Introduction to the Standard Model
Problem sheet 6

Deadline: Monday 1 June 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: currents, chirality

1. (6 P) Let $q(x) = (q_{\alpha,i,f}(x))$ be the quark field, where $\alpha \in \{1,\dots,4\}$ is the Dirac index, $i \in \{1,2,3\}$ the colour index, and $f \in \{1,\dots,N_f\}$ the flavour index. Write the following expressions explicitly in the index notation, including all indices:
   a) $Uq(x)$, where U is an element of colour SU(3),
   b) $\Omega q(x)$, where Ω is an element of flavour SU(N_f),
   c) $\gamma^\mu \partial_\mu q(x)$,
   d) $\bar{q}(x)q(x)$,
   e) $\bar{q}(x)\gamma^\mu q(x)$,
   f) $\bar{q}(x)\gamma^\mu T_a q(x)$, where T_a is a generator of flavour SU(N_f).

2. Calculate the Noether currents associated with the following transformations of the quark field in QCD:
   a) (2 P) flavour transformations $q' = \Omega q$, $\Omega \in \mathrm{SU}(N_f)$,
   b) (1 P) phase transformation $q' = \mathrm{e}^{-\mathrm{i}\alpha} q$.
   c) (4 P) Calculate the divergences $\partial_\mu j_a^\mu$ of the flavour currents

$$j_a^\mu(x) = \sum_{f,f'} \bar{q}_f(x)\gamma^\mu (T_a)_{ff'} q_{f'}(x),$$

   for free quarks with quark masses m_f, using the field equations.
   Hint: if all quark masses are equal, the expression should be zero.

3. Let $P_L = \frac{1}{2}(\mathbb{1} - \gamma_5)$ and $P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$.
   a) (2 P) Show that P_L and P_R are complementary projectors, i. e. $P_L^+ = P_L$, $P_R^+ = P_R$, $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L + P_R = \mathbb{1}$, $P_L P_R = P_R P_L = 0$.
   b) (3 P) For a Dirac field $\psi(x)$ define $\psi_L(x) = P_L\psi(x)$ and $\psi_R(x) = P_R\psi(x)$. Calculate $\overline{\psi_L}$ and $\overline{\psi_R}$. Express $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$ in terms of ψ_L and ψ_R.
   c) (3 P) Show that

$$\mathrm{e}^{-\mathrm{i}\alpha\gamma_5} = \cos(\alpha)\,\mathbb{1} - \mathrm{i}\sin(\alpha)\gamma_5 = \mathrm{e}^{-\mathrm{i}\alpha} P_R + \mathrm{e}^{\mathrm{i}\alpha} P_L.$$

   d) (2 P) Show that for the quark field $q(x)$

$$\exp(-\mathrm{i}\,\omega^a T_a \gamma_5)\, q(x) = \mathrm{e}^{-\mathrm{i}\omega^a T_a} q_R(x) + \mathrm{e}^{\mathrm{i}\omega^a T_a} q_L(x),$$

   where T_a are flavour symmetry generators.
   e) (2 P) Let $q'(x) = \exp(-\mathrm{i}\,\omega^a T_a \gamma_5)\, q(x)$ as in (d). Does $\bar{q}'(x)q'(x) = \bar{q}(x)q(x)$ hold?
   f) (2 P) Calculate the Noether current associated with the transformation in (e).