1 Linear stability analysis

- Linear stability analysis is a concept to analyze the local stability of an fixed point.
- Idea: Test the stability by disturbing the equilibrium state!
- Consider a fixed point at x^* ($\dot{x}^* = f(x^*) = 0$) and a small disturbance ϵ so that $x = x^* + \epsilon$ or $\epsilon = x - x^*$
- How can we quantify the stability of x^* ? $\rightarrow$ Look at the dynamics of ϵ!

$$\dot{\epsilon} = \dot{x} - \underbrace{\dot{x^*}}_{=0} = x$$
$$\dot{\epsilon} = f(x) = f(x^* + \epsilon)$$

- Since we consider ϵ as small, we can use a Taylor expansion and neglect higher oder terms

$$f(x^* + \epsilon) = \underbrace{f(x^*)}_{=0} + f'(x^*)\epsilon + \mathcal{O}(\epsilon^2)$$
$$= f'(x^*)\epsilon + \mathcal{O}(\epsilon^2)$$

- If $f'(x^*)$ one can make a statement on the stability of x^*
- Without $\mathcal{O}(\epsilon^2)$ the analytical solution for $t_0 = 0$ is

$$\epsilon(t) = \epsilon_0 e^{f'(x^*)t}$$

i) $f'(x^*) < 0 \rightarrow$ for $t \rightarrow \infty$ we have $\epsilon \rightarrow 0 \Leftrightarrow$ stable

ii) $f'(x^*) < 0 \rightarrow \epsilon$ grows $\Leftrightarrow$ unstable – no statement for $t \rightarrow \infty$ possible, since after a while the $\mathcal{O}(\epsilon^2)$ terms are no more negligible!

- $f'(x^*) = 0$? In general no statement possible $\rightarrow$ look at phase portrait

Example for linear stability: Look at $\dot{x} = \sin x$

$$\sin x^* = 0 \quad \Rightarrow x^* = k\pi$$

$$f'(x^*) = \cos x^* = \begin{cases} -1, \text{for } k \text{ even} \rightarrow \text{stable} \\ 1, \text{for } k \text{ even} \rightarrow \text{unstable} \end{cases}$$

Time scale: For $f'(x^*) \neq 0$ the typical time scale on which the perturbation decays/growth is

$$\tau = \frac{1}{|f'(x^*)|}$$

If $f'(x^*) = 0$ we have a special situation (*critical slowing down*).