QFT Exercises 1

Due on 30.10.14

Topics covered: Klein-Gordon and Dirac equations.
As always, we use natural units: $\hbar = c = 1$.

1. (20%) Show that

$$j_\mu = -\frac{i}{2}\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) - qA_\mu\phi^*\phi \tag{1}$$

satisfies the continuity equation when ϕ is a solution to the Klein-Gordon equation in an external electromagnetic potential A_μ.

2. (30%) Evaluate the commutator of the Dirac Hamiltonian, $H_D = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$, with the following operators:

   a) $\mathbf{p} = -i\nabla$
   b) $\mathbf{L} = \mathbf{r}\times\mathbf{p}$
   c) $\mathbf{L^2}$
   d) $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} = \frac{i}{2}\boldsymbol{\gamma}\times\boldsymbol{\gamma}$ and $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$
   e) $\mathbf{J} = \mathbf{L} + \mathbf{S}$
   f) $\mathbf{J^2}$
   g) $\boldsymbol{\Sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}$
   h) $\boldsymbol{\Sigma}\cdot\mathbf{n}$, where $\mathbf{n}$ is a unit vector

3. (20%) Show that if $\psi(x)$ is a solution of the Dirac equation in an electromagnetic field, then it satisfies

$$\left[\left(\partial_\mu - ieA_\mu\right)\left(\partial^\mu - ieA^\mu\right) - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2\right]\psi(x) = 0, \tag{2}$$

which is a generalized form of the Klein-Gordon equation. In this equation $\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$ and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

4. (30%) Find, up to a normalization constant, the explicit expressions for the basic Dirac spinors $u^{(r)}(\vec{p})$ and $v^{(r)}(\vec{p})$ with $r = 1, 2$.