SU(3)

$$[T_a, T_b] = i\, f_{abc}\, T_c \qquad a,b,c=\{1, \ldots 8\}$$

Casimir operators: $C_F = 2\,(N^2-1)\,/(2N) = 4/3$ $\qquad C_A = N = 3$

Gell-Mann matrices $\qquad T_k = \frac{\lambda_k}{2} \qquad Tr(\lambda_a\ \lambda_b) = 2\,\delta_{ab}$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(3) contains three SU(2) subgroups corresponding to symmetries between

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \longleftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix} \longleftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Isospin

$$\lambda_1 = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

V-spin

$$\lambda_4 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \lambda_5 = \begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

U-spin

$$\lambda_6 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \quad \lambda_7 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$$

For each subgroup, raising and lowering operators can be defined, cf. $J_{\pm}$ for SU(2)

$$T_{\pm} = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_{\pm} = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_{\pm} = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$

SU(2): 1 diagonal matrix σ_3 -> rank-1 group
SU(3): 2 diagonal matrices λ_3 , λ_8 -> rank-2 group

$$[T_3, T_8] = 0$$

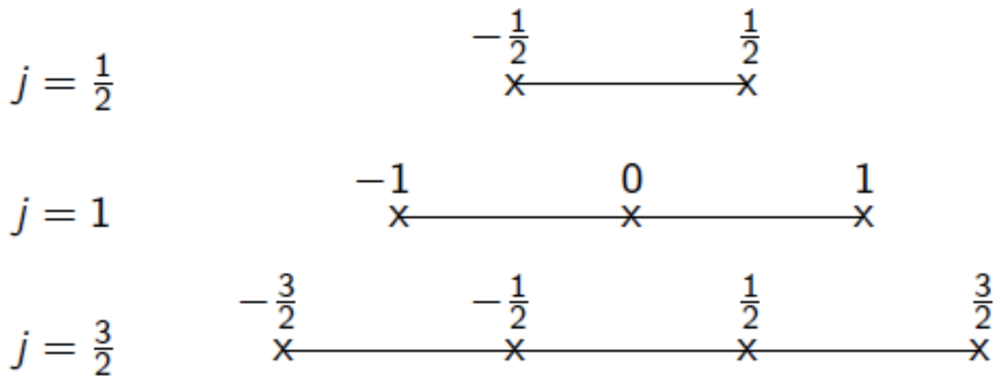
Define I_3 $Y = 2/\sqrt{3}\, T_8$

isospin hypercharge

-> states in SU(3) irreducible repr. can be labelled by eigenvalues of I_3 and Y

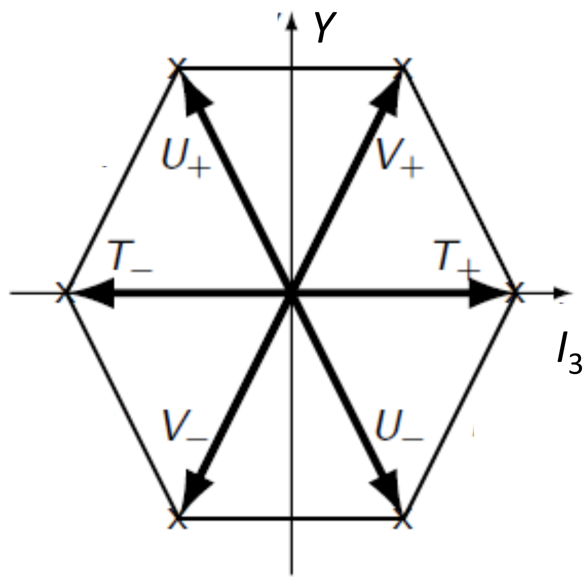
Graphical representation of the multiplets

SU(2)



One diagonal generator J_3, $J_\pm$ moves a state to the right or left <-> shifts between states labelled by eigenvalues of J_3

SU(3)



Two diagonal generators: T_3, T_8
$I_\pm$, $V_\pm$, $U_\pm$ move a state on the plane <-> shifts between eigenvalues of I_3 and Y

Similarly to what we have shown for $J_\pm$ in SU(2), one can show that:
$V_\pm$ shifts eigenvalues of I_3 by ± ½ and Y by ± 1,
$U_\pm$ shifts eigenvalues of I_3 by ± (-½) and Y by ± 1,
and $I_\pm$ shifts eigenvalues of I_3 by ± 1 and Y by 0

SU(3) of flavour

Historically, after the discovery of pion (1947) many more strongly interacting particles (hadrons) were identified.

- Mesons: hadrons with integer spin
- Baryons: hadrons with half-integer spin

Classification according to spin and parity: observed that particles within the same families have similar masses

Additionally, observed that some particles decay with a very long lifetime, despite being massive enough to decay into lighter objects

Proposed a new quantum number, strangeness *S*, conserved in strong interactions

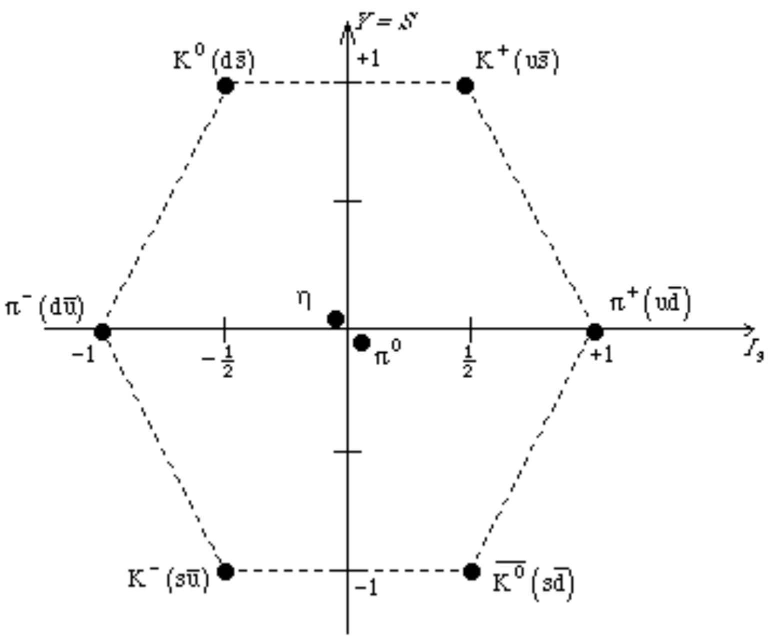
Gell-Mann-Nishijima relation

Y = *hypercharge*
B = *baryon number*

$$Q = I_3 + Y/2 \qquad Y = B + S$$

Extension of the isospin SU(2) isospin symmetry to SU(3) flavour symmetry
-> needed to accommodate hypercharge U(1) symmetry in addition to isospin

- Accommodates the idea of isospin and hypercharge as conserved quantities (by strong interactions)
- Allows for correspondence between ($I_{3,}$ Y) plots of mesons and baryons with the same spin and parity and the representations of the SU(3) group (*Gell-Mann and Ne'eman*)



BUT: Fundamental representation not identified with any known particles at that point

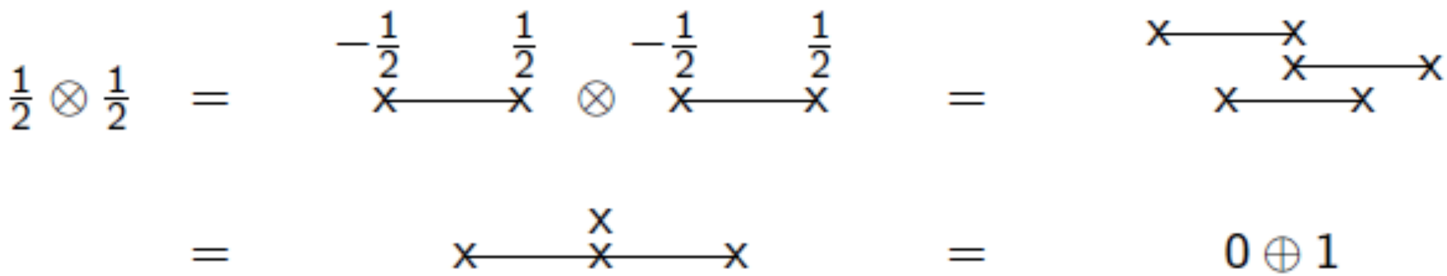
Quark model (*Gell-Mann and Zweig*)

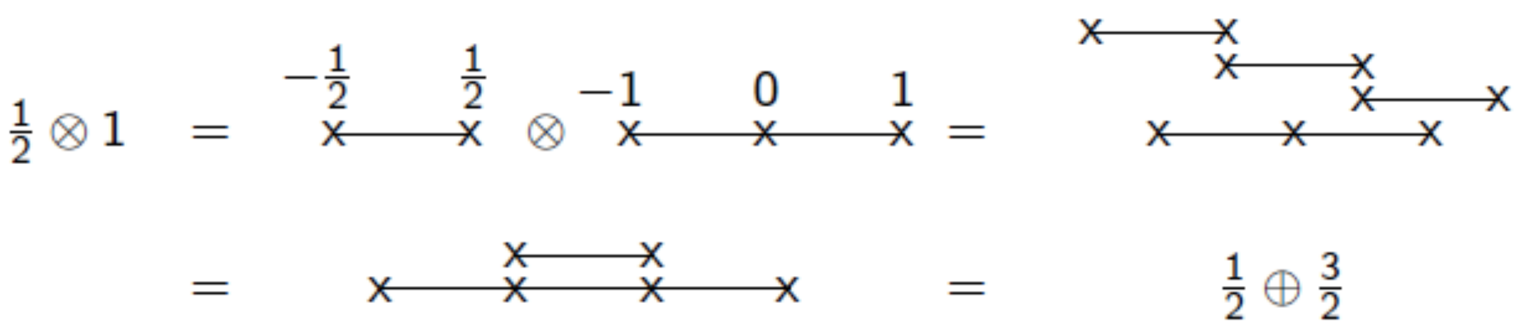
- All hadrons built out of spin-1/2 quarks which transform as members of fundamental representation
- There are three types of quarks (flavours) in the fundamental representation: up, down, strange
- Mesons (B = 0) are *q qbar* bound states
- Baryons (B = 1) are *qqq* bound states

Note: SU(3) flavour not as good as SU(2) isospin symmetry, mass splitting within a multiplet ~ 10%. Also, only 3 flavours were known in 60's of the last century, now we know there are 6 flavours.

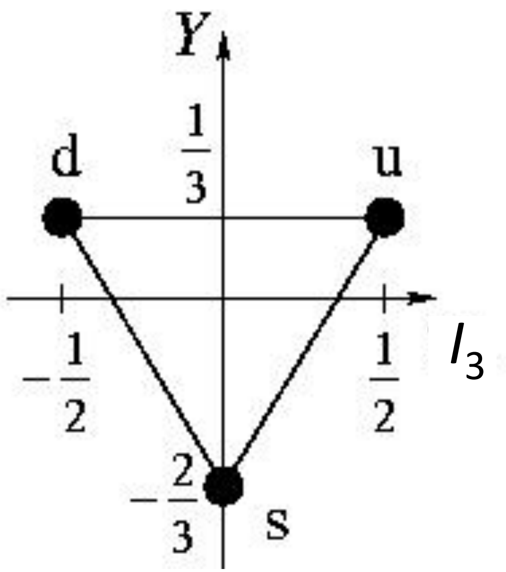
Graphical illustration of SU(2) product representations

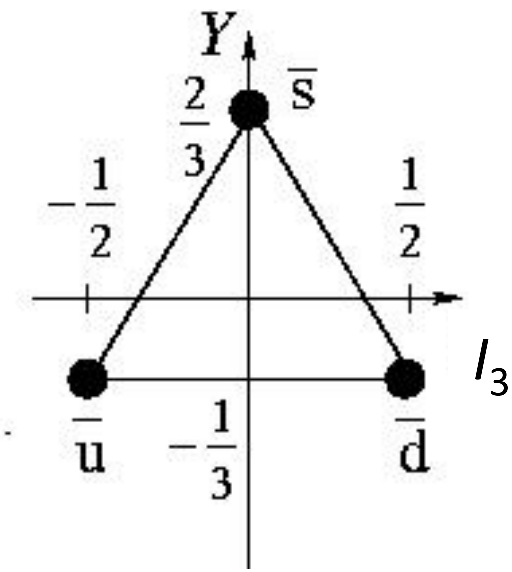
Eigenvalues of J_3 generators are additive in a product representation!





Quarks and antiquarks SU(3) representations (triplets)





$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I_3 u = \frac{1}{2} u$$

$$I_3 d = -\frac{1}{2} d$$

$$I_3 s = 0$$

$$I_3 \bar{u} = -\frac{1}{2} \bar{u}$$

$$I_3 \bar{d} = \frac{1}{2} \bar{d}$$

$$I_3 \bar{s} = 0$$

$$Y u = \frac{1}{3} u$$

$$Y d = \frac{1}{3} d$$

$$Y s = -\frac{2}{3} s$$

$$Y \bar{u} = -\frac{1}{3} \bar{u}$$

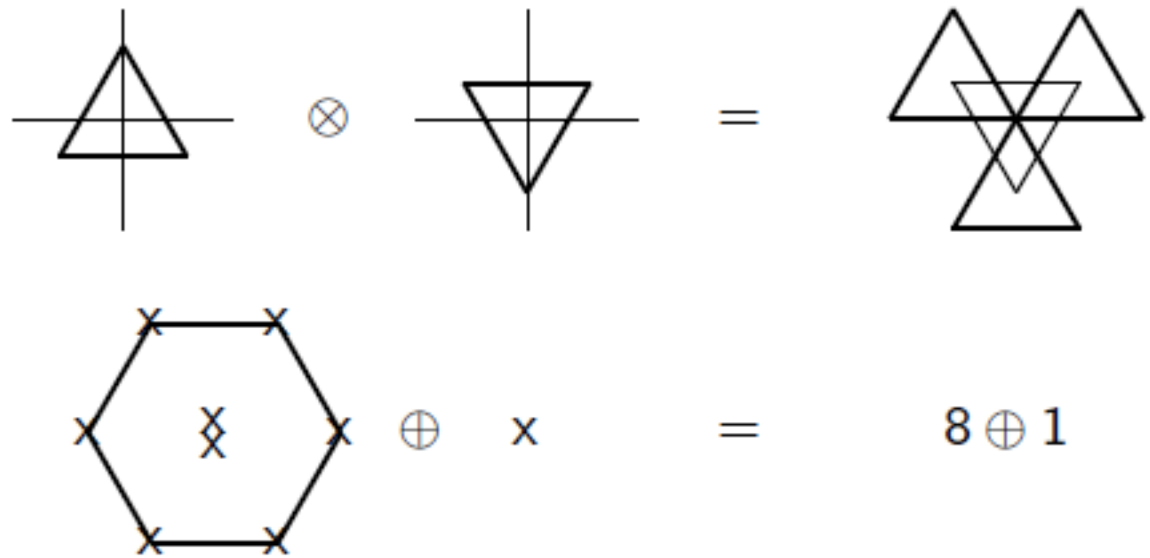
$$Y \bar{d} = -\frac{1}{3} \bar{d}$$

$$Y \bar{s} = \frac{2}{3} \bar{s}$$

Graphical illustration of SU(3) product representations

Mesons:

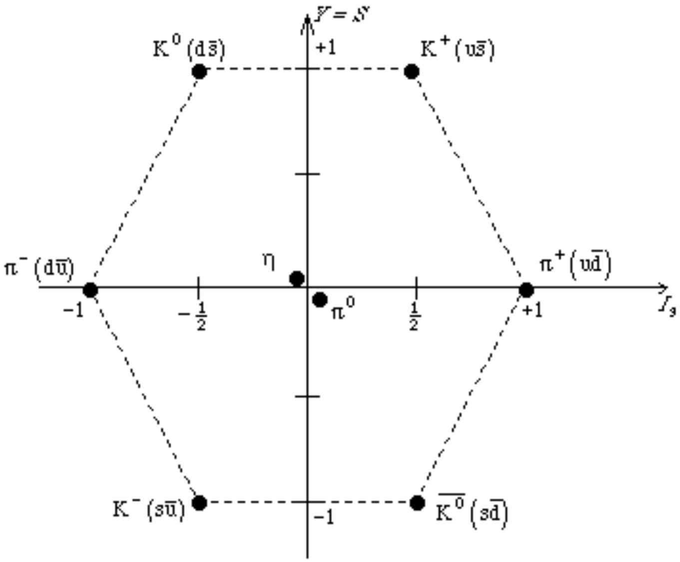
$$\bar{3} \otimes 3 =$$



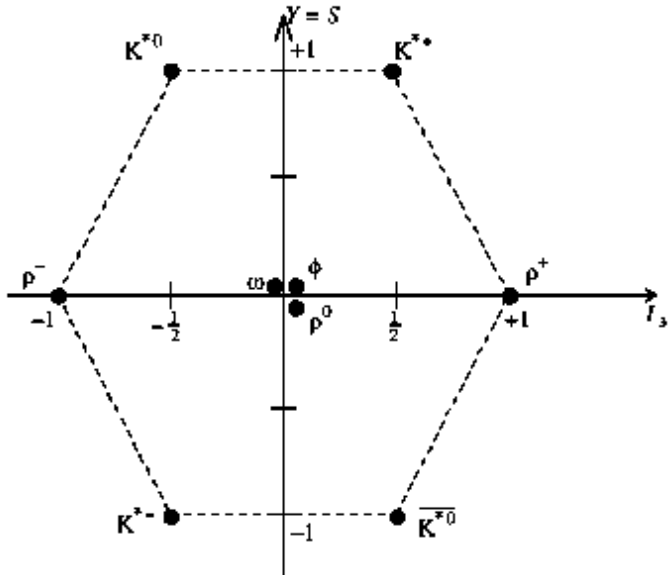
$$= 8 \oplus 1$$

Graphical illustration of SU(3) product representations

Meson octets:



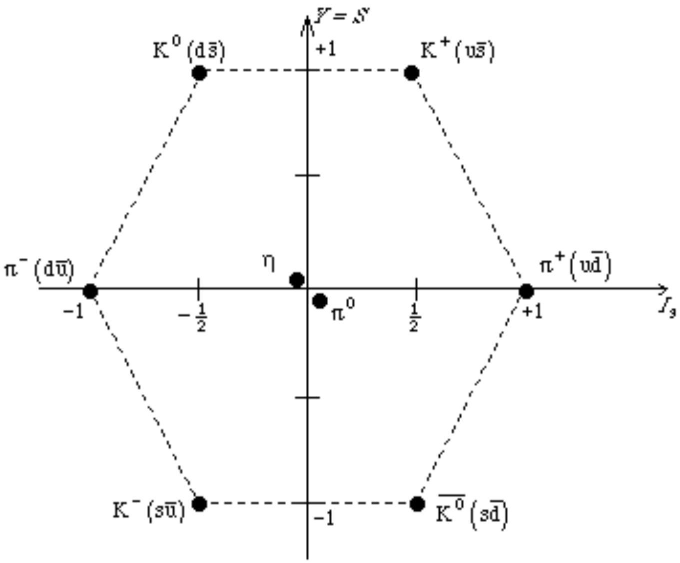
$J^P = 0^-$

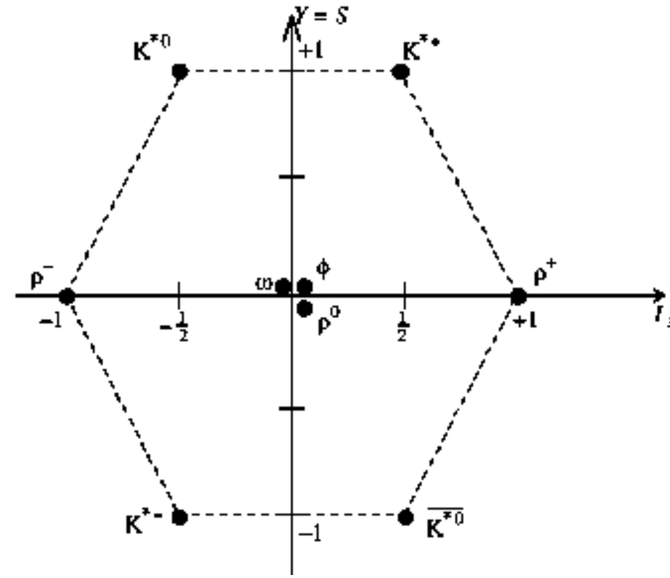


$J^P = 1^-$

Graphical illustration of SU(3) product representations

Meson octets:





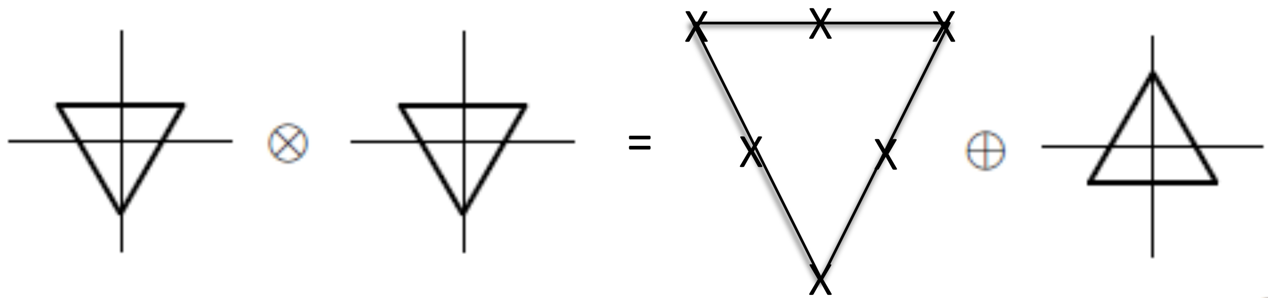
$$\pi^{\pm} : 140\,\mathrm{MeV} \qquad \pi^{0} : 135\,\mathrm{MeV}$$
$$K^{\pm} : 494\,\mathrm{MeV} \qquad K^{0}/\overline{K}^{0} : 498\,\mathrm{MeV}$$
$$\eta : 549\,\mathrm{MeV}$$

$$\rho^{\pm} : 770\,\mathrm{MeV} \qquad \rho^{0} : 770\,\mathrm{MeV}$$
$$K^{*\pm} : 892\,\mathrm{MeV} \qquad K^{*0}/\overline{K^{*}}^{0} : 896\,\mathrm{MeV}$$
$$\omega : 782\,\mathrm{MeV} \qquad \phi : 1020\,\mathrm{MeV}$$

Graphical illustration of SU(3) product representations

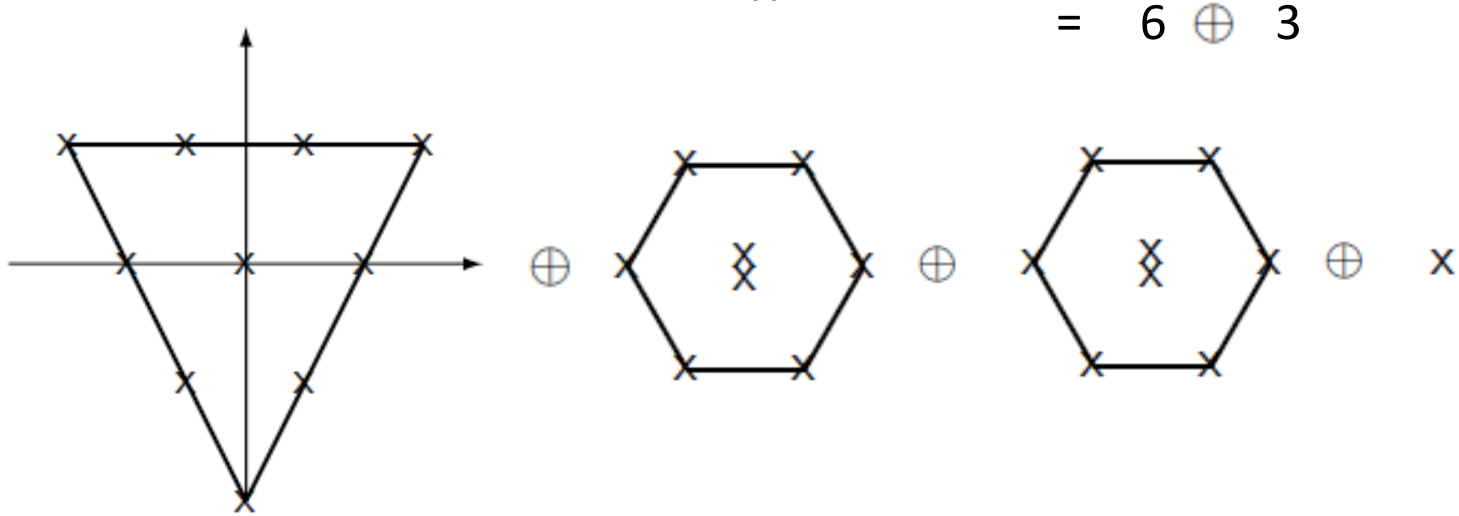
Baryons:

$3 \otimes 3 =$



$= 6 \oplus \bar{3}$

$3 \otimes 3 \otimes 3 =$



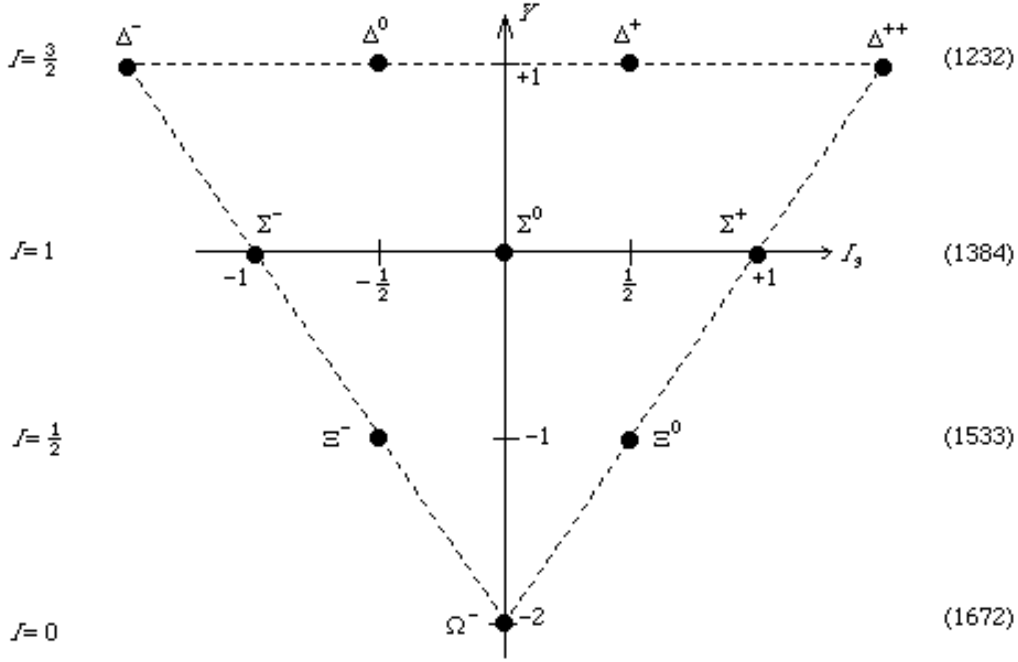
$= 10 \oplus 8 \oplus 8 \oplus 1$

See e.g. Georgi, Lie Algebras in Particle Physics

General method for SU(N) → Young tableaux

Graphical illustration of SU(3) product representations

e.g. SU(3) decouplet



$J^P = 3/2^{+}$

Problem: $\Delta^{++} = u\Uparrow u\Uparrow u\Uparrow$ spatial (ground state!) and spin (=3/2) wavefunctions totally symmetric

➔ violation of Fermi-Dirac statistics

SU(3) of colour : EXACT