Introduction to the Standard Model
Problem sheet 3

Deadline: Monday 4 May 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: SU(2) and SU(3) algebra

1. Consider the following Lagrangian, which describes interactions between pions and nucleons:
$$\mathscr{L}_I = \bar{N}[\frac{1}{\sqrt{2}}(I_+ \ \pi^+ + I_- \ \pi^-) + I_3 \ \pi^0]N,$$
where proton p and neutron n are combined in the iso-spinor $N = \begin{pmatrix} p \\ n \end{pmatrix}$.

   a) (1 P) Show that
$$\frac{1}{\sqrt{2}}(I_+ \ \pi^+ + I_- \ \pi^-) + I_3 \ \pi^0 = I_1 \ \pi_1 + I_2 \ \pi_2 + I_3 \ \pi_3,$$
   where $I_k = \frac{1}{2}\tau_k$ and $\pi^+ = \frac{1}{\sqrt{2}}(\pi_1 - \mathrm{i} \ \pi_2)$, $\pi^- = \frac{1}{\sqrt{2}}(\pi_1 + \mathrm{i} \ \pi_2)$, $\pi^0 = \pi_3$.

   b) (3 P) Show that the Lagrangian is invariant under SU(2) transformations.
   Hint: it is sufficient to consider infinitesimal transformations.

2. (2 P) Prove the identity $[D_\mu, D_\nu] = \mathrm{i}qF_{\mu\nu}$, where D_μ is the covariant derivative, q is the electric charge and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

3. (3 P) Consider the Lagrangian density for free nucleons
$$\mathscr{L} = \bar{N}[\mathrm{i}\gamma^\mu \partial_\mu - m]N.$$
It is invariant under U(2) transformations. This group can be decomposed as U(2) $\cong$ SU(2)$\otimes$ U(1). Consider its subgroup U(1), given by the transformations
$$N(x) \longrightarrow N'(x) = \mathrm{e}^{-\mathrm{i}\theta} N(x).$$
Find the corresponding Noether current and charge. What is the physical interpretation of this charge?

4. (3 P) Consider the Lagrangian of the previous problem (3). Show that it has an additional (different) U(1) symmetry given by the transformations
$$N(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix} \longrightarrow N'(x) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha} p(x) \\ n(x) \end{pmatrix}.$$
Find the related Noether current and charge. What is the physical interpretation of this charge?

Please notice the reverse side of this sheet.

5. The standard basis for the generators of SU(3) in its fundamental representation is

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad T_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- (1 P) Evaluate the following structure constants of SU(3): f_{123}, f_{126}, f_{147}, f_{246}, f_{257}, f_{367}, f_{456}, f_{678}.
- (1 P) Check the orthogonality condition $\mathrm{Tr}\,(T_a T_b) = \lambda_F\,\delta_{ab}$ for 4 cases $a \neq b$, and evaluate the constant λ_F for this representation.
- (2 P) Show that f_{abc} is totally antisymmetric.
  Hint: First derive $f_{abc} = -2\mathrm{i}\,\mathrm{Tr}\,([T_a, T_b]T_c)$, then use the cyclic property of the trace in a clever way.