Introduction to Quantum Field Theory
Problem sheet 7

Deadline: Wednesday 13 January 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Noether currents, Discrete symmetries, Dilatation.

1. (3 P) Calculate the energy momentum tensor of the free complex scalar field. Use it to obtain the Hamiltonian and the momentum operators (in terms of fields and conjugate momenta).

2. (3 P) Consider the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi \,,$$

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is an SU(2) doublet. Prove that the Noether charges Q^a (see Problem sheet 6, exercise 2) satisfy the commutation relations of the Lie algebra of SU(2).

3. (4 P) The operators for space reflection $\mathcal{P}$, charge conjugation $\mathcal{C}$, and time reversal $\mathcal{T}$ act on the free complex scalar field as:

$$\mathcal{P}\ \phi(x^0, \vec{x})\ \mathcal{P}^{-1} = \phi(x^0, -\vec{x}),$$
$$\mathcal{C}\ \phi(x)\ \mathcal{C}^{-1} = \phi^\dagger(x),$$
$$\mathcal{T}\ \phi(x^0, \vec{x})\ \mathcal{T}^{-1} = \phi(-x^0, \vec{x}).$$

Show that:

$$\mathcal{P}\ a^\dagger(\vec{k})\ \mathcal{P}^{-1} = a^\dagger(-\vec{k}),$$
$$\mathcal{P}\ b^\dagger(\vec{k})\ \mathcal{P}^{-1} = b^\dagger(-\vec{k}),$$
$$\mathcal{C}\ a^\dagger(\vec{k})\ \mathcal{C}^{-1} = b^\dagger(\vec{k}),$$
$$\mathcal{C}\ b^\dagger(\vec{k})\ \mathcal{C}^{-1} = a^\dagger(\vec{k}),$$
$$\mathcal{T}\ a^\dagger(\vec{k})\ \mathcal{T}^{-1} = a^\dagger(-\vec{k}),$$
$$\mathcal{T}\ b^\dagger(\vec{k})\ \mathcal{T}^{-1} = b^\dagger(-\vec{k}).$$

From these equations one might get the impression that $\mathcal{P}$ and $\mathcal{T}$ act in the same way on the Fock space. Why is this wrong ?

4. Dilatations are space-time transformations acting as:

$$x' = \rho\, x, \quad \rho > 0,$$
$$\phi'(x) = \rho^{-1}\, \phi(\rho^{-1} x),$$

where $\phi(x)$ is a scalar field.

Please notice the reverse side of this sheet.

a) (3P) Which of the action terms

$$\int d^4x\ (\partial_\mu\phi)(\partial^\mu\phi), \quad \int d^4x\ m^2\phi^2, \quad \int d^4x\ \frac{g}{24}\,\phi^4$$

is invariant under dilatations?

b) (1P) For $\rho = \mathrm{e}^\lambda$ find the infinitesimal ($\lambda \equiv \epsilon$) transformations δx^μ and $\delta\phi(x)$.

c) (2P) Show that for a real scalar field the Noether current of dilatations is given by:

$$j^\mu = \Theta^{\mu\nu}x_\nu + \phi\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

d) (2P) Calculate $\partial_\mu j^\mu$ for the interacting real scalar field with

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\,\phi^2 - \frac{g}{24}\,\phi^4,$$

using the field equations. In which case are dilations symmetry transformations?