Introduction to the Standard Model
Problem sheet 8

Deadline: Monday 15 June 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: β-function, static potential

1. The β-function of QED is of the form $\beta(e) = \beta_0\, e^3 + \mathcal{O}(e^5)$, where $\beta_0 = 1/(12\pi^2)$, and e is the electric charge.
   a) (2 P) Show that
   $$e_R^2(Q) = \frac{e_R^2(\mu)}{1 - \beta_0\, e_R^2(\mu) \ln(Q^2/\mu^2)},$$
   if the β-function is approximated by its lowest order term.
   b) (1 P) Estimate the value of Q, where $e_R^2(Q)$ would diverge in this approximation, if μ is taken to be the electron mass m_e, and $e_R^2(m_e)/4\pi \approx 1/137,036$.

2. β-function fixed point
   Consider a theory with β-function $\beta(g) = a_1 g - a_2\, g^3$, with $a_1, a_2 > 0$.
   a) (2 P) Draw a sketch of $\beta(g)$. On the graph of $\beta(g)$ indicate the flow of g for increasing momentum scale Q.
   b) (3 P) Find the limiting value g_c of g for $Q \to \infty$. Show that
   $$g_R(Q) - g_c \propto \left(\frac{Q^2}{\mu^2}\right)^{-\gamma} \qquad \text{as} \quad Q \to \infty,$$
   and calculate the exponent γ.

3. (1 P) The static quark-antiquark potential rises linearly at large distances, $V(r) \sim k\, r$, where the string tension has the value $k \approx 160\,000$ N. Estimate the string breaking distance r_B by assuming $V(r_B) \approx 2m_\pi$.

4. The Fourier transform of the quark-antiquark potential, $\tilde{V}(\vec{k}\,) = \int d^3x\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}\, V(\vec{r}\,)$, is in leading order of perturbation theory given by
   $$\tilde{V}^{(0)}(\vec{k}\,) = -\frac{4}{3} g^2 \frac{1}{\vec{k}^{\,2}}\,.$$
   a) (1 P) Write down $V^{(0)}(\vec{r}\,)$. (No calculation required.)
   b) (3 P) In the one-loop approximation the most important part of $\tilde{V}(\vec{k}\,)$ is obtained from its leading order expression by replacing g^2 by
   $$g^2 \left(1 - \beta_0\, g^2 \ln(\vec{k}^{\,2}/\mu^2)\right).$$
   Show that the corresponding potential is given by
   $$V(\vec{r}\,) = V^{(0)}(\vec{r}\,)\left(1 + \beta_0\, g^2 \ln(a\mu^2 r^2)\right),$$
   where a is some constant.
   Hint: Use the following integral.
   c) (2 P) Show by dimensional analysis that
   $$\int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\, (\vec{k}^{\,2})^{-(1+\alpha)} = \frac{C(\alpha)}{4\pi}\, r^{-1+2\alpha} \qquad (\alpha \geq 0), \quad \text{with} \quad C(0) = 1.$$
   d) (6 bonus points) This is only for the really ambitious students: show that in the previous integral
   $$C(\alpha) = \left(\cos(\pi\alpha)\, \Gamma(1+2\alpha)\right)^{-1}.$$