Introduction to the Standard Model
Problem sheet 5

Deadline: Monday 18 May 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: Gauge theories

1. Consider the Lagrangian density for the Maxwell field $A_\mu(x)$:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2.$$

   a) (2 P) Show by using partial integration in the action that it can be written equivalently in the form $\mathscr{L} = \frac{1}{2}A^\mu M_{\mu\nu}A^\nu$. Write down the expression for $M_{\mu\nu}$.

   b) (3 P) Calculate the photon propagator in momentum space $\tilde{D}_{F,\mu\nu}(k) = \left(\tilde{M}^{-1}\right)_{\mu\nu}(k)$.

2. Consider the Lagrangian

$$\mathscr{L} = \partial_\mu\phi^+\partial^\mu\phi - \phi^+ M\phi - \frac{g}{4}(\phi^+\phi)^2$$

   for a two-component complex scalar field $\phi(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}$ with mass matrix $M = \begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}$.

   a) (2 P) Write down the field equation for ϕ.

   b) (4 P) Find the global symmetry group for the case $m_1 = m_2$. Find a basis of its generators T_a that contains the SU(2) generators as a subset, and which fulfils $\mathrm{Tr}\,(T_aT_b) = \frac{1}{2}\delta_{ab}$. Find the conserved Noether currents.

   c) (2 P) Find the global symmetry group for the case $m_1 \neq m_2$. Find a basis of its generators T_a as a subset of those found in (b).

   d) (3 P) Calculate the divergences $\partial_\mu j^\mu$ of the Noether currents found in (b), but for the case $m_1 \neq m_2$.
   Remember: Use the field equations.

   e) (2 P) Consider the case $m_1 \neq m_2$. Modify the Lagrangian such that it is invariant under local transformations corresponding to the symmetry group found in (c) by coupling to a number of different "Maxwell fields". Write the covariant derivative with the help of the generators T_a.

3. (2 P) Prove the identity

$$[D_\mu, D_\nu] = -\mathrm{i}gT^aF^a_{\mu\nu},$$

   where D_μ is the covariant derivative for a non-Abelian gauge theory , g the coupling constant and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu$ is the field strength tensor.

4. Consider the Lagrangian for an N-component complex scalar field interacting with non-Abelian gauge fields:

$$\mathscr{L} = (D_\mu\phi)^+D^\mu\phi - m^2\phi^+\phi - \lambda(\phi^+\phi)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}.$$

   a) (2 P) Derive the field equations for the scalar field $\phi(x)$.

   b) (3 P) Derive the field equations for the gauge fields and show that they can be written as $\partial_\mu F^{\mu\nu} - \mathrm{i}g[A_\mu, F^{\mu\nu}] = j^\nu$. Write down the expression for the current $j^\nu = j^{a,\nu}T_a$, formed out of the scalar fields and the gauge field.