Kronecker limit formula of mirabolic Eisenstein series over function fields

Fu-Tsun Wei

(National Central University)

Conference on the Arithmetic of Function Fields

dedicated to the memory of David Goss

University of Münster, 27 June 2017

Outline

- Number field case.
- “Non-holomorphic” Eisenstein series on Drinfeld half space.
- Function field analogue of Kronecker limit formula.
- Application I: Colmez-type formula of Drinfeld modules.
- Application II: Special values of automorphic L-functions.

Number field case

Given $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$, recall the non-holomorphic Eisenstein series:

$$E(z,s) := \sum_{c,d\in\mathbb{Z}}{}' \frac{\operatorname{Im}(z)^s}{|cz+d|^{2s}}, \quad \operatorname{Re}(s) > 1.$$

It is clear that

$$E\left(\frac{az+b}{cz+d}, s\right) = E(z,s), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Number field case

The Eisenstein series $E(z,s)$ has meromorphic continuation to the complex s-plane satisfying a functional equation:

$$\pi^{-s}\Gamma(s)E(z,s) =: \widetilde{E}(z,s) = \widetilde{E}(z,1-s),$$

where

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx.$$

Kronecker limit formula

$$E(z,0) = -1 \quad \text{and} \quad \frac{\partial}{\partial s}E(z,s)\Big|_{s=0} = -\ln(\operatorname{Im}(z)|\Delta(z)|^{\frac{2}{12}}).$$

Number field case

Here $\Delta(z)$ is the modular discriminant function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right)^{24},$$

which satisfies

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Function field setting

- $k := \mathbb{F}_q(t)$.
- $A := \mathbb{F}_q[t]$.
- $|a/b| := q^{\deg a - \deg b}$ for $a, b \in A$ with $b \neq 0$.
- $k_\infty := \mathbb{F}_q((t^{-1}))$.
- $O_\infty := \mathbb{F}_q[[t^{-1}]]$.
- $\mathbb{C}_\infty$: the completion of an algebraic closure of k_∞.

Drinfeld period domain

Let

$$\mathfrak{H}^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) - \bigcup k_\infty\text{-rational hyperplanes.}$$

We have a left action of $\mathrm{GL}_r(k_\infty)$ on $\mathfrak{H}^r$ defined by:

$$g \cdot z := (\sum_{j=1}^{r} a_{1j} z_j : \cdots : \sum_{j=1}^{r} a_{rj} z_j)$$

for $z = (z_1 : \cdots : z_r) \in \mathfrak{H}^r$ and $g = (a_{ij})_{1 \leq i,j \leq n} \in \mathrm{GL}_r(k_\infty)$

Gekeler's Imaginary parts

Given $z = (z_1 : \cdots : z_{r-1} : z_r = 1) \in \mathfrak{H}^r$, for $1 \leq i < r$ we let

$$\mathrm{Im}(z)_i := \inf_{u_{i+1},\ldots,u_r \in k_\infty} \left| z_i + u_{i+1} z_{i+1} + \cdots + u_{r-1} z_{r-1} + u_r \right|,$$

and $\mathrm{Im}(z) := \prod_{1 \leq i < r} \mathrm{Im}(z)_i$.

Gekeler's Imaginary parts

Given $z = (z_1 : \cdots : z_{r-1} : z_r = 1) \in \mathfrak{H}^r$, for $1 \leq i < r$ we let

$$\mathrm{Im}(z)_i := \inf_{u_{i+1},\ldots,u_r \in k_\infty} |z_i + u_{i+1}z_{i+1} + \cdots + u_{r-1}z_{r-1} + u_r|,$$

and $\mathrm{Im}(z) := \prod_{1 \leq i < r} \mathrm{Im}(z)_i$.

Lemma

Given $z \in \mathfrak{H}^r$ and $g = \begin{pmatrix} & * & * \\ c_1 & \cdots c_{r-1} & d \end{pmatrix} \in \mathrm{GL}_r(k_\infty)$, we have

$$\mathrm{Im}(g \cdot z) = \frac{|\det g|}{|c_1 z_1 + \cdots + c_{r-1} z_{r-1} + d|^r} \cdot \mathrm{Im}(z).$$

"Non-holomorphic" Eisenstein series

For $z \in \mathfrak{H}^r$, we define

$$\begin{aligned}\mathbb{E}(z,s) &:= \sum_{c_1,\dots,c_{r-1},d\in A}{}' \frac{\mathrm{Im}(z)^s}{|c_1z_1+\cdots+c_{r-1}z_{r-1}+d|^{rs}} \\ &= \zeta_A(rs)\cdot \sum_{\gamma\in \mathrm{P}_r(A)\backslash \mathrm{GL}_r(A)} \mathrm{Im}(\gamma\cdot z)^s.\end{aligned}$$

Here $\mathrm{P}_r := \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in \mathrm{GL}_r \,\middle|\, a \in \mathrm{GL}_{r-1} \right\}$. Then

$$\mathbb{E}(\gamma\cdot z,s) = \mathbb{E}(z,s), \quad \forall \gamma \in \mathrm{GL}_r(A).$$

Kronecker (first) limit formula

Theorem 1 (W.)

(1) $\mathbb{E}(z,s)$ converges absolutely for $\mathrm{Re}(s) > 1$ and has meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$.

(2) Suppose $\mathrm{Im}_i(z) = q^{-\ell}$ with $\ell \in \mathbb{Z}$ for $1 \leq i < r$, we have the functional equation

$$(q^{rs}-1)^{-1} \cdot \mathbb{E}(z,s) =: \widetilde{\mathbb{E}}(z,s) = \widetilde{\mathbb{E}}(z^\vee, 1-s),$$

where $z^\vee \in \mathfrak{H}^r$ is a "dual" point of z.

(3) For every $z \in \mathfrak{H}^r$, we have $\mathbb{E}(z,0) = -1$ and

$$\frac{\partial}{\partial s}\mathbb{E}(z,s)\Big|_{s=0} = -\ln\left(\mathrm{Im}(z) \cdot |\Delta(z)|^{\frac{r}{q^r-1}}\right).$$

Drinfeld modules and lattices

Let $\Lambda \subset \mathbb{C}_\infty$ be an A-lattice of rank r. The exponential function associated to Λ is:

$$\exp_\Lambda(w) := w \cdot \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{w}{\lambda}\right), \quad \forall w \in \mathbb{C}_\infty.$$

There exists a unique polynomial

$$\phi_t^\Lambda(x) = tx + g_1(\Lambda)x^q + \cdots + g_{r-1}(\Lambda)x^{q^{r-1}} + \Delta(\Lambda)x^{q^r} \in \mathbb{C}_\infty[x]$$

satisfying $\exp_\Lambda(tw) = \phi_t\big(\exp_\Lambda(w)\big)$. Note that ϕ_t^Λ induces an $\mathbb{F}_q$-algebra homomorphism $\phi^\Lambda : A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$, called **the Drinfeld *A*-module associated to Λ.**

Drinfeld discriminant function

For $z = (z_1 : \cdots : z_{r-1} : 1) \in \mathfrak{H}^r$, set $\Lambda_z := Az_1 + \cdots + Az_{r-1} + A$ and

$$\Delta(z) := \Delta(\Lambda_z).$$

Then for every $\gamma = \begin{pmatrix} & * & * \\ c_1 & \cdots c_r & d \end{pmatrix} \in \mathrm{GL}_r(A)$ we have

$$\Delta(\gamma \cdot z) = (c_1 z_1 + \cdots c_{r-1} z_{r-1} + d)^{q^r - 1} \Delta(z).$$

Product expansion

$$\Delta(z) = t \cdot \prod_{w \in \frac{1}{t}\Lambda_z / \Lambda_z}{}' exp_{\Lambda_z}(w)^{-1}.$$

Meromorphic continuation of $\mathbb{E}(z,s)$

We may write $\mathbb{E}(z,s)$ as follows:

Proposition

For $z \in \mathfrak{H}^r$, we have

$$\begin{aligned}&\mathbb{E}(z,s)\\ &= \frac{\operatorname{Im}(z)^s}{q^{rs}-q^r} \cdot \sum_{w \in \frac{t^{-1}\Lambda_z}{\Lambda_z}}{}' \left[\frac{1}{|w|^{rs}} + \sum_{\substack{\lambda\in\Lambda_z \\ |\lambda|\leq|w|}}{}' \left(\frac{1}{|\lambda-w|^{rs}} - \frac{1}{|\lambda|^{rs}}\right)\right].\end{aligned}$$

Stieltjes-type formula of $\mathbb{E}(z, s)$

Thus we obtain the following formula of Taylor coefficients of $\mathbb{E}(z, s)$ at $s = 0$:

Proposition

$$\frac{\partial^n}{\partial s^n}\left(\frac{q^{rs} - q^r}{\operatorname{Im}(z)^s} \cdot \mathbb{E}(z, s)\right)\Bigg|_{s=0}$$
$$= (-r)^n \cdot \sideset{}{'}\sum_{w \in t^{-1}\Lambda_z/\Lambda_z} \left[\ln^n |w| + \sideset{}{'}\sum_{\substack{\lambda \in \Lambda_z \\ |\lambda| \leq |w|}} \left(\ln^n |\lambda - w| - \ln^n |\lambda|\right)\right].$$

Here $\ln^n x := (\ln x)^n$ for $x \in \mathbb{R}_{>0}$.

Question: A "modular" interpretation of higer derivatives of $\mathbb{E}(z, s)$ at $s = 0$?

Jacobi-type Eisenstein series

Given $z \in \mathfrak{H}^r$ and $w \in \mathbb{C}_\infty - \Lambda_z$, let

$$\mathbb{E}(z, w, s) := \sum_{\lambda \in \Lambda_z} \frac{\mathrm{Im}(z)^s}{|\lambda - w|^{rs}}.$$

Then for $\gamma = \begin{pmatrix} & * & * \\ c_1 & \cdots c_{r-1} & d \end{pmatrix} \in \mathrm{GL}_r(A)$ one has

$$\mathbb{E}(\gamma \cdot z, \frac{w}{c_1 z_1 + \cdots + c_{r-1} z_{r-1} + d}, s) = \mathbb{E}(z, w, s).$$

Kronecker (second) limit formula

The following equality

$$\mathbb{E}(z,w,s)-\mathbb{E}(z,s)=\frac{\mathrm{Im}(z)^s}{|w|^{rs}}+\sum_{\substack{0\neq\lambda\in\Lambda_z\\|\lambda|\leq|w|}}\left(\frac{\mathrm{Im}(z)^s}{|\lambda-w|^{rs}}-\frac{\mathrm{Im}(z)^s}{|\lambda|^{rs}}\right)$$

leads us to:

Corollary

For $z\in\mathfrak{H}^r$ and $w\in\mathbb{C}_\infty-\Lambda_z$, we have $\mathbb{E}(z,w,0)=0$ and

$$\frac{\partial}{\partial s}\mathbb{E}(z,w,s)\Big|_{s=0}=-\frac{r}{q^r-1}\ln\left|\Delta(z)\cdot\exp_{\Lambda_z}(w)^{q^r-1}\right|.$$

Taguchi height of Drinfeld modules

Let F be a finite extension of k. Given a Drinfeld A-module ϕ of rank r over F, write

$$\phi_t(x) = tx + \left(\sum_{i=1}^{r-1} g_i(\phi)x^{q^i}\right) + \Delta(\phi)x^{q^r}.$$

For each place w of F with $w \nmid \infty$, put

$$\mathrm{ord}_w(\phi) := \left\lfloor \min\left\{\frac{\mathrm{ord}_w(g_1(\phi))}{q-1}, \ldots, \frac{\mathrm{ord}_w(g_{r-1}(\phi))}{q^{r-1}-1}, \frac{\mathrm{ord}_w(\Delta(\phi))}{q^r-1}\right\}\right\rfloor.$$

The local height of ϕ at w is defined by

$$h_{\mathrm{Tag},w}(\phi/F) := -\frac{[\mathbb{F}_w : \mathbb{F}_q]}{[F:k]} \cdot \mathrm{ord}_w(\phi).$$

Taguchi height of Drinfeld modules

For each place $\tilde{\infty}$ of F with $\tilde{\infty} \mid \infty$, we embed F into $\mathbb{C}_\infty$ via $\tilde{\infty}$ and let $\Lambda_{\phi,\tilde{\infty}} \subset \mathbb{C}_\infty$ be the rank r A-lattice in $\mathbb{C}_\infty$ associated to ϕ. Let

$$h_{\mathrm{Tag},\tilde{\infty}}(\phi/F) := -\frac{[F_{\tilde{\infty}} : k_\infty]}{[F : k]} \cdot \log_q D(\Lambda_{\phi,\tilde{\infty}}),$$

where for every rank r A-lattice $\Lambda \subset \mathbb{C}_\infty$, put

$$D(\Lambda) := |\alpha| \cdot \mathrm{Im}(z)^{1/r}$$

if we write $\Lambda = \alpha\Lambda_z$ for $\alpha \in \mathbb{C}_\infty^\times$ and $z \in \mathfrak{H}^r$.

Taguchi height of Drinfeld modules

Definition

The Taguchi height of ϕ over F is defined by

$$h_{\mathrm{Tag}}(\phi/F) := \sum_{w \nmid \infty} h_{\mathrm{Tag},w}(\phi/F) + \sum_{\tilde{\infty} \mid \infty} h_{\mathrm{Tag},\tilde{\infty}}(\phi/F).$$

Let F' be a finite extension over F. Then

$$h_{\mathrm{Tag}}(\phi/F') \leq h_{\mathrm{Tag}}(\phi/F),$$

and the equality holds when ϕ has stable reduction everywhere. Define

$$h_{\mathrm{Tag}}^{\mathrm{st}}(\phi) := \ln q \cdot \lim_{\substack{F': \\ [F':F]<\infty}} h_{\mathrm{Tag}}(\phi/F'),$$

called the *stable Taguchi height of ϕ*.

CM Drinfeld modules

Given a rank r Drinfeld A-module ϕ over $\bar{k}$, the endomorphism ring $\mathrm{End}_A(\phi/\bar{k})$ can be identified with an A-order $\mathcal{O}$ of an "imaginary" field K with $[K:k] \leq r$. We call ϕ a **CM** Drinfeld module if $[K:k] = r$. Let $\mathfrak{I}$ be an ideal of $\mathcal{O}$ which is isomorphic to Λ_ϕ (as $\mathcal{O}$-modules). Then

Proposition

$$\begin{aligned} h_{\mathrm{Tag}}^{\mathrm{st}}(\phi) &= -\ln D(\mathfrak{I}) \\ &\quad - \frac{1}{r \# \operatorname{Pic}(\mathcal{O})} \sum_{[\mathfrak{A}] \in \operatorname{Pic}(\mathcal{O})} \ln \left(\|\mathfrak{A}\| \cdot |\Delta(\mathfrak{A}\mathfrak{I})|^{\frac{r}{q^r-1}} \right) \end{aligned}$$

where $\|\mathfrak{A}\| := \#(\mathcal{O}/a\mathfrak{A}) \cdot |a|^{-r}$ for $a \in A - \{0\}$ with $a\mathfrak{A} \subset \mathcal{O}$.

Colmez-type formula

Let

$$\begin{aligned}\zeta_{\mathfrak{I}}(s) &:= \sum_{\substack{\text{invertible } \mathcal{O}\text{-ideal} \\ \mathfrak{A} \subset \mathfrak{I}}} \frac{1}{\|\mathfrak{A}\|^s} \\ &= \frac{1}{\#\mathcal{O}^\times} \cdot \sum_{[\mathfrak{A}] \in \mathrm{Pic}(\mathcal{O})} \frac{\|\mathfrak{A}\|^s}{D(\mathfrak{A}\mathfrak{I})^{rs}} \cdot \mathbb{E}(\mathfrak{A}\mathfrak{I}, s).\end{aligned}$$

We then arrive at the following result.

Theorem 2 (W.)

$$h_{\mathrm{Tag}}^{\mathrm{st}}(\phi) = -\ln D(\mathfrak{I}) - \frac{1}{r} \cdot \frac{\zeta_{\mathfrak{I}}'(0)}{\zeta_{\mathfrak{I}}(0)}.$$

Colmez-type formula

Note that when K/k is separable and tamely ramified at ∞, one has

$$D(\mathcal{O}) = \|\mathfrak{d}(\mathcal{O}/A)\|^{1/2r},$$

where $\mathfrak{d}(\mathcal{O}/A) \subset A$ is the discriminant of $\mathcal{O}/A$. Suppose further that $\mathfrak{I}$ is an invertible $\mathcal{O}$-ideal, our Colmez-type formula can be reformulated as:

$$h_{\mathrm{Tag}}^{\mathrm{st}}(\phi) = -\frac{1}{r}\left(\ln \|\mathfrak{d}(\mathcal{O}/A)\|^{1/2} + \frac{\zeta_{\mathcal{O}}'(0)}{\zeta_{\mathcal{O}}(0)}\right).$$

Asymptotic behavior

Let $\zeta_K(s) := (1 - q^{-f_K(\infty)}s)\zeta_{O_K}(s)$, the complete zeta function of K. Write

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$$

and put $\gamma_K := c_0/c_{-1}$, so called the Euler-Kronecker constant of K. Let ϕ_o be a rank r Drinfeld A-module over $\bar{k}$ with CM by O_K. Then the above Colmez-type formula can be written as:

$$h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_o) = \frac{g_K - 1}{r} \ln q_K - \frac{1}{2} \ln q + \frac{\gamma_K}{r},$$

where g_K is the genus of K and q_K is the cardinality of the constant field of K.

Asymptotic behavior

It is known that (by Ihara 2006)

$$\gamma_K = O_{r,k}\big(\ln(g_K - 1)\big) \qquad \text{as } g_K \gg 0.$$

Therefore we obtain the following asymptotic formula:

$$h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_o) = \frac{g_K - 1}{r} \ln q_K + O_{r,k}\big(\ln(g_K - 1)\big) \qquad \text{as } g_K \gg 0.$$

Adelic settings

- $\mathbb{A}$: the adele ring of k.
- $O_{\mathbb{A}}$: the maximal compact subring of $\mathbb{A}$.
- $|\cdot|_{\mathbb{A}}$: norm on idele group $\mathbb{A}^{\times}$.
- $\mathbb{A}^{\infty}$: the finite adele ring of k. In particular, one has $\mathbb{A} = \mathbb{A}^{\infty} \times k_{\infty}$.
- $S(\mathbb{A}^r)$: the space of Schwartz functions on $\mathbb{A}^r$.
- $S((\mathbb{A}^{\infty})^r)$: the space of Schwartz functions on $(\mathbb{A}^{\infty})^r$.

Mirabolic Eisensteins series

Let $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a unitary Hecke character. Given $\varphi \in S(\mathbb{A}^r)$, for $g \in \mathrm{GL}_r(\mathbb{A})$ we put

$$\Phi_\varphi(g, s, \chi) := |\det g|_{\mathbb{A}}^s \int_{\mathbb{A}^\times} \varphi((0, ..., 0, a^{-1})g)\chi(a)|a|_{\mathbb{A}}^{-rs} d^\times a,$$

which converges absolutely for $\mathrm{Re}(s) > 1/r$ and has meromorphic continuation to the whole s-plane. Define

$$\mathcal{E}(g, s, \chi, \varphi) := \sum_{\gamma \in P_r(k)\backslash \mathrm{GL}_r(k)} \Phi_\varphi(\gamma g, s, \chi).$$

Mirabolic Eisensteins series

Proposition

(1) The Eisenstein series $\mathcal{E}(g, s, \chi, \varphi)$ converges absolutely for $\mathrm{Re}(s) > 1$ and has meromorphic continuation to the whole s-plane.

(2) Let $\varphi^\vee$ be the Fourier transform of φ: For $x = (x_1, ..., x_r) \in \mathbb{A}^r$, put

$$\varphi^\vee(x) := \int_{\mathbb{A}^r} \varphi(y)\psi(\sum_{i=1}^{r} x_i y_i)\, dy.$$

Then $\mathcal{E}(g, s, \chi, \varphi)$ satisfies the following functional equation:

$$\mathcal{E}(g, s, \chi, \varphi) = \mathcal{E}({}^t g^{-1}, 1 - s, \chi^{-1}, \varphi^\vee).$$

Mirabolic Eisensteins series

Note that we may write

$$\mathcal{E}(g, s, \chi, \varphi) = |\det g|_{\mathbb{A}}^{s} \int_{k^\times \backslash \mathbb{A}^\times} \left(\sum_{x \in k^r}{}' \varphi(a^{-1}xg) \right) \chi(a) |a|_{\mathbb{A}}^{-rs} d^\times a.$$

Mirabolic Eisensteins series

Note that we may write

$$\mathcal{E}(g,s,\chi,\varphi) = |\det g|_{\mathbb{A}}^{s} \int_{k^\times\backslash\mathbb{A}^\times} \left(\sum_{x\in k^r}{}' \varphi(a^{-1}xg) \right) \chi(a)|a|_{\mathbb{A}}^{-rs} d^\times a.$$

In particular, take $\chi = 1$, $g = g_\infty \in \mathrm{GL}_r(k_\infty)$, and $\varphi = \mathbf{1}_{O_{\mathbb{A}}^r}$, we get

$$\begin{aligned} & \mathcal{E}(g_\infty, s, 1, \mathbf{1}_{O_{\mathbb{A}}^r}) \\ = \; & \frac{|\det g_\infty|^s}{q-1} \int_{k_\infty^\times} \left(\sum_{x\in A^r}{}' \mathbf{1}_{O_\infty^r}(a_\infty^{-1} x g_\infty) \right) |a_\infty|^{-rs} d^\times a_\infty \\ = \; & \frac{1}{(q-1)(1-q^{-rs})} \cdot \sum_{x\in A^r}{}' \frac{|\det g_\infty|^s}{\nu_{L_{g_\infty}}(x)^{rs}} \end{aligned}$$

Building map from $\mathfrak{H}^r$ to the $\mathcal{B}^r(\mathbb{R})$

Here $L_{g_\infty} := O_\infty^r \cdot g_\infty^{-1} \subset k_\infty^r$, and for every O_∞-lattice $L \subset k_\infty^r$ we put

$$\nu_L(x) := \inf\{|a| : a \in k_\infty,\ x \in aL\}, \quad \forall x \in k_\infty^r.$$

Recall that the realization of the Bruhat-Tits building $\mathcal{B}_r$ of $\mathrm{PGL}_r(k_\infty)$ can be identified with the equivalent classes of the norms on k_∞^r, and we have a Building map $\mathfrak{H}^r \to \mathcal{B}_r(\mathbb{R})$ defined by

$$z \longmapsto [\nu_z],\ \text{where } \nu_z := (x \mapsto |x_1 z_1 + \cdots + x_{r-1} z_{r-1} + x_r|).$$

$\mathbb{E}(z, s)$ and $\mathcal{E}(g_\infty, s, 1, \mathbf{1}_{O_{\mathbb{A}}^r})$

Recall that

$$\mathbb{E}(z, s) = \sum_{c_1,\dots,c_{r-1},d\in A}' \frac{\operatorname{Im}(z)^s}{|c_1z_1 + \cdots + c_{r-1}z_{r-1} + d|^{rs}}$$
$$= \sum_{x\in A^r}' \frac{\operatorname{Im}(z)^s}{\nu_z(x)^{rs}}.$$

Therefore:

Proposition

Given $z \in \mathfrak{H}^r$, suppose $\nu_z = \nu_{L_{g_\infty}}$ for some $g_\infty \in \mathrm{GL}_r(k_\infty)$. We then have

$$\mathbb{E}(z, s) = (q-1)(1-q^{-rs}) \cdot \mathcal{E}(g_\infty, s, 1, \mathbf{1}_{O_{\mathbb{A}}^r}).$$

"Adelic" Kronecker limit formula

Let $\mathfrak{H}^r_{\mathbb{A}} := \mathfrak{H}^r \times \mathrm{GL}_r(\mathbb{A}^\infty)$. For $z_{\mathbb{A}} = (z, g^\infty) \in \mathfrak{H}^r_{\mathbb{A}}$, put $\mathrm{Im}(z_{\mathbb{A}}) := \mathrm{Im}(z) \cdot |\det g^\infty|_{\mathbb{A}}$. Given a Schwartz function $\varphi^\infty \in S((\mathbb{A}^\infty)^r)$, set

$$\mathbb{E}(z_{\mathbb{A}}, s; \varphi^\infty) := \sum_{x \in k^r}{}' \varphi^\infty(xg^\infty) \cdot \frac{\mathrm{Im}(z_{\mathbb{A}})^s}{\nu_z(x)^{rs}}, \quad \mathrm{Re}(s) > 1.$$

It is clear that

$$\mathbb{E}(\gamma \cdot z_{\mathbb{A}}, s; \varphi^\infty) = \mathbb{E}(z_{\mathbb{A}}, s; \varphi^\infty), \quad \forall \gamma \in \mathrm{GL}_r(k).$$

"Adelic" Kronecker limit formula

Note that

$$\varprojlim_{a \in A - \{0\}} (k^r / aA^r) = (\mathbb{A}^\infty)^r.$$

We may take a sufficiently "large" $a \in A$ so that $\varphi^\infty(\,\cdot\, g^\infty)$ corresponds to a function on k^r / aA^r with finite support. Then:

Lemma

$$\begin{aligned} \mathbb{E}(z_{\mathbb{A}}, s; \varphi^\infty) &= \frac{|\det g^\infty|_{\mathbb{A}}^s}{|a|^{rs}} \cdot \Bigg(\varphi^\infty(0) \cdot \mathbb{E}(z, s) \\ &\quad + \sum_{0 \neq \alpha \in k^r / aA^r} (\varphi^\infty)(\alpha g^\infty) \cdot \mathbb{E}(z, \alpha_z, s) \Bigg). \end{aligned}$$

"Adelic" Kronecker limit formula

For a unitary Hecke character $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ with $\chi(k_\infty^\times) = 1$, we define

$$\mathbb{E}(z_\mathbb{A}, s, \chi, \varphi^\infty) := \frac{1}{1 - q^{-rs}} \cdot \int_{k^\times \backslash \mathbb{A}^{\infty,\times}} \overline{\chi(a)} \cdot \mathbb{E}\big(z_\mathbb{A} a, s; \varphi^\infty\big) d^\times a.$$

Extending mirabolic Eisenstein series

Proposition

Let $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a unitary Hecke character with $\chi(k_\infty^\times) = 1$. Given a Schwartz function $\varphi^\infty \in S((\mathbb{A}^\infty)^r)$, we have

$$\mathbb{E}(z_{\mathbb{A}}, s, \chi, \varphi^\infty) = \mathcal{E}(g, s, \chi, \varphi^\infty \otimes \mathbf{1}_{O_\infty^r})$$

for every $z_{\mathbb{A}} \in \mathfrak{H}_{\mathbb{A}}^r$ with

$$\lambda(z_{\mathbb{A}}) = [g] \in \mathcal{B}_{\mathbb{A}}^r(\mathbb{Z}) = \mathrm{GL}_r(\mathbb{A})/k_\infty^\times \mathrm{GL}_r(O_\infty).$$

Our Kronecker limit formula enables us to connect the special value of $\mathbb{E}(z_{\mathbb{A}}, s, \chi, \varphi^\infty)$ at $s = 0$ for the Schwartz function φ^∞ with the "Drinfeld-Siegel units" on $\mathfrak{H}_{\mathbb{A}}^r$.

Drinfeld-Siegel units

Given a $\mathbb{Z}$-valued Schwartz function $\varphi^\infty \in S((\mathbb{A}^\infty)^r)$, take $a \in A - \{0\}$ so that φ^∞ can be viewed as a function on k^r/aA^r. put

$$u(z;\varphi^\infty) := \Delta(a\Lambda_z)^{\varphi^\infty(0)} \cdot \prod_{\alpha \in k^r/aA^r}{}' \left(\Delta(a\Lambda_z)\exp_{a\cdot\Lambda_z}(\alpha_z)^{(q^r-1)}\right)^{\varphi^\infty(\alpha)}.$$

Note that $u(z;\varphi^\infty)$ is independent of the chosen $a \in A$.

Drinfeld-Siegel units

The Drinfeld-Siegel unit associated to φ^∞ on $\mathfrak{H}^r_{\mathbb{A}}$ is

$$\mathfrak{u}(z_{\mathbb{A}};\varphi^\infty) := u(z;\rho(g^\infty)\varphi^\infty), \quad \forall z_{\mathbb{A}} = (z, g^\infty) \in \mathfrak{H}^r_{\mathbb{A}}.$$

Lemma

For $z_{\mathbb{A}} = (z, g^\infty) \in \mathfrak{H}^r_{\mathbb{A}}$ and $\gamma = \begin{pmatrix} & * & * \\ c_1 & \cdots c_{r-1} & d \end{pmatrix} \in \mathrm{GL}_r(k)$, we have

$$\mathfrak{u}(\gamma\cdot z_{\mathbb{A}};\varphi^\infty) = (c_1 z_1 + \cdots + c_{r-1}z_{r-1} + d)^{(q^r-1)\varphi^\infty(0)} \cdot \mathfrak{u}(z_{\mathbb{A}};\varphi^\infty).$$

In particular, $\mathfrak{u}(z_{\mathbb{A}};\varphi^\infty)$ can be viewed as a function on $\mathrm{GL}_r(k)\backslash\mathfrak{H}^r_{\mathbb{A}}$ if $\varphi^\infty(0) = 0$.

"Adelic" Kronecker limit formula

Theorem 3 (W.)

Given a $\mathbb{Z}$-valued Schwartz function $\varphi^\infty \in S((\mathbb{A}^\infty)^r)$ and $z_{\mathbb{A}} \in \mathfrak{H}^r_{\mathbb{A}}$, we have

$$\mathbb{E}(z_{\mathbb{A}}, 0; \varphi^\infty) = -\varphi^\infty(0)$$

and

$$\frac{\partial}{\partial s}\mathbb{E}(z_{\mathbb{A}}, s; \varphi^\infty)\Big|_{s=0} = -\ln\left(\operatorname{Im}(z_{\mathbb{A}})^{\varphi^\infty(0)} \cdot |\mathfrak{u}(z_{\mathbb{A}}; \varphi^\infty)|^{\frac{r}{q^r-1}}\right).$$

Lerch-type formula

Moreover, let $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a unitary Hecke character with $\chi(k_\infty^\times) = 1$. Suppose χ is non-trivial or $\varphi^\infty(0) = 0$. Then

Corollary

$$\begin{aligned} &\mathbb{E}(z_{\mathbb{A}}, 0, \chi, \varphi^\infty) \\ = \;& \frac{1}{1-q^r} \cdot \int_{k^\times \backslash \mathbb{A}^{\infty,\times}} \overline{\chi(a)} \cdot \log_q \left|\mathfrak{u}\left(z_{\mathbb{A}} a; \varphi^\infty\right)\right| d^\times a. \end{aligned}$$

Hayes-Stickelberger element

Suppose $r = 1$. The Drinfeld half space $\mathfrak{H}^1$ consists of one single point, say z_o. Given a proper ideal $\mathfrak{m} \lhd A$, let $H_\mathfrak{m}$ be the ray class field of k of conductor $\mathfrak{m}$. Let

$$\varphi_\mathfrak{m}^\infty := \mathbf{1}_{1+\mathfrak{m}\cdot O_{\mathbb{A}^\infty}} \in S(\mathbb{A}^\infty).$$

Then $\mathfrak{u}_\mathfrak{m} := \mathfrak{u}((z_o, 1), \varphi^\infty)$ lies in $H_\mathfrak{m}^\times$, and the Hayes-Stickelberger element of $H_\mathfrak{m}$ is:

$$\omega_\mathfrak{m} = -\sum_{\sigma \in \mathrm{Gal}(H_\mathfrak{m}/k)} \log_q |\mathfrak{u}_\mathfrak{m}^\sigma| \cdot \sigma^{-1} \in \mathbb{Z}[\mathrm{Gal}(H_\mathfrak{m}/k)].$$

Rankin-Selberg L-function

Let Π and Π' be two automorphic cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$ with central characters denoted by ω and ω', respectively. Put $\chi := (\omega \cdot \omega')^{-1}$. We define a multi-linear functional $\mathcal{P}^{\mathrm{RS}} : \Pi \times \Pi \times S((\mathbb{A}^\infty)^r)_{\mathbb{Z}} \to \mathbb{C}$ by

$$\mathcal{P}^{\mathrm{RS}}(f, f', \varphi^\infty) := \int_{\mathbb{A}^\times \mathrm{GL}_r(k) \backslash \mathrm{GL}_r(\mathbb{A})} f(g) f'(g) \eta_\chi(g; \varphi^\infty) dg.$$

Here

$$\eta_\chi(g; \varphi^\infty) := \frac{1}{1 - q^r} \int_{k^\times \backslash \mathbb{A}^{\infty,\times}} \overline{\chi(a)} \cdot \log_q \left| \mathfrak{u}(z_{\mathbb{A}} a; \varphi^\infty) \right| d^\times a$$

for every $z_{\mathbb{A}} \in \mathfrak{H}^r_{\mathbb{A}}$ whose image via the building map is $[g] \in \mathcal{B}^r_{\mathbb{A}}(\mathbb{Z})$.

Rankin-Selberg L-function

On the other hand, write $\Pi = \otimes_v \Pi_v$ and $\Pi' = \otimes_v \Pi'_v$. For each place v, put

$$L_v^{\mathrm{RS}}(s; f_v, f'_v, \varphi_v) := \frac{1}{L_v(s, \Pi \times \Pi')}$$
$$\cdot \int_{N_r(k_v)\backslash \mathrm{GL}_r(k_v)} W_{f_v}(g_v) W'_{f'_v}(g_v) \varphi_v\big((0, ..., 0, 1)g_v\big) |\det g_v|_v^s dg_v,$$

which is holomorphic at $s = 0$ and $L_v^{\mathrm{RS}}(0; f_v, f'_v, \varphi_v) = 1$ when v is “nice”.

Rankin-Selberg L-function

Define

$$\mathscr{P}_v^{\mathrm{RS}}(f_v, f'_v, \varphi_v) := L_v^{\mathrm{RS}}(0; f_v, f'_v, \varphi_v)$$

and consider the following multi-linear functional on $\Pi \times \Pi' \times S((\mathbb{A}^\infty)^r)_{\mathbb{Z}}$:

$$\mathscr{P}^{\mathrm{RS}} := \mathscr{P}_\infty^{\mathrm{RS}}(\cdot, \cdot, \mathbf{1}_{O_\infty^r}) \otimes \Big(\otimes_{v \neq \infty} \mathscr{P}_v^{\mathrm{RS}} \Big).$$

We then have

Theorem 4 (W.)

Given two cuspidal representations Π and Π' with $\Pi' \not\cong \widetilde{\Pi}$, we have

$$\mathcal{P}^{\mathrm{RS}} = L(0, \Pi \times \Pi') \cdot \mathscr{P}^{\mathrm{RS}}.$$

Godement-Jacquet L-function

Let Π be an automorphic cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$ with a unitary central character ω. For $f_1, f_2 \in \Pi$ and $\Phi \in S(\mathrm{Mat}_r(\mathbb{A}))$, the associated **Godement-Jacquet L-function** is

$$L(s; f_1, f_2, \Phi) := \int_{\mathrm{GL}_r(\mathbb{A})} \Phi(g) \cdot |\det g|_{\mathbb{A}}^{s} \cdot \langle \Pi(g) f_1, f_2 \rangle_{\mathrm{Pet}} dg.$$

It is known that $L(s; f_1, f_2, \Phi)$ converges absolutly for $\mathrm{Re}(s) > 1/r$ and has analytic continuation to the whole complex s-plane with the following functional equation:

$$L(s; f_1, f_2, \Phi) = L(r - s, \tilde{f}_1, \tilde{f}_2, \widehat{\Phi}).$$

Here $\tilde{f}(g) := f({}^t g^{-1})$ and

$$\widehat{\Phi}(X) := \int_{\mathrm{Mat}_r(\mathbb{A})} \Phi(Y) \psi\big(\mathrm{Tr}(X \cdot {}^t Y)\big) dY.$$

Godement-Jacquet L-function

Wite $\Pi = \otimes \Pi_v$ and take a pairing $\langle \cdot, \cdot \rangle_v$ for each place v so that

$$\langle \cdot, \cdot \rangle_{\mathrm{Pet}} = \prod_v \langle \cdot, \cdot \rangle_v.$$

For factorizable $f_1, f_2 \in \Pi$ and $\Phi \in S(\mathrm{Mat}_r(\mathbb{A}))$, we then have

$$L(s; f_1, f_2, \Phi) = L(s - \frac{r-1}{2}, \Pi) \cdot \prod_v L_v^o(s; f_{1,v}, f_{2,v}, \Phi_v),$$

where

$$L_v^o(s; f_{1,v}, f_{2,v}, \Phi_v) := \frac{1}{L_v(s - (r-1)/2, \Pi)} \cdot \int_{\mathrm{GL}_r(k_v)} \Phi_v(g_v) |\det g_v|_v^s \cdot \langle \Pi_v(g_v) f_{1,v}, f_{2,v} \rangle_v dg_v.$$

Godement-Jacquet L-function

Let $\mathscr{P}_v^{\mathrm{GJ}}(f_{1,v}, f_{2,v}, \Phi_v) := L_v^o(0; f_{1,v}, f_{2,v}, \Phi_v)$ and

$$\mathscr{P}^{\mathrm{GJ}} := \mathscr{P}_\infty^{\mathrm{GJ}}(\cdot, \cdot, \mathbf{1}_{\mathrm{Mat}_r(O_\infty)}) \otimes \left(\otimes_{v \neq \infty} \mathscr{P}_v^{\mathrm{GJ}}\right),$$

a multi-linear functional on $\Pi \times \Pi \times S(\mathrm{Mat}_r(\mathbb{A}^\infty))_{\mathbb{Z}}$.

Godement-Jacquet L-function

Let $\mathscr{P}_v^{\mathrm{GJ}}(f_{1,v}, f_{2,v}, \Phi_v) := L_v^o(0; f_{1,v}, f_{2,v}, \Phi_v)$ and

$$\mathscr{P}^{\mathrm{GJ}} := \mathscr{P}_\infty^{\mathrm{GJ}}(\cdot, \cdot, \mathbf{1}_{\mathrm{Mat}_r(O_\infty)}) \otimes \left(\otimes_{v \neq \infty} \mathscr{P}_v^{\mathrm{GJ}}\right),$$

a multi-linear functional on $\Pi \times \Pi \times S(\mathrm{Mat}_r(\mathbb{A}^\infty))_{\mathbb{Z}}$.

On the other hand, identifying $\mathrm{Mat}_r(k)$ with k^{r^2}, the left and right multiplications of GL_r on Mat_r then induces an embedding

$$\mathrm{GL}_r^2 = \mathrm{GL}_r \times \mathrm{GL}_r \hookrightarrow \mathrm{GL}_{r^2}.$$

Godement-Jacquet L-function

The mirabolic Eisenstein series on $\mathrm{GL}_{r^2}(\mathbb{A})$ restricting to $\mathrm{GL}_r^2(\mathbb{A})$ can be written as follows: for $\Phi \in S(\mathrm{Mat}_r(\mathbb{A}))$ and $(g_1, g_2) \in \mathrm{GL}_r^2(\mathbb{A})$, we have

$$\mathcal{E}((g_1, g_2), s, \omega^{-1}, \Phi) = |\det(g_1 g_2)|_{\mathbb{A}}^{rs} \cdot \int_{a \in k^\times \backslash \mathbb{A}^\times} \sum_{0 \neq X \in \mathrm{Mat}_r(k)} \Phi(a^{-1} \cdot {}^t g_2 X g_1) \omega^{-1}(a) |a|_{\mathbb{A}}^{r^2 s} d^\times a.$$

Theorem (Piatetski-Shapiro and Rallis)

For $f_1, f_2 \in \Pi$ and $\Phi \in S(\mathrm{Mat}_r(\mathbb{A}))$ we have

$$L(rs; f_1, f_2, \Phi) = \int_{\left(\mathbb{A}^\times \mathrm{GL}_r(k) \backslash \mathrm{GL}_r(\mathbb{A})\right)^2} f_1(g_1) \mathcal{E}((g_1, g_2), s; \omega^{-1}, \Phi) \overline{\tilde{f}_2(g_2)} dg_1 dg_2.$$

Godement-Jacquet L-function

Let $\mathcal{P}^{\mathrm{GJ}} : \Pi \times \Pi \times S(\mathrm{Mat}_r(\mathbb{A}^\infty))_{\mathbb{Z}} \to \mathbb{C}$ be the following integral:

$$\mathcal{P}^{\mathrm{GJ}}(f_1, f_2, \Phi) := \frac{1}{1-q^r} \cdot \int_{\left(\mathbb{A}^\times \mathrm{GL}_r(k) \backslash \mathrm{GL}_r(\mathbb{A})\right)^2} f_1(g_1)\eta_{\omega^{-1}}((g_1, g_2); \Phi^\infty)) \cdot \overline{\tilde{f}_2(g_2)} dg_1 dg_2.$$

Then:

Theorem 5 (W.)

$$\mathcal{P}^{\mathrm{GJ}} = L(\frac{1-r}{2}, \Pi) \cdot \mathscr{P}^{\mathrm{GJ}}.$$

The end.

Thank you for your attention.