Introduction to Quantum Field Theory
Problem sheet 1

Deadline: Wednesday 4 November 2015 (12 am)
at Dr. Giudice's office (KP 301)

Topics covered: Klein-Gordon and Dirac equation.

1. (2 P) Show that
$$j_\mu = -\frac{\mathrm{i}}{2}\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) - qA_\mu\phi^*\phi$$
satisfies the continuity equation when ϕ is a solution to the Klein-Gordon equation in an external electromagnetic potential A_μ and q is the charge of the particle.

2. The free Klein-Gordon equation is a second-order differential equation for a single function $\psi(\vec{r}, t)$.
   a) (2 P) Show that it is equivalent to two coupled differential equations that are of first order in time t and can be written as
   $$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = H\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1}$$
   and write down H (set $\hbar = c = 1$).
   b) (2 P) Find the two formal eigenvalues h_1 and h_2 of the 2×2 matrix H and write the differential equations in the decoupled form
   $$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\hat{\psi}_a = h_a\hat{\psi}_a \qquad (a = 1, 2). \tag{2}$$

3. (3 P) Calculate the following determinant:
$$\det\left(\gamma^\mu p_\mu - mc\mathbf{1}\right). \tag{3}$$

4. (3 P) Prove the following identities without using any particular representation of the gamma matrices:
   a) $\not{p}^2 = p^2$,
   b) $\gamma_\mu\gamma^\mu = 4$,
   c) $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$,
   d) $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}$,
   e) $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\eta\gamma^\mu = -2\gamma^\eta\gamma^\beta\gamma^\alpha$,
   f) $\sigma_{\mu\nu}\sigma^{\mu\nu} = 12$.

   Here the definitions $\not{p} \doteq \gamma^\mu p_\mu$ and $\sigma_{\mu\nu} \doteq (\mathrm{i}/2)[\gamma_\mu, \gamma_\nu]$ are used.