Introduction to Quantum Field Theory
Problem sheet 3

Deadline: Wednesday 18 November 2015 (12 am)
at Dr. Giudice’s office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Parity operator, Dirac equation, Classical Field Theory.

1. (2 P) The parity operator $P$ in Dirac theory is defined by the conditions:
   \[ P \gamma^0 P^{-1} = \gamma^0, \quad P \gamma^i P^{-1} = -\gamma^i \quad (i = 1, 2, 3), \quad P^4 = 1. \]
   This is solved by $P = \gamma^0$. There are other solutions for $P$: determine the most general one.

2. (2 P) Given the spin operator $\vec{S} = \frac{\hbar}{2} \vec{\Sigma}$, calculate $\vec{S}^2$ and determined the spin $s$ from $\vec{S}^2 = \hbar^2 s(s + 1) \mathbf{1}$.

3. (4 P) Calculate the commutators $[H, \vec{L}]$ and $[H, \vec{S}]$ and check that $[\vec{L} + \vec{S}, H] = 0$. $H$ is the Dirac Hamiltonian operator.

4. (3 P) Derive the field equations and the Hamiltonian from the following Lagrangian for a one-dimensional elastic string:
   \[ \mathcal{L} = \mu \frac{\ddot{\phi}}{2} - \kappa \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \lambda \phi^4. \]

5. (4 P) Let $\phi(x)$ be a complex scalar field with Lagrange density
   \[ \mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 |\phi|^2 - \frac{g}{6} |\phi|^4. \]
   a) Consider $\phi$ and $\phi^*$ as two distinct fields and derive the field equations for them. Find the Hamiltonian.
   b) Let the two real fields $\varphi_1$ and $\varphi_2$ be defined through
   \[ \phi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2). \]
   Write $\mathcal{L}$ in terms of $\varphi_1$ and $\varphi_2$. Derive the field equations for $\varphi_1$ and $\varphi_2$ and show that they are equivalent to those from a).