Quantum $j$ invariant and Real Multiplication program for global function fields

Luca Demangos

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Hilbert’s 12th problem

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1. The Hilbert class field $H_K$ of $K$: the maximal abelian extension of $K$, unramified at every place.
2. The maximal abelian extension $K^{ab}$ of $K$. 
The case $K = \mathbb{Q}$

Theorem of Kronecker-Weber

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**Theorem of Kronecker-Weber**

The abelian closure of $\mathbb{Q}$ is

$$\mathbb{Q}^{ab} = \mathbb{Q}(\langle \zeta_n \rangle_{n \in \mathbb{N} \setminus \{0\}}).$$
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In particular, when $K = \mathbb{Q}$, we have that $H_\mathbb{Q} = \mathbb{Q}$. 
Imaginary quadratic number fields

For $K$ a number field the only case essentially known so far, other than $K = \mathbb{Q}$, is $K = \mathbb{Q}(\mu)$, $\mu$ imaginary quadratic.
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In this case the solution comes by the theory of complex multiplication.

Let $d \in \mathbb{N} \setminus \{0\}$, squarefree, $\mu := \sqrt{-d}$, $\Lambda_\mu := \langle 1, \mu \rangle \mathbb{Z}$ is a CM lattice in $\mathbb{C}$. To $\Lambda_\mu$ one associates therefore the CM elliptic curve:

$$E_\mu : Y^2 = 4X^3 - 60G_4(\mu)X - 140G_6(\mu);$$

where:

$$G_n(\mu) := \sum_{\lambda \in \Lambda_\mu \setminus \{0\}} \lambda^{-n};$$

are the corresponding Eisenstein series.
Imaginary quadratic number fields

Let:

\[ j(\mu) := \frac{12^3}{1 - \frac{49}{20} \frac{G_6(\mu)^2}{G_4(\mu)^3}}; \]

be the value taken in \( \mu \) by the \( j \)-invariant:

\[ j : Mod \to \mathbb{C}; \]

defined on the moduli space:

\[ Mod = (\mathbb{C} \setminus \mathbb{R})/GL_2(\mathbb{Z}); \]

of elliptic curves defined over \( \mathbb{C} \).
Imaginary quadratic number fields

\[ j(\mu) \in \bar{K} \]
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**Theorem of Fueter-Weber**

1. There exists \( \alpha \in \mathbb{Q}(\mu) \setminus \mathbb{Q} \) such that:

\[ H_K = K(j(\alpha)); \]

2. \( K^{ab} = H_K(h(E_{\alpha \text{tors}})); \)

where \( h : E_{\alpha} \to \mathbb{P}_1 \) a Weber function.
More precisely, by calling $g_2(\mu) := 60G_4(\mu)$ and $g_3(\mu) := 140G_4(\mu)$, and $z := [\wp_z : \wp'_z : 1]$ the general point of $E_\mu$:

The Weber function $h$ is in particular:

$$h(z) = \begin{cases} 
\frac{g_2(\mu)g_3(\mu)}{\Delta(\mu)} \wp_z, & \text{if } j(\mu) \neq 0, 1728 \\
\frac{g_2(\mu)^2}{\Delta(\mu)} \wp'_z, & \text{if } j(\mu) = 1728 \\
\frac{g_3(\mu)}{\Delta(\mu)} \wp_z^3, & \text{if } j(\mu) = 0
\end{cases}$$

where $\Delta(\mu) = g_2(\mu)^2 - 27g_3(\mu)^3 \neq 0$. 

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\[ \begin{align*}
K & \overset{\text{unramified}}{\longrightarrow} H_K & \overset{\text{roots/torsion}}{\longrightarrow} K^{ab}
\end{align*} \]
Imaginary quadratic number fields

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K \xrightarrow{unramified} H_K \xrightarrow{roots/torsion} K^{ab}
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For example, if \( K = \mathbb{Q} \) the first step is skipped as \( H_{\mathbb{Q}} = \mathbb{Q} \), so that:

\[
\mathbb{Q}^{ab} = \mathbb{Q}(S^1_{\text{tors.}}) = \mathbb{Q}(e^{2\pi i \mathbb{Q}})
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directly (\( S^1 \) is the unit circle in \( \mathbb{C} \)).
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directly (\( S^1 \) is the unit circle in \( \mathbb{C} \)).

(Here the role of the Weierstrass \( \wp \)—function is played by the exponential \( e^{2\pi i} : \mathbb{R}/\mathbb{Z} \to S^1 \)).
Real Multiplication program

Let $K$ be a real quadratic number field. $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{N} \backslash \{0\}$, squarefree. The set $\langle 1, \theta \rangle \mathbb{Z}$ is a dense subgroup in $\mathbb{R}$. It is therefore not a lattice any longer. The corresponding quotient: $\Pi(\theta) := \mathbb{R} / \langle 1, \theta \rangle \mathbb{Z}$ is a quantum object. It is NOT Hausdorff and it is called a quantum torus.
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Real Multiplication program
Y. Manin’s RM project

Eisenstein series can not be defined over a subgroup dense in $\mathbb{R}$. The project of developing a theory of "real multiplication" for quantum tori has been proposed by Y. Manin in order to solve Hilbert’s 12th problem for real quadratic number fields in the spirit of Fueter-Weber theorem.
Let us define $\|x\| := \inf_{n \in \mathbb{Z}} \{d(x, n)\}$, for all $x \in \mathbb{R}$. For all $\epsilon > 0$ it is defined:
C. Castaño Bernard - T. M. Gendron results

Let us define $\|x\| := \inf_{n \in \mathbb{Z}} \{d(x, n)\}$, for all $x \in \mathbb{R}$. \(\forall \epsilon > 0\) it is defined:

$$\Lambda_\epsilon(\theta) := \{n \in \mathbb{Z}, \|n\theta\| < \epsilon\}.$$

Such a set is not trivially \{0\} (and actually infinite) if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ by Kronecker’s Theorem.
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$$\zeta_{\theta, \epsilon}(k) := \sum_{\lambda \in \Lambda_\epsilon(\theta) \setminus \{0\}} \lambda^{-k};$$

$$j_\epsilon(\theta) := \frac{12^3}{1 - \frac{49}{20} \frac{\zeta_{\theta, \epsilon}(6)^2}{\zeta_{\theta, \epsilon}(4)^3}}.$$
Theorem (T. M. Gendron, C. Castaño Bernard)

The limit:

$$j^{qt}(\theta) := \lim_{\epsilon \to 0^+} j_\epsilon(\theta)$$

produces multiple values and it is a modular invariant defined on the moduli space of quantum tori.

Definition

*The quantum $j$–invariant is the multi-valued function:*

$$j^{qt} : \mathbb{R}/GL_2(\mathbb{Z}) \to \mathbb{R} \cup \{\infty\}.$$
PARI-GP suggests that if $\theta$ is a fundamental unit of $O_K$ with fundamental discriminant $D$:

1. $\#\{j^{qt}(\theta)\} = D$;

2. $\infty \notin j^{qt}(\theta)$.

If $\theta = (1 + \sqrt{5})/2$ T. M. Gendron and C. Castaño Bernard computed explicit values.
PARi-GP suggests that if $\theta$ is a fundamental unit of $\mathcal{O}_K$ with fundamental discriminant $D$:

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Values of $j_\varepsilon(\phi)$, $\phi = \text{the golden mean}$

$0.8501..., j_\varepsilon(\phi)$

$\varepsilon = \phi^{-t}$

$0.8188..., t \to \infty (\varepsilon \to 0)$
C. Castaño Bernard - T. M. Gendron results

Conjecture

Let $\theta$ be irrational real quadratic. One then has that:

1. $\# j^{qt}(\theta) < \infty$ and $\infty \notin j^{qt}(\theta)$;
2. $j^{qt}(\theta) \subset \overline{\mathbb{Q}}$;
3. If $\theta$ is a fundamental unit:

$$H_K = K(N(j^{qt}(\theta)))$$

for $N$ some weighted norm.
Definitions

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We know that if $X = \mathbb{P}_1(\mathbb{F}_q)$ and $P = \infty$:

$$A := A_P = \mathbb{F}_q[T] \quad k := \mathbb{F}_q(T)$$

$$\nu_\infty = \nu_{1/T} := -\deg_T(\cdot) \quad |\cdot| = |\cdot|_{1/T} := q^{\deg_T(\cdot)}$$

$$k_\infty = \mathbb{F}_q((1/T)) \quad C := (\overline{k_\infty})_\infty$$
Definitions

\[ A \leftrightarrow \mathbb{Z}; \]
\[ k \leftrightarrow \mathbb{Q}; \]
\[ k_\infty \leftrightarrow \mathbb{R}; \]
\[ C \leftrightarrow \mathbb{C}. \]
Definitions

The Hilbert class field $H_K$ has infinite degree over $K$ as $\overline{\mathbb{F}_q}/\mathbb{F}_q$ is clearly abelian and unramified.
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If we restrict the definition of $H_K$ to be geometric, then there are $h_K = \# Cl_K$ distinct abelian, unramified extensions of $K$ which are all maximal.
Definitions

**Definition**

A *real* algebraic function field is an algebraic extension of $k$ which is contained in $k_{\infty}$. 
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$K/k$ real quadratic. So this field extension corresponds to a degree 2 morphism:

$$\pi : X \to \mathbb{P}_1(\mathbb{F}_q).$$
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Note that:

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\pi^{-1}(\infty) = \{\infty_1, \infty_2\}.
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Definitions

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*Relative integers* $\mapsto \mathcal{O}_K := \mathbb{F}_q[X \setminus \pi^{-1}(\infty)]$.

*Absolute integers* $\mapsto A_{\infty_1} := \mathbb{F}_q[X \setminus \{\infty_1\}]$. 
Definitions

Clearly:

\[ \mathcal{O}_K \supseteq A_{\infty_1}. \]
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We call **relative Hilbert class field** $H_{\mathcal{O}_K}$ the maximal abelian unramified extension of $K$ which *splits completely at* $\pi^{-1}(\infty)$.
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Clearly, if \( K/k \) is complex (which means \( \pi^{-1}(\infty) = \{\infty\} \)), we have that \( H_{A_\infty} = H_{\mathcal{O}_K} \).
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If \( K/k \) is real (which means \( \pi^{-1}(\infty) = \{\infty_1, \infty_2\} \)), we have that \( H_{A_{\infty_1}} \nsubseteq H_{\mathcal{O}_K} \).
Real quadratic function fields

Let us take $f \in k_\infty \cap \bar{k}$, quadratic, so that:

$$K = k(f).$$
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We may assume without loss of generality that \( f \) is a fundamental unit:

\[ f^2 = a(T)f + b, \quad a(T) \in A, \quad b \in \mathbb{F}_{q^*}. \]

Let \( d := \deg_T(discr(f)) = \deg_T(a(T)) \).
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Let $d := \deg_T(\text{discr}(f)) = \deg_T(a(T))$. Therefore

$$|f| = |a(T)| = q^d \text{ and } |f'| = |b/f| = q^{-d}. $$

$$f \in A_{\infty_1}, \notin A_{\infty_2}, \ f^{-1} \in A_{\infty_2}, \notin A_{\infty_1}, \ T \in \mathcal{O}_K, \notin A_{\infty_1}, \notin A_{\infty_2}. $$
Real quadratic function fields

\[ \mathcal{O}_K = \mathbb{F}_q[T, f] \supset A_{\infty_1} = \mathbb{F}_q[f, Tf, \ldots, T^{d-1}f]. \]
Real quadratic function fields

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\[ f \in \mathcal{O}_K^*, \notin A_{\infty_1}^*, \notin A_{\infty_2}^*. \] Therefore the group of units of \( \mathcal{O}_K \) is infinite and no \( \mathcal{O}_K \)-lattice can be embedded in \( \mathcal{C} \). While on the other hand \( A_{\infty_1}^* = \mathbb{F}_q^* \).
Real quadratic function fields

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$$f \in \mathcal{O}_K^*, \notin A_{\infty_1}^*, \notin A_{\infty_2}^*.$$ Therefore the group of units of $\mathcal{O}_K$ is infinite and no $\mathcal{O}_K$–lattice can be embedded in $\mathcal{C}$. While on the other hand $A_{\infty_1}^* = \mathbb{F}_q^*.$

We also note that both $A_{\infty_1}$ and $\mathcal{O}_K$ are Dedekind domains, so the ideal classes over them form actually a group.
Real quadratic function fields

$A_{\infty_1}$—lattices can be embedded in $\mathcal{C}$. This gives rise to a rank 1 theory which provides a generalization of the Theorem of Kronecker-Weber for any finite extension of $\mathbb{F}_q(T)$. 
Real quadratic function fields

$A_{\infty_1}$—lattices can be embedded in $C$. This gives rise to a rank 1 theory which provides a generalization of the Theorem of Kronecker-Weber for any finite extension of $\mathbb{F}_q(T)$.

**Definition**

An *Hayes module* is a rank 1 *sign-normalized* Drinfeld module:

$$\Psi : A_{\infty_1} \to \overline{K}\{\tau\}.$$
Real quadratic function fields

\( A_{\infty 1} \) —lattices can be embedded in \( \mathcal{C} \). This gives rise to a **rank 1** theory which provides a generalization of the Theorem of Kronecker-Weber for any finite extension of \( \mathbb{F}_q(T) \).

**Definition**

An **Hayes module** is a **rank 1** sign-normalized Drinfeld module:

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\Psi : A_{\infty 1} \rightarrow \overline{K}\{\tau}\).
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There are \( h_{A_{\infty 1}} \) distinct isomorphism classes of Hayes modules over \( A_{\infty 1} \).
Theorem of Hayes

1. $H_{A_{\infty 1}}$ is the smallest field of definition of a Hayes module over $A_{\infty 1}$.

2. For every given modulus $m$ in $A_{\infty 1}$ the (narrow) absolute ray class field of conductor dividing $m$ is:

$$K_{\infty 1}^m = H_{A_{\infty 1}}(\Psi[m]).$$
Kronecker’s Theorem for function fields

For any $h \in k_\infty \setminus k$ the image of $Ah$ is dense in $S^1 := k_\infty / A$. 
Definitions

Kronecker’s Theorem for function fields
For any $h \in k_\infty \backslash k$ the image of $Ah$ is dense in $S^1 := k_\infty / A$.

Definition
The corresponding quantum torus is the quotient:

$$k_\infty / \langle 1, h \rangle_A.$$
Definitions

**Definition**

Let $h \in k_\infty$, $\epsilon > 0$. We define:

$$\Lambda_\epsilon(h) := \{ a \in A, \| ah \| < \epsilon \};$$
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Let $h \in k_\infty$, $\epsilon > 0$. We define:

$$\Lambda_\epsilon(h) := \{ a \in A, \| ah \| < \epsilon \};$$

where $\|x\| := \inf_{a \in A} \{|x - a|\}$, for any given $x \in k_\infty$. 

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\[ \zeta_{h, \epsilon}(n) := \sum_{a \in \Lambda_{\epsilon}(h)^+ \setminus \{0\}} a^{-n}, \ \forall n \in \mathbb{N}. \]
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**Definition**

\[ j_\epsilon(h) := \frac{T^q - T}{1 - \frac{T^{q^2} - T^q}{T^{q^2} - T} \frac{\zeta_{h, \epsilon}(q^2 - 1)}{\zeta_{h, \epsilon}(q - 1)^{q+1}}}, \ \forall \epsilon > 0. \]
Definitions

\[ \zeta_{h, \epsilon}(n) := \sum_{a \in \Lambda_{\epsilon}(h)^+ \setminus \{0\}} a^{-n}, \quad \forall n \in \mathbb{N}. \]

\[ j_{\epsilon}(h) := \frac{T^q - T}{1 - \frac{T}{T q^2 - T} \frac{T q^2 - T}{T q^2 - T} \frac{\zeta_{h, \epsilon}(q^2 - 1)}{\zeta_{h, \epsilon}(q - 1) q + 1}}, \quad \forall \epsilon > 0. \]

(Analogue of E. U. Gekeler’s \( j \)-invariant for rank 2 Drinfeld modules).
Definitions

Definition

We call quantum $j$-invariant the following multi-valued function:

$$j^{qt} : k_\infty / GL_2(A) \rightarrow k_\infty \cup \{\infty\};$$

$$j^{qt}(h) := \lim_{\epsilon \rightarrow 0^+} j_\epsilon(h).$$
Definitions

Theorem

1. \( \forall 0 < \epsilon < 1, \ \forall h \in k \setminus k, \ |j_\epsilon(h)| = q^{2q-1}. \)

In particular, \( \infty \not\in j^{qt}(h). \)

2. \( h \in k \iff j^{qt}(h) = \infty. \)
Definitions

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2

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Theorem

1. \( h \text{ is quadratic} \implies \#j^{qt}(h) < +\infty. \)

2. \( h \text{ is quadratic unit} \implies \#j^{qt}(h) = d = \deg_T(discr(h)). \)
Definitions

**Theorem**

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2. \( h \) is quadratic unit \( \implies \#j^{qt}(h) = d = \deg_T(\text{discr}(h)). \)

Main theorem

Let $K/k$ be quadratic and real, $h \in \mathcal{O}^{\ast}_K$ unit. Then:

$$H_{\mathcal{O}_K} = K(N(j^{qt}(h))) ;$$

$$N(j^{qt}(h)) := \prod_{\alpha \in j^{qt}(h)} \alpha.$$
Class field generation

Main theorem

Let $K/k$ be quadratic and real, $h \in \mathcal{O}_{K}^{*}$ unit. Then:

$$H_{\mathcal{O}_{K}} = K(N(j^{qt}(h)));$$

$$N(j^{qt}(h)) := \prod_{\alpha \in j^{qt}(h)} \alpha.$$ 

Class field generation

Step 1: Diophantine approximation

Let $h \in k_\infty \setminus k$. Let assume $h^2 = a(T)h + b$, $b \in \mathbb{F}_q^*$. 

\[ q_0 := 1, \quad q_1 := a, \quad \ldots, \quad q_i := aq_{i-1} + bq_{i-2}; \]

(best approximation of $h$ if $b = 1$ by continued fraction).

We have that

\[ || q_n h || = |q_n h - q_{n+1}| = |h'_{n+1}| || h - h' || q^{-n-(n+1)/2} d. \]

This follows by Binet's formula:

\[ q_n = h_n + (−h')_n + \sqrt{D}, \forall n \in \mathbb{N}. \]
Class field generation

Step 1: Diophantine approximation

Let $h \in k_\infty \setminus k$. Let assume $h^2 = a(T)h + b$, $b \in \mathbb{F}_q^*$. We define the sequence:

\[ q_0 := 1, \quad q_1 := a, \quad \ldots, \quad q_i := aq_{i-1} + bq_{i-2}; \]

(best approximation of $h$ if $b = 1$ by continued fraction).
Class field generation

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We have that $\|q_nh\| = |q_nh - q_{n+1}| = \frac{|h'^{n+1}| |h-h'|}{|a(T)|} = q^{-(n+1)d}$. 

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We have that $||q_nh|| = |q_nh - q_{n+1}| = \frac{|h'^{n+1}||h-h'|}{|a(T)|} = q^{-(n+1)d}$.

This follows by Binet’s formula:

$$q_n = \frac{h^{n+1} + (-h')^{n+1}}{\sqrt{D}}, \quad \forall n \in \mathbb{N}.$$
Class field generation

\{ T^{d-1} q_0, \ldots, T q_0, q_0, T^{d-1} q_1, \ldots, T q_1, q_1, \ldots \} is \mathbb{F}_q - \text{basis of } A, where the order is by \textbf{decreasing errors}:

\[ \| T^l q_n h \| = | T^l q_n h - T^l q_{n+1} | = q^{l-(n+1)d} < 1. \]
Class field generation

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We fix:

\[ \epsilon := q^{-Nd-l}, \quad l = 0, \ldots, d - 1. \]
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\left\| T^l q_n h \right\| = \left| T^l q_n h - T^l q_{n+1} \right| = q^{l-(n+1)d} < 1.

We fix:

\epsilon := q^{-Nd-l}, \quad l = 0, \ldots, d - 1.

One can prove that:

\Lambda_\epsilon(h) = \mathbb{F}_q \langle q_N, q_N T, \ldots, q_N T^{d-l-1}, q_{N+1}, q_{N+1} T, \ldots, q_{N+1} T^{d-1}, \ldots \rangle.
Class field generation

\{ T^{d-1} q_0, ..., T q_0, q_0, T^{d-1} q_1, ..., T q_1, q_1, ... \} is \( \mathbb{F}_q \)-basis of \( A \), where the order is by decreasing errors:

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By Binet’s formula, if \( N \to +\infty \), so \( \epsilon \to 0^+ \), we have that:

\[ q_{N+i} \approx \frac{h^{N+1+i}}{\sqrt{D}}. \]
Class field generation

Let:

\[ a_i := (h, hT, ..., hT^i)_{A_{\infty 1}} \]

for any \( i = 0, ..., d - 1 \).
Class field generation

Let:

$$a_i := (h, hT, \ldots, hT^i)_{A_{\infty 1}};$$

for any $i = 0, \ldots, d - 1$.

So, by defining:

$$\zeta_{a_{d-l-1}}(n) := \sum_{\alpha \in a_{d-l-1}^+ \setminus \{0\}} \alpha^{-n};$$
Class field generation

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So, by defining:

\[ \zeta_{a_{d-l-1}}(n) := \sum_{\alpha \in \Lambda^+ a_{d-l-1} \setminus \{0\}} \alpha^{-n}; \]

we obtain:

\[ \zeta_{h,\epsilon}(n) = \sum_{a \in \Lambda_{\epsilon}(h)^+ \setminus \{0\}} a^{-n} \xrightarrow{\epsilon \to 0^+} \frac{h^N}{\sqrt{D}} \{ \zeta_{a_{d-l-1}}(n) \}_{l=0,\ldots,d-1}. \]

By homogeneity of the degree \( 1 - q^2 \) we have cancellation of \( (h^N/\sqrt{D})^{1-q^2} \) in the expression of \( j^{qt}(h) \), so we obtain:
Class field generation

Step 2: Description of $j^{qt}$

$$j^{qt}(h) = \left\{ \frac{T^q - T}{1 - \frac{T{q^2} - T}{T{q^2} - Tq} \frac{\zeta_i(q^2 - 1)}{\zeta_i(q - 1)q^2}} \right\}_{i=0,\ldots,d-1} = \{j(a_i)\}_{i=0,\ldots,d-1}.$$
Class field generation

Step 2: Description of $j^{qt}$

$$j^{qt}(h) = \begin{cases} \frac{T^q - T}{1 - \frac{T^q - T}{T^{q^2 - T}} \frac{\zeta_a(q^2 - 1)}{\zeta_a(q - 1)^{q + 1}}} \end{cases} \quad = \{j(a_i)\}_{i=0,\ldots,d-1}.$$  

Theorem of Goss

$$\exists r \in C \setminus \{0\}, \forall a \in Cl_{A_{\infty}}, \forall n \equiv 0 \mod (q - 1):$$

$$\frac{\zeta_a(n)}{r^n} \in H_{A_{\infty}}.$$
Class field generation

Step 2: Description of $j^{qt}$

$$j^{qt}(h) = \left\{ \frac{T^q - T}{1 - \frac{Tq^2 - T}{Tq^2 - Tq} \frac{\zeta_{a_i}(q^2 - 1)}{\zeta_{a_i}(q - 1)q^1}} \right\}_{i=0,\ldots,d-1} = \{j(a_i)\}_{i=0,\ldots,d-1}.$$ 

Theorem of Goss

$$\exists r \in C \setminus \{0\}, \forall a \in Cl_{A_{\infty 1}}, \forall n \equiv 0 \mod (q - 1) : \frac{\zeta_a(n)}{r^n} \in H_{A_{\infty 1}}.$$ 

This generalizes Euler’s formula:

$$\frac{\zeta(2n)}{\pi^{2n}} \in Q = H_Q.$$
Class field generation

Again by homogeneity of the degree $1 - q^2$ we factor out $r^1 - q^2$. 
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$$j(a) \in H_{A_{\infty 1}}, \forall a \in Cl_{A_{\infty 1}}.$$

Which means that:

$$j^{qt}(h) \subset H_{A_{\infty 1}}.$$
Step 3: Class field generation

\[ G(H_{A_{\infty,1}}/K) \overset{\pi}{\longrightarrow} G(H_{O_K}/K) \]

\[ \overset{\simeq}{\longrightarrow} \]

\[ Cl_{A_{\infty,1}} \overset{\pi}{\longrightarrow} Cl_{O_K} \]

where:

\[ \pi : a \mapsto aO_K. \]
Class field generation

Step 3: Class field generation

\[ \begin{align*}
G(H_{A_{\infty}} / K) & \xrightarrow{\pi} G(H_{O_k} / K) \\
\xrightarrow{\cong} & \xrightarrow{\cong} \\
Cl_{A_{\infty}} & \xrightarrow{\pi} Cl_{O_k}
\end{align*} \]

where:
\[ \pi : a \mapsto aO_K. \]

It is easy to see that:
\[ \{a_i\}_{i=0, \ldots, d-1} = \{(f, fT, \ldots, fT^i)\}_{i=0, \ldots, d-1} \subset Ker(\pi). \]
**Class field generation**

**Step 3: Class field generation**

\[
G(H_{A_{\infty 1}} / K) \xrightarrow{\pi} G(H_{\mathcal{O}_K} / K) \\
\downarrow \cong \downarrow \cong \\
Cl_{A_{\infty 1}} \xrightarrow{\pi} Cl_{\mathcal{O}_K}
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where:

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\pi : a \mapsto a\mathcal{O}_K.
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It is easy to see that:

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\((f \in \mathcal{O}_K^*)\).
Proposition

\[ h_{A_{\infty_1}} = dh_{\mathcal{O}_K}. \]
Class field generation

Proposition

\[ h_{A_{\infty 1}} = dh_{\mathcal{O}_K}. \]

Let \( Cl_K := \text{Div}(K)/\text{Princ}(K) \) the class group of \( K \).

Proposition

\[ Cl_{A_{\infty 1}} \simeq Cl_K/\langle \infty_1 \rangle, \quad Cl_{\mathcal{O}_K} \simeq Cl_K/\langle \infty_1, \infty_2 \rangle. \]
Class field generation

Proposition

$$h_{A_{\infty_1}} = dh_{O_K}.$$  

Let $Cl_K := \text{Div}(K)/\text{Princ}(K)$ the class group of $K$.

Proposition

$$Cl_{A_{\infty_1}} \cong Cl_K/\langle \infty_1 \rangle, \quad Cl_{O_K} \cong Cl_K/\langle \infty_1, \infty_2 \rangle.$$  

Corollary

$$Ker(\pi) = \{[a_0], \ldots, [a_{d-1}]\} = \langle [a_{d-1}] \rangle = \langle \infty_2 \rangle \cong \mathbb{Z}/d\mathbb{Z}.$$  

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Therefore:

\[ j^{qt}(h) = \{ j(a), \ a \in \text{Ker}(\pi) \}. \]
Therefore:

$$j^{qt}(h) = \{j(a), \ a \in \text{Ker} (\pi)\}.$$

By Artin reciprocity:

$$\text{Ker} (\pi) \simeq G(H_{A_{\infty}} / H_{O_K}).$$
Therefore:

\[ \forall \sigma \in G(H_{A_{\infty 1}}/H_{O_K}), \quad N(j^{qt}(h))^\sigma = \prod_{a_i \in \text{Ker}(\pi)} j(a_i)^\sigma = \prod_{a_i \in \text{Ker}(\pi)} j(a_i)^{\sigma - 1} = N(j^{qt}(h)). \]
Class field generation

Therefore:

\[ \forall \sigma \in G(H_{A_{\infty_1}}/H_{O_K}), \ N(j^{qt}(h))^\sigma = \prod_{a_i \in Ker(\pi)} j(a_i)^\sigma = \prod_{a_i \in Ker(\pi)} j(a_i a_\sigma^{-1}) = N(j^{qt}(h)). \]
Therefore:

\[
\forall \sigma \in G(H_{A_{\infty}/H_{\mathcal{O}_K}}), \quad N(j^{qt}(h))\sigma = \prod_{a_i \in \text{Ker}(\pi)} j(a_i)\sigma = \prod_{a_i \in \text{Ker}(\pi)} j(a_ia_{\sigma}^{-1}) = N(j^{qt}(h)).
\]

It follows that:

\[
N(j^{qt}(h)) \in H_{\mathcal{O}_K}.
\]
Class field generation

Theorem

\[ \#Orb_G(H_{\mathcal{O}_K}/K)(N(j^{qt}(h))) = h_{\mathcal{O}_K} = [H_{\mathcal{O}_K} : K]. \]
Class field generation

Theorem

\[ \#\text{Orb}_G(H_\mathcal{O}_K/K)(N(j^{qt}(h))) = h_\mathcal{O}_K = [H_\mathcal{O}_K : K]. \]

As \( G(H_{A_{\infty 1}}/K) \) is abelian: \( \implies K(N(j^{qt}(h)))/K \) is Galois extension.
Class field generation

**Theorem**

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As \( G(H_{A_{\infty 1}}/K) \) is abelian: \[ \implies \] \( K(N(j^{qt}(h)))/K \) is Galois extension.

**Corollary**

\[ K(N(j^{qt}(h))) = H_{\mathcal{O}_K}. \]
Class field generation

The $\epsilon$–exponential $\exp_\epsilon : C \rightarrow C$ defined in the intuitive way:

$$\exp_\epsilon(z) := z \prod_{\lambda \in \Lambda_\epsilon(h) \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right);$$

does not give any deep information because:

$$\lim_{\epsilon \rightarrow 0^+} \exp_\epsilon(z) = z.$$
Class field generation

The $\epsilon-$exponential $\exp_\epsilon : \mathbb{C} \to \mathbb{C}$ defined in the intuitive way:

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does not give any deep information because:

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Given $\Lambda$ a rank 1 lattice, the corresponding rank 1 Drinfeld module becomes sign normalized by rescaling $\Lambda$ by a suitable factor $\xi(\Lambda) \in \mathbb{C}$ explicitly computed as a function of $\Lambda$. We define a transcendental $\epsilon-$scaling factor $\xi_\epsilon \in \mathbb{C}$ associated to $\Lambda_\epsilon(h)$ in complete analogy.
For every $\epsilon > 0$ the normalized $\epsilon-$exponential $\hat{\exp}_\epsilon : \mathbb{C} \to \mathbb{C}$ is defined as follows:

$$\hat{\exp}_\epsilon(z) := \xi_\epsilon \exp_\epsilon(\xi_\epsilon^{-1} z).$$
Class field generation

**Definition**

For every $\epsilon > 0$ the normalized $\epsilon$-exponential $\hat{\exp}_{\epsilon} : \mathbb{C} \rightarrow \mathbb{C}$ is defined as follows:

$$\hat{\exp}_{\epsilon}(z) := \xi_{\epsilon} \exp_{\epsilon}(\xi_{\epsilon}^{-1} z).$$

**Definition**

$$\exp^{qt}(z) := \lim_{\epsilon \to 0^+} \hat{\exp}_{\epsilon}(z).$$
Class field generation

Theorem

The quantum exponential is a multi-valued function:

\[ \exp^q_t : \mathbb{C} \rightarrow \mathbb{C}; \]

such that:

\[ \exp^q_t(z) = \{ e_0(z), ..., e_{d-1}(z) \}; \]

where \( e_0, ..., e_{d-1} : \mathbb{C} \rightarrow \mathbb{C} \) are the exponential functions associated to the Hayes modules \( \Psi_0, ..., \Psi_{d-1} \), such that:

\[ e_i : \mathbb{C}/\xi_i \alpha_i \simeq \Psi_i; \]

for \( i = 0, ..., d - 1 \), where:

\[ \lim_{\epsilon \to 0^+} \xi_\epsilon = \{ \xi_0, ..., \xi_{d-1} \}. \]
Let $\mathcal{M}$ be an ideal of $\mathcal{O}_K$.

**Definition**

\[ \mathcal{I}^m := \{ \mathcal{A} \subseteq \mathcal{O}_K, \ (\mathcal{A}, M) = 1 \}. \]
\[ \mathcal{P}^m := \{ a\mathcal{O}_K \in \mathcal{I}^m, \ a \equiv 1 \mod M \}. \]
\[ \mathcal{P}_1^m := \{ a\mathcal{O}_K \in \mathcal{P}^m, \ a \text{ positive (} \text{sgn}(a) = 1) \}. \]
\[ \mathcal{P}_2^m := \{ a\mathcal{O}_K, \ a^\sigma \text{ positive, } 1 \neq \sigma \in G(K/k) \}. \]
\[ \mathcal{P}_+^m := \{ a\mathcal{O}_K, \ a^\sigma \text{ positive, } \forall \sigma \in G(K/k) \}. \]
**Class field generation**

**Definition**

\[ \text{Cl}^m := \mathcal{I}^m / \mathcal{P}^m. \]

\[ \text{Cl}_i^m := \mathcal{I}^m / \mathcal{P}_i^m, \quad i = 1, 2. \]

\[ \text{Cl}^+_m := \mathcal{I}^m / \mathcal{P}^+_m. \]
Class field generation

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\[ Cl^m := \mathcal{I}^m / \mathcal{P}^m. \]
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\[ Cl^+_m := \mathcal{I}^m / \mathcal{P}^+_m. \]

Completely analogous definitions are given for \( m := M \cap A_{\infty 1} \), except that there is no more need to distinguish positivities as the place above \( \infty \) which is considered is only one.
**Class field generation**

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Completely analogous definitions are given for \( m := \mathcal{M} \cap A_{\infty 1} \), except that there is no more need to distinguish positivities as the place above \( \infty \) which is considered is only one.

**Definition**

\( K^m \) and \( K_1^m \) are the unique abelian extensions of \( K \) having their Galois groups over \( K \) isomorphic via reciprocity to \( Cl^m \) and \( Cl_1^m \), respectively.
We call:

\[ Z := G(K^m/K_1^m). \]
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**Theorem**

*Z acts transitively on the set of $A_{\infty_1}$-modules of $m$-torsion points* 
\{\psi_0[m], ..., \psi_{d-1}[m]\}. 

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## Definition

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Class field generation

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\[ t^{qt} := \text{Orb}_Z(t_0), \quad t_0 \in \Psi_0[m] \subset K^m. \]

**Theorem**

\[ K_1^m = H_{\mathcal{O}_K}(\langle \text{Tr}_Z(t^{qt}), \forall t^{qt} \rangle). \]
Class field generation

Definition

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Theorem

\[ K_1^m = H_{O_K}(\langle Tr_Z(t^{qt}), \forall t^{qt} \rangle). \]

The analogous result is shown as well for \( K_2^m \) by repeating the arguments on \( A_{\infty_2} \) and replacing \( f \) with \( f' \). Then:

\[ K^m = K_1^m K_2^m. \]

This gives an explicit description of \( K^m \) in terms of \( \exp^{qt} \) and \( t^{qt} \). Submitted soon