QFT Exercises 2

Due on 06.11.14

Topics covered: Dirac equation.
As always, we use natural units: $\hbar = c = 1$.

1. (25%) Show that

$$\sum_{r=1}^{2} u^{(r)}(\mathbf{p})\bar{u}^{(r)}(\mathbf{p}) \propto (\not{p} + m), \tag{1}$$

$$\sum_{r=1}^{2} v^{(r)}(\mathbf{p})\bar{v}^{(r)}(\mathbf{p}) \propto (\not{p} - m), \tag{2}$$

   where u, v are Dirac spinors. Find also the proportionality factors.

2. (25%) Prove that

$$\frac{\mathbf{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|} u^{(r)}(\mathbf{p}) = (-1)^{r+1} u^{(r)}(\mathbf{p}) \tag{3}$$

$$\frac{\mathbf{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|} v^{(r)}(\mathbf{p}) = (-1)^{r} v^{(r)}(\mathbf{p}) \tag{4}$$

3. (25%) Without using any particular representation of the Dirac spinors, prove the following identities ($\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$):

$$2m\, \bar{u}(\mathbf{p_1})\gamma_\mu u(\mathbf{p_2}) = \bar{u}(\mathbf{p_1})\left[(p_1 + p_2)_\mu + i\sigma_{\mu\nu}(p_1 - p_2)^\nu\right] u(\mathbf{p_2}) \tag{5}$$

$$\bar{u}(\mathbf{p'})\sigma_{\mu\nu}(p + p')^\nu u(\mathbf{p}) = i\bar{u}(\mathbf{p'})(p' - p)_\mu u(\mathbf{p}) \tag{6}$$

4. (25%) Prove the following identities without using any particular representation of the gamma matrices:

   a) $\not{p}^2 = p^2$,
   b) $\gamma_\mu \gamma^\mu = 4$,
   c) $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$,
   d) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4g^{\alpha\beta}$,
   e) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\eta \gamma^\mu = -2\gamma^\eta \gamma^\beta \gamma^\alpha$.
   f) $\sigma_{\mu\nu}\sigma^{\mu\nu} = 12$
   g) $\sigma_{\mu\nu}\gamma^\alpha \sigma^{\mu\nu} = 0$