Drinfeld modular forms in higher rank : Analytic theory

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Basic Setup

F : Global function field

∞ : fixed place

$A = \{x \in F \mid x \text{ regular away from } \infty\}$

F_∞ = completion at ∞

$C_\infty = \hat{\bar{F}}_\infty$

$r \geq 2$

Drinfeld period domain

$$\Omega^r = \left\{ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{bmatrix} \in \mathbb{C}_\infty^r \;\middle|\; \begin{array}{l} \omega_1, \dots, \omega_r \ F_\infty\text{-lin. indep} \\ \text{normalisation} : \omega_r = \xi \end{array} \right\}$$

fixed constant (e.g. $\xi = \bar{\pi}$, Carlitz period)

$GL_r(F_\infty)$ acts from the left on Ω:
$\gamma \in GL_r(F_\infty)$

$$\gamma(\omega) := j(\gamma, \omega)^{-1} \gamma\omega$$

matrix product

$$j(\gamma, \omega) = \xi^{-1} \cdot (\text{last entry of } \gamma\omega)$$

factor of automorphy.

Def: A weak modular form of weight k, type m for $\Gamma \subset GL_r(F)$ is a holom. $f : \Omega^r \to \mathbb{C}_\infty$ satisfying

arithmetic subgroup

$$f(\gamma(\omega)) = (\det \gamma)^m j(\gamma, \omega)^{-k} f(\omega)$$

Parameter at infinity:

$\Gamma \subset GL_r(F)$ arithmetic subgroup

$$\Gamma_u = \Gamma \cap \left(\begin{array}{c|c} 1 & * \\ \hline 0 & I_{r-1} \end{array} \right) \qquad (\text{translations in } \Gamma)$$

$$\Lambda' := \left\{ v' \in F^{r-1} \;\middle|\; \left(\begin{array}{c|c} 1 & v' \\ \hline 0 & I_{r-1} \end{array} \right) \in \Gamma_u \right\}$$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega' \end{pmatrix} \qquad \omega' \in \Omega^{r-1}$$

$$\Lambda' \omega' = \left\{ \sum_{i=2}^{r} v_i \omega_i \in \mathbb{C}_\infty \right\} \subseteq \mathbb{C}_\infty \quad \text{strongly discrete}$$

Finite intersection with balls of finite radius

has associated exp. function $e_{\Lambda'\omega'}$

$$\boxed{u_{\omega'}(\omega_1) = \frac{1}{e_{\Lambda'\omega'}(\omega_1)}}$$

$\Lambda' = A$

Example: $r = 2$, $A = \mathbb{F}_q[t]$, $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ $\omega_2 = \bar{\pi}$

$$u_{\omega'}(\omega_1) = \frac{1}{e_{A\bar{\pi}}(\omega_1)} = \frac{1}{\bar{\pi} e_1(z)}, \quad z = \frac{\omega_1}{\bar{\pi}} \in \Omega'$$

Theorem: let $f : \Omega^r \to \mathbb{C}_\infty$ be Γ_u-invariant holom,
then $\exists!\, f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ holom s.t.

$$f\begin{pmatrix} \omega_1 \\ \omega' \end{pmatrix} = \sum_{n \in \mathbb{Z}} f_n(\omega')\, u_{\omega'}(\omega_1)^n$$

(converges on a "neighborhood of infinity" of Ω^r)

Further if f is a W.M.F. then each $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$
is a W.M.F. of weight $k-n$ and type m
for $\Gamma_L \subset GL_{r-1}(F)$

$$\Gamma \cap \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma_L \end{array}\right)$$

Def: f is holom at ∞ if $f_n \equiv 0 \;\forall n < 0$
vanishes at ∞ if $f_n \equiv 0 \;\forall n \leq 0$
is meromorphic ∞ if $f_n \equiv 0 \;\forall n \ll 0$

Def: $f : \Omega^r \to \mathbb{C}_\infty$ is a modular form of wt k and type m for Γ if it is a WMF and

$$(*) \begin{cases} (\det \gamma)^m j(\gamma, \omega)^{-k} f(\gamma(\omega)) \\ \text{is holomorphic at } \infty \text{ for all } \gamma \in GL_r(F) \end{cases}$$

Remark: It suffices to check $(*)$ for γ representatives of

$$\Gamma \backslash GL_r(F) / P \qquad P = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ \vdots & \\ 0 & \end{array} \right)$$

Examples: $\Lambda \subset F^r$ projective A-module of rank r

$\omega \in \Omega^r$, $\Lambda\omega \subseteq \mathbb{C}_\infty$ lattice

$$GL(\Lambda) = \{\gamma \in GL_r(F) \mid \Lambda\gamma = \Lambda\}$$

Eisenstein series: $\mathfrak{n} \subset A$ non-zero ideal
$v \in (\mathfrak{n}^{-1}\Lambda/\Lambda)$

$$\Gamma(\mathfrak{n}) = \ker\left(GL(\Lambda) \to \mathrm{Aut}(\mathfrak{n}^{-1}\Lambda/\Lambda)\right)$$

$k \geq 1$

$$E_{k,v,\Lambda}(\omega) = \sum_{\substack{\lambda \in \Lambda \\ (\lambda \neq -v)}} \frac{1}{((\lambda+v)\omega)^k}$$

is a MF of wt k, type $m=0$, for $GL(\Lambda)$

<u>Coefficient Forms</u>: $\omega \in \Omega^r \rightsquigarrow \varphi^\omega$ Drinfeld module associated to $\Lambda\omega$

$a \in A$

$$\varphi_a^\omega(X) = \sum_{i=0}^{r\deg a} \underbrace{g_i(a,\omega)}_{\substack{\text{modular form} \\ k = q^i - 1}} X^{q^i}$$

$$= aX \prod_{0 \neq v \in (a^{-1}\Lambda/\Lambda)} \left(1 - X E_{1,v,\Lambda}(\omega)\right)$$

$$\Delta_r(a,\omega) = g_{r\deg(a)}(a,\omega)$$ Discriminant function

$$= a \prod_{v \in (a^{-1}\Lambda/\Lambda)}' E_{1,v,\Lambda}(\omega)$$

$$h_r(a,\omega) = \sqrt[q-1]{-a} \prod_{[v] \in (a^{-1}\Lambda/\Lambda)/\mathbb{F}_q^*}' E_{1,v,\Lambda}(\omega)$$

is cusp form of weight $k = \dfrac{q^{r\deg a} - 1}{q-1}$

and type $m = \deg a$