Numerical Simulation of Supersymmetric Yang-Mills Theory

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The DESY-Münster-Roma Collaboration performed the first large-scale simulation of supersymmetric Yang-Mills (SYM) theory. Theoretical expectations about the behaviour of non-perturbative dynamics near the supersymmetric limit at zero gaugino mass have been confronted with numerical results.

1 Introduction

In recent decades research on the structure of matter has revealed that all known matter is composed of a small number of fundamental constituents, the so-called quarks and leptons. Moreover, the physics of these particles is governed by four types of forces: gravitation, electro-magnetic forces, the weak and the strong interactions. In subatomic physics only the latter three play a role. The particles and their interactions are theoretically summarized in the so-called Standard Model. Almost all known experimental data in the presently available energy range can be described by the Standard Model.

In spite of this, many theorists are convinced that by the increase of the energy of particle accelerators within the next ten years new phenomena will emerge which go "beyond the Standard Model". In particular, the large number of free parameters in the Standard Model is not entirely satisfactory. There are also experimental facts which point towards some extension of the Standard Model: the recently established non-zero mass of neutrinos and the observed baryon matter excess in the Universe which cannot be explained in the minimal Standard Model. In addition, the present theoretical framework has to be extended, if gravitation is to be described in a manner consistent with quantum theory.

Many of the possible extensions of the Standard Model beyond the presently known energy range are based on *supersymmetry*. Supersymmetry is a concept beyond the usual notion of symmetry. It connects bosons, particles with integral spin, to fermions, which have half-integral spin[1]. In case of perfect supersymmetry the bosons and fermions are grouped in *supermultiplets*, in which the members have equal masses. Since such mass degeneracies are not observed in the present experiments one has to assume that supersymmetry is broken. An important consequence of supersymmetry breaking is the heavy mass of the supersymmetric partners of the presently known elementary particles. This would explain why they are not observed up to now.

Although supersymmetry does not seem to immediately solve the problem of proliferation of free parameters, the only known framework which has the potential of incorporating the quantum theory of gravity is based on supersymmetric string theories. It is generally assumed that the scale where supersymmetry becomes manifest is near to the presently explored electroweak scale and that the supersymmetry is spontaneously broken.

The Standard Model of elementary particles is based on gauge field theories. Supersymmetric extensions involve supersymmetric gauge theories. A common method to investigate field theories is perturbation theory, which amounts to an expansion of the quantities of interest in powers of a small parameter. However, many interesting features are of a non-perturbative nature. An attractive possibility for spontaneous supersymmetry breaking is to exploit non-perturbative mechanisms in supersymmetric gauge theories. This is the basis of a strong theoretical interest for investigating supersymmetry non-perturbatively.

The motivation for a study of non-perturbative features of supersymmetric gauge theories partly also comes from the desire to understand relativistic quantum field theories better in general: the supersymmetric points in the parameter space of all quantum field theories are very special since they correspond to situations of a high degree of symmetry. The seminal work of Seiberg and Witten[2] and others showed that there is a possibility to approach non-perturbative questions in four dimensional quantum field theories by starting from exact solutions in some highly symmetric points and treat the symmetry breaking as a small perturbation. Beyond this, the knowledge of non-perturbative dynamics in supersymmetric quantum field theories can also be helpful in understanding the greatest puzzle of the Standard Model, with or without supersymmetric extensions, namely the existence of a large number of seemingly free parameters. As we know from Quantum Chromodynamics, the theory of the strong interactions of quarks, strong interactions in non-abelian gauge theories are capable of reproducing a large number of dynamically generated parameters for quantities characterizing bound states from a small number of input parameters. This is a possible solution also for the parameters of the Standard Model if new strong interactions are active beyond the electroweak symmetry breaking scale.

The simplest supersymmetric gauge theory is the supersymmetric extension of Yang-Mills theory, SYM. It describes the carriers of gauge interactions, the gauge particles, together with their supersymmetric partners, the gauginos. Gauginos are massless Majorana fermions, which are in the adjoint representation of the gauge group. The DESY-Münster-Roma Collaboration has mainly investigated the simplest non-abelian SYM theory with gauge group SU(2)[3–5]. (Some first results have also been obtained with SU(3) gauge group[6].) A non-trivial problem to be solved was to perform the numerical simulations with sufficiently light (Majorana) fermions. A suitable simulation algorithm has been developed[7] which is applicable for light dynamical gauginos and, more generally, also for a broad class of fermionic quantum field theories including Quantum Chromodynamics (QCD)[8].

Our investigations have revealed the basic non-perturbative features of supersymmetric Yang-Mills theory, which will be described below.

2 SYM Theory on the Lattice

The numerical simulation of SYM theory is performed on a hypercubic lattice in four dimensional Euclidean space-time. The fourth coordinate, besides the three space coordinates, is the imaginary time. In the path integral formulation we need the Euclidean action which gives the weight of lattice field configurations in the path integral. For a Majorana fermion in the adjoint representation the fermionic part of the Wilson lattice action is

$$S_f \equiv \tfrac{1}{2}\overline{\lambda}Q\lambda$$

$$\equiv \tfrac{1}{2}\sum_x \left\{ \overline{\lambda}^a_x \lambda^a_x - K\sum_{\mu=1}^4 \left[\overline{\lambda}^a_{x+\hat{\mu}} V_{ab,x\mu}(1+\gamma_\mu)\lambda^b_x + \overline{\lambda}^r_x V^T_{ab,x\mu}(1-\gamma_\mu)\lambda^b_{x+\hat{\mu}} \right] \right\} . \tag{1}$$

Here K is the hopping parameter which determines the gaugino mass, γ_μ denotes a Dirac matrix and $V_{x\mu}$ is the gauge field variable in the adjoint representation of the gauge group. The gaugino field λ satisfies the Majorana condition

$$\overline{\lambda}_x = \lambda_x C \, , \tag{2}$$

with the charge conjugation Dirac matrix C.

The full lattice action is the sum of the pure gauge part and fermionic part:

$$S = S_g + S_f \, . \tag{3}$$

The standard Wilson action for the gauge field S_g is a sum over the plaquettes

$$S_g = \beta \sum_{pl} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{pl} \right) \, , \tag{4}$$

with the gauge coupling given by $\beta \equiv 2N_c/g^2$.

The path integral over the fermion field λ can be carried out and the result is a *Pfaffian*:

$$\int [d\lambda] e^{-\frac{1}{2}\overline{\lambda}Q\lambda} = \int [d\lambda] e^{-\frac{1}{2}\lambda M \lambda} = \mathrm{Pf}(M) \, , \tag{5}$$

where M is the antisymmetric matrix defined as

$$M \equiv CQ = -M^T \, . \tag{6}$$

The square of the Pfaffian $\mathrm{Pf}(M)$ is equal to the determinant of the fermion matrix Q:

$$\det(Q) = \det(M) = [\mathrm{Pf}(M)]^2 \, . \tag{7}$$

In order to perform Monte Carlo simulations of SYM theory one needs a positive measure on the gauge field which allows for importance sampling of the path integral. Therefore the sign of the Pfaffian has to be taken into account separately. Since the absolute value of the Pfaffian is the non-negative square root of the determinant the effective gauge field action is[9]:

$$S_{CV} = \beta \sum_{pl} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{pl} \right) - \frac{1}{2} \log\det Q[U] \, . \tag{8}$$

The factor $\frac{1}{2}$ in front of $\log\det Q$ shows that we effectively have a flavour number $N_f = \frac{1}{2}$ of adjoint fermions. The omitted sign of the Pfaffian can be taken into account by reweighting the expectation values according to

$$\langle A \rangle = \frac{\langle A \, \mathrm{signPf}(M) \rangle_{CV}}{\langle \mathrm{signPf}(M) \rangle_{CV}} \, , \tag{9}$$

where $\langle \ldots \rangle_{CV}$ denotes expectation values with respect to the effective gauge action S_{CV}.

The sign of the Pfaffian may lead in principle to a *sign problem* if the contributions with opposite sign in the reweighting formula (9) cancel each other. This cancellation may lead to an intolarable increase of the statistical error. We have shown[3–5] that the monitoring of the sign of the Pfaffian can be done with reasonable numerical effort and for positive gaugino masses practically no sign problem occurs because the positive contributions dominate.

3 Low Energy Dynamics in SYM Theory

On the basis of its similarity to QCD one can assume that the basic features of SYM dynamics are simular to QCD: confinement of the coloured degrees of freedom and spontaneous chiral symmetry breaking. As in QCD, a central feature of low-energy dynamics is the realization of the global chiral symmetry. There is only a single Majorana adjoint “flavour” therefore the global chiral symmetry of SYM is an abelian symmetry $U(1)_\lambda$. This is, however, only a symmetry of the classical Lagrangian and it is not fully realized in the quantum field theory. Therefore it is an *anomalous symmetry*. The remnant symmetry in the quantum theory is Z_{2N_c} for a gauge group $\mathrm{SU}(N_c)$.

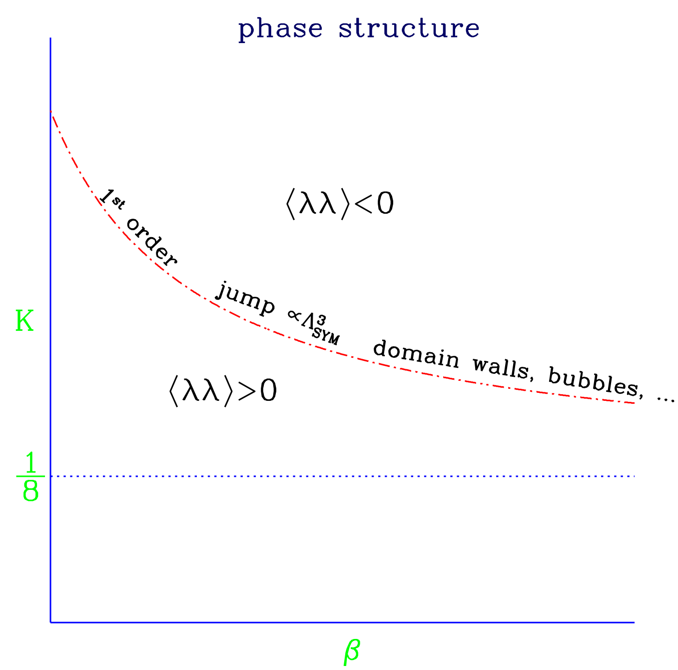

Figure 1. Expected phase structure of Yang-Mills theory with a Majorana fermion in adjoint representation in the (β, K)-plane. The dashed-dotted line $K = K_{cr}(\beta)$ is a first order phase transition (or cross-over) at zero gaugino mass.

Similarly to QCD the discrete chiral symmetry Z_{2N_c} is expected to be spontaneously broken to Z_2 by the non-zero *gaugino condensate* $\langle\lambda\lambda\rangle \neq 0$. The consequence of this spontaneous chiral symmetry breaking pattern is the existence of a first order phase transition at zero gaugino mass $m_{\tilde{g}} = 0$ (see figure 1). For instance, in case of $N_c = 2$ at $m_{\tilde{g}} = 0$ there exist two degenerate ground states with opposite signs of the gaugino condensate. The symmetry breaking is linear in $m_{\tilde{g}}$, therefore the two ground states are exchanged at $m_{\tilde{g}} = 0$ and there is a first order phase transition. In fact, the Z_{2N_c} chiral symmetry is only expected to be exact in the continuum limit $\beta \to \infty$ (at zero lattice spacing). It is possible that for finite β there is a cross-over which becomes a genuine first order phase transition only at $\beta = \infty$.

The DESY-Münster Collaboration performed a first lattice investigation of gaugino condensation in SYM theory with the SU(2) gauge group[3]. The distribution of $\langle\lambda\lambda\rangle$ has been studied for fixed gauge coupling $\beta = 2.3$ as a function of the hopping parameter K, which determines the bare gaugino mass, on a $6^3 \cdot 12$ lattice.

A first order phase transition (or cross-over) shows up on small to moderately large lattices as a metastability expressed by a two-peak structure in the distribution of the gaugino condensate. By tuning K one should be able to achieve that the two peaks are equal (in height or area). This is a possible definition of the phase transition point in finite volumes. By increasing the volume the tunneling between the two ground states becomes less and less probable and at some point practically impossible.

The observed distributions are shown in figure 2. One can see that the distributions cannot always be described by a single Gaussian which would correspond to a single phase. The volume dependence of this signal has not been studied up to now. It is left for further studies which could distinguish between a phase transition and a cross-over.

Confinement of the fundamental colour charge is shown by the linear increase of the potential between static colour sources. This *fundamental static potential* has been extracted, as usual, from Wilson loops and shows indeed a linear rise in agreement with expectations[3].

The consequence of confinement is that the low energy spectrum in SYM theory consists of colourless *hadronic bound states* as in QCD. In the supersymmetric limit the states have to belong to *supermultiplets* with degenerate mass. For non-zero gaugino mass the supersymmetry is softly broken and the supermultiplets are split up in mass. In analogy with the effective chiral Lagrangian of QCD it is possible to derive low energy effective Lagrangians also in SYM theory which reflects the symmetries and is based on an assumption about the nature of the lowest mass states. The simplest assumption gives the Veneziano-Yankielowicz effective Lagrangian[10] but generalizations with more involved supermultiplet structure are possible[11].

The spectrum of SYM theory with SU(2) gauge group has been studied in numerical simulations by the DESY-Münster Collaboration[4]. A summary of the results on the masses of light bound states is shown in figure 3. The results at the smallest gaugino mass are consistent with two light supermultiplets split up in mass[11] but further investigations in a larger lattice volume are necessary for a more definite statement.

4 Realization of Supersymmetry

An important feature of lattice regularization is that some symmetries are broken for non-zero lattice spacing and are expected to be recovered in the continuum limit. The details of the lattice formulation, which also influence the degree of symmetry breaking, are not

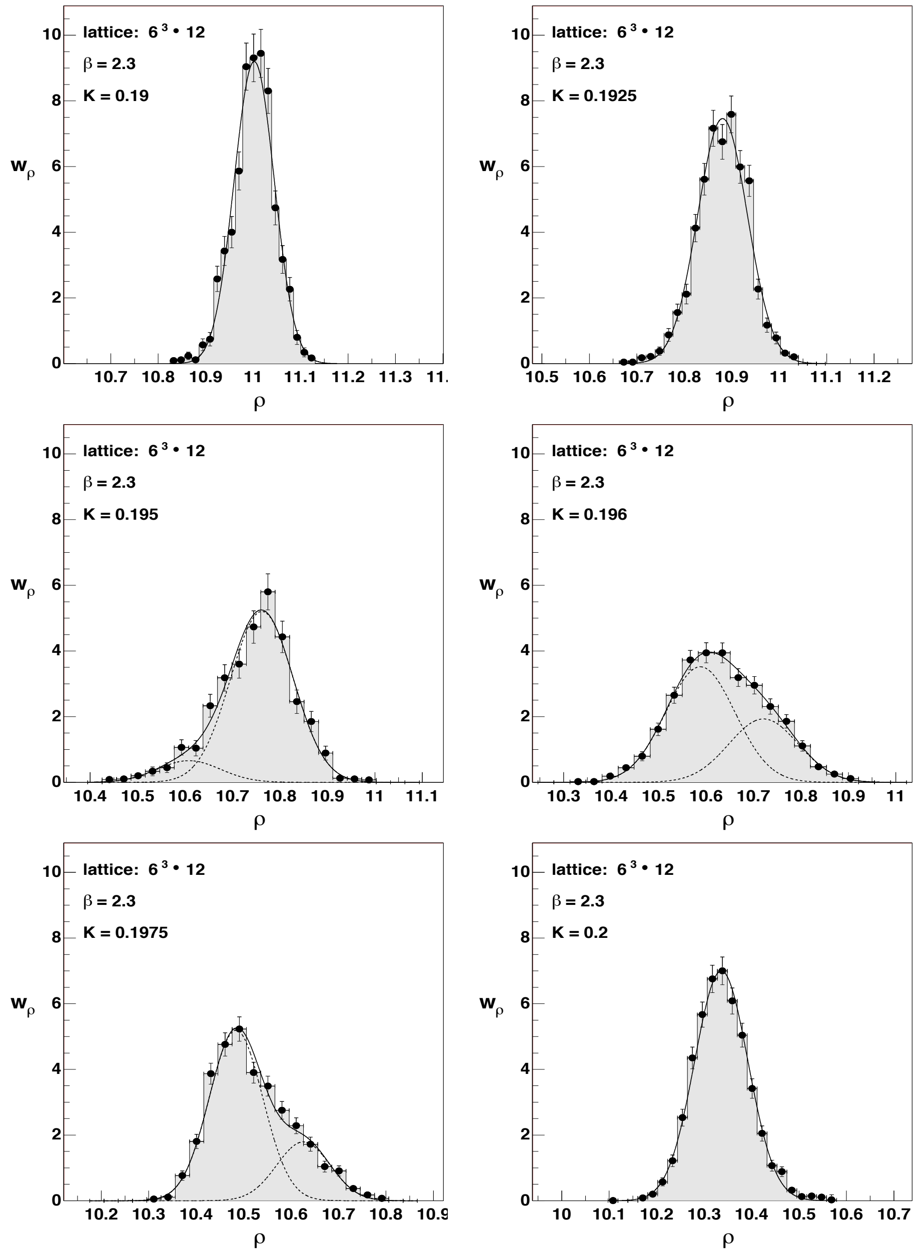

Figure 2. The probability distributions of the gaugino condensate $\rho \equiv \langle\lambda\lambda\rangle$ for different hopping parameters at $\beta = 2.3$ on a $6^3 \cdot 12$ lattice. The dashed lines show the Gaussian components.

relevant in the continuum limit because of the *universality of critical points*. A basic set of symmetries broken by the lattice and restored in the continuum limit is the (Euclidean)

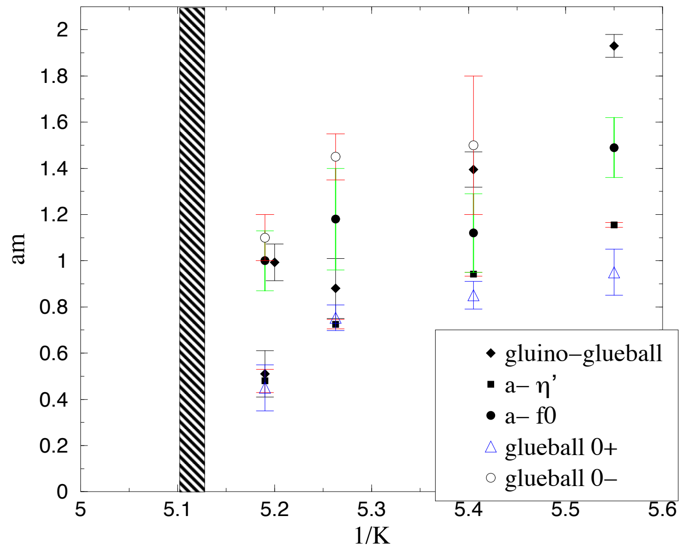

Figure 3. The lightest bound state masses in lattice units as function of the bare gaugino mass parameter $1/K$ for fixed $\beta = 2.3$. The shaded area at $K = 0.1955(5)$ is where zero gaugino mass and supersymmetry are expected.

Lorentz symmetry including rotations and translations. It is clear that on any regular lattice these symmetries are always broken. Internal symmetries as, for instance, global chiral symmetry are sometimes broken and sometimes conserved on the lattice, depending on the actual formulation. From the point of view of symmetry realization supersymmetry is expected to work similarly to the Lorentz symmetry: at finite lattice spacing it is broken but it becomes restored in the continuum limit. This similarity is quite natural since there is an intimate relation of supersymmetry to the Lorentz symmetries of space-time, shown, for instance, by the fact that the anticommutators of the supersymmetry charges give translations.

In the framework of quantum field theory the symmetries can be exploited by the corresponding Ward-Takahashi (WT) identities. In the case of SYM theory this way of realizing supersymmetry was first considered by Curci and Veneziano[9]. At zero gaugino mass both supersymmetry and anomalous chiral symmetry has to be manifest in the corresponding WT identities. Generally speaking one expects the existence of a renormalized supercurrent operator $S_{R\mu}$ satisfying the Ward-Takahashi-type identity

$$\partial_\mu S_{R\mu} = 2 m_R \chi_R \, . \tag{10}$$

Here m_R is the renormalized gaugino mass and χ_R is a suitably defined spinorial density. At zero gaugino mass $m_R = 0$ the supercurrent is conserved and the supersymmetry is exact.

The DESY-Münster-Roma Collaboration studied the SUSY WT-identities in numerical simulations[5]. Omitting $\mathcal{O}(a)$ terms which vanish in the continuum limit $a \to 0$ the lattice version of the SUSY WT-identity can be written as[12]

$$\langle \nabla_\mu S_\mu(x)\mathcal{O}(y)\rangle + \frac{Z_T}{Z_S}\langle \nabla_\mu T_\mu(x)\mathcal{O}(y)\rangle = \frac{m_S}{Z_S}\langle \chi(x)\mathcal{O}(y)\rangle \; . \qquad (11)$$

$S_\mu(x)$ is a suitably defined supercurrent and $T_\mu(x)$ is an admixture with the same quantum numbers as $S_\mu(x)$. Eq. (11) has to be valid for every gauge invariant function $\mathcal{O}(y)$ which is defined in a point y in such a way that there are no common points with the functions defined at x. Considering (11) with different $\mathcal{O}(y)$ a system of linear equations is obtained for the two unknowns

$$z_{TS} \equiv \frac{Z_T}{Z_S} \; , \qquad z_{MS} \equiv \frac{m_S}{Z_S} \; . \qquad (12)$$

(Note that z_{MS} is proportional to the renormalized gaugino mass $m_R \equiv m_S/Z_m$.) Since there are many different possible $\mathcal{O}(y)$'s, the expectation that (12) has, in the continuum limit, a unique solution pair (z_{TS}, z_{MS}) is highly non-trivial. Its numerical investigation can strongly support (or possibly contradict) our expectations about the realization of supersymmetry in SYM theory.

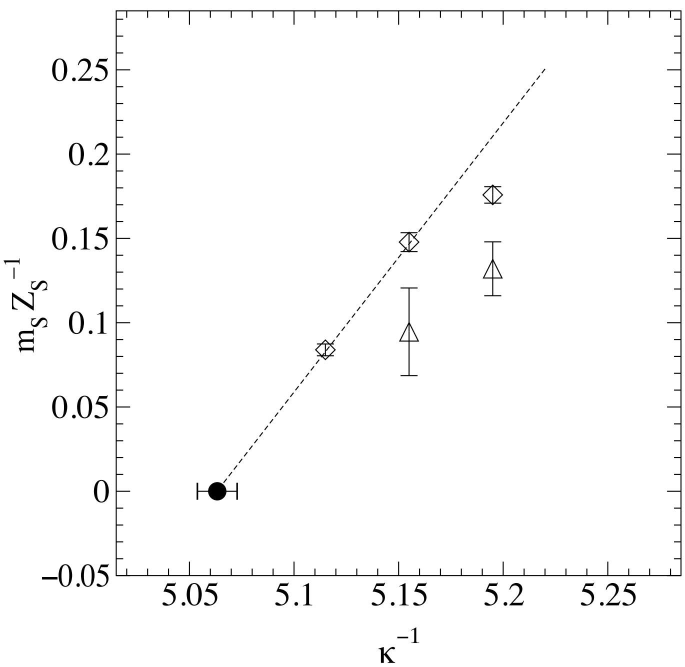

Figure 4. The variable z_{MS} as a function of the bare gaugino mass. An extrapolation to zero defines the point where supersymmetry is realized in the continuum limit.

The numerical data of the DESY-Münster-Roma Collaboration (see for instance in figure 4) demonstrate the feasibility of implementing lattice SUSY WT identities in order to verify supersymmetry restoration in a non-perturbative framework.

5 Conclusions

In the context of our investigations of supersymmetric Yang-Mills theories on a lattice we have developed an efficient algorithm for simulating dynamical Majorana fermions, corresponding to $N_f = 1/2$ fermion flavours, on massively parallel supercomputers. Basic features of the model have been studied on lattices of different sizes. This includes the phase structure, gaugino condensate, sign of the Pfaffian, confinement potential, spectrum of low-lying bound states, and supersymmetric Ward-Takahashi identities.

Most of the numerical calculations presented in this report have been performed on the Cray T3E of the NIC, Jülich. The total CPU time since 1997 was about $2 \cdot 10^6$ hours.

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