Complex Langevin dynamics: localised distributions

Gert Aarts[1], **Pietro Giudice**[2], Erhard Seiler[3]

g.aarts@swansea.ac.uk, p.giudice@uni-muenster.de, ehs@mppmu.mpg.de

[1]Department of Physics, College of Science, Swansea University, UK

[2]Universität Münster, Institut für Theoretische Physik, Münster, Germany

[3]Max-Planck-Institut für Physik (Werner-Heisenberg-Institut) München, Germany

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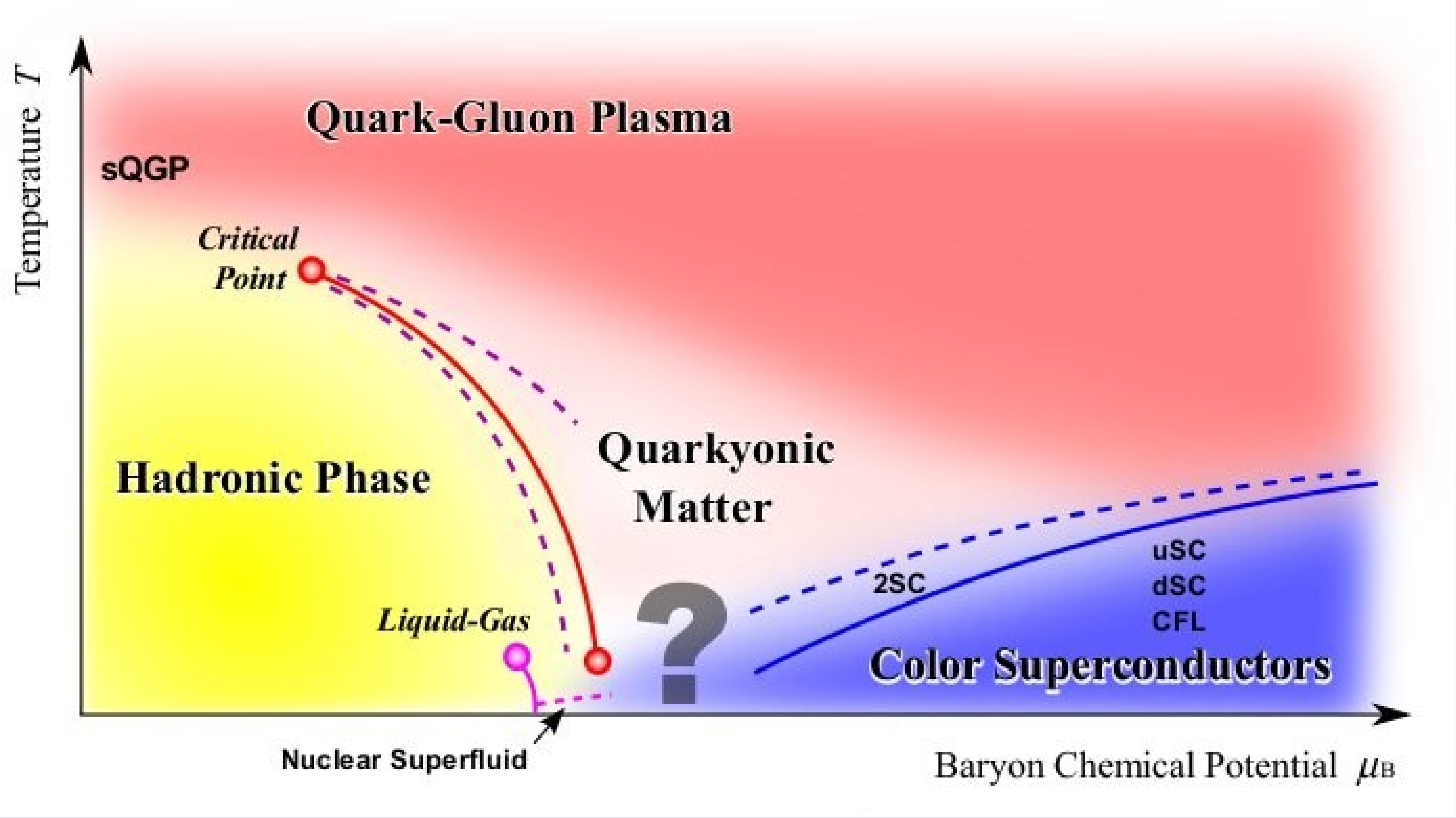
Our dream



Temperature T
Quark-Gluon Plasma
sQGP
Critical Point
Hadronic Phase
Quarkyonic Matter
?
2SC
uSC
dSC
CFL
Color Superconductors
Liquid-Gas
Nuclear Superfluid
Baryon Chemical Potential μ_B

Introduction

- Complex Langevin dynamics provides a way to simulate theories with complex actions ⟶ no importance sampling, no sign problem!
- It opens the way to QCD simulations at $\mu_B \neq 0$
- The method was introduced in 1983 by G.Parisi and J.R.Klauder but shortly after it was clear that correct results **are not** garanteed
- We do not have a FULL UNDERSTANDING of the problem yet!
- A combination of analytical and numerical results, also on simple models, can help us!
- Recently the importance of the properties of the probability distribution (generated by the Langevin process) in the complexified configuration space has been clarified: the distribution has to drop very rapidly (in particular in the imaginary direction) ⟶ this can be formalised in a criterion for correctness [G. Aarts, F. A. James, E. Seiler and I. -O. Stamatescu, Eur. Phys. J. C 71 (2011) 1756]

The goal of this work

- Here we study the probability distribution (by brute force and solving the Fokker-Planck Equation, FPE) and then we relate the results to the criterion for correctness

- We have a complete characterisation of the dynamics by studying:
  - Classical flow
  - Criterion for correctness
  - Explicit solution of the FPE

- We show moreover that:
  - If the distribution has support only on a strip of the complexified configuration space, then correct results are obtained !

The model + CL

- The toy model: $Z = \int_{-\infty}^{\infty} dx\, e^{-S}, \qquad S = \frac{1}{2}\sigma x^2 + \frac{1}{4}\lambda x^4, \qquad \sigma \in \mathbb{C}, \lambda \in \mathbb{R}$
- Analytic solution: $Z = \sqrt{\frac{4\xi}{\sigma}} e^{\xi} K_{-\frac{1}{4}}(\xi) \quad \xi = \sigma^2/(8\lambda) \quad \Rightarrow \langle x^n \rangle$
- Complex Langevin (CL) equation: $\dot{z} = -\partial_z S(z) + \eta$
- Complexification: $z = x + iy, \qquad \eta = \eta_R + i\eta_I, \qquad \sigma = A + iB$
- CL is now: $\dot{x} = K_x(x,y) + \eta_R, \qquad \dot{y} = K_y(x,y) + \eta_I$
- Drift:

$$K_x \equiv -Re\partial_z S(z) = -Ax + By - \lambda x \left(x^2 - 3y^2\right)$$
$$K_y \equiv -Im\partial_z S(z) = -Ay - Bx - \lambda y \left(3x^2 - y^2\right)$$

- Noise:

$$\langle \eta_R(t)\eta_R(t')\rangle = 2N_R\delta(t-t'), \quad \langle \eta_I(t)\eta_I(t')\rangle = 2N_I\delta(t-t'), \quad N_R - N_I = 1$$

Criterion for correctness

- Averaging over the noise we can determine the expectation values $\langle O \rangle_\eta$
- The probability distribution $P(x,y;t)$ describes how the configuration space is sampled and its evolution in time is given by the Fokker-Planck Equation:

$$\dot{P}(x,y;t) = L^T P(x,y;t), \qquad L^T = \partial_x \left(N_R \partial_x - K_x\right) + \partial_y \left(N_I \partial_y - K_y\right)$$

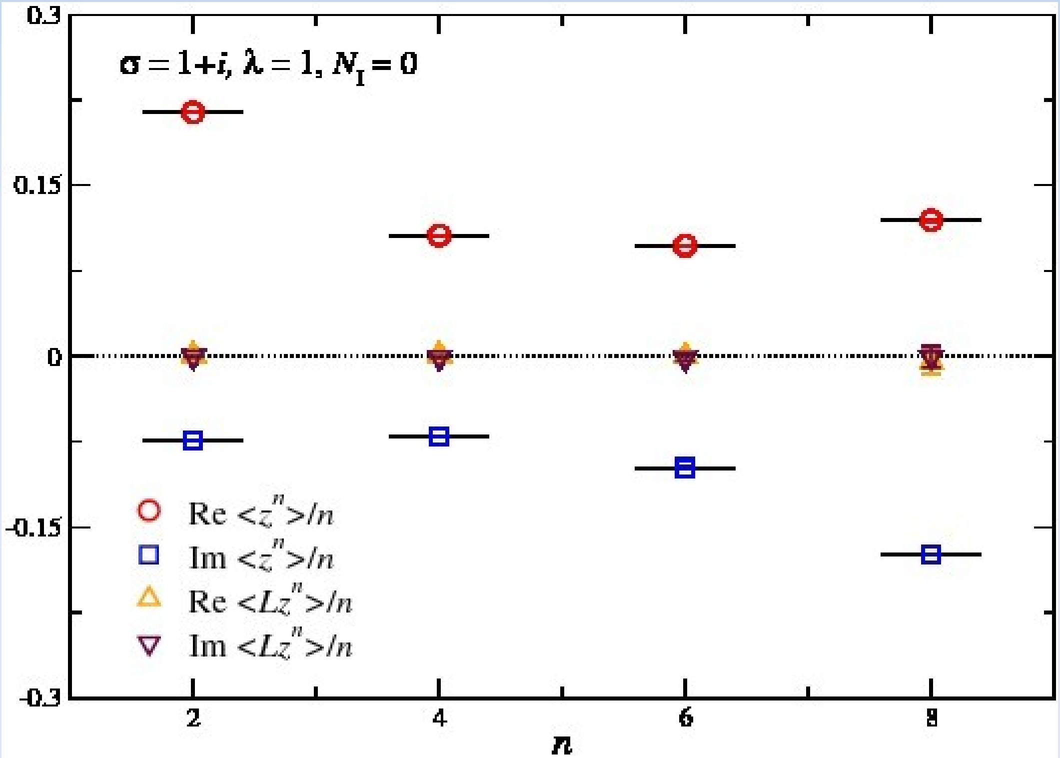
- The expectation value is given by:

$$\langle O \rangle_{P(t)} = \int dx dy \, P(x,y;t) O(x+iy)$$

- But we know that: $\langle O \rangle_{\rho(t)} = \int dx \, \rho(x,t) O(x), \quad \rho(x) = e^{-S(x)}$
- Therefore we want that: $\langle O \rangle_{\rho(t)} = \langle O \rangle_{P(t)}$
- Introducing the Langevin Operator: $\tilde{L} = \left[\partial_z - (\partial_z S(z))\right] \partial_z$,
  the criterion for correctness is given by:
  $C_O \equiv \langle \tilde{L} O(z) \rangle = 0$
  (to be satisfied for **a complete set** of observables)

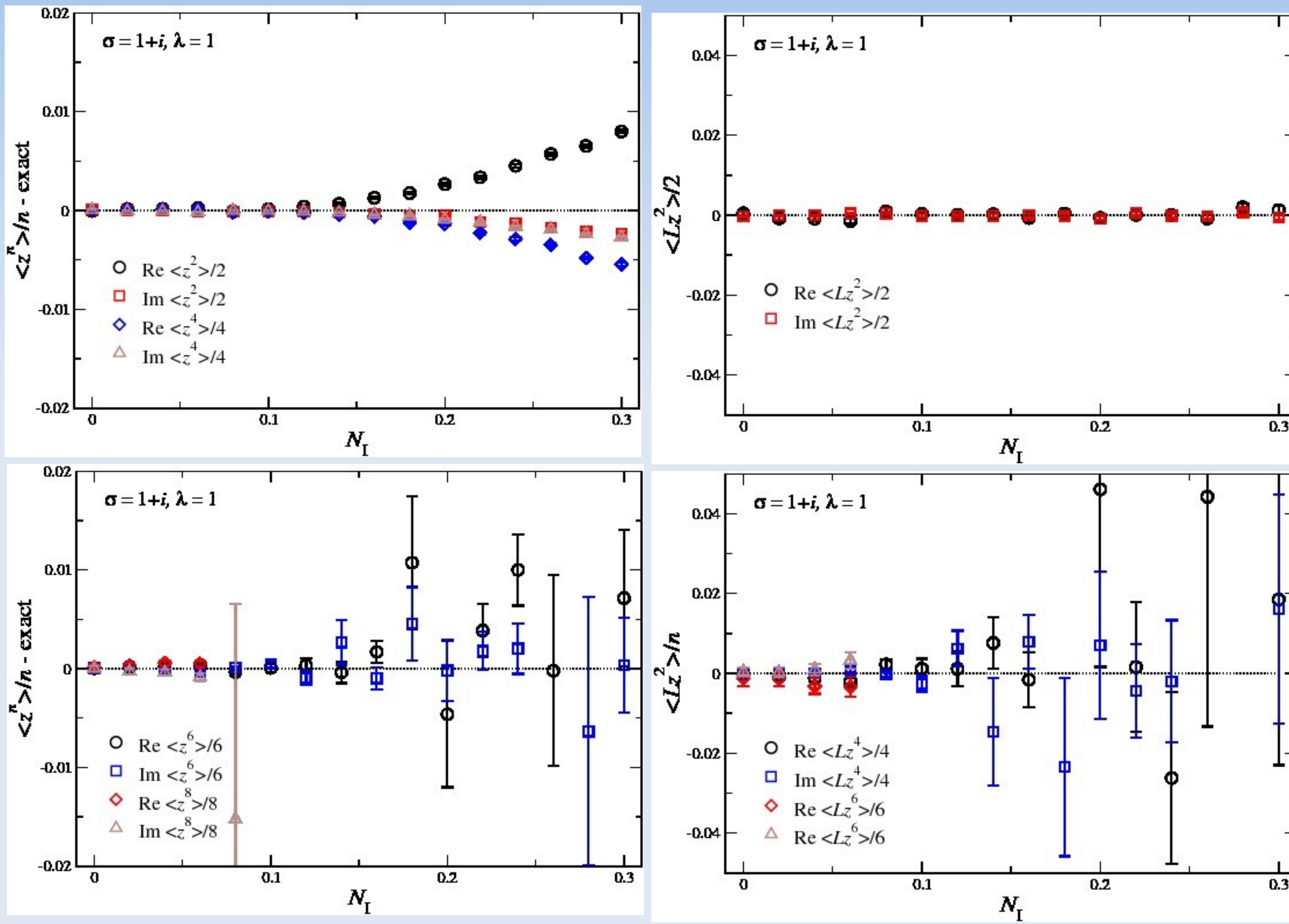
Real noise $(N_I = 0)$

- We consider the observables: $O_n(z) = \frac{1}{n} z^n$ and the criterion: $C_n \equiv \frac{1}{n} \left\langle \tilde{L} z^n \right\rangle = 0$



Perfect agreement and criterion for correctness satisfied!

Complex noise



- Note: C_2 always consistent with zero; strong fluctuation for large N_I

Solving FPE

- We want to solve the FP equation: $\dot{P}(x,y;t) = L^T P(x,y;t)$
- To do that we solve the eigenvalue problem: $-L^T P_\kappa(x,y) = \kappa \ P_\kappa(x,y)$
- If we have a unique ground state P_0 with eigenvaue $\kappa = 0$, then the solution is: $P(x,y;t) = P_0(x,y) + \sum_{\kappa \neq 0} e^{-\kappa \, t} P_\kappa(x,y)$
- In [A.Duncan, M.Niedermaier, Annals Phys.329 (2013) 93] P(x,y) is expanded in a basis of Hermite functions:

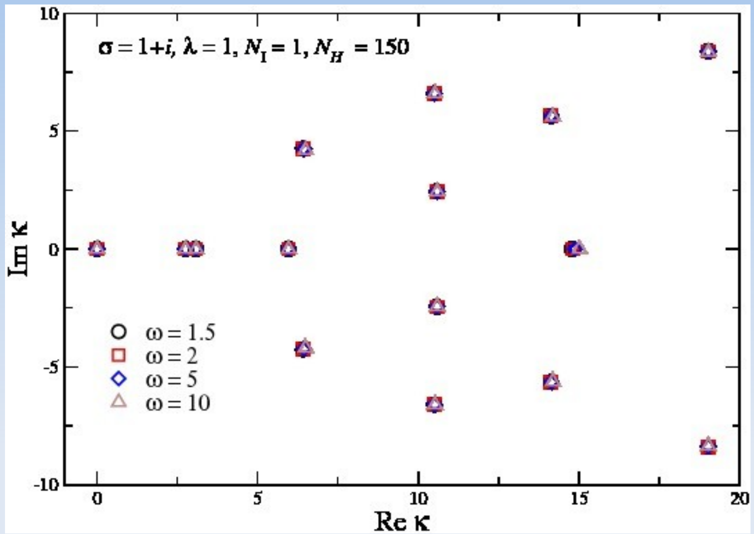
$$P(x,y) = \sum_{k=0}^{N_H-1} \sum_{l=0}^{N_H-1} c_{kl} H_k\left(\sqrt{w}x\right) H_l\left(\sqrt{w}y\right)$$

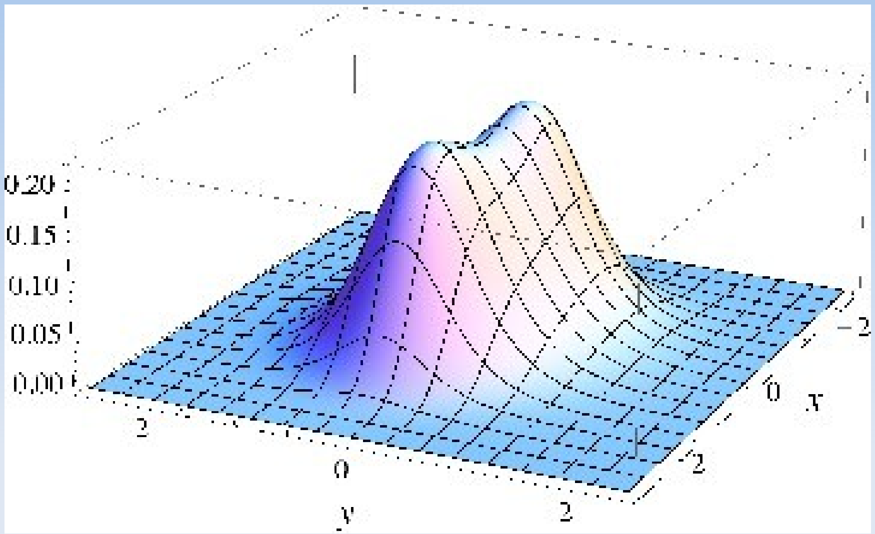
- This was done introducing creation and annihilation operators, a and b:

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p_x = -i\partial_x = i\sqrt{\frac{\omega}{2}}\left(a^\dagger - a\right),$$
$$y = \frac{1}{\sqrt{2\omega}}\left(b + b^\dagger\right), \quad p_y = -i\partial_y = i\sqrt{\frac{\omega}{2}}\left(b^\dagger - b\right)$$

- We determine the matrix elements: $\langle kl|L^T|mn\rangle$ where $|mn\rangle = \frac{1}{\sqrt{m!n!}} a^{\dagger m} b^{\dagger n} |0\rangle$ and therefore $H_m(\sqrt{\omega}x) = \langle x|m\rangle$
- Note: ω and N_H

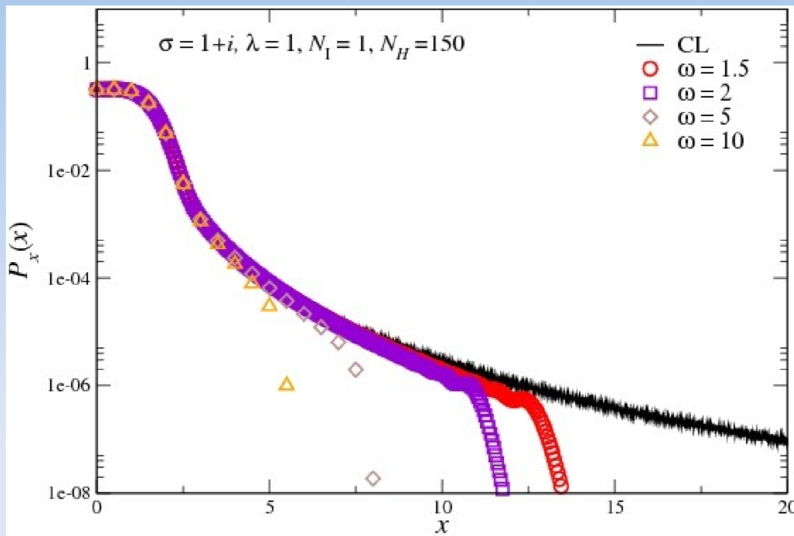
Complex noise (eigenvalues & 3d distr.)

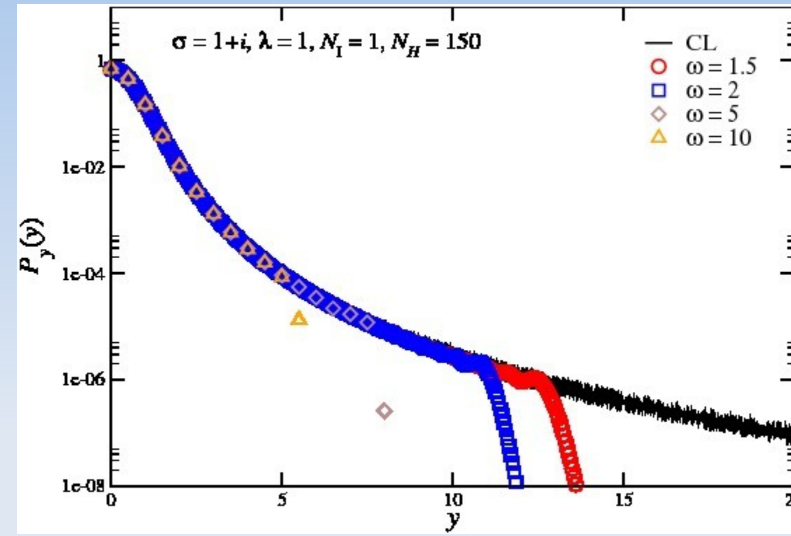




- The eigenvalues around the origin are independent of ω
- Ground state: $\omega = 1.5$ and $N_H = 150$
- We find that there is an interval for ω for which:
  - There is always an eigenvalue consistent with zero
  - The other eigenvalues are in the right half-plane
  - The ground state is stable under variation of ω and N_H

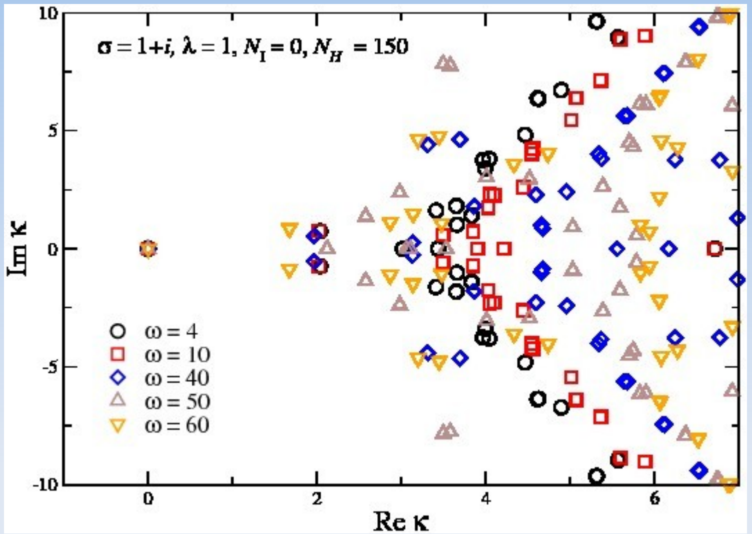
Complex noise (integrat. distr. & power decay)

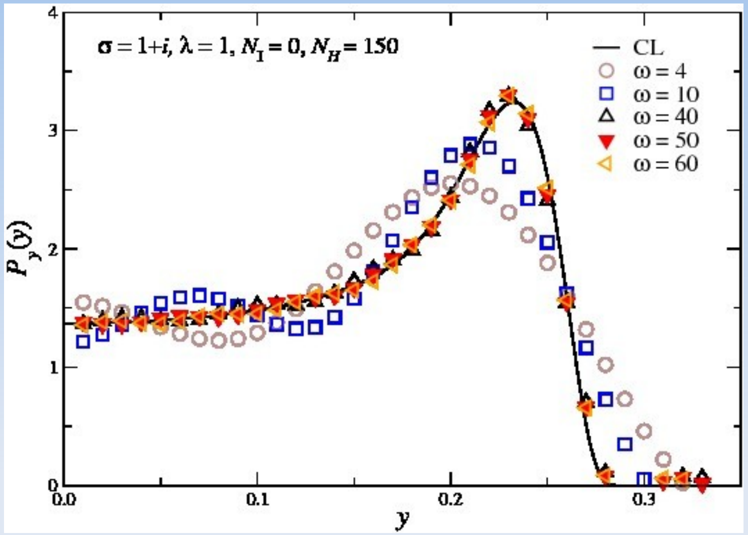




- Partially integrated distr: $P_x(x) = \int_{-\infty}^{\infty} dy\, P(x,y), \quad P_y(y) = \int_{-\infty}^{\infty} dx\, P(x,y)$
- Manifestation of the truncation in N_H
- We observe a power decay with power 5: $P_x(x) \sim \frac{1}{|x|^5}, \quad P_y(y) \sim \frac{1}{|y|^5}$
- This suggests: $P(x,y) \sim \frac{1}{(x^2+y^2)^3}$

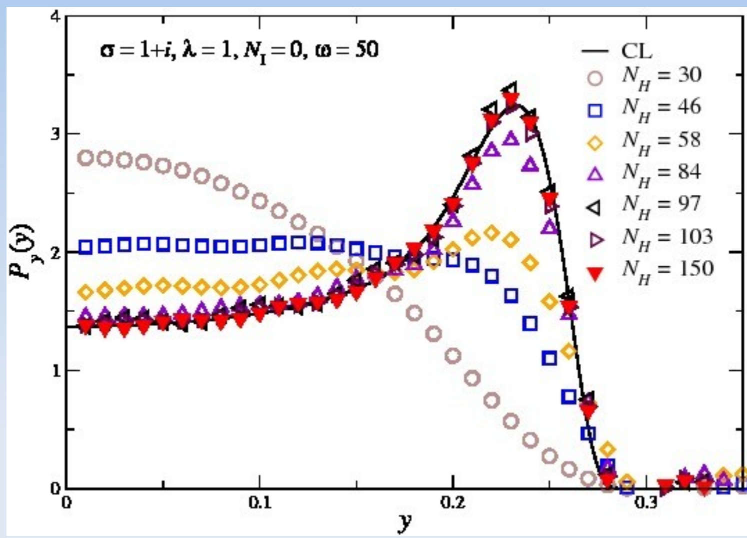
Real noise (eigenvalues & distr.)

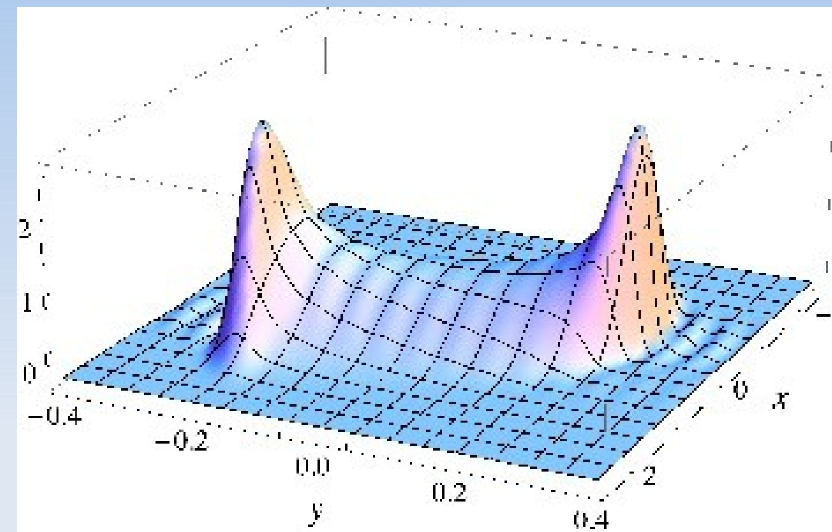


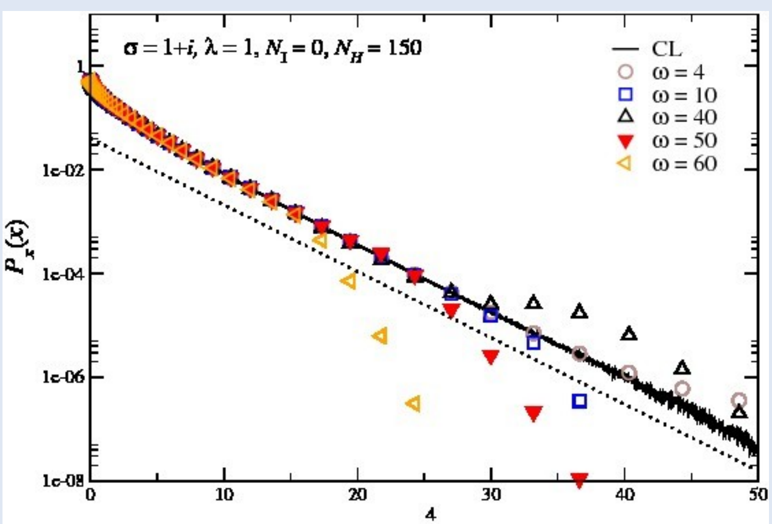


- There is an eigenvalue at the origin but in general they depend on ω
- From $P_y(y)$ we see convergence only for large values of ω
- Distribution very localised, drops to zero around $y \approx 0.28$

Real noise (truncation & 3d plot)

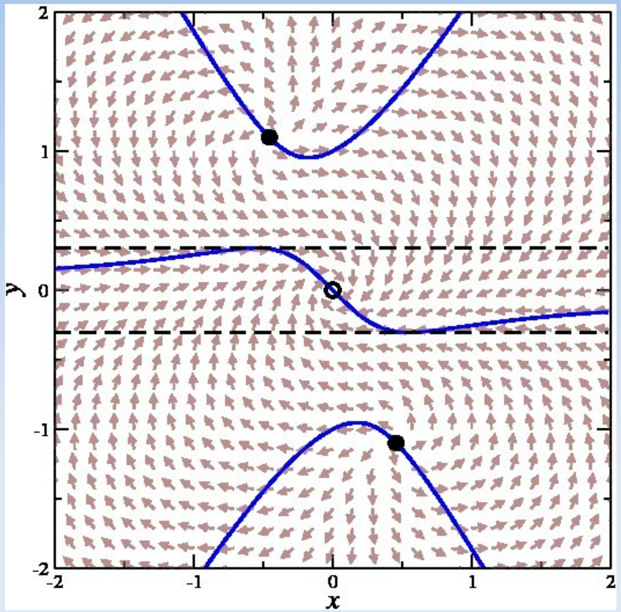




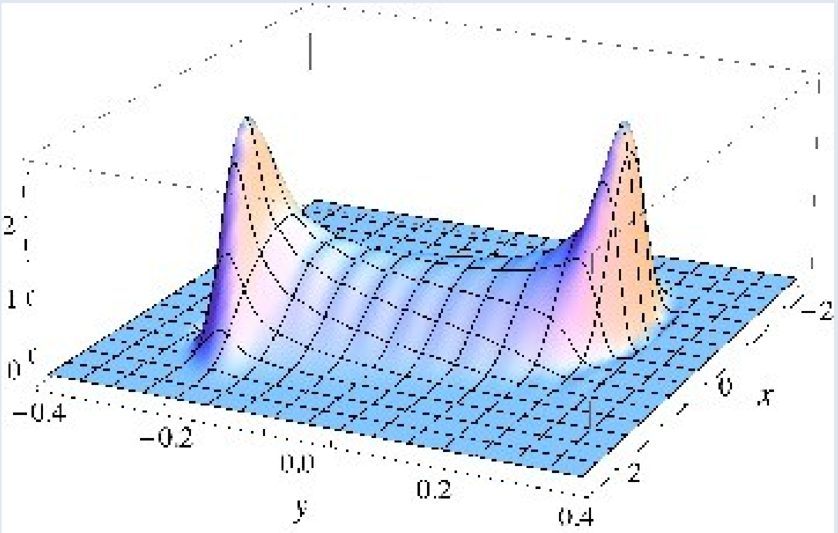


- $P_x(x)$ exponential decay!
- $P_x(x) \sim e^{-ax^4}, \quad a \sim 0.295.$

Classical flow



- $(K_x(x,y), K_y(x,y))$ for $\sigma = 1 + i$ and $\lambda = 1$
- 3 fixed points (where $K_x(x,y) = K_y(x,y) = 0$):
  - An attractive point at (x,y)=(0,0)
  - Two repulsive points at $(\pm 0.455, \mp 1.10)$
- Blue lines where $K_y(x,y)$ changes sign
- Dynamics confined between the dashed lines!!! (we have: $-0.3029 < y < 0.3029$)



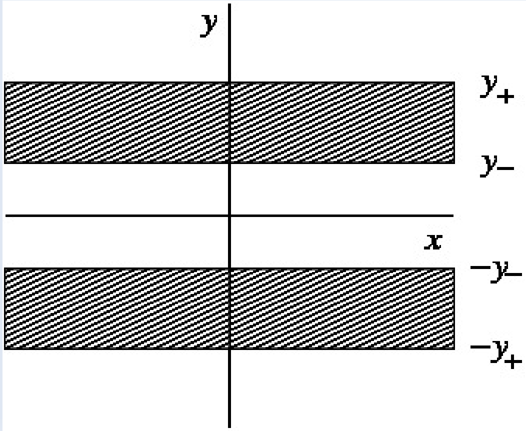
Conservation law

- The classical flow result can be made more rigorous
- We note that the FPE takes the form of a conservation law:

$$\dot{P}(x,y;t) = \partial_x J_x(x,y;t) + \partial_y J_y(x,y;t)$$

$$J_x = (N_R \partial_x - K_x)\, P, \qquad J_y = (N_I \partial_y - K_y)\, P$$

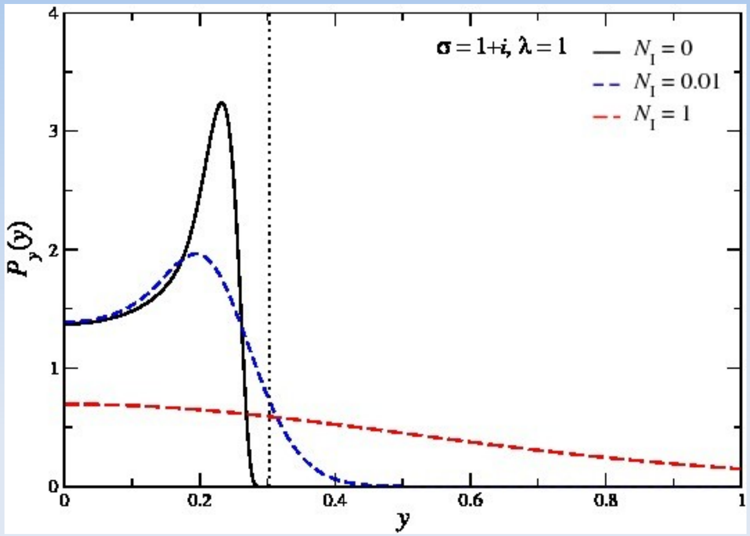
- We can now introduce the charge $Q(y,t) = \int_{-\infty}^{\infty} dx\, J_y(x,y;t)$
- Assuming sufficient decay, i.e. $K_{x,y}(x,y)P(x,y) \to 0$ and real noise we have: $Q(y) = \int_{-\infty}^{\infty} dx\, K_y(x,y)P(x,y) = 0$
- Since $P(x,y)$ is not negative, if $K_y(x,y)$ has a definite sign as a function of x for a given y, then $P(x,y)$ has to vanish for this y value

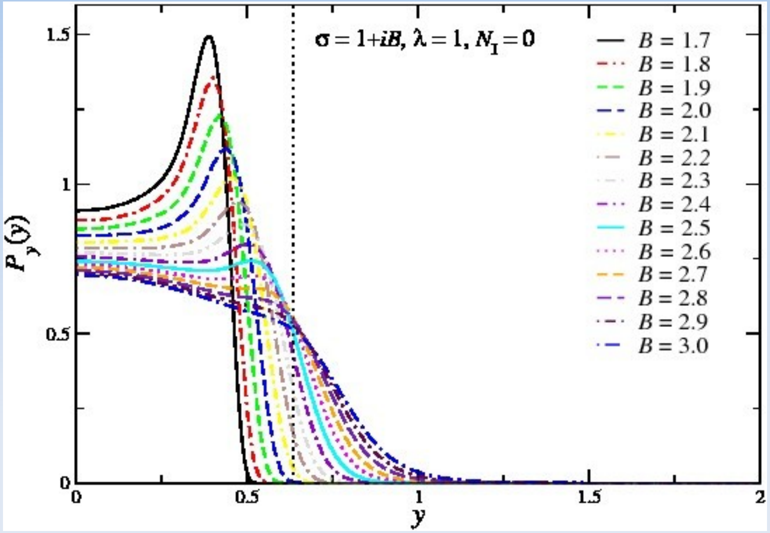


- The distribution is strictly zero in the two strips provided that $3A^2 > B^2$ and $N_I = 0$
- Where:

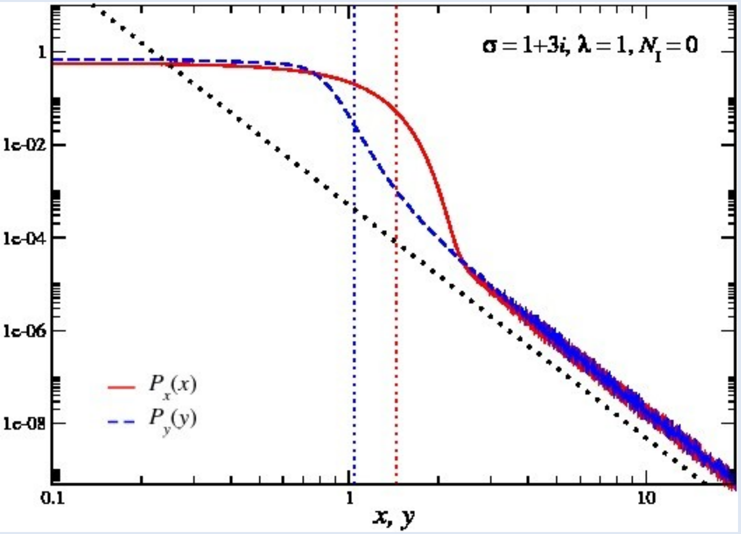
$$y_\pm^2 = \frac{A}{2\lambda}\left(1 \pm \sqrt{1 - \frac{B^2}{3A^2}}\right)$$

Absence of strips

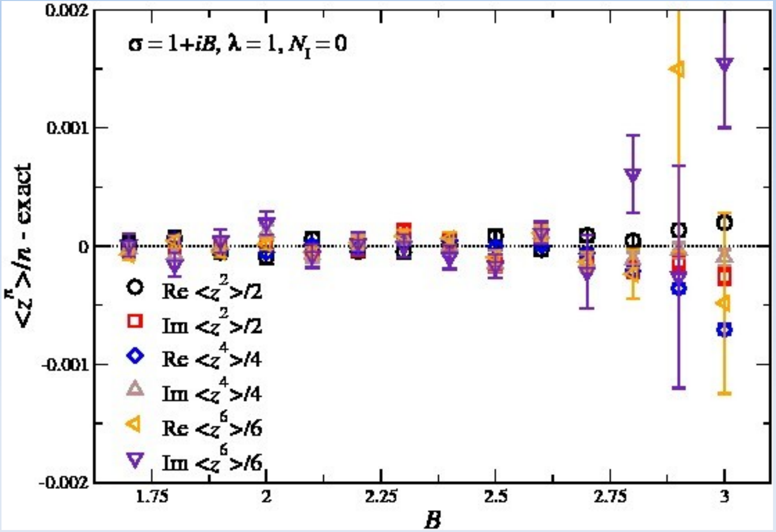


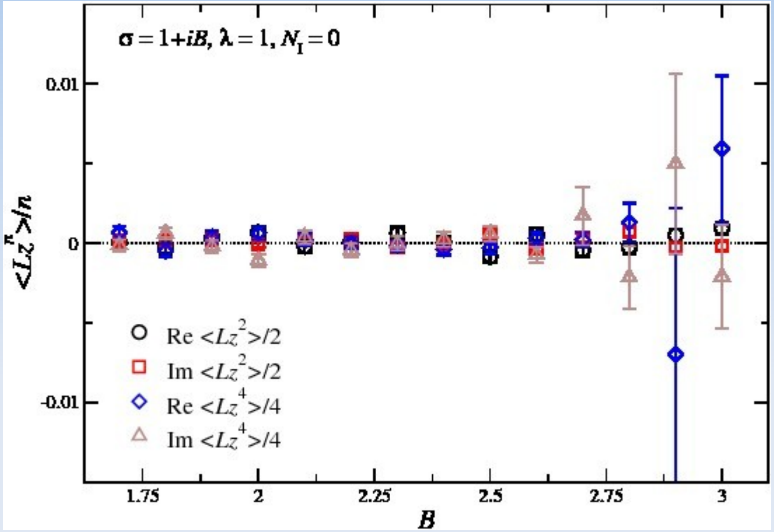


- For complex noise there are no strips!
- Always power decay: $P_y(y) \sim 1/|y|^5$
- For real noise no strip if $3A^2 < B^2$
- Increasing B similar to increasing N_I



Criterion for correctness vs B





- Also from here we see that the effect of increasing B is very similar to increasing the value of N_I

Universal decay behaviour

- It is possible to understand the universal power decay!
- Starting from FPE: $\dot{P}(x,y;t) = L^T P(x,y;t)$
- And substituting the Ansatz: $P(x,y) = \frac{c}{(x^2+y^2)^\alpha}$
- We find that:

$$\alpha \frac{x^2 - y^2 + 2\alpha(N_R x^2 + N_I y^2)}{(x^2+y^2)^2} + A(1-\alpha) + \lambda(3-\alpha)(x^2-y^2) = 0$$

- At large x and y, only the last term dominates and we have: $\alpha = 3$
- And therefore: $P_x(x) \sim \frac{1}{|x|^5}, \quad P_y(y) \sim \frac{1}{|y|^5}$

Conclusions

- In order to justify the results obtained with CL the probability distribution has to be sufficiently localised
- Here we have studied the properties of the distribution via a number of methods: classical flow, histogram by brute force, explicit solution of FPE, criterion for correctness
- We have found:
  - For real noise as $3A^2 > B^2$, the distribution has support only in a strip and it has an exponential decay in the real direction; criterion for correctness satisfied and correct results obtained!
  - When $3A^2 < B^2$ or the noise is complex the distribution is NOT localised; the distribution has a power law: $P(x,y) \sim (x^2+y^2)^{-3}$, because of this slow decay high moments are not well-defined; criterion for correctness suffer of large fluctuations: signal of failure!
- A consistent picture of the dynamics can be obtained already from a combination of partially integrated distribution and criterion for correctness
- These tools are readily available to study SU(N) gauge theories (plus gauge cooling...)