# QCD at finite temperature and density an effective lattice theory approach

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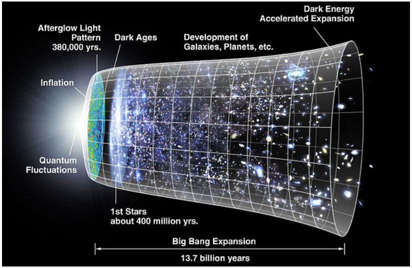
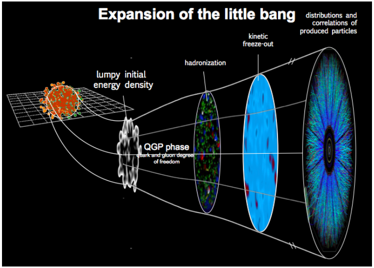
O. Philipsen, J. Langelage, S. Lottini, M. Neuman, M. Fromm . . .

# State of matter at high temperature

"Big bang" "Little bang"

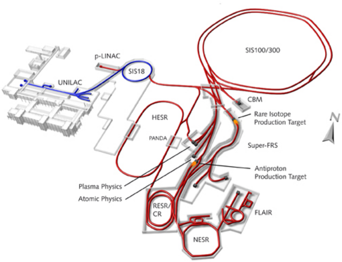


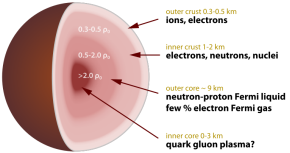



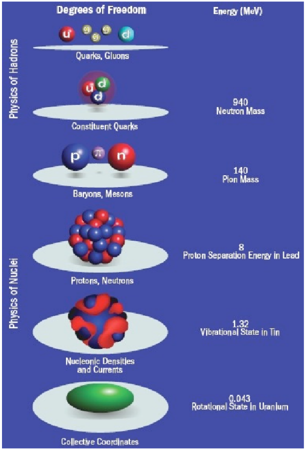

[1201.0784]

- strong interactions at finite $T$ and finite $\mu$
- complicated dynamics: time evolution, influence of electromagnetic fields
- simplification: equation of state for the strongly interacting matter

# State of matter at high density

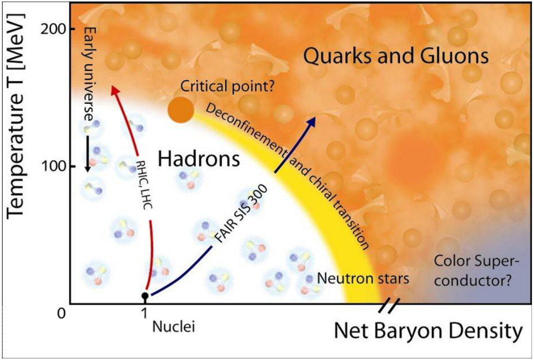



[Reddy, Schladming 2013]

# QCD at finite temperature and density




- critical temperature: confinement $\leftrightarrow$ deconfinement
- critical temperature: chiral symmetry restoration
- properties of the phases: $\epsilon(T)$, $p(T)$, screening length, ...

# QCD thermodynamics

$$S[A,\bar\psi,\psi;\mu] = \int_0^{\frac{1}{T}} \left[ \frac{1}{4}\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \sum_f \bar\psi_f \left( \not{D} + m_f + \gamma_0 \mu \right) \psi_f \right]$$

QCD partition function

$$Z(T;\mu) = \mathrm{Tr}(e^{-(H-\mu Q)/T}) = \int_{\mathrm{bc.}} \mathcal{D}A\mathcal{D}(\bar\psi,\psi)\; e^{-S[A,\bar\psi,\psi;\mu]}$$

- temperature $\rightarrow$ boundary conditions:
  bosons: periodic
  fermions: antiperiodic

Thermodynamic quantities

$$f = -\frac{T}{V}\log Z\;; \quad p = \frac{\partial T \log Z}{\partial V}; n = \frac{T}{V}\frac{\partial \log Z}{\partial \mu} \quad \ldots$$

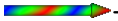
# Effective theories from perturbative investigations

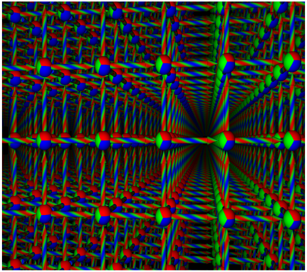
- structure of perturbation theory different at finite $T$: zero mode Matsubara frequencies
- resummation to effective mass: reorganization of perturbation theory
- no thermal mass for colour magnetic fields: can not be treated perturbatively (Linde problem)
- "Helsinki approach" integrate out perturbative degrees of freedom, non-perturbative effective three dimensional theory (scale separation $\frac{g^2 T}{\pi} << gT << \pi T$)
- fails below $5T_c$, breaks center symmetry
- here: three dimensional effective theory starting form the low temperature confined phase

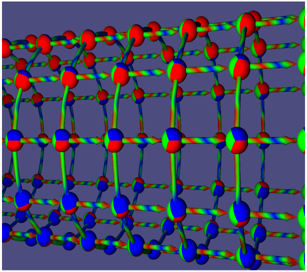
# QCD thermodynamics on the lattice

Discretized QCD partition function

$$Z(T) = \mathrm{Tr}(e^{-H/T}) = \int_{\mathrm{bc.}} \prod_n dU(n,\mu) d(\bar{\psi}_n, \psi_n)\; e^{-S[U,\bar{\psi},\psi]}$$

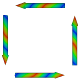
- gauge fields: links  $= U = \mathcal{P} e^{ig \int_x^{x+\hat{\mu}} dx^\mu A_\mu}$
- matter fields $\psi$: 
- temperature: lattice boundary conditions $T = \frac{1}{L_t} = \frac{1}{aN_t}$



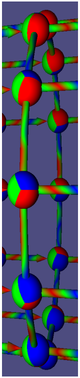


# The QCD lattice action

$$S = S_{\mathrm{YM}} + S_{\mathrm{ferm}} = -\frac{\beta}{6} \sum_p (\mathrm{Tr}\, U_p + \mathrm{Tr}\, U_p^\dagger) + \sum_f \bar{\psi}_f (\mathrm{D[U]} + m_f) \psi_f$$

- $U_p =$ 
- Path integral: Haar measure $dU$
- invariant under gauge transformations
  $U_\mu(x) \to \Omega(x)^{-1} U_\mu(x) \Omega(x+\mu)$
- fermions are integrated out:
  $S_{\mathrm{ferm.\ eff.}} = -\log(\prod_f \det(D[U] + m_f))$

# The two faces of the Polyakov loop: Gauge observable

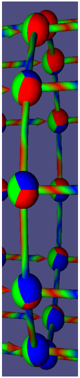


Polyakov loop:

$$L(\mathbf{x}) = \mathrm{Tr}\, W(\mathbf{x}) = \mathrm{Tr}[\prod_{\tau=0}^{N_t} U_0(\mathbf{x},\tau)] = \mathcal{P} e^{ig\int_0^{\frac{1}{T}} d\tau A_0(\mathbf{x},\tau)}$$

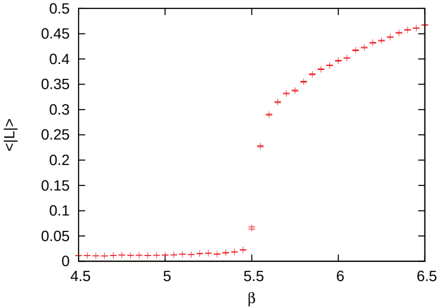
- Gauge invariant quantity
- naturally obtained with constant background gauge field $A_0$
- resembles gauge dynamics / interaction with gauge background

# The two faces of the Polyakov loop: Heavy quark

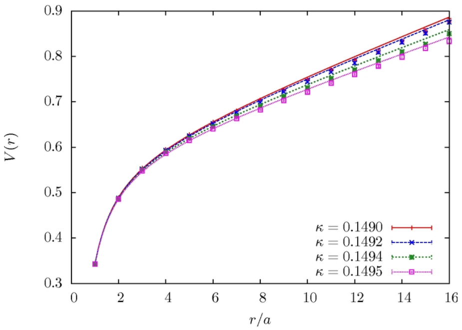


Polyakov loop: world line of a static quark

- $L$ puts infinitely heavy quark in the theory
- $\langle L \rangle = \exp(-(F_Q - F_0)/T)$
- confinement (no free quarks) $F_Q \to \infty$: $\langle L \rangle = 0$
- deconfinement: $\langle L \rangle \neq 0$
- $-\log < L(0)L(R) >$ at $T \to 0$: static quark-antiquark potential




- first order phase transition at $\beta_c$




- short range: Coulomb like $V(R) = -\frac{c}{R}$
- long range: linear rise with string tension $V(R) = \sigma R$

# $Z(N_c)$ center symmetry

Hidden symmetry of the pure gauge theory

- multiply all links on one timeslice with center element $z_n = e^{i\frac{2\pi}{N_c}n}\mathbb{1}$
- $S_{\mathrm{YM}}$ invariant
- $L$ picks up phase: order parameter for $Z(N_c)$ symmetry breaking
- confinement $\leftrightarrow$ spontaneous symmetry breaking
- with fermions: symmetry "washed out"

# Non-perturbative effective theories

Guided by phenomenological observations:

- MIT-Bag model
- Hadron resonance gas model

Guided by symmetries (chiral symmetry, center symmetry)

- NJL, PNJL model
  $\rightarrow$ models gauge dynamics by Polyakov loops

Advantage: simple description of relevant properties and processes

Here: Simple effective model follows naturally from the strong coupling expansion of lattice QCD

# Effective action for the Polyakov loop

$$e^{-S_{\text{eff}}[L]} = \int [dU_i] \prod_p e^{\frac{\beta}{6}\mathrm{Tr}\left(U_p + U_p^\dagger\right)} , \quad Z = \int [dL] e^{-S_{\text{eff}}[L]}$$

- integrating out spatial links
- final result depends only on Polyakov lines $L$
- dimensional reduction from $3+1$D to $3D$
- $S_{\text{eff}}$ expanded in terms of interactions / interaction distances
- numerical methods:
  inverse MC, demon methods [Heinzl, Kästner, Wozar, Wipf, Wellegehausen],
  relative weights [Langfeld,Greensite]

Here: strong coupling approach, can be applied also when Monte-Carlo fails

# Strong coupling expansion in lattice gauge theory

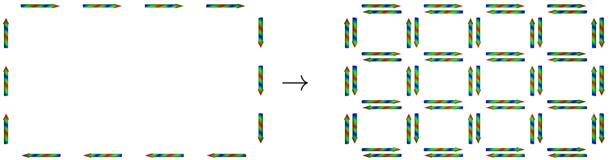
$$Z = \int [dU_\mu] \prod_p e^{\frac{\beta}{6}\mathrm{Tr}\left(U_p + U_p^\dagger\right)}$$

- expansion in $\beta = 6/g^2$ (opposite to weak coupling)
- similar to high temperature expansion in statistical physics
- at low orders: simple integration rules for products of plaquette contributions

$$\int dU\, U = \int dU\, U^\dagger = 0; \quad \int dU\, UU^\dagger = \frac{1}{3}\mathbb{1}$$

# Static quark-antiquark potential in strong coupling limit

simplest example: $\langle$ Wilson loop $\rangle$

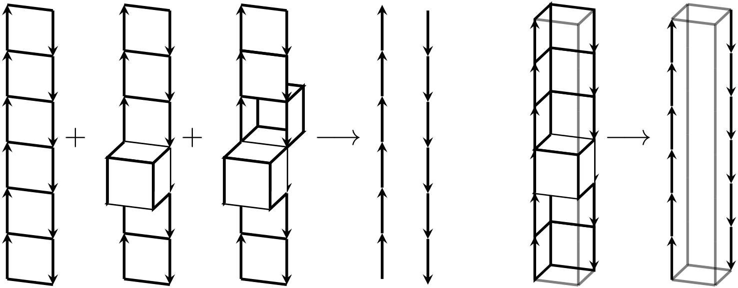


$\propto \left(\frac{\beta}{18}\right)^{\mathcal{R}\mathcal{T}}$

- first contribution: Loop filled with plaquettes
- confinement:
  $V(\mathcal{R}) = -\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \log\langle W\rangle = -\sigma\mathcal{R}$
- extension: $O = \sum_n O_n \beta^n$
- more convenient expansion parameter $u = \frac{\beta}{18} + \ldots$

# Effective action from strong coupling

Integrating out spacial links to get effective theory




[Polonyi, Szlachanyi]

# Effective action from strong coupling and simulations

$$S_{\text{eff}} = \lambda_1 S_{\text{distance } 1} + \lambda_2 S_{\text{distance } \sqrt{2}} + \dots$$

- ordering principle for the interactions
  higher representations and long distances are suppressed
  $(u^{N_t};\ u^{2N_t};\ u^{2N_t+2})$
- effective couplings exponentiate:
  $\lambda_1 = u^{N_t} \exp(N_t P(u))$ (resummation)
- collect similar terms to log (resummation)

$$S_{\text{nearest neighbors}} = \sum_{<ij>} (\lambda_1 \Re L_i L_j^* - (\lambda_1 \Re L_i L_j^*)^2 + \dots)$$
$$= \sum_{<ij>} \log(1 + \lambda_1 \Re L_i L_j^*)$$

# Numerical lattice simulations of the effective theory

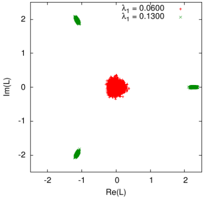
Remaining path integral of the effective theory

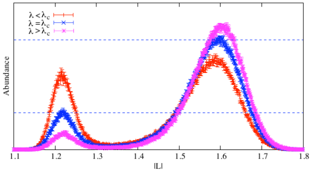
$$\int [dL] e^{-S_{\text{eff}}[L]}$$

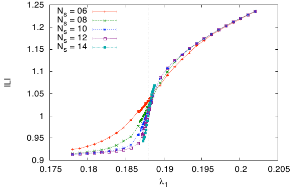
- several effective model studies consider only mean field for the effective theory
- here: numerical Monte-Carlo simulation
- full non-perturbative dynamics of the effective degrees of freedom

# Correct representation of the $Z(N_c)$ symmetry and phase transition

- effective action: $Z(N_c)$-symmetric combinations of $L$
- $Z(N_c)$ gets spontaneously broken at larger values of $\lambda_1$
- first order phase transition

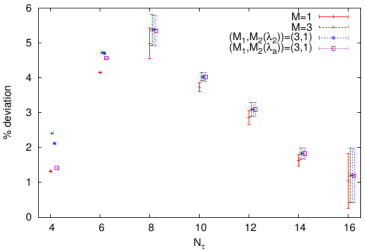


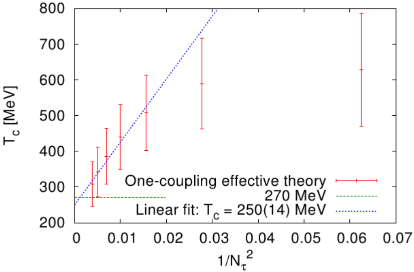



# Confinement - deconfinement phase transition

Strong coupling relation $\lambda(\beta)$: mapping back $\lambda_c$ to $\beta_c$

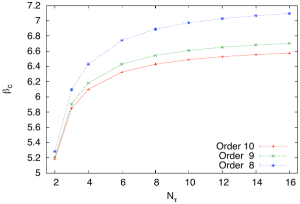


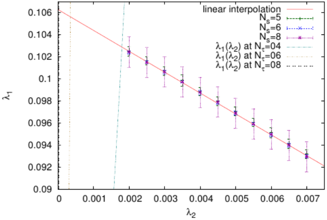

Yang-Mills relation $a(\beta) \Rightarrow T_c = \frac{1}{a(\beta_c)N_t}$
Extrapolation of continuum limit from strong coupling result!

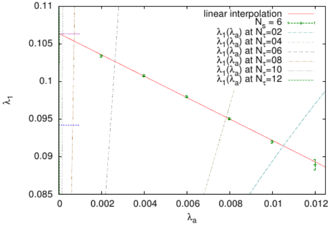
# Corrections in the expansion

Two possible corrections:
$\lambda_1 = u^{N_t} exp(N_t(4u^4 + 12u^5 + \ldots))$
$S_{\text{eff}} = \lambda_1 S_{\text{distance } 1} + \lambda_2 S_{\text{distance } \sqrt{2}} + \ldots$




Next to nearest neighbors, adjoint rep. ...



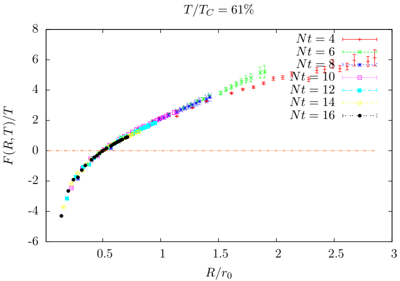




... suppressed with $u^{N_t}$, not important for the phase transition in continuum limit.

# Is Yang-Mills theory that simple?

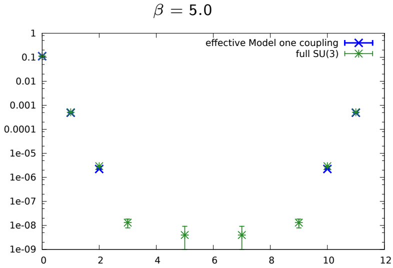
Free energy of
static quark-antiquark pair

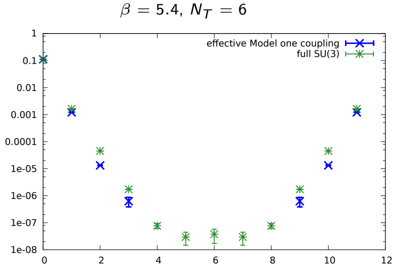
$$\langle L(\vec{0})L^{\dagger}(\vec{R})\rangle = \exp(-F(|\vec{R}|, T)/T)$$




- precise check of long and short range correlations
- short range (like zero temperature), long range: temperature dependent string tension $\sigma(T)$

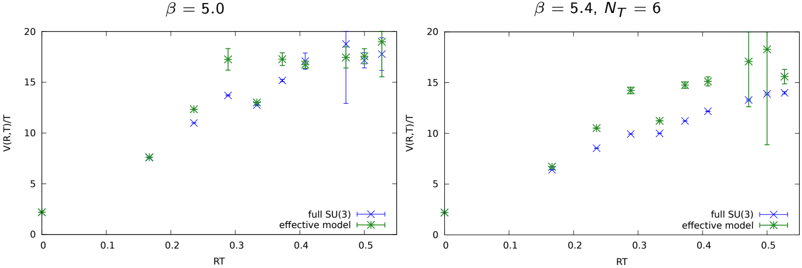
# Precise measurements



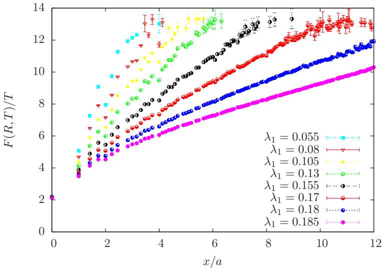

- Multilevel and Mulithit algorithm: error below $10^{-9}$
- deviations close to critical $\beta$; but still reasonable agreement

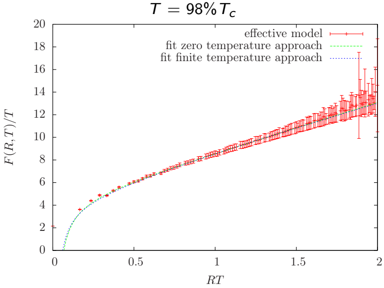
# "Lattice structure" in the effective theory




- identification $\lambda_1(\beta) \leftrightarrow \beta$ less restoration of rotational invariance
- remnant lattice structure

# String tension from the effective theory

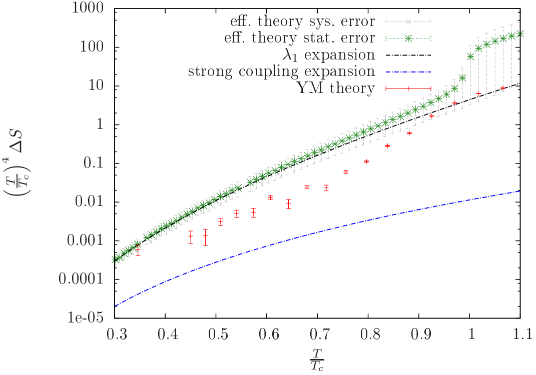



- identification $\lambda_c \leftrightarrow \beta_c$ restoration of rotational invariance in both theories
- string tension is overestimated!

# Yang-Mills thermodynamics from the effective theory

- can not compute $Z(T)$ directly, use derivatives
- $\Delta S(\beta) = \frac{N_t^4}{V}\left( \frac{d\log Z}{d\beta}\Big|_T - \frac{d\log Z}{d\beta}\Big|_{T=0}\right)$
- $\frac{f_r}{T^4}\Big|_{\beta_0}^{\beta} = -\int_{\beta_0}^{\beta} d\beta' \Delta S(\beta')$



# Conclusions of pure Yang-Mills results

- effective theory captures main features of the phase transition
- continuum results can be extrapolated from the effective theory
- some measurements depend on suppressed long range interactions – not handled precise enough in strong coupling approach ($\rightarrow$ can be improved)
- beside the phase transition, might be able to extract thermodynamic properties – especially in low temperature region

Effective theory nice tool to explore regions inaccessible by ordinary simulations, especially to investigate phase transitions.

# QCD on the lattice: fermions

$$\sum_{x,y} \bar\psi(x)(\mathrm{D}+m)_{x,y}\psi(y) = \sum_{x} \Big[(m+4r)\bar\psi(x)\psi(x)$$
$$-\frac{1}{2}\sum_{\mu}\bar\psi(x)\big((1-\gamma_\mu)U_\mu(x)\psi(x+\hat\mu)+(1+\gamma_\mu)U_\mu^\dagger(x-\hat\mu)\psi(x-\hat\mu)\big)\Big]$$
$$= C\sum_{x,y}\bar\psi(x)\Big[\delta_{x,y}-\kappa H\Big]\psi(y)$$

- derivatives $\rightarrow$ gauge invariant difference operators
- hopping parameter $\kappa = \frac{1}{2m+8}$
- naive expectation: $\kappa < 0.125$
- real simulations: $\kappa < \kappa_c(\beta)$, but still small

# Hopping parameter expansion

$$S_q = -\log\left[\prod_f \det(D-m)\right] = -N_f \mathrm{Tr}\log(1-\kappa H) = N_f \sum_l \frac{\kappa^l}{l}\mathrm{Tr} H^l$$

- expansion around $\kappa = 0$, infinitely heavy quarks
- $H$: spacial ($S = (1-\gamma_i)U_i$), temporal ($T = (1-\gamma_0)U_0$) hops
- expansion represented in terms of closed loops of hops
- effective action: integrating out spacial links in strong coupling expansion

$\Rightarrow$ expansion in $u$ and $\kappa$

# Static determinant

First contribution: no spacial hops!

$$\det(D-m) \approx \det(1+T^- + T^+)$$

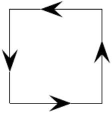
$$\det(1+T^- + T^+) = \prod_n (1+hL_n+h^2L_n^\dagger+h^3)^2(1+\bar{h}L_n^\dagger+\bar{h}^2L_n+\bar{h}^3)^2$$

- $\bar{h} = h = (2\kappa)^{N_t} +$ gauge corrections
- as expected: quarks break center symmetry!

# Higher orders: Let them propagate!

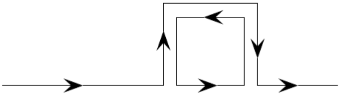
different contributions at higher order in $u$ and $\kappa$

- gluon-like contributions from quarks



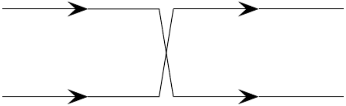
$\Rightarrow \lambda(\beta, \kappa)$ (shift of $\beta$)

- gluon modifications of quark lines



$\Rightarrow h(\kappa, \beta)$

- interaction between quarks

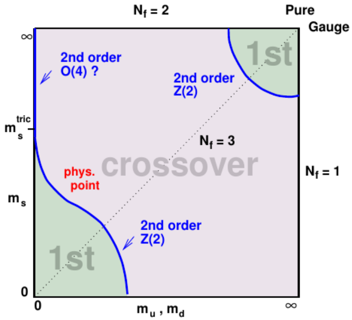


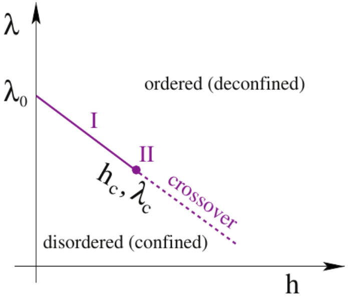
$\Rightarrow L_i L_j$ interaction

So far included: $\kappa^2, \kappa^4$ corrections

# Heavy quark QCD results

- reproduce phase transition in heavy quark limit




- mapping critical values of effective theory to QCD: around 5% error ($N_t = 4$)

# Lattice QCD and finite density

$$Z(T,\mu) = \mathrm{Tr}(e^{-(H-\mu Q)/T})$$

- continuum physics: extra term $\mu\bar{\psi}\gamma_0\psi$
- on the lattice modification of $\mathrm{D}$

$$\bar{\psi}(x)\big((r-\gamma_0)e^{a\mu}U_0(x)\psi(x+\hat{0}) + (r+\gamma_0)e^{-a\mu}U_0^\dagger(x-\hat{0})\psi(x-\hat{0})\big)$$

- $\gamma_5\mathrm{D}(\mu)^\dagger\gamma_5 = \mathrm{D}(-\mu^*) \Rightarrow \det(\mathrm{D}(\mu)) = \det(\mathrm{D}(-\mu^*))^*$
- complex measure $\Rightarrow$ lattice methods fail at large $\mu$

$\Rightarrow$ any information about the model at finite $\mu$ is helpful

$\Rightarrow$ need playground to test methods and find possible effects.

# Effective model at finite density

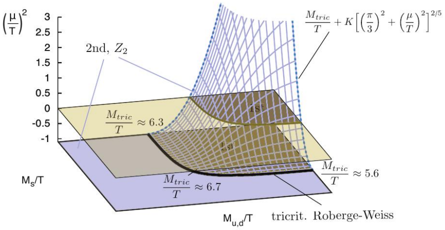
general form of the action

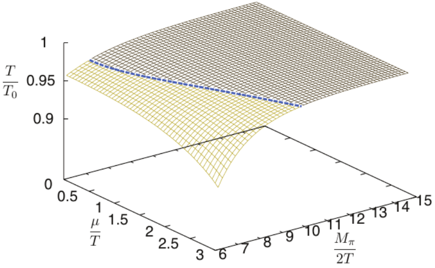
$$\sum_i \lambda_i S_{\text{center symm.}i} + \sum_i h_i S_{\text{as}i} + \sum_i \bar{h}_i S^\dagger_{\text{as}i}$$

- finite $\mu$ introduces factor $e^{\pm a\mu}$ for temporal up/down hops $\Rightarrow h \neq \bar{h}$
- $h(\mu) = \bar{h}(-\mu) \Rightarrow$ mild sign problem
- simple model: can cure the sign problem by reweighting
- alternative algorithm: complex Langevin (works!)

# Results with quark matter and finite density

Phase transition at finite densities

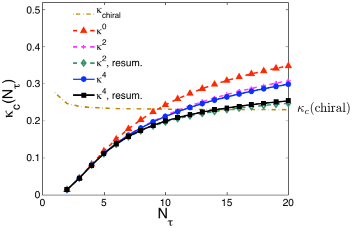



$N_f = 2,\ N_t = 6$

# Is QCD that simple?

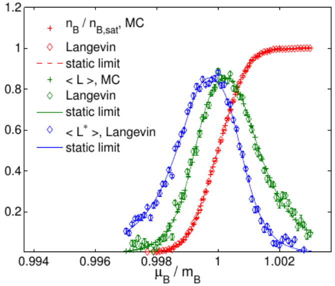
- Yang-Mills part: same limitations as in pure gauge theory
- additional limitation for QCD: truncation of hopping parameter expansion
- conf.-deconf. phase transition: limited to small $N_t$

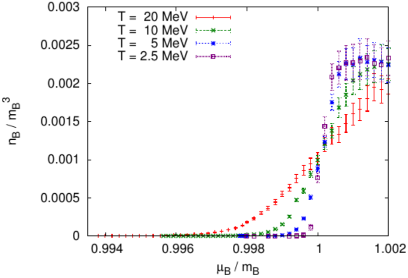



Gauge corrections $u(\beta)^{N_t} \Rightarrow$ small at low $T = \frac{1}{a(\beta)N_t}$!
Heavy quark region: $m_\pi \approx 20$ GeV

# Results with quark matter and finite density

Nuclear transition:




continuum extrapolation $\Rightarrow n_B \approx 0.16\text{fm}^{-3}$

# Conclusions and outlook

- systematic derivation of effective theory: spatial strong coupling expansion
- reproduces main features of QCD (phase transitions!)
- quarks are included in a hopping expansion
- phase transitions in the heavy-dense limit
- great progress towards higher orders in $\kappa \Rightarrow$ lower masses
- further improvement: corrections of long range interactions in pure gauge part