The Super-Poincaré-Algebra

Seminar
Kern- und Teilchentheorie

Wintersemester 1999/2000

Tobias Galla

outline:

- Revision: Supersymmetric Quantum Mechanics
- Graduated Algebras
- The Poincaré-Algebra and its Casimir-Operators
- The Super-Poincaré-Algebra
- Representations of the Super-Poincaré -Algebra

The SUSY-Oscillator

system of bosons and fermions

$$\begin{aligned} |n_B n_F> &= |n_B>|n_F> \\ n_B &= 0, 1, 2, \ldots \\ n_F &= 0, 1 \end{aligned}$$

SUSY-Operators:

$$\begin{aligned} Q_+|n_B n_F> &\sim |n_B - 1, n_F + 1> \\ Q_-|n_B n_F> &\sim |n_B + 1, n_F - 1>, \end{aligned}$$

where

$$\begin{aligned} Q_+ &= b^- f^+ \\ Q_- &= b^+ f^-. \end{aligned}$$

Since $(f^+)^2 = (f^-)^2 = 0$ we have

$$(Q_+)^2 = (Q_-)^2 = 0$$

i.e. Q_+ and Q_- are nilpotent.

Requirement:

Q-transformations do not change the system's energy:

$$[H, Q_\pm] = 0$$

ansatz:

$$H = \{Q_+, Q_-\}$$

problem: $Q_\pm$ are not Hermitian, but

$$\begin{aligned} Q_1 &= Q_+ + Q_- \\ Q_2 &= -i(Q_+ - Q_-) \end{aligned}$$

are.

The Hamiltonian reads

$$H = Q_1^2 = Q_2^2$$

and fulfills

$$[H, Q_1] = [H, Q_2] = 0.$$

SUSY-Oscillator

$$\begin{aligned} Q_+ &= \sqrt{\hbar\omega} b^- f^+ \\ Q_- &= \sqrt{\hbar\omega} b^+ f^- \end{aligned}$$

hamiltonian:

$$\begin{aligned} H &= \{Q_+, Q_-\} \\ &= \hbar\omega \{b^- f^+, b^+ f^-\} \\ &= \hbar\omega (b^+ b^- + f^+ f^-) \\ &= H_B + H_F \\ &= Q_1^2 = Q_2^2. \end{aligned}$$

energies:

$$\begin{aligned} H|n_B n_F> &= \hbar\omega (N_B + N_F)|n_B n_F> \\ E &= \hbar\omega (n_B + n_F) \end{aligned}$$

Algebra:

$$\begin{aligned} [H, Q_i] &= 0 \\ \{Q_i, Q_j\} &= 2H\delta_{ij} \end{aligned}$$

Lie-groups and Lie-algebras:

determined by the commutators of the generators of the algebra.

SUSY-Algebra: anti-commutators involved !

example:

$$\begin{aligned} [H, Q_i] &= 0 \\ \{Q_i, Q_j\} &= 2H\delta_{ij} \end{aligned}$$

even (bosonic) operators: H

odd (fermionic) operators: Q_i

$$\begin{aligned} [B, B] &\sim B \quad \text{(Lie-Algebra)} \\ [B, F] &\sim F \\ \{F, F\} &\sim B \end{aligned}$$

mathematical framework:

graduated (Lie-) algebras

definition:
Let $\mathcal{G}$ be a vector space with a bilinear product $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (i.e. an algebra).

$\mathcal{G}$ is called $\mathbb{Z}/(2\mathbb{Z})$-graduated, if

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$$

for subspaces $\mathcal{G}_0$, $\mathcal{G}_1$ and if

$$\begin{aligned} \mathcal{G}_0 \circ \mathcal{G}_0 &\subset \mathcal{G}_0 \\ \mathcal{G}_0 \circ \mathcal{G}_1 &\subset \mathcal{G}_1 \\ \mathcal{G}_1 \circ \mathcal{G}_1 &\subset \mathcal{G}_0 \end{aligned}$$

i.e.

$$|g \circ h| \equiv |g| + |h| \text{ mod } 2$$

where $|g| = i$ for $g \in \mathcal{G}_i$, $i \in \{0, 1\}$.

If we have furthermore

the generalized Jacobi-identity

$$\begin{aligned} & (-1)^{|u_k||u_m|} u_k \circ (u_\ell \circ u_m) \\ + \ & (-1)^{|u_\ell||u_k|} u_\ell \circ (u_m \circ u_k) \\ + \ & (-1)^{|u_m||u_\ell|} u_m \circ (u_k \circ u_\ell) \\ = \ & 0 \end{aligned}$$

and supersymmetry

$$u_k \circ u_\ell = (-1)(-1)^{|u_k||u_\ell|}(u_\ell \circ u_k)$$

then we speak of $\mathcal{G}$ as a

$$\mathbb{Z}/(2\mathbb{Z})\text{-graduated super-algebra}$$

supersymmetry in this context means:

$\circ$ is antisymmetric on $\mathcal{G}_0 \times \mathcal{G}_0$ $([\cdot,\cdot])$

$\circ$ is symmetric on $\mathcal{G}_1 \times \mathcal{G}_1$ $(\{\cdot,\cdot\})$

The Poincaré-Algebra

revision:

Lorentz-group:
6 parameters
3 rotations in 3-dimensional space
3 boosts

translation-group
4 parameters

$\rightsquigarrow$ Poincaré-group
10 parameters

generators: $M^{\mu\nu} = -M^{\nu\mu}$, P^{μ},

where

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

Poincaré-Algebra:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [P^\mu, M^{\rho\sigma}] &= i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= -i\,(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} \\ &\quad -g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}) \end{aligned}$$

general transformation

$$\Lambda(\omega, a) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - ia_\mu P^\mu}$$

Casimir-Operators

(operators that commute with all generators)

$$\begin{aligned} P^2 &= P_\mu P^\mu \\ W^2 &= W_\mu W^\mu \end{aligned}$$

$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\mu M^{\rho\sigma}$ is the Pauli-Lubanski-vector

Consider massive representations:

In the rest frame $P = (mc, 0, 0, 0)$:

$$
\begin{aligned}
P^2 &= m^2 c^2 \\
W &= (0, mcS^1, mcS^2, mcS^3)
\end{aligned}
$$

$\rightsquigarrow$ representation is characterized by the mass m and the spin s

$$
\begin{aligned}
P^2 &= m^2 c^2 \\
W^2 &= -m^2 c^2 s(s+1)
\end{aligned}
$$

Elementary particles are irreducible representations of the Poincaré-group !

Supersymmetric Extension

aim:
add an internal symmetry to the Poincaré-group:

1967 no-go-theorem (Coleman, Mandula):
This is not possible within the framework of Lie-algebras.

1971 Golfand, Likhtman:
It is possible provided one uses graduated Lie-algebras.

1975 Haag, Lopuszanski, Sohnius
$\mathbb{Z}/(2\mathbb{Z})$-graduation is the only extension which is in accordance with relativistic QFT.

$\rightsquigarrow$ $\mathbb{Z}/(2\mathbb{Z})$-graduation of the Poincaré-algebra extension: $4N$-SUSY-Generators Q_a, $a = 1, ..., 4$

for $N = 1$ they are assumed to form a spinor

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$

with Majorana property

$$Q = C\bar{Q}^t, \text{ resp. } \bar{Q} = Q^t C$$

SUSY-algebra:

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$$

using the Jacobi-identity one gets:

$\mathcal{G}_0 \times \mathcal{G}_0 \to \mathcal{G}_0$: Poincaré-algebra

$\mathcal{G}_0 \times \mathcal{G}_1 \to \mathcal{G}_1$:

$$\begin{aligned} [P^\mu, Q_a] &= 0 \\ [M^{\mu\nu}, Q_a] &= -\Sigma^{\mu\nu}_{ab} Q_b \end{aligned}$$

$\mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_0$: $\{Q_a, \bar{Q}_b\} = 2\gamma^\mu_{ab} P_\mu$

note: $\{Q, \bar{Q}\} \sim P$
i.e. 2 SUSY-transformations=translation !

The fermion-boson-rule:

$$(-1)^{N_F} Q = -Q(-1)^{N_F}$$

and the same for $\bar{Q}$.

Now apply $\Delta = \mathrm{Tr}(-1)^{N_F}$ (Witten-Index):

$$\begin{aligned}\Delta\{Q,\bar{Q}\} &= \mathrm{Tr}\{(-1)^{N_F} Q\bar{Q}\} + \mathrm{Tr}\{(-1)^{N_F}\bar{Q}Q\} \\ &= 0\end{aligned}$$

Since $\{Q,\bar{Q}\} \sim P$ we have $\mathrm{Tr}\{(-1)^{N_F} P_\mu\} = 0$ and therefore

$$\Delta = \mathrm{Tr}(-1)^{N_F} = 0$$

$\rightsquigarrow$ Irreducible representations contain an equal number of bosonic and fermionic states

Casimir-operators

$[P^2, Q] = 0$,
i.e. $P^2 = P_\mu P^\mu$ is still a Casimir-operator

$[W^2, Q] \neq 0$,
i.e. $W^2 = W_\mu W^\mu$ has to be replaced

define

$$Y^\mu := W^\mu - \frac{1}{2}\bar{Q}\gamma^\mu\gamma_5 Q$$

then Y^μ is a generalized angular momentum (one finds $[Y_\mu, Y_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\sigma Y^\sigma$), which we call superspin

$$C^{\mu\nu} := Y^\mu P^\nu - Y^\nu P^\mu$$

$\rightsquigarrow$

$$C^2 := C_{\mu\nu}C^{\mu\nu},$$

which is our second Casimir

eigenvalues in the rest frame

$$C^2 = -2m^4 y(y+1),\ y = 0, \frac{1}{2}, 1, ...$$

massive representations of the Super-Poincaré-algebra are characterized by

$$m, y, y^3, s^3, \vec{p}$$

for fixed (m, y)

$$|state >:= |p, y^3, s^3 >$$

where

$$\begin{aligned}
P^\mu|p, y^3, s^3 > &= p^\mu|p, y^3, s^3 > \\
C^2|p, y^3, s^3 > &= -2m^4 y(y+1)|p, y^3, s^3 > \\
Y^3|p, y^3, s^3 > &= my^3|p, y^3, s^3 > \\
W^3|p, y^3, s^3 > &= ms^3|p, y^3, s^3 >
\end{aligned}$$

<u>literature:</u>

- H. Kalka, G. Soff, *Supersymmetrie*, Teubner, 1997
- D. Bailin, A. Love, *Supersymmetric Gauge Field Theory and String Theory*, Inst. of Physics Publishing, 1994
- M.F. Sohnius, *Introducing Supersymmetry*, Physics Reports, 128 (1985) 39
- I.L. Buchbinder, S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, Inst. of Physics Publishing, 1995
- R.U. Sexl, H.K. Urbantke, *Relativität, Gruppen, Teilchen*, Springer, 1976