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A sorting and transformation game

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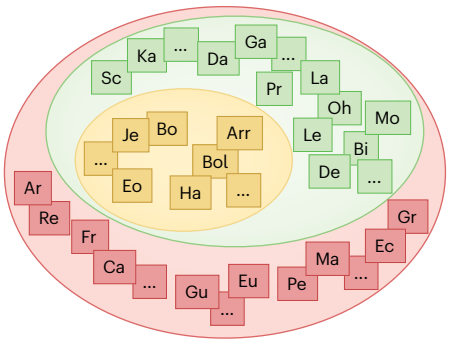
Uwe Thiele **discusses how one might sort dimensionless numbers based on their thermodynamic character, thereby proposing a subgroup structure.**

When modeling processes in the natural sciences and engineering, an important step is the non-dimensionalization of the developed mathematical models. Important reasons to do so are the resulting reduction in the number of parameters and the easy recognition of pertinent limiting cases, scaling laws and related crossover behavior. The process of non-dimensionalization is based on the introduction of typical scales for independent variables (for example, time and spatial coordinates), dependent variables (such as density, velocity and temperature) and other important quantities (such as forces, pressure and energy) into the governing equations and boundary and initial conditions. In disciplines employing physico-chemical hydrodynamics, the remaining non-dimensional parameters often obtain specific names. Examples that have been discussed in past issues' By the Numbers include Marangoni, Péclet, Rayleigh and Reynolds numbers. The *Wikipedia* page on 'Dimensionless numbers in fluid mechanics' lists 84 of them[1]. Other fields also use identical procedures of non-dimensionalization, but normally they either do not name the resulting numbers (instead they say which elementary constants they boldly "set to one") or they call them factors, for example, the Boltzmann factor is the ratio of some specific energy to the thermal energy.

One may easily be confused by the large variety of non-dimensional numbers and by their magic capacity to transform into each other. The latter results, on the one hand, from their group property, which means multiply two and you get another one – Prandtl and



Grashof numbers give the Rayleigh number – and, on the other hand, from the power of specification. If you replace the general velocity scale by a specific velocity scale, the number transforms. For instance, if you introduce the intrinsic velocity scale given by the surface tension of a liquid divided by its dynamic viscosity into the Reynolds number, you obtain the Laplace number.

Here, I tentatively propose a way to sort the many numbers by their (non)-equilibrium character, thereby establishing a subgroup structure. The first class are equilibrium numbers that only involve equilibrium quantities such as densities, surface tensions, heat capacities and elastic moduli, and also length scale(s) and possibly elementary constants. Such numbers can normally be understood as the ratio of energies or of length scales. Examples include Arrhenius, Boltzmann, Bond and Hamaker numbers. Also, three-phase contact angles belong in this class, such as Young's angle because its cosine directly corresponds to a ratio of interface energies. The equilibrium numbers form a subgroup of more exhaustive groups as the product of two again gives an equilibrium number.

The second class additionally contains relaxational numbers that only involve transport coefficients, that is, quantities such as viscosity, diffusion constants or reaction rates that capture the relaxation toward equilibrium, along with quantities involved in the first class. Such numbers can normally be understood as a ratio of timescales of different dissipative processes. Examples include the Damköhler, Galilei, Kapitza, Laplace, Lewis, Ohnesorge and Prandtl numbers. They form a subgroup of the third class that additionally contains out-of-equilibrium numbers. These incorporate quantities that reflect persistent imposed (thermodynamic) driving and also quantities involved in the second class. The former include general velocities and fluxes as well as differences of pressures and temperatures. Most numbers in the above-mentioned list of 84 are out-of-equilibrium numbers, such as Archimedes, capillary, Euler, Grashof, Marangoni, Péclet, Rayleigh and Reynolds numbers.

One may further refine the scheme through the definition of pure relaxational numbers (which only contain transport coefficients, such as Prandtl and Lewis numbers) and pure out-of-equilibrium numbers (which only contain driving quantities; none in the list of 84, even Eckert and Euler numbers, contain additional constants). The resulting group structures and magic transformations between individual numbers or sets of them might be seen as a delightful but futile *Glass Bead Game*[2]; however, in this case, the game can provide insights into deeper relations between equilibrium, relaxational and out-of-equilibrium behavior as well as about the similarity between different systems.

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Competing interests

The author declares no competing interests.