Introduction to the Standard Model
Problem sheet 10

Deadline: Monday 29 June 2015 (12 am)
at Dr. Giudice's office (KP 301) and Dr. Piemonte's office (KP 412)

Topics covered: spontaneous symmetry breaking, σ-models

1. The so-called linear σ-model of Gell-Mann and Lévy is an effective field theory for nucleons interacting with pions. The nucleons are represented by the isospin doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$. The four-component real scalar field ϕ_r, $r = 0, 1, 2, 3$, contains the pions $\pi_k \equiv \phi_k$, $k = 1, 2, 3$, and an additional scalar field $\sigma \equiv \phi_0$. The Lagrangian is

$$\mathscr{L} = \bar{N}\left[\mathrm{i}\gamma^\mu \partial_\mu + g\left(\sigma + \mathrm{i}\pi_k \tau_k \gamma_5\right)\right] N + \frac{1}{2}\partial_\mu \phi_r \, \partial^\mu \phi_r - \frac{m^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4,$$

where τ_k are the isospin Pauli matrices.

   a) (1 P) Write the nucleon-pion interaction term $\bar{N}\left(\sigma + \mathrm{i}\pi_k \tau_k \gamma_5\right) N$ in terms of the chiral components N_L and N_R.

   b) (1 P) Under transformation with elements $(U_L, U_R) \in \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ the nucleons transform as $N_L \to U_L N_L$, $N_R \to U_R N_R$. How has

$$\Phi \doteq \sigma \mathbf{1} + \mathrm{i}\pi_k \tau_k$$

   to transform, if the nucleon-pion interaction term is to be invariant under chiral symmetry transformations?

   c) (1 P) Show that these transformations act as rotations on ϕ, and therefore the potential for ϕ is also invariant.
   Remark: $\mathrm{SU}(2) \otimes \mathrm{SU}(2)/Z_2$ is isomorphic to $\mathrm{SO}(4)$.

   d) (1 P) What are the masses of the nucleons, the pions and σ in the case of unbroken symmetry, $m^2 > 0$.

   e) (3 P) Consider the case $m^2 < 0$ in the classical approximation. Assume that spontaneous symmetry breaking occurs and that the groundstate is represented by the fields $\sigma(x) = v$, $\pi_k(x) = 0$. Calculate v. Expand the fields around the ground state and calculate the masses of the nucleons, the pions and the σ particle.

   f) (1 P) Which subgroup of the chiral symmetry group $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ remains unbroken?

   g) (2 P) In the scenario $m^2 < 0$, add an explicit symmetry breaking term to the Lagrangian:

$$\mathscr{L}' = \mathscr{L} + c\,\sigma(x),$$

   where the constant c is small. Calculate the masses of the pseudo-Goldstone bosons π_k and of the σ-particle up to terms linear in c.

2. The non-linear σ-model is described by a n-component real scalar field $\phi_k(x)$, $k = 1, \ldots, n$, with fixed length,

$$\sum_{k=1}^{n} \phi_k^2 = f_\pi^2.$$

   Its Lagrangian is

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi\,.$$

   Assume that the groundstate is represented by

$$\phi_n(x) \equiv \sigma(x) = f_\pi, \qquad \pi_k(x) \equiv \phi_k(x) = 0, \quad k = 1, \ldots, n-1.$$

   a) (2 P) Express the Lagrangian in terms of the pion fields $\pi_k(x)$. What is the mass of the pions?

   b) (1 P) What is the symmetry group of the Lagrangian and what is its remaining unbroken subgroup? Check their dimensions versus the number of Goldstone bosons.

   c) (1 P) Write down the quartic self-interaction term of the pions.