Introduction to the Standard Model
Problem sheet 7

Deadline: Monday 8 June 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: spontaneously broken chiral symmetry, β-function

1. Spontaneous symmetry breaking

   If a symmetry is unbroken, then vacuum expectation values $\langle 0|\mathcal{O}|0\rangle$ of observables are invariant under the symmetry transformations.

   Consider QCD with $N_f = 2$ quark flavours and equal quark masses $m_u = m_d$. Assume that the quark condensates $\langle 0|\bar{u}u|0\rangle$ and $\langle 0|\bar{d}d|0\rangle$ are non-zero.

   a) (2 P) Explain why this implies a spontaneous breaking of chiral $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ symmetry in the case of vanishing quark masses $m_u = m_d = 0$.

   b) (2 P) Show that $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle$, if flavour SU(2) is unbroken.

2. Let $\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 + \ldots$ be the renormalisation group β-function of QCD in a particular scheme.

   a) (3 P) From
   $$Q\frac{\mathrm{d}}{\mathrm{d}Q}\, g_R(Q) = \beta(g_R(Q)) \qquad \text{(see lecture)},$$
   show that
   $$g_R^2(Q) = \frac{g_R^2(\mu)}{1 + \beta_0\, g_R^2(\mu)\ln(Q^2/\mu^2)},$$
   if the β-function is approximated by its lowest order term.

   b) (3 P) Taking into account higher orders, show that
   $$\ln(Q^2/\mu^2) = F(g_R(Q)) - F(g_R(\mu)),$$
   where
   $$F(g) = \frac{1}{\beta_0\, g^2} + \frac{2\beta_1}{\beta_0^2}\ln g + \mathcal{O}(g^2).$$

   c) (4 P) Let $\tilde{g}$ be a different renormalised coupling, defined in another scheme, such that
   $$\tilde{g} = g + cg^3 + \mathcal{O}(g^5).$$
   The corresponding β-function, given by
   $$Q\frac{\mathrm{d}}{\mathrm{d}Q}\,\tilde{g}(Q) = \tilde{\beta}(\tilde{g}(Q))$$
   is of the form
   $$\tilde{\beta}(\tilde{g}) = -\tilde{\beta}_0\,\tilde{g}^3 - \tilde{\beta}_1\,\tilde{g}^5 - \tilde{\beta}_2\,\tilde{g}^7 + \ldots$$
   Show that
   $$\tilde{\beta}_0 = \beta_0, \quad \tilde{\beta}_1 = \beta_1\,,$$
   and that $\tilde{\beta}_2 \neq \beta_2$ in general.