:

$$\zeta_{h,\epsilon}(n) = \sum_{a \in \Lambda_\epsilon(h)^+ \setminus \{0\}} a^{-n} \xrightarrow[\epsilon \to 0^+]{} \frac{h^N}{\sqrt{D}} \{\zeta_{\mathfrak{a}_{d-l-1}}(n)\}_{l=0,...,d-1} .$$

By homogeneity of the degree $1 - q^2$ we have cancellation of $(h^N/\sqrt{D})^{1-q^2}$ in the expression of $j^{qt}(h)$, so we obtain:

Class field generation

Step 2: Description of j^{qt}

$$j^{qt}(h) = \left\{ \frac{T^q - T}{1 - \frac{T^{q^2} - T}{T^{q^2} - T^q} \frac{\zeta_{\mathfrak{a}_i}(q^2-1)}{\zeta_{\mathfrak{a}_i}(q-1)^{q+1}}} \right\}_{i=0,\ldots,d-1} = \{j(\mathfrak{a}_i)\}_{i=0,\ldots,d-1}.$$

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Theorem of Goss

$$\exists r \in \mathcal{C} \setminus \{0\},\ \forall \mathfrak{a} \in Cl_{A_{\infty_1}},\ \forall n \equiv 0\ mod\ (q-1):$$

$$\frac{\zeta_{\mathfrak{a}}(n)}{r^n} \in H_{A_{\infty_1}}.$$

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This generalizes Euler's formula:

$$\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q} = H_{\mathbb{Q}}.$$

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Which means that:

$$j^{qt}(h) \subset H_{A_{\infty_1}}.$$

Class field generation

Step 3: Class field generation

$$\begin{array}{ccc} G(H_{A_{\infty_1}}/K) & \xrightarrow{\pi} & G(H_{\mathcal{O}_K}/K) \\ \downarrow \simeq & & \downarrow \simeq \\ Cl_{A_{\infty_1}} & \xrightarrow{\pi} & Cl_{\mathcal{O}_K} \end{array}$$

where:

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It is easy to see that:

$$\{\mathfrak{a}_i\}_{i=0,\dots,d-1} = \{(f, fT, \dots, fT^i)\}_{i=0,\dots,d-1} \subset Ker(\pi).$$

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$(f \in \mathcal{O}_K{}^*)$.

Class field generation

Proposition

$$h_{A_{\infty_1}} = d h_{\mathcal{O}_K}.$$

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$$h_{A_{\infty_1}} = dh_{\mathcal{O}_K}.$$

Let $Cl_K := Div(K)/Princ(K)$ the class group of K.

Proposition

$$Cl_{A_{\infty_1}} \simeq Cl_K/\langle \infty_1 \rangle, \;\; Cl_{\mathcal{O}_K} \simeq Cl_K/\langle \infty_1, \infty_2 \rangle.$$

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Corollary

$$Ker(\pi) = \{[\mathfrak{a}_0], ..., [\mathfrak{a}_{d-1}]\} = \langle[\mathfrak{a}_{d-1}]\rangle = \langle\infty_2\rangle \simeq \mathbb{Z}/d\mathbb{Z}.$$

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By Artin reciprocity:

$$Ker(\pi) \simeq G(H_{A_{\infty_1}}/H_{\mathcal{O}_K}).$$

Class field generation

Therefore:

$$\forall \sigma \in G(H_{A_{\infty_1}}/H_{\mathcal{O}_K}),\ N(j^{qt}(h))^{\sigma} = \prod_{\mathfrak{a}_i \in Ker(\pi)} j(\mathfrak{a}_i)^{\sigma} =$$

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It follows that:

$$N(j^{qt}(h)) \in H_{\mathcal{O}_K}.$$

Class field generation

Theorem

$$\sharp Orb_{G(H_{\mathcal{O}_K}/K)}(N(j^{qt}(h))) = h_{\mathcal{O}_K} = [H_{\mathcal{O}_K} : K].$$

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As $G(H_{A_{\infty_1}}/K)$ is **abelian**: $\Longrightarrow K(N(j^{qt}(h)))/K$ is Galois extension.

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Corollary

$$K(N(j^{qt}(h))) = H_{\mathcal{O}_K}.$$

Class field generation

The $\epsilon-$exponential $\exp_\epsilon : \mathcal{C} \to \mathcal{C}$ defined in the intuitive way:

$$\exp_\epsilon(z) := z \prod_{\lambda \in \Lambda_\epsilon(h) \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right);$$

does not give any deep information because:

$$\lim_{\epsilon \to 0^+} \exp_\epsilon(z) = z.$$

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Given Λ a rank 1 lattice, the corresponding rank 1 Drinfeld module becomes sign normalized by rescaling Λ by a suitable factor $\xi(\Lambda) \in \mathcal{C}$ explicitly computed as a function of Λ.
We define a transcendental $\epsilon-$scaling factor $\xi_\epsilon \in \mathcal{C}$ associated to $\Lambda_\epsilon(h)$ in complete analogy.

Class field generation

Definition

For every $\epsilon > 0$ *the normalized* $\epsilon-$*exponential* $\widehat{\exp}_\epsilon : \mathcal{C} \to \mathcal{C}$ *is defined as follows:*

$$\widehat{\exp}_\epsilon(z) := \xi_\epsilon \exp_\epsilon(\xi_\epsilon^{-1} z).$$

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Definition

$$\exp^{qt}(z) := \lim_{\epsilon \to 0^+} \widehat{\exp}_\epsilon(z).$$

Class field generation

Theorem

*The **quantum exponential** is a multi-valued function:*

$$\exp_h^{qt} : \mathcal{C} \multimap \mathcal{C};$$

such that:

$$\exp_h^{qt}(z) = \{e_0(z), \dots, e_{d-1}(z)\};$$

where $e_0, \dots, e_{d-1} : \mathcal{C} \to \mathcal{C}$ *are the exponential functions associated to the Hayes modules* $\Psi_0, \dots, \Psi_{d-1}$*, such that:*

$$e_i : \mathcal{C}/\xi_i \mathfrak{a}_i \simeq \Psi_i;$$

for $i = 0, \dots, d-1$*, where:*

$$\lim_{\epsilon \to 0^+} \xi_\epsilon = \{\xi_0, \dots, \xi_{d-1}\}.$$

Class field generation

Let $\mathfrak{M}$ be an ideal of $\mathcal{O}_K$.

Definition

$$\mathcal{I}^{\mathfrak{M}} := \{\mathfrak{A} \subseteq \mathcal{O}_K, \ \ (\mathfrak{A}, \mathfrak{M}) = 1\}.$$

$$\mathcal{P}^{\mathfrak{M}} := \{a\mathcal{O}_K \in \mathcal{I}^{\mathfrak{M}}, \ \ a \equiv 1 \ \ mod \ \ \mathfrak{M}\}.$$

$$\mathcal{P}_1^{\mathfrak{M}} := \{a\mathcal{O}_K \in \mathcal{P}^{\mathfrak{M}}, \ \ a \ positive \ (sgn(a) = 1)\}.$$

$$\mathcal{P}_2^{\mathfrak{M}} := \{a\mathcal{O}_K, \ \ a^{\sigma} \ \ positive, 1 \neq \sigma \in G(K/k)\}.$$

$$\mathcal{P}_+^{\mathfrak{M}} := \{a\mathcal{O}_K, \ \ a^{\sigma} \ \ positive, \forall \sigma \in G(K/k)\}.$$

Class field generation

Definition

$$Cl^{\mathfrak{M}} := \mathcal{I}^{\mathfrak{M}} / \mathcal{P}^{\mathfrak{M}}.$$

$$Cl_i^{\mathfrak{M}} := \mathcal{I}^{\mathfrak{M}} / \mathcal{P}_i^{\mathfrak{M}}, \quad i = 1, 2.$$

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Completely analogous definitions are given for $\mathfrak{m} := \mathfrak{M} \cap A_{\infty_1}$, except that there is no more need to distinguish positivities as the place above ∞ which is considered is only one.

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Completely analogous definitions are given for $\mathfrak{m} := \mathfrak{M} \cap A_{\infty_1}$, except that there is no more need to distinguish positivities as the place above ∞ which is considered is only one.

Definition

$K^{\mathfrak{m}}$ and $K_1^{\mathfrak{M}}$ are the unique abelian extensions of K having their Galois groups over K isomorphic via reciprocity to $Cl_+^{\mathfrak{m}}$ and $Cl_1^{\mathfrak{M}}$, respectively.

Class field generation

We call:

$$Z := G(K^{\mathfrak{m}}/K_1^{\mathfrak{M}}).$$

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Theorem

Z acts transitively on the set of $A_{\infty_1}-$modules of $\mathfrak{m}-$torsion points $\{\Psi_0[\mathfrak{m}], \ldots, \Psi_{d-1}[\mathfrak{m}]\}$.

Class field generation

Definition

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The analogous result is shown as well for $K_2^{\mathfrak{M}}$ by repeating the arguments on A_{∞_2} and replacing f with f'. Then:

$$K^{\mathfrak{M}} = K_1^{\mathfrak{M}} K_2^{\mathfrak{M}}.$$

This gives an explicit description of $K^{\mathfrak{M}}$ in terms of $\exp^{qt}$ and t^{qt}.
Submitted soon